The Motivic Euler characteristics and the Motivic Segal-Becker Theorem

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Outline

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2 The Motivic trace and the Motivic Euler characteristic

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3 The Motivic Segal-Becker Theorem for Algebraic K-theory

The classical Becker-Gottlieb transfer

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- Key fact: $\chi_{\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})} = 1$, where G is a compact Lie group and $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ is the normalizer of a maximal torus in G.
- see J. Becker and D. Gottlieb, *The transfer map and fiber bundles*, Topology, **14**, (1975), 1-12.

The Motivic Analogue

- The motivic analogue is a conjecture due to Morel, that a suitable motivic Euler characteristic in the Grothendieck-Witt group is 1, for $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})$, G a split linear algebraic group. In the first part of the talk we sketch an affirmative solution to this conjecture assuming that the base field k contains a $\sqrt{-1}$.

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- In the second part of the talk we will apply the above result to discuss what we call *The Motivic Segal-Becker Theorem for Algebraic K-theory.*

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$$\Sigma^{\infty}_{\mathbf{T}} \xrightarrow{c} \Sigma^{\infty}_{\mathbf{T}}\mathrm{X}_+ \wedge \mathrm{D}(\Sigma^{\infty}_{\mathbf{T}}\mathrm{X}_+) \xrightarrow{\tau} \mathrm{D}(\Sigma^{\infty}_{\mathbf{T}}\mathrm{X}_+) \wedge \Sigma^{\infty}_{\mathbf{T}}\mathrm{X}_+$$
$$\xrightarrow{id \wedge f} \mathrm{D}(\Sigma^{\infty}_{\mathbf{T}}\mathrm{X}_+) \wedge \Sigma^{\infty}_{\mathbf{T}}\mathrm{X}_+ \xrightarrow{\mathrm{e}} \Sigma^{\infty}_{\mathbf{T}}.$$

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  In positive characteristic p, one has to invert p. When $f = id_{\mathrm{X}}$, the resulting trace is $\chi_{mot}(\mathrm{X})$: also denoted $\tau_{\mathrm{X}+}$.
- By Morel's work this identifies with a class in the Grothendieck-Witt group of the field k: $GW(k)$.

The conjecture of Morel (as stated by Levine)

- G: a split linear algebraic group over k, T: a split maximal torus and $\mathrm{N_G(T)}$ its normalizer in G. Then the conjecture states that $\chi_{mot}(\mathrm{G/N_G(T)}) = 1$ in $GW(k)$ with the characteristic exponent of the field inverted.

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- In fact, this is the *strong form* of the conjecture. The *weak form* of the conjecture is simply the statement that $\chi_{mot}(\mathrm{G/N_G(T)})$ is a *unit* in $GW(k)$ with the characteristic exponent of the field inverted.

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- see M. Levine, *Motivic Euler Characteristics and Witt-valued Characteristic classes*, Nagoya Math Journal,(2019).

Theorem 1 and Corollary 1

- **Theorem** (J-P: 2020). Assume G, $N_G(T)$ as above. Then $\chi_{mot}(G/N_G(T)) = 1$ in $GW(k)$ with the characteristic exponent of the field inverted assuming k has a square root of -1.

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- In A: 2021, A provided an idependent proof of the weak-form of the conjecture. Shortly thereafter, J-P provided a means to deduce the weak-form of the conjecture from the strong form, at least in many cases, thereby also simplifying the arguments in A's proof.
- **Corollary** (A, J-P: 2021). Assume G, $N_G(T)$ as above. Then $\chi_{mot}(G/N_G(T))$ is a unit in $GW(k)$ with the characteristic exponent of the field inverted, with no further assumptions on k.

The above papers

- R. Joshua and P. Pelaez, *Additivity and Double coset formulae for the Motivic and Étale Becker-Gottlieb transfer*, Preprint, (2020).

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- R. Joshua and P. Pelaez, *Additivity and Double coset formulae for the Motivic and Étale Becker-Gottlieb transfer*, Preprint, (2020).
- A. Ananyevskiy, *On the $\mathbb{A}^1$-Euler Characteristic of the variety of maximal tori in a reductive group*, arXiv:2011.14613v2 [math.AG] 24 May 2021.

