MOTIVIC SPANIER-WHITEHEAD DUALITY AND MOTIVIC BECKER-GOTTLIEB TRANSFER

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Abstract. In this paper, we develop a theory of Spanier-Whitehead duality in the context of motivic homotopy theory. Notable among the applications of this theory is a variant of the classical Becker-Gottlieb transfer in the framework of motivic homotopy theory, with several potential applications, and which will be dealt with in detail in a sequel. A variant of this theory in the context of étale homotopy theory was already developed by the second author several years ago: we will explore the connections between these two theories as well.

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1. Introduction

Spanier-Whitehead duality in algebraic topology is a classical result formulated and established by E. H. Spanier and J. H. C. Whitehead in the 1950s (see [SpWh55], [SpWh58] and [Sp59]): it was shown there that finite CW complexes have dual complexes if one works in the stable category. This lead to the theory of spectra and much of stable homotopy theory followed. In the 60s, Atiyah (see [At]) showed that the Thom-spaces of the normal bundles associated to the imbedding of compact C^{∞}-manifolds in high dimensional Euclidean spaces provided a Spanier-Whitehead dual for the manifold. A key application of this classical Spanier-Whitehead duality is the notion of a transfer map for fibrations which need not be covering spaces, due to Becker and Gottlieb, see [BG76]. The transfer turned out to be a versatile tool in algebraic topology: see for example, [BG75], [Seg] and [Sn].

Though the homotopy theory of algebraic varieties in the context of motives and algebraic cycles started only with the work of Voevodsky and Morel (see [MV]), a closely related theory that only considers algebraic varieties from the point of view of the étale topology has been in existence for over 40 years starting with [AM]. David Cox in his thesis (see [Cox]) and Eric Friedlander (see [Fr]) developed the very important notion of étale tubular neighborhoods in the context of étale homotopy. The second author's Ph. D thesis (see [JT], [J86], [J87]) developed the theory of Spanier-Whitehead duality in the context of étale homotopy theory, following upon the work of Cox and Friedlander. He also used this to construct a transfer map as a map of stable étale homotopy types for proper smooth maps between algebraic varieties over algebraically closed fields.

In recent years there has been renewed interest in the homotopy theory of algebraic varieties due to the work of Voevodsky (see [Voev], [MV]) on the Milnor conjecture which introduced several new techniques and the framework of motivic homotopy theory as in [MV]. It is therefore, natural to ask if a suitable theory of Spanier-Whitehead duality exists in the framework of motivic homotopy theory and if it could be used to construct an analogue of classical Becker-Gottlieb transfer. The first author meanwhile has been interested in descent questions for algebraic K-theory and formulated a possible approach to understanding these questions using a motivic variant of the Becker-Gottlieb transfer.

The present paper is the second in a series of papers devoted to exploring these descent questions for algebraic K-theory. In [CJ1], we had already built the frame-work of equivariant stable motivic homotopy theory: all our results in this paper and subsequent ones will be stated in this framework. In the present paper, we establish a general theory of Spanier-Whitehead duality and apply this to produce a Becker-Gottlieb type transfer for certain classes of algebraic varieties which are closely related to the linear varieties studied in [Tot] and [J01]. A general framework for constructing the Becker-Gottlieb transfer using a variant of Spanier-Whitehead duality was discussed in [DP] long before stable motivic homotopy theory was invented. A key idea needed here is the notion of objects that are finite in a suitable sense so that they are *strongly dualizable.*

A theory of Atiyah-style duality in the motivic context already appears in [Voev, Proposition 2.7]. However, this requires the schemes considered be projective and smooth and therefore does not apply to schemes of the form $G/N(T)$, where G is a reductive group and $N(T)$ the normalizer of a maximal torus in G. We prove the required finiteness by first introducing the notion of *stable motivic homotopy type* for schemes in general and then by showing that the stable motivic homotopy types of linear schemes and schemes like $G/N(T)$ after completion away from the residue characteristic are closely related to the completion of the corresponding étale topological types. We then observe that these completed stable étale homotopy types are finite.

We will adopt the following terminology throughout.

- We will restrict to separated schemes of finite type over a base field k_0. For such a scheme X, X_{mht} (X_{eht}) will denote the pro-object of simplicial sheaves given by the rigid hypercoverings of X in the Nisnevich topology (étale topology, respectively). X_{et} will denote the usual *étale topological type* which is obtained from X_{eht} by applying the connected component functor. Then $\Sigma_{\bar{\mathbb{P}}^1_{et}} \bar{X}_{et,+}$, $\Sigma_{S^2} \bar{X}_{et,+}$ will denote the stable étale topological types defined with respect to the Galois-equivariant suspension spectra given by $\Sigma_{\bar{\mathbb{P}}^1_{et}}$ and Σ_{S^2} as defined in 2.1: see also 3.4. $\Sigma_{\mathbb{P}^1} X_{mht,+}$ will denote the corresponding Galois-equivariant stable motivic homotopy type defined in section 3.
- Let l denote a fixed prime different from $char(k_0)$ and let Z/l_{∞} and $\widetilde{Z/l}_{\infty}$ denote the Z/l-completions in the sense of [CJ1, section 4]. h^{Nis} (h^{et}) will denote the generalized (equivariant) cohomology spectrum computed on the Nisnevich site (étale site, respectively) as defined in section 5 and H denotes a subgroup of Gal of finite index. h_{Nis} and h_{et} will denote the corresponding generalized (equivariant) homology spectra.

One of our main results may now be summarized in the following theorem. The version of the theorem given below is stated in the context of equivariant stable homotopy theory, equivariant with respect to the action of the Galois group Gal of an algebraic closure $\bar{k}$ of the given field k_0: see section 2. For several applications it suffices to consider the non-equivariant version of this theorem. Therefore, the reader may omit the Galois group actions (and hence the subgroup H), at least on first reading. Moreover, even when allowing the Galois group action, we let it act only on the suspension coordinates as explained in Example 2.1.

Theorem 1.1. *(See Theorem 7.3 and Corollary 7.4.) Let X denote a linear scheme over the base field k_0 (see Definition 4.4) which is also smooth and where the strata are isomorphic to product of affine spaces and split tori over k_0. We may also let $X = G/N(T)$ where $N(T)$ is the normalizer of a maximal torus in G, with G a split connected reductive group G, split over k_0 and T (B) is a fixed maximal torus of G (B a Borel subgroup containing T, respectively). Let $\bar{X}$ denote the scheme $X \underset{Spec\, k_0}{\times} Spec\, k$, where k is the algebraic closure of k_0. Let the derived smash product of spectra be defined as in* (2.1.2).

(i) Then there exists a weak-equivalence of completed spectra $Z/l_\infty(\Sigma_{S^2}) \simeq Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})$ and a natural map

$$\Sigma_{\bar{\mathbb{P}}^1_{et}} \bar{X}_{et,+} \overset{L}{\underset{\Sigma_{\bar{\mathbb{P}}^1_{et}}}{\wedge}} \Sigma_{\mathbb{P}^1} \rightarrow \Sigma_{\mathbb{P}^1} X_{mht,+}$$

that induces weak-equivalences

$$h_{Nis}(X, \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}), H) \overset{\simeq}{\rightarrow} h_{et}(\bar{X}, Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}), H) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})^H}{\wedge}} \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})^H.$$

(ii) Next let X denote linear schemes of the following form: (a) G/B or (b) G/T where G is a split connected reductive group G, split over k_0 and T (B) is a fixed maximal torus of G (B a Borel subgroup containing T, respectively). We may also let $X = G/N(T)$ where $N(T)$ is the normalizer of a maximal torus in G, with G as above. Then one obtains the weak-equivalences:

$$Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}} \bar{X}_{et,+})^H \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})^H}{\wedge}} \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})^H \simeq h^{et}(\bar{X}, Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}), H) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})^H}{\wedge}} Z/l_\infty(\Sigma^\infty_{\mathbb{P}^1})^H$$

$$\simeq h^{Nis}(X, \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})^H) \simeq (\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \overset{L}{\underset{\Sigma_{\mathbb{P}^1}}{\wedge}} \Sigma_{\mathbb{P}^1} X_+)^H$$

of spectra in $\mathbf{Spt}_{\mathrm{s}}(\mathrm{k_0}, \mathrm{Gal}, \mathbb{P}^1)$. *Here we have used $Z/l_\infty \Sigma_{\bar{\mathbb{P}}^1_{et}} \bar{X}_{et,+}$ to denote the homotopy limit of the obvious pro-object denoted by the same symbol.*

(iii) If X is any one of the schemes as in (ii), then it admits a lifting to a linear scheme $\bar{X}_{\bar{\mathbb{C}}}$ defined over $\mathbb{C}$ and one also obtains the weak-equivalences:

$$Z/l_\infty(\Sigma_{S^2} \bar{X}_{\mathbb{C}}) \simeq Z/l_\infty(\Sigma_{S^2} \bar{X}_{\mathbb{C}})_{et}$$

This theorem shows that the motivic stable homotopy types of the schemes appearing above, when completed at l is finite and hence strongly dualizable by taking their function-spectra mapping into appropriately completed sphere-spectra. We make use of the above theorem to construct **the stable transfer map** (see Definition 8.6)

$$tr(f) : \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1}(BG_+) \rightarrow \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1}(EG \underset{G}{\times} G/N(T)_+) \simeq \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1}(BN(T)_+)$$

so that the composition $\pi \circ tr(f)$ identifies with the identity map in the stable motivic homotopy category, where $\pi : \Sigma_{\mathbb{P}^1}(EG \underset{G}{\times} G/N(T)_+) \rightarrow \Sigma_{\mathbb{P}^1}(BG_+)$ is the obvious projection.

An important corollary of the last theorem and the stable transfer map is the following.

Theorem 1.2. *Let G denote a split connected reductive group G, split over k_0 and let T denote a maximal torus with $N(T)$ denoting its normalizer. Then the map $\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1} BN(T)) \rightarrow \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1} BG)$ is a split epimorphism in* $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \mathbb{P}^1)$. *In particular, the above map induces a split injection on any contravariant functor defined on* $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \mathbb{P}^1)$.

