Math 366 (Joshua, Autumn, 2009): Test II.

Show sufficient details for full credit. Each problem is worth 40 points. Time 50 minutes

I. Are the following equalities of sets always true, given three subsets $A$, $B$ and $C$ of a given set $S$. If you think yes, the provide a complete proof. If you think no, provide a counter-example.

(i) $(A - B) - C = A - (B \cup C)$

$LHS = (A \cap B^c) \cap (B \cap C)^c = (A \cap B^c) \cap (B^c \cup C)$

$= (A \cap B^c \cap B^c) \cup (A \cap B^c \cap C) = (A \cap B^c) \cup (A \cap B^c \cap C) = A \cap B^c$

(ii) $(A - B) - (B - C) = A - B$
II. Which of the following numbers are prime? Justify your answer.

(i) $667 = 29 \cdot 23$ Therefore, it is not prime.

(ii) 557. This is prime.
III. Prove each of the following statements if they are true and if they are false give a counter example.

(i) For all real numbers $x$ and $y$, $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$

Take $x = 1.1$, $y = 1.2$. Then $\lceil x \rceil = 2$ and $\lceil y \rceil = 2$. So the RHS = 4. But $x.y = 1.32$, so that $\lceil x.y \rceil = 2$.

(ii) For all real numbers $x$, $\lceil x + 1 \rceil = \lceil x \rceil + 1$

Let $n = \lceil x \rceil$. Then $n - 1 < x \leq n$, so that $n < x + 1 \leq n + 1$. Therefore, $\lceil x + 1 \rceil = \lceil x \rceil + 1$. 
IV. (i) Prove using induction that $7^n - 1$ is divisible by 6 for each integer $n \geq 0$.

Let $P(n) : 7^n - 1$ is divisible by 6. Clearly $P(0)$ is true as 0 is divisible by 6. Next suppose $P(n)$ is true. Then $7^{n+1} - 1 = 7^n - 1 + 7^n = 7^n(7 - 1) + 7^n - 1$. By $P(n)$, $7^n - 1$ is divisible by 6. Therefore, $7^{n+1} - 1$ is also divisible by 6.

(ii) Suppose $e_0, e_1, \cdots, e_i, \cdots$ is a sequence defined by $e_0 = 1$, $e_1 = 2$, $e_2 = 3$ and $e_k = e_{k-1} = e_{k-2} + e_{k-3}$, for all $k \geq 3$. Prove that $e_n \leq 3^n$ for all integers $n \geq 0$.

Let $P(n) : e_n \leq 3^n$. Then $P(0)$ and $P(1)$ are true since $e_0 = 1 \leq 3^0 = 1$ and $e_1 = 2 \leq 3^1 = 3$. Next suppose $P(k)$ is true for all $k \leq n$. Then $e_{n+1} = e_n + e_{n-1} + e_{n-2} \leq 3^n + 3^{n-1} + 3^{n-2} \leq 3.3^n = 3^{n+1}$. Therefore, $P(n + 1)$ also holds.