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- The second part is very geometric and depends on a decomposition of the scheme $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})$.

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$$\tau_{\mathrm{X/U}} = \tau_{\mathrm{X_1/U_1}} + \tau_{\mathrm{X_2/U_2}} - \tau_{(\mathrm{X_1 \cap X_2})/(\mathrm{U_1 \cap U_2})} \, in \, \mathcal{SH}(k)$$

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  in char 0 and after inverting the characteristic p, in case $p > 0$.
- $i : \mathrm{Z} \rightarrow \mathrm{X}$ a closed immersion of smooth schemes with $j : \mathrm{U} \rightarrow \mathrm{X}$ denoting the corresponding open complement. $\mathcal{N}$: the normal bundle associated to i, $\mathrm{Th}(\mathcal{N})$: its Thom-space. Then, in $\mathcal{SH}(k)$ when $char(k) = 0$:

  $\tau_{\mathrm{X}+} = \tau_{\mathrm{U}+} + \tau_{\mathrm{X/U}}$, and assuming $\sqrt{-1} \, in \, k, \tau_{\mathrm{X/U}} = \tau_{\mathrm{Th}(\mathcal{N})} = \tau_{\mathrm{Z}+}$.

Outline of the proof of Theorem 1: III: **Theorem 2** continued

- $\{S_\alpha | \alpha\}$: a stratification of the smooth scheme X into finitely many locally closed and smooth subschemes S_α. Then assuming $\sqrt{-1} \in k$, we obtain in $\mathcal{SH}(k)$ when $char(k) = 0$:

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- We will not discuss the proof of the Mayer-Vietoris statement nor the proof of the first statement of additivity in any detail: these follow largely the proof of the corresponding statements in algebraic topology, such as discussed in [LMS].
- Additivity (the second statement). The only new aspect is the need for the $\sqrt{-1}$ in k: makes use of the multiplicative property of the trace as discussed next.

Outline of the proof of Theorem 1: III: Multiplicative property of the trace

- Assume $\mathrm{F_i}$, $i = 1, 2$ are simplicial presheaves, $\mathrm{F_2}$ pointed and let $f_i : \mathrm{F_i} \rightarrow \mathrm{F_i}$, $i = 1, 2$ denote a given map.

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$$\tau_{\mathrm{F}}(f) = \tau_{\mathrm{F}_{1+}}(f_{1+}) \wedge \tau_{\mathrm{F}_2}(f_2).$$

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- Then
$$\tau_{\mathrm{F}}(f) = \tau_{\mathrm{F}_{1+}}(f_{1+}) \wedge \tau_{\mathrm{F}_2}(f_2).$$
- By making use of this property, and the Mayer-Vietoris property, one reduces to the case when the normal bundle to Z in X is trivial. So $\tau_{\mathrm{X/U}} \simeq \tau_{\mathrm{Z}+} \wedge \tau_{\mathbf{T}^c}$. Then the assumption that k contains $\sqrt{-1}$, shows the class of $\tau_{\mathbf{T}}$ identifies with 1 in $GW(k)$.

Outline of the Proof of Theorem 1: IV: Decomposition of the variety $G/N_G(T)$

Proposition

(See [Proposition 4.10, Th86] or [(3.6), BP].) T: *a split torus acting on a smooth scheme* X, *all defined over the perfect base field* k. *Then the following hold.*

X *admits a decomposition into a disjoint union of finitely many locally closed,* T*-stable subschemes* X_j *so that*

$$X_j \cong (T/\Gamma_j) \times Y_j.$$

Each Γ_j: *a sub-group-scheme of* T, *each* Y_j: *is regular and on which* T *acts trivially with the above isomorphism being* T*-equivariant.*

Outline of the Proof of Theorem 1: IV: Decomposition of the variety $G/N_G(T)$

One may derive this from the generic torus slice theorem proved in [Proposition 4.10, Th86], which says: T: a split torus acting on a reduced separated scheme X of finite type over a perfect field, then the following are satisfied:

- there is an open subscheme U which is regular and stable under the T-action

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- there is an open subscheme U which is regular and stable under the T-action
- a geometric quotient U/T exists, which is a regular scheme of finite type over k
- U is isomorphic as a T-scheme to $T/\Gamma \times U/T$ where Γ is a diagonalizable subgroup scheme of T and T acts trivially on U/T.