The above theorem, in fact enables, one to restrict the structure group from G to $N(T)$ (and then to T by ad-hoc arguments) in several situations. Taking $G = GL_n$, this becomes a *splitting principle* reducing problems on vector bundles to corresponding problems on line bundles. Such applications are explored fully in [CJ2].

Recall that in [Voev, Proposition 2.7], it is shown that for every projective smooth variety X, there exists a vector bundle $\mathcal{N}$ together with a Thom-Pontrjagin collapse map: $TP : T^{d+n} \to Th(\mathcal{N})$. Here T^{d+n} is a $\mathbf{T}$-sphere of dimension $d+n$ (and identified with the projective space $\mathbb{P}^{d+n}_{k_0}$) and $Th(\mathcal{N})$ is the Thom-space of the vector bundle $\mathcal{N}$. As another application of the theory developed in this paper, we are able to provide an independent proof that this collapse map provides a version of *Atiyah-duality* (see [At]) in the motivic context when X is assumed to be both linear and smooth projective. This is independent of [Voev] which contains a (rather difficult to follow) proof for general smooth projective schemes.)

Theorem 1.3. *Let X denote a projective smooth linear scheme over a field k_0. Then there exists a vector bundle $\mathcal{N}$ over X together with a map This provides a map $\mu : \Sigma^\infty_{\mathbf{T}} T^{d+n} \to \Sigma^\infty X_+ \wedge Th(\mathcal{N})$. This map induces a map from the stable $\mathbf{T}$-homotopy of X_+ to the stable $\mathbf{T}$-cohomotopy of $Th(\mathcal{N})$ which induces a weak-equivalence after smashing with the completed sphere spectrum $\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})$.*

Here is a *summary of the paper*. We will freely make use of the framework of Galois equivariant motivic homotopy theory developed in [CJ1]. For the reader's convenience, we recall the main features of this theory in section two. The third section explores the connection between étale topological type and the motivic homotopy type of algebraic varieties, the key link being provided by hypercoverings. (In fact the notion of motivic homotopy type introduced here seems to be new.) The fourth section explores the relation between cellular objects in the motivic stable homotopy category and linear varieties. This is followed by a detailed definition and study of the basic properties of generalized motivic cohomology and homology in section 5. In section 6, we discuss the general theory of Spanier-Whitehead duality in the context of function spectra. Section 7, which is a key section, is devoted to a proof that the motivic stable homotopy type of many algebraic varieties (for example linear varieties) is finite. This is used in section 8 to construct a special case of the Becker-Gottlieb transfer and we conclude in section 9 by discussing the variant of Atiyah-duality as in Theorem 1.3

2. A quick review of Galois equivariant motivic stable homotopy theory

We will fix a field k_0 and let Gal denote the Galois group of an algebraic closure $\bar{k}$ over k_0. Then $\mathrm{PSh(Sm/k_0)}$ will denote the category of simplicial presheaves on the category $\mathrm{Sm/k_0}$ of smooth schemes of finite type over k_0. *The projective model structure* on $\mathrm{PSh(Sm/k_0)}$ is where the fibrations and weak-equivalences are defined object-wise and the cofibrations are defined using the left-lifting property with respect to trivial fibrations. We next localize $\mathrm{PSh(Sm/k_0)}$ by inverting maps associated to an elementary distinguished square in the Nisnevich topology. The resulting category will be $\mathrm{PSh(Sm/k_0)}_{mot}$. If instead one inverts maps of the form $U_\bullet \to U$ where $U \,\varepsilon\, \mathrm{Sm}/k_0$ and $U_\bullet \to U$ is a hypercovering in the given topology, the resulting category will be $\mathrm{PSh(Sm/k_0)}_{des}$. The objects of the categories $\mathrm{PSh(Sm/k_0)}_{mot}$ and $\mathrm{PSh(Sm/k_0)}_{des}$ will often be called *sheaves upto homotopy* for obvious reasons. The topology will always be one of the following: the Zariski, the étale or the Nisnevich topology.

Next one considers the orbit category $\mathcal{O}_{Gal} = \{Gal/H \mid |Gal/H| < \infty\}$. A morphism $Gal/H \to Gal/K$ corresponds to $\gamma \,\varepsilon\, Gal$, so that $\gamma.H\gamma^{-1} \subseteq K$. One may next consider the category $\mathrm{PSh}^{\mathcal{O}^o_{Gal}}_{mot}$ of $\mathcal{O}^o_{Gal}$ -diagrams with values in $\mathrm{PSh(Sm/k_0)}_{mot}$ and also the category $\mathrm{PSh}^{\mathcal{O}^o_{Gal}}_{des}$ of $\mathcal{O}^o_{Gal}$ -diagrams with values in $\mathrm{PSh(Sm/k_0)}_{des}$. Let $\mathcal{C}$ denote either $\mathrm{PSh}^{\mathcal{O}^o_{Gal}}_{mot}$ or $\mathrm{PSh}^{\mathcal{O}^o_{Gal}}_{des}$.

Let $\mathcal{C}'$ denote a $\mathcal{C}$-enriched full-subcategory of $\mathcal{C}$ consisting of *objects closed under the monoidal product $\wedge$, all of which are assumed to be cofibrant and containing the unit S^0*. Let $\mathcal{C}'_0$ denote a $\mathcal{C}$-enriched sub-category of $\mathcal{C}'$, which may or may not be full, but closed under the monoidal product $\wedge$ and containing the unit S^0. Then *the basic model of equivariant motivic stable homotopy category* will be the category $[\mathcal{C}'_0, \mathcal{C}]$. This is the category whose objects are *$\mathcal{C}$-enriched covariant functors from $\mathcal{C}'_0$ to $\mathcal{C}$*: see [Dund1, 2.2].

We let $\mathrm{Sph}(\mathcal{C}'_0)$ denote the $\mathcal{C}$-category defined by taking the objects to be the same as the objects of $\mathcal{C}'_0$ and where $\mathcal{H}om_{Sph(\mathcal{C}'_0)}(T_U, T_V) = T_W$ if $T_V = T_W \wedge T_U$ and $*$ otherwise. Since T_W is a sub-object of $\mathcal{H}om_{\mathcal{C}}(T_U, T_V)$, it follows that $\mathrm{Sph}(\mathcal{C}'_0)$ is a sub-category of $\mathcal{C}'_0$. Now an enriched functor in $[\mathrm{Sph}(\mathcal{C}'_0), \mathcal{C}]$ is simply given by a collection $\{X(T_V) | T_V \,\varepsilon\, \mathrm{Sph}(f\mathcal{C}_0)\}$ provided with a compatible collection of maps $T_W \wedge X(T_V) \to X(T_W \wedge T_V)$. We let $Spectra(\mathcal{C}) = [\mathrm{Sph}(\mathcal{C}'_0), \mathcal{C}]$.

Several possible choices for the category $\mathcal{C}'_0$ are discussed in detail in [CJ1, Examples 3.3]. However, for the purposes of this paper it seems preferable to restrict to the following choices.

Example 2.1. The *main choice* of the subcategory $\mathcal{C}'$ above is the following.

(i)Let $\mathbf{T} = \mathbb{P}^1_k$ for some *fixed*, not necessarily finite extension k of k_0. Let K denote a normal subgroup of Gal with finite index. Let $\mathbf{T}_K$ denote the $\wedge$ of Gal/K copies of $\mathbf{T}$ with Gal/K acting by permuting the various factors above. We let $\mathcal{C}'$ denote the full sub-category of $\mathcal{C}$ generated by these objects under finite applications of $\wedge$ as K is allowed to vary subject to the above constraints. *This sub-category of $[\mathcal{C}', C]$ will be denoted $[\mathcal{C}', C]_{\mathbb{P}^1_k}$ and the corresponding category of spectra will be denoted $Spectra(\mathcal{C}, \mathbb{P}^1_k)$.* If we fix the normal subgroup K of Gal, the resulting category of $\mathbb{P}^1_k$-spectra will be denoted $Spectra(\mathcal{C}, Gal/K, \mathbb{P}^1_k)$: here the subcategory $\mathcal{C}'$ will denote the full sub-category of $\mathcal{C}$ generated by these objects under finite applications of $\wedge$.

More generally, given any object $P \,\varepsilon\, \mathcal{C}$ together with an action of Gal/K for some finite quotient group of Gal, one may define the categories $[\mathcal{C}', C]_P$ and the category, $Spectra(\mathcal{C}, P)$ of P-*spectra* similarly by replacing $\mathbb{P}^1_k$ above by P. (For example, P could be $F(\mathbb{P}^1_k)$ for a functor $F: \mathcal{C} \to \mathcal{C}$

(ii) Under the same hypotheses as in (i), we may also define $\mathcal{C}'$ as follows. Let $\mathbf{T} = S^n$ (for some fixed positive integer n) denote the usual simplicial n-sphere. Let $\mathbf{T}_K$ denote the $\wedge$ of Gal/K copies of T with Gal/K acting by permuting the various factors. In case $K = Gal$ and $n = 1$, we obtain a spectrum in the usual sense and indexed by the non-negative integers. Such spectra will be called *ordinary spectra*: when the constituent simplicial presheaves are all simplicial abelian presheaves, such spectra will be called *ordinary abelian group spectra*.

Definition 2.2. $\mathbb{P}^1$-motivic and étale spectra. (i) If $\mathcal{C} = \mathrm{PSh}_{mot}^{\mathcal{O}^o_{Gal}}$ and $\mathcal{C}'_0$ is chosen as in Examples 2.1(i), the resulting category of spectra, $Spectra(\mathcal{C})$, with the stable model structure will be called $\mathbb{P}^1$-*motivic spectra* and denoted $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \mathbb{P}^1)$.

