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I. Derived categories. Let \((X, \mathcal{A})\) denote a ringed space with a sheaf of commutative rings \(\mathcal{A}\). Let \(\text{Mod}(X, \mathcal{A})\) denote the category of sheaves of \(\mathcal{A}\)-modules and let \(C^+(X, \mathcal{A})\) denote the category of bounded below complexes of sheaves of \(\mathcal{A}\)-modules. Let \(K^+(X, \mathcal{A})\) denote the corresponding homotopy category with the same objects, but where morphisms are chain homotopy classes of maps. Let \(D^+(X, \mathcal{A})\) denote the category with the same objects as \(K^+(X, \mathcal{A})\) but where a morphism \(K \rightarrow L\) is given by diagram \(K \leftarrow K' \rightarrow L\) with the map \(K \leftarrow K'\) a map that induces isomorphisms on cohomology sheaves.

(i) Show that this defines a category (i.e. there is a well-defined composition of morphisms which is associative.)

(ii) Show that the category of sheaves of \(\mathcal{A}\)-modules has enough injectives.

(iii) Let \(\mathcal{A}\) denote an Abelian category and let \(F: \text{Mod}(X, \mathcal{A}) \rightarrow \mathcal{A}\) denote a left-exact additive functor. Show that one can define the derived functor \(\mathcal{R}F: D^+(X, \mathcal{A}) \rightarrow D^+(\mathcal{A})\) as a functor so that \(\mathcal{R}^iG(\mathcal{F}(I)) = 0\) for all \(i > 0\) and \(\mathcal{R}^0F = F\). (Here \(D^+(\mathcal{A})\) is the category defined similar to \(D^+(X, \mathcal{A})\) starting with \(C^+(\mathcal{A})\) in the place of \(C^+(X, \mathcal{A})\).

II. Derived functors: Assume that \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{C}\) are Abelian categories and \(F: \mathcal{A} \rightarrow \mathcal{B}\), \(G: \mathcal{B} \rightarrow \mathcal{C}\) are left-exact functors. Assume that both \(\mathcal{A}\) and \(\mathcal{B}\) have enough injectives and that if \(I\) is an injective object of \(\mathcal{A}\), \(\mathcal{R}^iG(F(I)) = 0\) for all \(i > 0\). Then show that there is a natural isomorphism of functors \(\mathcal{R}G \circ \mathcal{R}F \simeq \mathcal{R}(G \circ F)\). (Hint: show that if \(K^\bullet\) is a bounded below complex in \(\mathcal{B}\) so that \(\mathcal{R}^iG(K^n) = 0\) for all \(i > 0\) and all \(n\), then for any quasi-isomorphism \(K^\bullet \rightarrow I^\bullet\) by a complex of injectives in \(\mathcal{B}\), the induced map \(G(K^\bullet) \rightarrow G(I^\bullet)\) is also a quasi-isomorphism. Here quasi-isomorphism between complexes means an isomorphism on taking the associated cohomology objects.)

III. Hartshorne Chapter III, 9.1.

IV. Hartshorne Chapter III, 9.3.