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The above references

- R. Thomason, *Comparison of equivariant algebraic and topological K-theory*, Duke Math. J. 53 (1986), no. 3, 795-825.

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- R. Thomason, *Comparison of equivariant algebraic and topological K-theory*, Duke Math. J. 53 (1986), no. 3, 795-825.
- M. Brion and E. Peyre, *Counting points of Homogeneous varieties over finite fields*, Journal für die reine und angewandte Mathematik (Crelle's Journal), **645**, (2010), 105-124.

Outline of the Proof of Theorem 1: IV: Theorem 3

Theorem 3 Assume k is perfect and contains a $\sqrt{-1}$ and $char(k) = 0$. The the following hold:

- $\tau_{\mathbb{G}_m+} = 0$ in $GW(k)$ and if T is a split torus, $\tau_{\mathrm{T}+} = 0$ in $GW(k)$. (Remark: compare with $\chi_{\mathrm{S}^1} = 0$.)

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- T: a split torus acting on a smooth scheme X. Then X^{T} is also smooth, and $\tau_{\mathrm{X}+} = \tau_{\mathrm{X}^{\mathrm{T}}+}$ in $GW(k)$.

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- T: a split torus acting on a smooth scheme X. Then X^{T} is also smooth, and $\tau_{\mathrm{X}+} = \tau_{\mathrm{X}^{\mathrm{T}}+}$ in $GW(k)$.
- If $char(k) = p > 0$, the corresponding assertions hold after inverting p.

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- But by $\mathbb{A}^1$-contractibility, $\tau_{\mathbb{A}^1+} = \tau_{\{0\}+}$. These prove $\tau_{\mathbb{G}_m+} = 0$ in $GW(k)$.
- The multiplicative property of the trace now shows $\tau_{\mathrm{T}+} = 0$ also. These prove (i).

Outline of the Proof of Theorem 1: IV: Proof of Theorem 3 (contd)

- X^T is the disjoint union of the subschemes X_j for which $\Gamma_j = T$.

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- X^{T} is the disjoint union of the subschemes $\mathrm{X_j}$ for which $\Gamma_j = \mathrm{T}$.
- $\tau_{\mathrm{X}+} = \Sigma_j \tau_{\mathrm{X_j}+} = \Sigma_j (\tau_{\mathrm{T}/\Gamma_{\mathrm{j}}+}) \wedge \tau_{\mathrm{Y_{j}}+}$.

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- Now statement (i) in the theorem shows that the j-th summand on the right-hand-side is trivial unless $\Gamma_j = \mathrm{T}$. But, then $\mathrm{X^T}$ is the disjoint union of such $\mathrm{X_j}$.

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- Now statement (i) in the theorem shows that the j-th summand on the right-hand-side is trivial unless $\Gamma_j = \mathrm{T}$. But, then X^{T} is the disjoint union of such $\mathrm{X_j}$.
- Finally the additivity of the the trace applied once more to X^{T} proves the sum of the non-trivial terms on the right-hand-side is $\tau_{\mathrm{X^T}+}$.

Outline of the Proof of Theorem 1

- Can reduce to the case G is connected as follows:

$$G/N_G(T) = \{gTg^{-1} | g \epsilon G\}$$

$$\cong \{g_o T g_o^{-1} | g_o \epsilon G^o\} = G^o/N_{G^o}(T).$$

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- $G/N_G(T)$: the variety of all split maximal tori in G, and T has an action on $G/N_G(T)$ (induced by the left translation action of T on G).

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- Since $\mathrm{G/N_G(T)} \cong \mathrm{G_{red}/N_{G_{red}}(T)}$, we may also assume G is reductive.
- $\mathrm{G/N_G(T)}$: the variety of all split maximal tori in G, and T has an action on $\mathrm{G/N_G(T)}$ (induced by the left translation action of T on G).
- Therefore, there is exactly a single fixed point, namely the coset $e\mathrm{N_G(T)}$, that is, $(\mathrm{G/N_G(T)})^{\mathrm{T}} = \{\mathrm{eN_G(T)}\} = \{\mathrm{Spec\,k}\}$.