(ii) If $\mathcal{C} = \mathrm{PSh}_{des}^{\mathcal{O}^o_{Gal}}$ with the étale topology and $\mathcal{C}'_0$ chosen as in Examples 2.1(ii), the resulting category of spectra, $Spectra(\mathcal{C})$ with with the stable model structure will be called $\mathbb{P}^1$-*étale spectra* and denoted $\mathbf{Spt}_{\mathrm{et}}(\mathrm{k_0}, \mathrm{Gal}, \mathbb{P}^1)$. In case $F: \mathcal{C} \to \mathcal{C}$ is a functor as in 2.1 (i), $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{F}(\mathbb{P}^1))$ ($\mathbf{Spt}_{\mathrm{et}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{F}(\mathbb{P}^1))$) will denote the corresponding category. Observe from Examples 2.1(i), that there are several possible choices for the sub-categories $\mathcal{C}'_0$.

If S^n for some fixed positive integer n is used in the place of $\mathbb{P}^1$ above, the resulting categories will be denoted $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{S^n})$ and $\mathbf{Spt}_{\mathrm{et}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{S^n})$.

The Galois group Gal will be suppressed when we consider spectra with trivial action by Gal.

At this point, it is important to recall the identification in [CJ1, 3.3.3] that identifies spectra in the above categories with diagrams of spectra indexed by $\mathcal{O}^o_{Gal}$. If X belongs to any of the above categories of spectra and K denotes a normal subgroup of Gal with finite index, X^K will denote the diagram X evaluated at G/K.

Remark 2.3. Clearly it is possible to use any of the categories $\mathcal{C} = \mathrm{PSh}_c(\mathrm{Sm/k_0}, Gal)$ $\mathrm{PSh}_{\mathbb{A}^1,c}(\mathrm{Sm/k_0}, Gal)$, or $\mathrm{PSh}_{des}^{\mathcal{O}^o_{Gal}}$ with $\mathcal{C}'_0$ chosen as in 2.1(i) or as in 2.1(ii). However, the simplicial presheaves forming the terms of these spectra are *not sheaves* upto homotopy. Hence the reason for adopting the above definitions. Clearly, for certain applications, it may be enough to consider such spectra.

Definition 2.4. $\mathbf{HSpt}_{mot}(k_0, Gal, \mathbb{P}^1)$ ($\mathbf{HSpt}_{et}(k_0, Gal, \mathbb{P}^1)$) will denote the corresponding stable homotopy category of $\mathbb{P}^1$-motivic (étale) spectra.

The Galois group Gal will be suppressed when we consider spectra with trivial action by Gal.

2.1. **Key properties of $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{F}(\mathbb{P}^1))$ and $\mathbf{Spt}_{\mathrm{et}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{F}(\mathbb{P}^1))$.** We summarize the following properties which have already been established above. Similar properties also hold for the categories $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{S^n})$ and $\mathbf{Spt}_{\mathrm{et}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{S^n})$ for any $n \geq 1$ though we do not state them explicitly.

(i) *Weak-equivalences and fibration sequence* A map $f: A \to B$ in any one of the above categories of Gal-equivariant spectra is a weak-equivalence if and only if the induced map $f^H: A^H \to B^H$ is a weak-equivalence of spectra for all subgroups $H \subseteq Gal$ of finite index. A diagram $F \to E \to B$ is a fibration sequence in any one of the above categories of Hal-equivariant spectra if and only if the induced diagrams $F^H \to E^H \to B^H$ are all fibration sequences of spectra for all subgroups H of finite index in Gal.

(ii) *Stably fibrant objects.* A spectrum E is fibrant in the above stable model structure if and only if each $E(T_V)$ is fibrant in $\mathcal{C}$= the appropriate unstable category of simplicial presheaves and the induced map $E(T_V) \to \mathcal{H}om_{\mathcal{C}}(T_W, (E(T_V \wedge T_W)))$ is a weak-equivalence of Galois-equivariant spaces for all $T_V, T_W \,\varepsilon\, \mathcal{C}'_0$.

(iii) Cofiber sequences A diagram $A \to B \to B/A$ is a cofiber sequence if and only if the the homotopy fiber of the map $B \to B/A$ is stably weakly-equivalent to A.

(iv) *Finite sums.* Given a finite collection $\{E_\alpha|\alpha\}$ of objects, the finite sum $\mathrm{V}_\alpha E_\alpha$ identifies with the product $\Pi_\alpha E_\alpha$ up to stable weak-equivalence.

(v) *Additive structure.* The corresponding stable homotopy categories is an additive category.

(vi) *Shifts.* Each $T_V \,\varepsilon\, \mathcal{C}'_0$ defines a shift-functor $E \to E[T_V]$, where $E[T_V](T_W) = E(T_V \wedge T_W)$. This is an automorphism of the corresponding stable homotopy category. The inverse of this automorphism will be denoted $E \to E[-T_V]$, where $E[-T_V](T_V \wedge T_W) = E(T_W)$.

(vii) *Cellular left proper simplicial model category structure.* All of the above categories of spectra have the structure of *cellular left proper simplicial model categories.* The category $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{F}(\mathbb{P}^1))$ and the corresponding category of spectra on the étale site are weakly finitely generated. The above categories are locally presentable, so that the corresponding model categories are combinatorial.

(viii) *Symmetric monoidal structure.* There is a symmetric monoidal structure on all the above categories of spectra, where the product is denoted $\wedge$. The sphere spectrum, (i.e. the inclusion of $\mathcal{C}'_0$ into $\mathcal{C}$) is the unit in this symmetric monoidal structure. Given a fixed spectrum E, the functor $F \mapsto E \wedge F$ has a right adjoint which will be denoted $\mathcal{H}om$ or often $\mathcal{H}om$. This is the internal hom in the above categories of spectra. The derived functor of this $\mathcal{H}om$ denoted $\mathcal{RH}om$ may be defined as follows. $\mathcal{RH}om(F, E) = \mathcal{H}om(C(F), Q(E))$, where $C(F)$ is a cofibrant replacement of F and $Q(E)$ is a fibrant replacement of E.

(ix) *Smash products and Ring spectra.* First we recall the construction of smash products of enriched functors from [Dund1, 2.3]. Given $F', F \,\varepsilon\, [\mathcal{C}'_0, \mathcal{C}]$, their *smash product* $F' \wedge F$ is defined as the Kan-extension along the monoidal product $\circ : \mathcal{C}'_0 \times \mathcal{C}'_0 \to \mathcal{C}'_0$ of the $\mathcal{C}$-enriched functor $F' \times F \,\varepsilon\, [\mathcal{C}'_0 \times \mathcal{C}_0, \mathcal{C}]$. Given spectra $\mathcal{X}$ and $\mathcal{Y}$ in $[Sph(\mathcal{C}'_0), \mathcal{C}]$, this also defines their smash product $\mathcal{X} \wedge \mathcal{Y}$. One also defines the *derived smash product* $\mathcal{X} \overset{L}{\wedge} \mathcal{Y}$ by $C(\mathcal{X}) \wedge \mathcal{Y}$, where $C(\mathcal{X})$ is a cofibrant replacement of $\mathcal{X}$ in the stable model structure on *Spectra*$(\mathcal{C})$. An *algebra* in $[\mathcal{C}'_0, \mathcal{C}]$ is an enriched functor $\mathcal{X}$ provided with an associative and unital pairing $\mu : \mathcal{X} \wedge \mathcal{X} \to \mathcal{X}$, i.e. for $T_V, T_W \,\varepsilon\, \mathcal{C}'_0$, one is given a pairing, $\mathcal{X}(T_V) \wedge \mathcal{X}(T_W) \to \mathcal{X}(T_V \wedge T_W)$ which is compatible as T_W and T_V vary and is also associative and unital.

A *ring spectrum* in *Spectra*$(\mathcal{C})$ is an algebra in $[Sph(\mathcal{C}'_0), \mathcal{C}]$ for some choice of $\mathcal{C}'_0$ satisfying the above hypotheses. A map of ring-spectra is defined as follows. If $\mathcal{X}$ is an algebra in $[Sph(\mathcal{C}'_0), \mathcal{C}]$ and $\mathcal{Y}$ is an algebra in $[Sph(\mathcal{D}'_0), \mathcal{C}]$ for some choice of subcategories C'_0 and D'_0, then a map $\phi : \mathcal{X} \to \mathcal{Y}$ of ring-spectra is given by the following data: (i) an enriched covariant functor $\phi : \mathcal{C}'_0 \to \mathcal{D}'_0$ compatible with $\wedge$ and (ii) a map of spectra $X \to \phi_*(Y)$ compatible with the ring-structures. (Here $\phi_*(Y)$ is the spectrum in *Spectra*$(\mathcal{C}, \mathcal{C}'_0)$ defined by $\phi_*(Y)(T_V) = Y(\phi(T_V))$.

Given a ring spectrum A, a left module spectrum M over A is a spectrum M provided with a pairing $\mu : A \wedge M \to M$ which is associative and unital. One defines right module spectra over A similarly. Given a left (right) module spectrum M (N, respectively) over A, one defines

(2.1.1) $$M \underset{A}{\wedge} N = coequalizer(M \wedge A \wedge N \rightrightarrows M \wedge N)$$

where the coequalizer is taken in the category of spectra and the two maps correspond to the module structures on M and N, respectively. Let $Mod(A)$ denote the category of left module spectra over A with morphisms being maps of left module spectra over A. Then the underlying functor $U : Mod(A) \to Spectra(\mathcal{C})$ has a left-adjoint given by the functor $F_A(N) = A \wedge N$. The composition $T = F_A \circ U$ defines a triple and we let $TM = \underset{\Delta}{\mathrm{hocolim}}\{T^n M|n\}$. Since a map $f : M' \to M$ in $Mod(A)$ is a weak-equivalence of spectra if and only if $U(f)$ is, one may observe readily that TM is weakly-equivalent to M. Therefore, one defines

(2.1.2) $$M \underset{A}{\overset{L}{\wedge}} N = TM \underset{A}{\wedge} N = coequalizer(TM \wedge A \wedge N \rightrightarrows M \wedge N)$$

Definition 2.5. (Stable homotopy groups). Let E denote a fibrant spectrum in any of the above categories of spectra and let $\mathbf{T}_V \,\varepsilon\, \mathcal{C}'$ denote any given object. Then $\mathbf{T}_V = \mathbf{T}_{K_1} \wedge \cdots \wedge T_{K_n}$ with K_i, $i = 1, \cdots, n$ a normal subgroup of Gal of finite index. Clearly, $K = \cap_{i=1}^n K_i$ is a normal subgroup of Gal with finite index in Gal and it acts trivially on $\mathbf{T}_V$. Recall that since E is fibrant, the obvious map $E(T_V) \to \Omega_{TW}(E(T_V \wedge T_W)) = \mathcal{H}om_{\mathcal{C}}(T_W, E(T_V \wedge T_W))$

is a weak-equivalence. Therefore, the k-th iteration of the above map, $E(T_V) \to \Omega^k_{T_W}(E(T_V \wedge T_W^{\wedge^k}))$ is also a weak-equivalence. Therefore, we define

$$\pi^K_{s+2|T_V|,T_V}(E(U)) = [\Sigma^{s+2|T_V|,T_V}U_+, E^K_{|U}] = \lim_{n\to\infty}[T_V \wedge T_W^{\wedge^n} \wedge S^s \wedge U_+, E^K_{|U}(T_W^{\wedge^n})],\ U\,\varepsilon\,(\mathrm{Sm/k_0, Gal})_?.$$

Here $[A, B]$ denotes Hom in the homotopy category associated to the corresponding category of Gal-equivariant spectra.

$$\mathbf{H}^{\mathrm{s+2|T_V|,T_V,K}}(\mathrm{U_+, E}) = \mathcal{H}\mathrm{om}_{\mathrm{Sp,Gal/K}}(\Sigma^{\mathrm{s+2|T_V|,T_V}}\mathrm{U_+}, \mathrm{E^K_{|U}}) =$$

$$= \lim_{n\to\infty}\mathcal{H}om_{Sp,,Gal/K}(T_V \wedge T_W^{\wedge^n} \wedge S^s \wedge U_+, E^K_{|U}(T_W^{\wedge^n})),\ U\,\varepsilon\,(\mathrm{Sm/k_0, Gal})_?$$

where $\mathcal{H}om_{Sp}$ denotes the internal hom-functor defined in [CJ1, 3.1.6 Mapping spectra] for Gal/K-equivariant objects and ? denotes either the Nisnevich or the étale sites. We will use the notation $\mathbf{H}^{\mathrm{s+2|T_V|,T_V,K}}_{\mathrm{mot}}(\mathrm{U_+, E})$ ($\mathbf{H}^{\mathrm{s+2|T_V|,T_V,K}}_{\mathrm{et}}(\mathrm{U_+, E})$) to denote the corresponding hypercohomology when $E\,\varepsilon\,\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0, Gal}, \mathbf{T})$ ($E\,\varepsilon\,\mathbf{Spt}_{\mathrm{et}}(\mathrm{k_0, Gal}, \mathbf{T})$, respectively).

When the Galois actions are ignored (or trivial), a general $T_W\,\varepsilon\,\mathcal{C}'_0$ is a finite iterated smash product of $\mathbb{P}^1$, so that in this case the above stable homotopy groups may be just indexed by two integers.

3. **Motivic and étale homotopy types**

3.1. **Nisnevich hypercoverings vs. étale hypercoverings.** We will briefly recall the definition of rigid coverings from [CJ1, Definition 2.11]. Let $X\,\varepsilon\,\mathrm{Sm/k_0}_?$ denote a fixed scheme. Let $\bar{X}$ denote a chosen conservative family of points of X appropriate to the site. Then a rigid covering of X in the topology ? then is a disjoint union of pointed separated maps $U_x, u_x \to X, x$ in the chosen site with each U_x connected and indexed by $x\,\varepsilon\,\bar{X}$.

A *rigid hypercovering* of X is then a simplicial scheme $U_\bullet$ together with a map $U_\bullet \to X$ so that U_0 is a rigid covering of X in the given topology ? and so that for each $n \geq 0$, the induced map $U_n \to (cosk^X_{n-1}(U_\bullet))_n$ is a rigid covering. (Here we use the convention that $(cosk^X_{-1}(U_\bullet))_0 = X$.)

Next it is important to observe the following:

(i) If $U \to X$ is a rigid covering of the scheme X in the Nisnevich topology, it is also a rigid covering in the étale topology. To see this, let $U_x, u_x \to X, x$ be a pointed map from a connected component of U lying over a point x of X. Since this is a rigid cover in the Nisnevich topology, the induced map $k(x) \to k(u)$ is an isomorphism. Let the imbedding of $k(u)$ into a separable closure denote the geometric point $\bar{u}$ and let the pre-composition of this imbedding with the isomorphism $k(x) \to k(u)$ denote the geometric point $\bar{x}$. Therefore, we see that the geometric point corresponding to u lies over the geometric point corresponding to x. Since this holds for all connected components of U, it follows that $U \to X$ is a rigid étale cover. It follows readily that any hypercover of a simplicial scheme $X_\bullet$ in the Nisnevich topology is a hypercover of $X_\bullet$ in the étale topology.

(ii) As in the étale case (see [Fr, Proposition 4.1]), one may readily show that there is at most one map between two rigid coverings of a scheme, when the coverings are in the Nisnevich topology. Therefore, the category of rigid coverings is clearly a left directed category in both the Nisnevich and étale topology. Given a simplicial scheme $X_\bullet$ with each $X_n\,\varepsilon\,\mathrm{Sm/k_0}$, we will denote by $\mathrm{HRR}_?(X_\bullet)$ the category of rigid hypercoverings of $X_\bullet$ in the topology ?, which denotes either one of the Nisnevich or étale topologies. It follows readily that $\mathrm{HRR}_?(X_\bullet)$ is a cofiltered category.

We will view each such rigid hypercovering as an object in $\mathrm{PSh(Sm}/k_0)$.

Definition 3.1. (Motivic and étale homotopy types) Given a simplicial scheme $X_\bullet$ as above, we let $X_{\bullet,mht} = \{U_\bullet\,\varepsilon\,\mathrm{HRR}_?(X_\bullet)\}$ when $? = Nis$ and $X_{\bullet,eht} = \{U_\bullet\,\varepsilon\,\mathrm{HRR}_?(X_\bullet)\}$ when $? = \acute{e}t$.

Remark 3.2. For the case $? = \acute{e}t$, $X_{\bullet,eht}$ is different from the étale topological type as is usually defined, where one also applies the connected component functor to the hypercoverings to produce an inverse system of simplicial sets. However, the disadvantage of the latter is that the étale cohomology of $X_\bullet$ can recovered from this étale topological type only with respect to locally constant sheaves. Since one clearly needs to consider cohomology with respect to complexes of sheaves which are generally not apriori locally constant (for example, the motivic complexes), it is important define the motivic homotopy type as we have done. A key property of the motivic homotopy type defined above is the following result.

Let X denote a scheme in Sm/k and let $U \rightarrow X$ denote an étale map. Let $\pi_0(\sqcup(U,\bar{u}))\wedge Spec\,\bar{k} = \sqcup_{\pi_0(\sqcup(U,\bar{u}))}(Spec\,\bar{k})$. One may map this to $\sqcup(U,\bar{u})$ by mapping the component $Spec\,\bar{k}$ indexed by $\bar{u}$ to $\bar{u}$. Applying this map to each term in a rigid étale hypercovering of each X one obtains a natural map $X_{et} \rightarrow X_{eht}$ of inverse systems of simplicial sheaves. Since every rigid Nisnevich hypercovering of X is a rigid étale hypercovering as observed above, one also obtains a natural map $X_{eht} \rightarrow X_{mht}$ of inverse systems of simplicial sheaves.

Proposition 3.3. *Let $X \,\varepsilon\, \mathrm{Sm/k_0}$. If P is a simplicial abelian presheaf belonging to $\mathrm{PSh}(\mathrm{Sm}/k_0)$ which is additive in the sense that $\Gamma(\sqcup_i U_i, P) = \Pi_i \Gamma(U_i, P)$ and $Ga(P)$ is a fibrant replacement of aP, then the obvious maps*

$$\underset{\rightarrow\,\mathrm{HRR}_?(X)}{lim}\Gamma(U_\bullet, P) \rightarrow \underset{\rightarrow\,\mathrm{HRR}_?(X)}{lim}\Gamma(U_\bullet, GaP) \longleftarrow \Gamma(X, GaP)$$

induce isomorphisms on taking cohomology. Moreover, the map $\Gamma(X, GaP) \rightarrow \Gamma(U_\bullet, GaP)$ induces an isomorphism on taking cohomology for every hypercovering $U_\bullet$. In particular, if P is an abelian sheaf on $(\mathrm{Sm}/k_0)_{et}$ ($(\mathrm{Sm}/k_0)_{Nis}$), there exists a natural quasi-isomorphism $\mathrm{hom}(X_{eht}, P) \simeq \Gamma(X, GaP)$ ($\mathrm{hom}(X_{mht}, P) \simeq \Gamma(X, GaP)$, respectively) where hom *denotes hom in the category of abelian presheaves.*

Next we proceed to compare between the sphere-spectra and the $\mathbb{P}^1$-spectra. First we digress to discuss pro-objects in model categories some detail.

3.2. **Pro-objects in a model category.** An *example* to keep in mind is the following which concerns étale realizations which we will consider in the next section. Let $(simpl.sets)_*$ denote the category of pointed simplicial sets. In view of the imbedding $(simpl.sets_*) \rightarrow \mathrm{Psh}(\mathrm{Sm}/k)$, we will view pro-objects in the former as pro-objects in the latter. (In fact one may make the following stronger observation. Let $\mathrm{PSh}_{mot}^{\mathcal{O}^o_{Gal}}$ denote the category of simplicial presheaves on the Nisnevich site that are sheaves upto homotopy: see 2. The category of fibrant pointed simplicial sets will be imbedded in this category in the obvious manner.) This will come in handy for the comparison results in the next sections.