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- Therefore, $\tau_{\mathrm{G/N_G(T)+}} = \tau_{\mathrm{(G/N_G(T))^T+}} = \tau_{\mathrm{Spec}\,k+} = id_{\Sigma^\infty_{\mathbf{T}}}$, and

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- $\chi_{mot}(\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})) = 1$.
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- $\chi_{mot}(\mathrm{G/N_G(T)}) = 1$.
- In positive characteristic p, the corresponding statement holds after inverting p.
- These complete the proof of the strong form of the conjecture.
- Brion and Peyre (see [BP]) have a proof in étale cohomology for the ordinary Euler characteristic. That, and discussion with Brion greatly helped us prove the conjecture.

Outline of the Proof of Corollary 1

Assume $char(k) = p > 0$ or k is *not* formally real.

- Let $\bar{k}$ denote the alg. closure of k. Clearly $\bar{k}$ has a $\sqrt{-1}$.

  Then we consider:
$$\begin{array}{ccc} GW(\bar{k})[p^{-1}] & \xrightarrow[\cong]{rk} & \mathbb{Z}[p^{-1}] \\ \uparrow & & \uparrow id \\ GW(k)[p^{-1}] & \xrightarrow{rk} & \mathbb{Z}[p^{-1}]. \end{array}$$

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- When k is formally real, one also needs to show the signature of the corresponding quadratic form is a unit.

Theorem 4: Splittings in generalized motivic cohomology theories:I

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- Then,

$$q^* : h^{\bullet,*}(E \underset{G}{\times} Y, M) \rightarrow h^{\bullet,*}(E \underset{G}{\times} (G \underset{N_G(T)}{\times} Y), M)$$

is also a split injection, where $h^{\bullet,*}(\quad, M)$ is a generalized motivic cohomology with respect M.

Theorem 4: Splittings in generalized motivic cohomology theories: idea of the proof

- In the above setting, one has a transfer map

$$tr : \Sigma^{\infty}_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \to \Sigma^{\infty}_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{G} \underset{\mathrm{N_G(T)}}{\times} \mathrm{Y}))_+$$

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- see G. Carlsson and R. Joshua, *Motivic and Étale Spanier-Whitehead duality and the Becker-Gottlieb transfer*, Preprint, (2020).

Examples of the torsors

- $\mathrm{B} = \mathrm{BG}^{gm,m}$= a degree m-approximation to the *geometric classifying space* of G and $\mathrm{E} = \mathrm{EG}^{gm,m}$= the corresponding universal principal G-bundle over $\mathrm{BG}^{gm,m}$.

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- B, E as above, with $\mathrm{Y} = Speck$. Now the transfer $tr : \Sigma_{\mathbf{T}}\mathrm{BG}_+ \rightarrow \Sigma_{\mathbf{T}}\mathrm{BN_G(T)}_+$.

The classical Segal-Becker theorem

- A classical result due to Graeme Segal from the early 1970s is a theorem that shows the classifying space of the infinite unitary group, namely BU, is a split summand of $\lim_{m\to\infty} \Omega^m_{S^1}((S^1)^m \wedge \mathbb{CP}^\infty)$.

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- see G. Segal, *The stable homotopy of complex projective space*, Quart. Jour. Math, **24**,(1973) 1-5.

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- J. Becker, *Characteristic Classes and K-theory*, Lect. Notes in Math., 428 *Algebraic and Geometrical methods in Topology*, (1974)), 132-143.
- Becker's proof makes strong use of the transfer and also provides a separate proof for the infinite unitary group.

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- For a pointed motivic space P, $\mathrm{Q(P)} = \lim_{n\rightarrow\infty} \Omega^n_{\mathbf{T}} \mathbf{T}^{\wedge n}(\mathrm{P})$.
- There is an $\Omega_{\mathbf{T}}$-motivic spectrum whose 0-th term is given by the motivic space BGL_∞. In fact, just take $f_1(\mathbf{K})$. (See [Lemma 2.2, VV01] and [Theorem 7.5.1, Lev08].)