Much of what is said here also applies in general to the category of pro-objects in any symmetric monoidal model category $\mathbf{C}$. There are several distinct model category structures put on $pro-\mathbf{C}$, when $\mathbf{C}$ is a cellular model category: see [EH], [Isak] and [F-I]. We will assume the cofibrations (weak-equivalences) are maps which are essentially level-wise cofibrations (weak-equivalences, respectively). Recall an object $X = \{X_i | i \,\varepsilon\, I\}$ in $pro-\mathbf{C}$ is an inverse system of objects in $\mathbf{C}$ indexed by a small cofiltered category I. Recall also that a map $f : \{X_i | i \,\varepsilon\, I\} \rightarrow \{Y_j | j \,\varepsilon\, J\}$ of pro-objects in $\mathbf{C}$ is an essentially level-wise cofibration (weak-equivalence), if after re-indexing the above pro-objects, the given map is isomorphic in $pro-\mathbf{C}$ to a level-map $g = \{g_k : X'_k \rightarrow Y'_k | k\}$ with each g_k a cofibration (weak-equivalence, respectively) in $\mathbf{C}$. The fibrations are defined using the right-lifting property.

It is shown in [AM] and also [EH, (2.1.5) Proposition] that any finite diagram D in $pro-\mathbf{C}$ with no loops can be replaced upto isomorphism in $pro-\mathbf{C}$ by a level-diagram, i.e. where the vertices and maps in the diagram are all indexed by the same small cofiltered category. Moreover it is shown in [EH, Chapter 2] that one may replace such a diagram upto isomorphism in $pro-\mathbf{C}$ by one which is indexed by a small cofinite strongly directed set. Now the fibrations in the above model structure have the following interpretation: (see [EH, (3.2.2) Theorem] and also [Isak, section 4]) a map $f : X \rightarrow Y$ in $pro-\mathbf{C}$ is a fibration in the above sense if and only if, after replacing f by a level map indexed by a cofinite strongly directed set, it satisfies a *matching space* condition as in [EH, (3.2.2) Theorem]. In particular it follows that every constant object, i.e. an object of $\mathbf{C}$ which is fibrant in $\mathbf{C}$, when viewed as a pro-object is fibrant in the above model structure.

Let J denote a cofinite strongly directed set. Then one may put a model structure on $\mathbf{C}^J$ where the cofibrations and weak-equivalences are defined level-wise and the fibrations are defined by the lifting property. Then the same matching space conditions characterize fibrations, so that the constant objects in $\mathbf{C}$ that are fibrant in $\mathbf{C}$ are fibrant in this model structure on $\mathbf{C}^J$. Moreover, in this setting, the inverse-limit $\lim_J : \mathbf{C}^J \rightarrow \mathbf{C}$ is a right-Quillen functor right adjoint to the constant functor $\mathbf{C} \rightarrow \mathbf{C}^J$. Therefore, it is shown in [EH, Chapter 4], that one may define the *homotopy inverse limit functor*

$$\underset{J}{\mathrm{holim}} : \mathbf{C}^J \rightarrow \mathbf{C} \text{ as } \lim_J \circ Q \tag{3.2.1}$$

where Q is a fibrant replacement functor. It is also shown in [EH, Chapter 4] that this homotopy inverse limit is weakly-equivalent to the Bousfield-Kan homotopy inverse limit which is defined using the projective model structure on $\mathbf{C}^J$.

Next we let $\mathbf{C} = (simpl.sets)_*$. We will imbed this into PSh(Sm/k$_0$). Let Z/l_∞ denote the completion functor considered earlier in this section. Let $\mathbf{T} = \mathbb{P}^1_k$ for some fixed finite Galois extension of k_0, (often just k_0 itself) $\bar{\mathbf{T}} = \mathbb{P}^1_{\bar{k}}$, where $\bar{k}$ is a fixed algebraic closure of k_0 and let $\bar{\mathbf{T}}_{W(k)} = \mathbb{P}^1_{W(\bar{k})}$, where $W(\bar{k})$ denotes the ring of Witt vectors of $\bar{k}$. Let $\bar{\mathbf{T}}_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$ denote the 1-dimensional complex projective space. One then has an imbedding of $W(\bar{k})$ into $\mathbb{C}$ and the residue field of $W(\bar{k}) = \bar{k}$, permitting a comparison of the Z/l-completed étale topological types of $\bar{\mathbf{T}}_k$, $\bar{\mathbf{T}}_{W(k)}$ and the Z/l-completion of $\bar{\mathbf{T}}_{\mathbb{C}}$. For a scheme X, X_{et} will denote the étale topological type as in [Fr, Definition 4.4]. Then we obtain the maps of pro-simplicial sets that induce isomorphism on the corresponding pro-homotopy-groups π_k for $k \geq 1$ and on the corresponding pro-homotopy sets π_0:

$$Z/l_n(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C},et}) \xrightarrow{\simeq} Z/l_n(\bar{\mathbf{T}}^{\wedge^m}_{W(k),et}) \xleftarrow{\simeq} Z/l_n(\bar{\mathbf{T}}^{\wedge^m}_{\bar{k},et}) \tag{3.2.2}$$

for all $n, m \geq 1$. (One may prove this using ascending induction on m and n.) By considering these pro-objects as indexed by a cofinite strongly directed set J, and using the injective model structure on $(simpl.sets)^J_*$, we may take their $\underset{J}{\text{holim}}$ using $\underset{J}{\lim} \circ Q$ as shown above. We will let

$Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C}})_{et} = holimZ/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C},et})$, $Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{W(k)})_{et} = holimZ/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{W(k),et})$ and

$Z/l_\infty((\bar{\mathbf{T}}_{\bar{k}})^{\wedge^m}_{et})_{et} = holimZ/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\bar{k},et})$

so that we obtain the maps $Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C}})_{et} \xrightarrow{\simeq} Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{W(k)})_{et} \xleftarrow{\simeq} Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\bar{k}})_{et}$.

It is also shown in [Fr, Theorem 8.4] that if one lets $\bar{\mathbf{T}}_{\mathbb{C},s.et} = \{Sing(U_\bullet(\mathbb{C}))|U_\bullet \,\varepsilon\, HRR(\bar{\mathbf{T}}_{\mathbb{C}}\}$, then one has maps

$\rho : Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C},s.et}) \to Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C},et})$ and $\tau : Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C},s.et}) \to Z/l_\infty Sing(\bar{\mathbf{T}}_{\mathbb{C}}(\mathbb{C})^{\wedge^m}) \cong Z/l_\infty((S^2)^{\wedge m})$

that induce isomorphism on the corresponding pro-homotopy-groups π_k for $k \geq 1$ and on the corresponding pro-homotopy sets π_0. (This is in fact shown there only for the case $m = 1$, but the general case may be deduced readily using similar arguments.) Therefore, we may take their homotopy inverse limits in the same manner as above to obtain maps $holim\,\rho : Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C}})_{s.et} = holimZ/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C},s.et}) \to Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C}})_{et} = holimZ/l_\infty\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C},et})$ and $holim\,\tau : Z/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C}})_{s.et} = holimZ/l_\infty(\bar{\mathbf{T}}^{\wedge^m}_{\mathbb{C},s.et}) \to Z/l_\infty(Sing(\bar{\mathbf{T}}_{\mathbb{C}}(\mathbb{C}))^{\wedge^m}) \cong Z/l_\infty((S^2)^{\wedge^m})$.

3.2.3. **Common ring spectra that play a role in the paper.** Next we briefly recall the definition of ring-spectra from [CJ1, 3.2]. An *algebra* in $[\mathcal{C}'_0, \mathcal{C}]$ is an enriched functor $\mathcal{X}$ provided with an associative and unital pairing $\mu : \mathcal{X} \wedge \mathcal{X} \to \mathcal{X}$, i.e. for $T_V, T_W \,\varepsilon\, \mathcal{C}'_0$, one is given a pairing, $\mathcal{X}(T_V) \wedge \mathcal{X}(T_W) \to \mathcal{X}(T_V \wedge T_W)$ which is compatible as T_W and T_V vary and is also associative and unital.

A *ring spectrum* in $Spectra(\mathcal{C})$ is an algebra in $[Sph(\mathcal{C}'_0), \mathcal{C}]$ for some choice of $\mathcal{C}'_0$ satisfying the above hypotheses. A map of ring-spectra is defined as follows. If $\mathcal{X}$ is an algebra in $[Sph(\mathcal{C}'_0), \mathcal{C}]$ and $\mathcal{Y}$ is an algebra in $[Sph(\mathcal{D}'_0), \mathcal{C}]$ for some choice of subcategories C'_0 and D'_0, then a map $\phi : \mathcal{X} \to \mathcal{Y}$ of ring-spectra is given by the following data: (i) an enriched covariant functor $\phi : \mathcal{C}'_0 \to \mathcal{D}'_0$ compatible with $\wedge$ and (ii) a map of spectra $X \to \phi_*(Y)$ compatible with the ring-structures. (Here $\phi_*(Y)$ is the spectrum in $Spectra(\mathcal{C}, \mathcal{C}'_0)$ defined by $\phi_*(Y)(T_V) = Y(\phi(T_V))$.

Next we will list a small collection of ring-spectra that will play a major role in the paper. In all these examples, the category $\mathcal{C}$ used in the definition of spectra (see section 2) will be $\mathrm{PSh}^{\mathcal{O}^o_{Gal}}_{mot}$. Using the observation that any constant sheaf is clearly $\mathbb{A}^1$-local, the category of pointed simplicial sets will be imbedded in this category in the obvious manner. Let $\mathbf{T} = \mathbb{P}^1_k$ for some finite Galois extension of k_0, $\bar{\mathbf{T}} = \mathbb{P}^1_{\bar{k}}$, where $\bar{k}$ is a fixed algebraic closure of k_0 and let $\bar{\mathbf{T}}_{W(\bar{k})} = \mathbb{P}^1_{W(\bar{k})}$, where $W(\bar{k})$ denotes the ring of Witt vectors of $\bar{k}$. In this situation, one may define pro-objects, $\mathbf{T}_{eht}$, $\mathbf{T}_{mht}$, $\bar{\mathbf{T}}_{eht}$ and $(\bar{\mathbf{T}}_{W(k)})_{eht}$ as in the next section.

Examples 3.4. (i) *The sphere spectrum* Σ_{S^n}. Here we take $\mathcal{C}'_0$ be the full subcategory generated by $\underset{Gal/K}{\wedge} S^n$ by taking iterated smash products. The value of Σ_{S^n} on $\underset{Gal/K}{\wedge} S^n$ is $\underset{Gal/K}{\wedge} S^n$.

(ii) $Z/l_\infty(\Sigma_{S^n})$. In this case the subcategory $\mathcal{C}'_0$ is the same as in the last example, but $Z/l_\infty(\Sigma_{S^n})(\underset{Gal/K}{\wedge} S^n) = Z/l_\infty(\underset{Gal/K}{\wedge} S^n)$. Clearly the map $\Sigma_{S^n} \to Z/l_\infty(\Sigma_{S^n})$ is a map of ring-spectra.

(iii) *The spectrum* $Z/l_\infty(\Sigma_{(\bar{\mathbf{T}}_{W(\bar{k})_{et}})})$. Here $\bar{k}$ is assumed to be separably closed and $W(\bar{k})$ is its ring of Witt vectors. The residue field at the generic point is of characteristic 0 which we imbed into the complex numbers. In this case the subcategory $\mathcal{C}'_0$ is the full subcategory with objects $\{\underset{Gal/K}{\wedge} Z/l_\infty((\bar{\mathbf{T}}_{W(\bar{k})_{et}}))|Gal/K\}$. $Z/l_\infty(\Sigma_{(\bar{\mathbf{T}}_{W(\bar{k})_{et}})}(\underset{Gal/K}{\wedge} Z/l_\infty((\bar{\mathbf{T}}_{W(\bar{k})_{et}}))) = Z/l_\infty(\underset{Gal/K}{\wedge}(\bar{\mathbf{T}}_{W(\bar{k})_{et}}))$. The maps $\underset{Gal/K}{\wedge} S^2 \to \underset{Gal/K}{\wedge} Z/l_\infty((\bar{\mathbf{T}}_{W(\bar{k})_{et}})$ induce a map of ring spectra $\Sigma_{S^2} \to Z/l_\infty(\Sigma_{S^2}) \to Z/l_\infty(\Sigma_{(\bar{\mathbf{T}}_{W(\bar{k})_{et}})})$.

(iv) *The spectrum* $Z/l_\infty(\Sigma_{(\bar{\mathbf{T}}_{et})})$. Here $\bar{k}$ is assumed to be again separably closed. This spectrum is defined as in the last example above, with $\bar{\mathbf{T}}_k$ replacing $\bar{\mathbf{T}}_{W(\bar{k})}$. Since the residue field of $W(\bar{k})$ is $\bar{k}$, we obtain an obvious map of ring spectra $Z/l_\infty(\Sigma_{(\bar{\mathbf{T}}_k)_{et}}) \to Z/l_\infty(\Sigma_{(\bar{\mathbf{T}}_{W(\bar{k})})_{et}})$.

(v) *The spectra* $\widetilde{Z/l}_\infty(\Sigma_{\bar{\mathbf{T}}_{eht}})$ *and* $\widetilde{Z/l}\infty(\Sigma_{\mathbf{T}_{eht}})$. For any pointed scheme X, X_{eht} is as defined above and as observed above, comes equipped with a map $X_{et} \to X_{eht}$ viewed as a map of inverse systems of simplicial sheaves. In this case the subcategory $\mathcal{C}'_0$ is the full subcategory with objects $\{\underset{Gal/K}{\wedge}\widetilde{Z/l}\infty(\bar{\mathbf{T}}_{eht})|Gal/K\}$. $\widetilde{Z/l}_\infty(\Sigma_{(\bar{\mathbf{T}}_{eht})})(\underset{Gal/K}{\wedge}\widetilde{Z/l}_\infty(\bar{\mathbf{T}}_{eht})) = \widetilde{Z/l}_\infty(\underset{Gal/K}{\wedge}(\bar{\mathbf{T}}_{eht}))$. The spectrum $\Sigma_{\mathbf{T}_{eht}}$ is defined similarly using $\mathbf{T}_{eht}$ in the place of $\bar{\mathbf{T}}_{eht}$. The maps $\bar{\mathbf{T}}_{et} \to \bar{\mathbf{T}}_{eht} \to \mathbf{T}_{eht}$ induce maps of ring spectra $Z/l_\infty(\Sigma_{\bar{\mathbf{T}}_{et}}) \to \widetilde{Z/l}_\infty(\Sigma_{\bar{\mathbf{T}}_{eht}}) \to \widetilde{Z/l}_\infty(\Sigma_{\mathbf{T}_{eht}})$.

(vi) *The spectrum* $\widetilde{Z/l}_\infty(\Sigma_{\mathbf{T}_{mht}})$. This is defined as in the last example with $\mathbf{T}_{mht}$ replacing $\mathbf{T}_{eht}$. There is an obvious map of ring spectra $\widetilde{Z/l}_\infty(\Sigma_{\mathbf{T}_{eht}}) \to \widetilde{Z/l}_\infty(\Sigma_{\mathbf{T}_{mht}})$ induced by the map $\mathbf{T}_{eht} \to \mathbf{T}_{mht}$. Observe also that the map of hypercoverings of $\mathbf{T}$ to the scheme $\mathbf{T}$ induces a map $\mathbf{T}_{mht} \to \mathbf{T}$ which induces a map of ring spectra $\widetilde{Z/l}_\infty(\mathbf{T}_{mht}) \to \widetilde{Z/l}_\infty(\mathbf{T})$. (The existence of this map is clear in view of the definition of the homotopy inverse limit of a pro-object as in 3.2.1.)

Proposition 3.5. *Assume the above situation. Then one obtains maps of spectra*

$$\Sigma_{S^2} \to Z/l_\infty\Sigma_{S^2} \to Z/l_\infty\Sigma_{\overline{\mathbf{T}}_{W(k)_{et}}} \leftarrow Z/l_\infty\Sigma_{\overline{\mathbf{T}}_{et}} \to \widetilde{Z/l}_\infty\Sigma_{\overline{\mathbf{T}}_{eht}} \to \widetilde{Z/l}_\infty\Sigma_{\overline{\mathbf{T}}_{mht}} \to \widetilde{Z/l}_\infty\Sigma_{\mathbf{T}_{mht}} \to \widetilde{Z/l}_\infty\Sigma_{\mathbf{T}}$$

which are compatible with their ring-structures. The first two maps are between spectra in $\mathbf{Spt}_{\rm mot}(\rm k_0, Gal, \Sigma_{S^2})$ *and the rest are maps between spectra in in* $\mathbf{Spt}_{\rm mot}(\rm k_0, Gal, Z/l_\infty\Sigma_{\overline{\mathbf{T}}_{et}})$. *Moreover, the second and third maps are weak-equivalences.*

Proof. This is essentially discussed in the above set of examples. The first map is the obvious map defined by the Z/l-completion and the next two maps are defined by the comparison maps in étale homotopy theory: see [AM] for example. Therefore these two maps are weak-equivalences. The next maps are induced by the maps $\bar{\mathbf{T}}_{et} \to \bar{\mathbf{T}}_{eht} \to \bar{\mathbf{T}}_{mht} \to \mathbf{T}_{mht} \to \mathbf{T}$, where the ' $^-$ ' denotes the objects defined over $Spec\,\bar{k}$. The details on the ring structure of the above spectra and the the verification that these maps are all compatible with the ring structures on the above spectra are discussed in the examples above. The observation that the second and third maps are weak-equivalences follows from comparison between the étale topological types and the topological types of complex algebraic varieties as in [**?**] or [Fr]. □

3.3. **Stable motivic and étale homotopy types.**

Definition 3.6. Let $X = X_\bullet$ denote a simplicial scheme with each $X_n \,\varepsilon\, \mathrm{Sm/k_0}$. Let $\bar{X}_\bullet = Spec\, k \underset{Spec\, k_0}{\times} X_\bullet$ where k denotes the chosen algebraic closure of k_0. The *stable étale topological type* will denote the S^2-suspension spectrum, $\Sigma_{S^2} X_{et,+}$ associated to $X_{et,+}$. By viewing a simplicial set as the simplicial presheaf represented by it, one may view $\Sigma_{S^2} X_{et,+}$ as an object in $\mathrm{pro}-\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0, Gal}, \Sigma_{\mathrm{S^2}})$.

With $\mathbf{T} = \mathbb{P}^1_{k_0}$, we obtain the suspension spectra with respect to $\widetilde{Z/l_\infty}\mathbf{T}_{eht}$, $\widetilde{Z/l_\infty}\mathbf{T}_{mht}$ (already considered in Examples 3.4):the object $\widetilde{Z/l_\infty}\Sigma_{\mathbf{T}_{eht}} \wedge X_{eht+}$ ($\widetilde{Z/l_\infty}\Sigma_{\mathbf{T}_{mht}} \wedge X_{mht+}$) is a pro-object of spectra, which may not be necessarily $\mathbb{A}^1$-local. We will view them as imbedded in $\mathrm{pro}-\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0, Gal}, \widetilde{\mathrm{Z/l_\infty}}\Sigma_{\mathbf{T}_{\mathrm{eht}}})$. $\widetilde{Z/l_\infty}\Sigma_{\mathbf{T}_{eht}} \wedge X_{eht+}$ ($\widetilde{Z/l_\infty}\Sigma_{\mathbf{T}_{mht}} \wedge X_{mht+}$) will be called the stable étale homotopy type of X (the motivic homotopy type of X, respectively). The corresponding object $\widetilde{Z/l_\infty}\Sigma_{\bar{\mathbf{T}}_{eht}} \wedge \bar{X}_{eht+}$ ($\widetilde{Z/l_\infty}\Sigma_{\bar{\mathbf{T}}_{mht}} \wedge \bar{X}_{mht+}$), which is a pro-object of $\widetilde{Z/l_\infty}\Sigma_{\bar{\mathbf{T}}_{eht}}$-spectra will be called the stable étale homotopy type of $\bar{X}$ (the stable motivic homotopy type of $\bar{X}$ respectively).