The framework for the Motivic Segal-Becker theorem:II

- Of key importance for us is the following map:

$$\lambda : \mathrm{Q}(\mathrm{B}\mathbb{G}_{\mathrm{m}}) \to \mathrm{Q}(\varinjlim_{\mathrm{n}} \mathrm{BGL}_{\mathrm{n}}) = \mathrm{Q}(\mathrm{BGL}_{\infty}) \xrightarrow{\mathrm{q}} \mathrm{BGL}_{\infty},$$

The framework for the Motivic Segal-Becker theorem:II

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- The map q is the obvious one induced by the fact that BGL_∞ is the 0-th space of an $\Omega_{\mathbf{T}}$-spectrum. The map

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is induced by the inclusion,

$$\mathbb{G}_{\mathrm{m}} \to \mathrm{GL}_n \to \mathrm{GL}_\infty,$$

where the first map is the diagonal imbedding.

The Motivic Segal-Becker theorem for Algebraic K-Theory; Theorem 5

- Assume $chark = 0$. Then the map λ induces a surjection for every pointed motivic space X that is a compact object in the unstable pointed motivic homotopy category:

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- For a pointed motivic space P, $\mathrm{Q(P)} = \lim_{\mathrm{n}\to\infty} \Omega^{\mathrm{n}}_{\mathbf{T}}(\mathbf{T}^{\wedge \mathrm{n}}(\mathrm{P}))$.
- Let $\mathrm{p_n} : \mathrm{BN_{GL_n}(T_n)} \to \lim_{\to \mathrm{n}} \mathrm{BN_{GL_n}(T_n)} \to \lim_{\to \mathrm{n}} \mathrm{BGL_n} = \mathrm{BGL_\infty}$ denote the map induced by the inclusion of $\mathrm{N_{GL_n}(T_n)}$ in GL_n. Let $\mathrm{p} = \lim_{\to \mathrm{n}} \mathrm{p_n}$.

The Motivic Segal-Becker theorem for Algebraic K-Theory: Proof: Step 1.2

Proposition

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Proposition

- *Assume* $char(k) = 0$*. Let* $\bar{q}_n = q \circ Q(p_n)$ *and* $\bar{q} = \lim_{\to n} \bar{q}_n$*. Then the map*

$$\bar{q} = q \circ \mathrm{Q(p)} : \mathrm{Q(BN_{GL_\infty}(T))} = \mathrm{Q}(\lim_{\to \mathrm{n}} \mathrm{BN_{GL_n}(T_n)})$$

$$\to \mathrm{Q}(\lim_{\to \mathrm{n}} \mathrm{BGL_n}) = \mathrm{Q(BGL_\infty)} \xrightarrow{\mathrm{q}} \mathrm{BGL_\infty}$$

induces a surjection for every compact object X*:*

$$[\mathrm{X}, \mathrm{Q}(\lim_{\to \mathrm{n}} \mathrm{BN_{GL_n}(T_n)})] \to [\mathrm{X}, \mathrm{BGL_\infty}].$$

The Motivic Segal-Becker theorem for Algebraic K-Theory: Proof: Step 1.2 (contd)

Proposition

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- *If $char(k) = p > 0$ and k is perfect, then the same conclusion holds after inverting the prime p.*

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- Clearly the map $\bar{q}$ provides a map of the corresponding spectra:

$$\Sigma^{\infty}_{\mathbf{T}}(BN_{GL_{\infty}}(T)) = \Sigma^{\infty}_{\mathbf{T}}(\lim_{n \to \infty} BN_{GL_n}(T_n)) \to \tilde{\mathbf{K}},$$

where $\tilde{\mathbf{K}}$ is the motivic $\Omega_{\mathbf{T}}$-spectrum whose 0-th space is BGL_{∞}.

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- Let ϕ denote the above map.
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- The proof makes strong use of the splitting provided by the transfer and the change of base points for $h^{*,\bullet}$ as on the next slide.