Next we may consider the following objects associated to X in the same category $\mathrm{pro}-\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0, Gal, Z/l_\infty}\Sigma_{\bar{\mathbf{T}}_{\mathrm{et}}})$: $Z/l_\infty\Sigma_{\bar{\mathbf{T}}_{et}} \wedge \bar{X}_{et+}$, $Z/l_\infty\Sigma_{\bar{\mathbf{T}}_{W(k)_{et}}} \wedge \bar{X}_{et+}$, $\widetilde{Z/l_\infty}\Sigma_{\bar{\mathbf{T}}_{eht}} \wedge \bar{X}_{eht+}$, $\widetilde{Z/l_\infty}\Sigma_{\bar{\mathbf{T}}_{mht}} \wedge \bar{X}_{mht+}$ all of which are associated to $\bar{X}$. The first two are suspension-spectra associated to $\bar{X}_{et}$ using different models of completed spheres: the first uses the completion of the étale topological type of $\bar{T}$ and the second uses the completion of the étale topological type of $\bar{\mathbf{T}}_{W(k)}$. The third (fourth) is the suspension-spectrum associated to $\bar{X}_{eht+}$ using the spectrum $\widetilde{Z/l}(\Sigma_{\bar{\mathbf{T}}_{eht}})$ ($\widetilde{Z/l_\infty}(\bar{X}_{mht+})$ using the sphere spectrum $\widetilde{Z/l_\infty}(\Sigma_{\bar{T}_{mht}})$, respectively). One also obtains $\widetilde{Z/l_\infty}(\Sigma_{T_{mht}}) \wedge X_{mht+}$.

Now one obtains the following maps between these objects (defined as in Proposition 3.5):

$$
\begin{aligned}
(3.3.1)\qquad \Sigma_{S^2} \wedge \bar{X}_{et+} &\to (Z/l_\infty\Sigma_{S^2}) \wedge \bar{X}_{et+} \to (Z/l_\infty\Sigma_{\bar{\mathbf{T}}_{W(k)_{et}}}) \wedge \bar{X}_{et+} \leftarrow (Z/l_\infty\Sigma_{\bar{\mathbf{T}}_{et}}) \wedge \bar{X}_{et+}\\
&\to (\widetilde{Z/l_\infty}\Sigma_{\bar{\mathbf{T}}_{eht}}) \wedge \bar{X}_{eht+} \to (\widetilde{Z/l_\infty}\Sigma_{\bar{\mathbf{T}}_{mht}}) \wedge \bar{X}_{mht+}\\
&\to (\widetilde{Z/l_\infty}\Sigma_{\mathbf{T}_{mht}}) \wedge X_{mht+} \to (\widetilde{Z/l_\infty}\Sigma_{\mathbf{T}}) \wedge X_{mht+}
\end{aligned}
$$

Proposition 3.7. *Let* $\mathbf{Spt}$ *denote any one of the categories of spectra appearing in the list of properties : 2.1.*

(i) For any spectrum $E \,\varepsilon\, \mathbf{Spt}$*, and any* $U_\bullet \to X_\bullet$ *which is a rigid hypercovering of* $X_\bullet$ *in the ?-topology, one obtains a weak-equivalence:* $\Gamma(U_\bullet, E) \simeq \Gamma(Spec\, k_0, \mathcal{H}om_{Sp}(\Sigma U_{\bullet,+}, E))$ *where* $\mathcal{H}om_{Sp}$ *denotes the internal hom in the category* $\mathbf{Spt}$ *as discussed in* [CJ1, [3.1.6, Mapping spectra] *and* $\Sigma U_{\bullet,+}$ *denotes the suspension-spectrum of* $U_{\bullet,+}$.

(ii) In case $E \,\varepsilon\, \mathbf{Spt}$ *is a constant presheaf of spectra,* $\Gamma(U_\bullet, E) = E^{\pi_0(U_\bullet)} \simeq \mathcal{H}om(\Sigma_T(\pi_0(U))_+, E)$ *where* $\mathcal{H}om$ *denotes the internal hom in the category* $\mathbf{Spt}$.

Proof. (i) This is rather obvious and therefore the proof is skipped. The main observation needed for (ii) is that, since E is assumed constant, $\Gamma(U_\bullet, E) \simeq E^{\pi_0(U_\bullet)}$. ☐

Theorem 3.8. *Let* $X \,\varepsilon\, \mathrm{Sm}/k_0$ *and let* $E \,\varepsilon\, \mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0, Gal}, \mathbf{T})$ *be a fibrant object. Let* K *denote a normal subgroup of Gal with finite index in Gal.*

(i) Then

$$\mathbf{H}^{\mathrm{s,T_V,K}}_{\mathrm{mot}}(\mathrm{X_+, E}) = \mathcal{H}\mathrm{om_{Sp,Gal/K}}(\Sigma^{\mathrm{s,T_V}} \wedge \mathrm{X_+}, \mathrm{E^K_{|X}}) = \mathcal{H}\mathrm{om_{Sp,Gal/K}}(\Sigma^{\mathrm{s,T_V}} \wedge \mathrm{U_{\bullet,+}}, \mathrm{E^K_{|X}})$$

for any $U_\bullet \,\varepsilon\, HRR_{Nis}(X)$*. Therefore, in particular,*

$$\mathbf{H}^{\mathrm{s+2|T_V|,T_V,K}}_{\mathrm{mot}}(\mathrm{X_+, E}) = \mathcal{H}\mathrm{om_{Sp,Gal/K}}(\Sigma^{\mathrm{s,T_V}} \wedge \mathrm{X_+}, \mathrm{E^K_{|X}}) = \underset{\to\, \mathrm{U_\bullet \,\varepsilon\, HRR_{Nis}(X)}}{lim} \mathcal{H}\mathrm{om_{Sp,Gal/K}}(\Sigma^{\mathrm{s,T_V}} \wedge \mathrm{U_{\bullet,+}}, \mathrm{E^K_{|X}})$$

(ii) Let $\bar{X} = X \underset{Spec\, k_0}{\times} Spec\, \bar{k}$ *where* $\bar{k}$ *denotes an algebraic closure of* k_0*. Let* l *denote a fixed prime different from* $char(k_0)$*. Then the augmentation maps*

$$\mathbf{H}^{\mathrm{s+2|T_V|,T_V}}_{\mathrm{et}}(\bar{\mathrm{X}}_+, \widetilde{\mathrm{Z/l}_\infty}(\mathrm{E})) = \mathrm{holim_n}\{\underset{\Delta}{holim}(\mathcal{H}\mathrm{om_{Sp}}(\Sigma^{\mathrm{s,T_V}} \wedge \bar{\mathrm{X}}_+, \mathrm{G}(\widetilde{\mathrm{Z/l}_n}(\mathrm{E})))|\mathrm{n}\}$$

$$\to holim_n\{\underset{\to\, U_\bullet \,\varepsilon\, \mathrm{HRR_{et}(\bar{X})}}{lim} \underset{\Delta}{holim}\, \mathcal{H}om_{Sp}(\Sigma^{s,T_V} \wedge U_{\bullet,+}, G((\widetilde{Z/l_n}(E))))|n\}$$

$$\leftarrow holim_n\{\lim_{\to U_\bullet \,\varepsilon\, \mathrm{HRR_{et}}(\bar{X})} \mathcal{H}om_{Sp}(\Sigma^{s,T_V} \wedge U_{\bullet,+}, \widetilde{Z/l_n}(E))|n\}$$

are all weak-equivalences, where $G(\quad)$ *denotes the fibrant replacement on the étale site provided by the Godement resolution.*

(iii) Under the same hypotheses as in (ii), the augmentation maps

$$\mathbf{H}_{\mathrm{et}}^{\mathrm{s+2|T_V|,T_V,K}}(\bar{\mathrm{X}}_+, \widetilde{\mathrm{Z/l}_\infty}(\mathrm{E})) = \mathrm{holim_n}\{\underset{\Delta}{holim}(\mathcal{H}\mathrm{om_{Sp,Gal/K}}(\Sigma^{\mathrm{s,T_V}} \wedge \bar{\mathrm{X}}_+, \mathrm{G}(\widetilde{\mathrm{Z/l_n}}(\mathrm{E})^{\mathrm{K}})))|\mathrm{n}\}$$

$$\to holim_m holim_n\{\lim_{\to U_\bullet \,\varepsilon\, \mathrm{HRR_{et}}(\bar{X})} \underset{\Delta}{holim}\, \mathcal{H}om_{Sp,Gal/K}(\Sigma^{s,T_V} \wedge U_{\bullet,+}, G(P^{(m)}(\widetilde{Z/l_n}(E)^K)))|n\}$$

$$\leftarrow holim_m holim_n\{\lim_{\to U_\bullet \,\varepsilon\, \mathrm{HRR_{et}}(\bar{X})} \mathcal{H}om_{Sp,Gal/K}(\Sigma^{s,T_V} \wedge U_{\bullet,+}, P^{(m)}\widetilde{Z/l_n}(E)^K)|n\}$$

are all weak-equivalences, where $G(\quad)$ *denotes a fibrant replacement on the étale site and* $P^{(m)}$ *denotes a functorial Postnikov truncation that kills the homotopy groups in dimension* $> m$.

Proof. We will prove both (i) and (ii) simultaneously by observing that the Nisnevich site has finite cohomological dimension whereas the étale site of a scheme of finite type over an algebraically closed field k has finite l-cohomological dimension for $l \neq char(k)$. Therefore, we will consider the non-equivariant case first, where there is no action of finite quotients of the Galois groups involved.

One may first replace E by an Ω-spectrum, so that we may replace E by its 0-th term, namely the value of the spectrum E on S^0. We will denote this by E_0. Since E_0 is now a fibrant simplicial presheaf one applies the canonical Postnikov truncation functors P_m to E. Recall $\pi_l(P^{(m)}\Gamma(V,E)) \cong \pi_m(\Gamma(V,E))$ if $l \leq m$ and $\cong 0$ if $l > m$ for all V in the given site: one may readily verify that these functors takes Gal-equivariant simplicial presheaves to Gal-equivariant simplicial presheaves. Replacing E by the fibration sequence $K(\pi_m(E), n) \to P^{(m)}(E) \to P^{(m-1)}(E)$ in the above augmentation maps provides a map of the corresponding spectral sequences. The finite cohomological dimension or finite l-cohomological dimension hypothesis and the torsion hypothesis on E show that the corresponding spectral sequences converge strongly. Therefore, in order to prove the first statement, it suffices to prove the theorem when E is replaced by an abelian presheaf and this has been already established in Proposition 3.3.