The Motivic Segal-Becker theorem for Algebraic K-Theory: Changing base points

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- Then the diagram

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- The right vertical map and the horizontal maps are all split monomorphisms. Therefore, the left vertical map is also a monomorphism.

The Motivic Segal-Becker theorem for Algebraic K-Theory: Proposition 2

Let $\lambda : \mathrm{Q}(\mathrm{B}\mathbb{G}_{\mathrm{m}}) \rightarrow \mathrm{Q}(\mathrm{BGL}_{\infty}) \rightarrow \mathrm{BGL}_{\infty}$.

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- *Assume char(k) = 0. X: any pointed motivic space. Then, there exists a map* $\zeta : \mathrm{Q}(\mathrm{BN}_{\mathrm{GL}_\infty}(\mathrm{T})) \to \mathrm{Q}(\mathrm{B}\mathbb{G}_{\mathrm{m}}) = \mathrm{Q}(\mathbb{P}^\infty)$, *making the triangle commute:*

$$\begin{array}{ccc} [\mathrm{X}, \mathrm{Q}(\mathrm{BN}_{\mathrm{GL}_\infty}(\mathrm{T}))] & \xrightarrow{\zeta_*} & [\mathrm{X}, \mathrm{Q}(\mathrm{B}\mathbb{G}_{\mathrm{m}})] \\ & \searrow^{\bar{q}_*} & \downarrow{\scriptstyle \lambda_*} \\ & & [\mathrm{X}, \mathrm{BGL}_\infty] \end{array}$$

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$$\mathrm{B\widetilde{N_{GL_{\mathit{n}}}(T_n)}^{gm,n}} = \mathrm{St_{2n,n}}/(\mathbb{G}_\mathrm{m} \times \mathrm{N_{GL_{n-1}}(T_{n-1})}).$$

- Now $\mathbb{G}_m \times \mathrm{N_{GL_{n-1}}(T_{n-1})}$ is a subgroup of index n in $\mathrm{N_{GL_n}(T_n)}$, so that the projection $r_n : \mathrm{B\widetilde{N_{GL_{\mathit{n}}}(T_n)}^{gm,n}} \rightarrow \mathrm{BN_{GL_{\mathit{n}}}(T_n)^{gm,n}}$ is a finite étale cover of degree n.

The Motivic Segal-Becker theorem for Algebraic K-Theory: Proof

- The map $\mathrm{St_{2n,n}} \to \mathrm{St_{2n,1}}$ sends an n-frame $(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ to $\mathbf{v}_1$. This factors through the quotient of $\mathrm{St_{2n,n}}/1 \times \mathrm{GL_{n-1}}$, where GL_{n-1} acts only on the last $n-1$-vectors in the n-frame $(\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{v}_n)$. Therefore, we obtain the map

$$\phi_n : \mathrm{St_{2n,n}}/(1 \times \mathrm{GL_{n-1}}) \to \mathrm{St_{2n,1}}.$$

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$$\mathrm{B}\mathbb{G}_\mathrm{m}^{\mathrm{gm,n}} = \mathrm{St_{2n,1}}/\mathbb{G}_\mathrm{m}.$$

The Motivic Segal-Becker theorem for Algebraic K-Theory: Proof

- $$u_n : \mathrm{BN}_{\widetilde{\mathrm{GL}_n}}(\mathrm{T_n})^{\mathrm{gm,n}} = \mathrm{St_{2n,n}}/(\mathbb{G}_\mathrm{m} \times \mathrm{N_{GL_{n-1}}}(\mathrm{T_{n-1}}))$$
$$\to \mathrm{St_{2n,n}}/(\mathbb{G}_\mathrm{m} \times \mathrm{GL_{n-1}}), \text{ and}$$
$$\bar{u}_n = \bar{\phi}_n \circ u_n : \mathrm{BN}_{\widetilde{\mathrm{GL}_n}}(\mathrm{T_n})^{\mathrm{gm,n}} = \mathrm{St_{2n,n}}/(\mathbb{G}_\mathrm{m} \times \mathrm{N_{GL_{n-1}}}(\mathrm{T_{n-1}}))$$
$$\to \mathrm{St_{2n,1}}/\mathbb{G}_\mathrm{m} = \mathrm{B}\mathbb{G}_\mathrm{m}^{\mathrm{gm,n}}.$$

The Motivic Segal-Becker theorem for Algebraic K-Theory: Proof

- $$u_n : \mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(\mathrm{T_n})}^{\mathrm{gm,n}} = \mathrm{St_{2n,n}}/(\mathbb{G}_\mathrm{m} \times \mathrm{N_{GL_{n-1}}}(\mathrm{T_{n-1}}))$$
$$\rightarrow \mathrm{St_{2n,n}}/(\mathbb{G}_\mathrm{m} \times \mathrm{GL_{n-1}}), \text{ and}$$
$$\bar{u}_n = \bar{\phi}_n \circ u_n : \mathrm{B}\widetilde{\mathrm{N}_{\mathrm{GL}_n}(\mathrm{T_n})}^{\mathrm{gm,n}} = \mathrm{St_{2n,n}}/(\mathbb{G}_\mathrm{m} \times \mathrm{N_{GL_{n-1}}}(\mathrm{T_{n-1}}))$$
$$\rightarrow \mathrm{St_{2n,1}}/\mathbb{G}_\mathrm{m} = \mathrm{B}\mathbb{G}_\mathrm{m}^{\mathrm{gm,n}}.$$
- These maps are compatible as n increases.

The Motivic Segal-Becker theorem for Algebraic K-Theory: Proof

- Recall

$$r_n : \mathrm{BN}_{\widetilde{\mathrm{GL}_n}(\mathrm{T_n})}^{\mathrm{gm,n}} = \mathrm{St_{2n,n}}/(\mathbb{G}_\mathrm{m} \times \mathrm{N_{GL_{n-1}}}(\mathrm{T_{n-1}}))$$

$$\rightarrow \mathrm{St_{2n,n}}/\mathrm{N_{GL_n}}(\mathrm{T_n}) = \mathrm{BN_{GL_n}}(\mathrm{T_n})^{\mathrm{gm,n}}$$

is finite étale of degree n. So let

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  is finite étale of degree n. So let

- $$\tau_n : \Sigma^{\infty}_{\mathbf{T}} \mathrm{BN}_{\mathrm{GL}_n}(\mathrm{T_n})^{\mathrm{gm,n}}_{+} \rightarrow \Sigma^{\infty}_{\mathbf{T}} (\mathrm{BN}_{\widetilde{\mathrm{GL_n}}(\mathrm{T_n})}^{\mathrm{gm,n}})_{+}$$

  denote the corresponding transfer.

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  denote the corresponding transfer.

- We have not yet defined this transfer, but will be defined shortly.

The Motivic Segal-Becker theorem for Algebraic K-Theory: Proof

- We let

$$\zeta_n : \Sigma^\infty_{\mathbf{T}} \mathrm{BN}_{\mathrm{GL}_n}(\mathrm{T_n})^{\mathrm{gm,n}}_+ \overset{\tau_{\mathrm{n}}}{\to} \Sigma^\infty_{\mathbf{T}} (\widetilde{\mathrm{BN}_{\mathrm{GL_n}}(\mathrm{T_n})^{\mathrm{gm,n}}})_+$$

$$\overset{\pi_n}{\to} \Sigma^\infty_{\mathbf{T}} \mathrm{B}\mathbb{G}^{\mathrm{gm,n}}_{\mathrm{m}} \overset{\mathrm{j_n}}{\to} \Sigma^\infty_{\mathbf{T}} \mathrm{B}\mathbb{G}_{\mathrm{m}}$$

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- The map π_n is the composition of the map $\Sigma^{\infty}_{\mathbf{T}} \bar{u}_{n+}$ followed by the map that sends the base point $+$ to the base point of $\mathrm{B}\mathbb{G}_{\mathrm{m}}^{\mathrm{gm,n}}$.

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- The maps $\{\zeta_n | n\}$ are compatible as n varies.
- To show the resulting triangle commutes, one needs to show that the pull-back by τ_n agrees with pushforward for finite étale maps in K-theory.

The Motivic Segal-Becker theorem for Algebraic K-Theory: transfers for projective smooth maps and Gysin maps

- It takes as much effort to define transfers for finite étale maps and establish the above property, as defining the transfer for all projective smooth maps and establish the required properties for them.

The Motivic Segal-Becker theorem for Algebraic K-Theory: transfers for projective smooth maps and Gysin maps

- It takes as much effort to define transfers for finite étale maps and establish the above property, as defining the transfer for all projective smooth maps and establish the required properties for them.
- Here $\mathrm{p} : \mathrm{E} \rightarrow \mathrm{B}$: a smooth fiber bundle between compact manifolds E and B. Then one may obtain a closed imbedding of E in $\mathrm{B} \times \mathbb{R}^{\mathrm{N}}$ for N sufficiently large. We will denote this imbedding by i. Therefore,

$$\mathrm{TP} : \mathrm{B}_{+} \wedge \mathrm{S}^{\mathrm{N}} \rightarrow \mathrm{Th}(\nu)$$

The Motivic Segal-Becker theorem for Algebraic K-Theory: transfers for projective smooth maps and Gysin maps

- E and B are quasi-projective varieties,

$$i : \mathrm{E} \to \mathrm{B} \times \mathbb{P}^{\mathrm{N}}$$

for a large enough N. Therefore, [Proposition 2.7, Lemma 2.10 and Theorem 2.11, VV03] provides the Voevodsky collapse

$$\mathrm{V} : \mathrm{B}_+ \wedge \mathbf{T}^{\mathrm{n}} \to \mathrm{Th}(\nu)$$

for a suitably large n, and where ν the *virtual normal bundle.*

The Motivic Segal-Becker theorem for Algebraic K-Theory: transfers for projective smooth maps and Gysin maps

- $\tau = \tau_{\mathrm{E/B}}$ denote the relative tangent bundle associated to $\mathrm{p} : \mathrm{E} \rightarrow \mathrm{B}$. Assume the relative dimension of p is d. Then it follows from [Proposition 2.7 through Theorem 2.11, Voev] that $\nu \oplus \tau$ is a trivial bundle on pull-back to $\widetilde{\mathrm{E}}$, where $\widetilde{\mathrm{E}}$ is a (functorial) affine replacement of E provided by the technique of Jouanolou.

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- Therefore, we may define the Becker-Gottlieb transfer in the above situation as follows:

$$tr : \mathrm{B}_+ \wedge \mathbf{T}^{\mathrm{n}} \overset{\mathrm{V}}{\to} \mathrm{Th}(\nu) \overset{\mathrm{i}_\nu}{\to} \mathrm{Th}(\nu \oplus \tau) \simeq \mathrm{E}_+ \wedge \mathbf{T}^{\mathrm{n}}$$

where i_ν is the map induced by the obvious inclusion $\nu \to \nu \oplus \tau$.

The Motivic Segal-Becker theorem for Algebraic K-Theory: transfers for projective smooth maps and Gysin maps

- $\mathrm{h}^{*,\bullet}$: a generalized motivic cohomology which is orientable. tr: the transfer as above. Then if $eu(\tau)$ denotes the Euler class of the bundle τ, we obtain the relation:

$$tr^*(\alpha) = \mathrm{p}_*(\alpha \cup \mathrm{eu}(\tau)), \alpha \varepsilon \mathrm{h}^{*,\bullet}(\mathrm{E})$$

where p_* denotes a Gysin map.

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  where p_* denotes a Gysin map.
- This Gysin map may defined using the Voevodsky collapse (as in the case of the Thom-Pontrjagin collapse) and then show it agrees with Gysin maps defined by other means

The Motivic Segal-Becker theorem for Algebraic K-Theory: transfers for projective smooth maps and Gysin maps

- If p : E → B denote a finite étale map between smooth quasi-projective schemes.

$$tr^* = \mathrm{p}_*$$

where tr^* denotes the map induced by the motivic Becker-Gottlieb transfer tr in the above cohomology theory and p_* denotes the Gysin map. Moreover, for Algebraic K-Theory, the Gysin map p_* agrees with the finite pushforward defined for coherent sheaves.

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