Finally take the homotopy inverse limit over n. Since this homotopy inverse limit commutes with the homotopy inverse limit defining the Godement resolution G and the homotopy inverse limit coming from the simplicial scheme, this completes the proof in this case, thereby proving the non-equivariant case of (i) and also (ii).

Next we consider the equivariant case of (i). We may once again assume that E is a fibrant simplicial presheaf. In this case, one next observes that the functorial Postnikov truncation being functorial, is also equivariant. For this we recall the equivariant site for the action of the finite group $G = Gal/K$ as in [CJ1, 2.3.5]. Observe that the objects in this site are schemes U provided with a G-action and the morphisms are maps of schemes that are G-equivariant. The points on this site correspond to the orbits under G of the points of the underlying (non-equivariant) site. Given any object U in this site, we obtain a G-equivariant map $G \times U = \sqcup_{g \,\varepsilon\, G} U \to U$, where the G-action on $G \times U$ is only on the factor G and the map on the summand indexed by g is multiplication by g. This map is readily seen to be a covering in this site. For any G-equivariant sheaf F, $Hom_G(G \times U, F) = Hom(U, F^G)$, where Hom_G denotes the external Hom in the category $AbSh(\mathrm{Sm}/k_0, G, ?)$ and Hom denotes the (external) Hom in the category $AbSh(\mathrm{Sm}/k_0, ?)$. The underlying non-equivariant site in (i) is the Nisnevich site which has finite cohomological dimension for any fixed U. The underlying non-equivariant site in (ii) is the étale site of a scheme U of finite type over an algebraically closed field $\bar{k}$ which has finite l-cohomological dimension for $l \neq char(\bar{k})$. Therefore, in both cases for any U in the site, there exists a large enough integer d depending on U, so that $Ext^n_G(G \times U, F) \cong H^n(U, F^G) = 0$ for $n > d$ and for all abelian sheaves F in case (i) and all l-torsion abelian sheaves F in cases (ii) and (iii). This verifies the hypotheses in [MV, Theorem 1.37, see also Definition 1.31], which proves that in case (i), one obtains a weak-equivalence $holim_m G(P^{(m)}E) \simeq G(E)$. and in case (ii), one obtains a weak-equivalence $holim_m G(P^{(m)}\widetilde{Z/l_n}E) \simeq G(\widetilde{Z/l_n}E)$. In view of these observations, it suffices to show that the maps in (i) ((iii)) are weak-equivalences after replacing E by $P^{(m)}E$ ($\widetilde{Z/l_n}E$ by $P^{(m)}\widetilde{Z/l_n}E$). Then we reduce to the case where E is replaced by $K(\pi_m(E), m)$ ($\widetilde{Z/l_n}E$ is replaced by $K(\pi_m(\widetilde{Z/l_n}E), m)$, respectively). This case then follows by [CJ1, Proposition 2.14] and Corollary 3.3. $\square$

Remark 3.9. One interpretation of the above theorem is that, under the above hypotheses,

$\mathbf{H}_{\mathrm{mot}}^{\mathrm{s}+2|\mathrm{T_V}|,\mathrm{T_V}}(\mathrm{X}_+,\mathrm{E}) \simeq \mathcal{H}\mathrm{om}_{\mathrm{Sp}}(\Sigma_{\mathbf{T}}^{\mathrm{s}+2|\mathrm{T_V}|,\mathrm{T_V}}(\mathrm{X}_{\mathrm{mht},+},\mathrm{E})$ and

$\mathbf{H}_{\mathrm{et}}^{\mathrm{s},\mathrm{T_V}}(\bar{\mathrm{X}}_+,\widetilde{\mathrm{Z}/\mathrm{l}}_\infty(\mathrm{E})) \simeq \mathcal{H}\mathrm{om}_{\mathrm{Sp}}(\Sigma_{\mathbf{T}}^{\mathrm{s}+2|\mathrm{T_V}|,\mathrm{T_V}}(\bar{\mathrm{X}}_{\mathrm{eht},+}),\widetilde{\mathrm{Z}/\mathrm{l}}_\infty(\mathrm{E}))$

i.e. The generalized motivic cohomology of X (the generalized étale cohomology of $\bar{X}$) with respect to the spectrum E ($\widetilde{Z/l}_\infty(E)$, respectively) may be computed from the motivic homotopy type of X (the étale homotopy type of $\bar{X}$, respectively).

4. Cellular objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0,\mathrm{Gal})$ and $\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0,\mathrm{Gal})$ vs. linear schemes

Throughout the rest of the paper we will restrict to $\mathbb{P}^1$-spectra. As such the suspension spectra associated a simplicial presheaf P will be often denoted simply by $\Sigma^{s+2|T_V|,T_V}P$, $s \geq 0$, $T_V \,\varepsilon\, \mathcal{C}_0'$.

Definition 4.1. Let k denote a fixed Galois extension of k_0: often this will be just k_0 itself. Let (Cells) $=$ $\{(Gal/H)_+ \wedge \Sigma^s_{\mathbb{P}^1} TV | V = \text{ an affine space over } k, \text{ which is a representation of } Gal/K,$
K, H subgroups of finite index in Gal with K normal in Gal, $n, m \geq 0\}$. (These are the $\mathbb{P}^1$-suspension spectra associated to Thom-spaces of finite dimensional representations of Gal/K.) Then the class of *cellular-objects* in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0,\mathrm{Gal})$ ($\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0,\mathrm{Gal})$) is the smallest class of objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0,\mathrm{Gal})$ ($\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0,\mathrm{Gal})$, respectively) so that (i) it contains (Cells) (ii) if X is weakly-equivalent in $Spt_{mot}(k_0,Gal)$ ($\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0,\mathrm{Gal})$) to a cellular object, then X is a cellular object and (iii) if $\{X_i | i \,\varepsilon\, I\}$ is a collection of cellular objects indexed by a small category I, then $\underset{I}{\mathrm{hocolim}} X_i$ is also cellular.

The above definition follows the approach taken in classical algebraic topology, where one defines a spectrum to be cellular if it is built out of the suspension spectra of cells by taking iterated homotopy colimits. Observe that, (Cells) now contains all suspension spectra of the form $(Gal/H)_+ \wedge TV$, where V any affine space: this is large enough to obtain the cellular approximation proved below.

On the other hand, the *linear schemes* considered in [Tot] and [J01] are schemes that are built out of affine spaces and tori and are particularly simple objects to study in the setting of algebraic geometry: see [J01]. Though it is not immediately apparent, these two notions are closely related: in fact we will show below that the suspension spectrum of every smooth linear scheme is cellular.

Proposition 4.2. *(Equivariant cellular approximation) (i) If* $E \,\varepsilon\, \mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0,\mathrm{Gal})$ *(*$\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0,\mathrm{Gal})$*) and is fibrant, there exists a natural* $E_c \to E$ *in* $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0,\mathrm{Gal})$ *(*$\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0,\mathrm{Gal})$*) with* E_c *cellular that induces an isomorphism on all presheaves of homotopy groups* $\pi^H_{t,s}$.

(ii) If $E \,\varepsilon\, \mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0,,\mathrm{Gal})$ *(*$\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0,\mathrm{Gal})$*) is cellular and* $\pi^H_{T_V,s}(E(U)) = 0$ *for all* T_V, s, $H \subseteq Gal$ *of finite index and* U *in the site, then* E *is contractible.*

Proof. Non-equivariant versions of these statements are proven in [DI, Proposition 7.3 and Lemma 7.6]. The same proof applies after taking the fixed points with respect to any subgroup $H \subseteq Gal$ of finite index. Finally make use of the adjunction between taking the fixed points with respect to a subgroup H and taking the product with Gal/H. □

In the above situation, the full subcategory of *finite cellular objects* or equivalently *finite* $\mathbb{P}^1$*-spectra* is the smallest class of objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0,\mathrm{Gal})$ ($\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0,\mathrm{Gal})$) containing (Cells) with the following properties: (i) it is closed under finite sums (ii) if X is weakly-equivalent to a finite cellular object, then X is a finite cellular object and (iii) it is closed under finite homotopy pushouts.

The following are some of the properties of cellular objects. (See [DI], where they are stated in the non-equivariant setting.)

Proposition 4.3. *(i) If* $X, Y \,\varepsilon\, \mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0,\mathrm{Gal})$ *(*$\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0,\mathrm{Gal})$*) are cellular, then so is* $X \wedge Y$ *and* $X \times Y$.

(ii) If $\{X_i | i \,\varepsilon\, I\}$ *is a family with each* X_i *cellular, then so is* $\bigvee_i X_i$ *which is the co-product of the* X_i.

(iii) The suspension-spectra $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}(\mathbb{A}^n_{k_1})_+$, $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}(\mathbb{A}^n_{k_1}-0)_+$ *are cellular. (Here* k_1 *is a finite Galois extension of* k_0*.)*

(iv) If X *is a scheme and* $U_* \to X$ *is a Nisnevich (étale) hypercover and each* $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}U_{n+}$ *is cellular, then* $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}X_+$ *is also cellular.*

(v) Let $\{U_i\}$ *be a Zariski open cover of the given scheme* X *so that the* $\mathbb{P}^1$*-suspension spectra of each intersection* $U_{\alpha_1\cdots\alpha_n}=U_{\alpha_1}\cap\cdots\cap U_{\alpha_n}$ *is cellular. Then* $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}X$ *is also cellular for all* s, T_V*.*

(vi) If $p\colon E\to B$ *is an algebraic fiber bundle with fiber* F *such that* $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}F$ *is cellular and* B *has a Zariski open cover satisfying the hypotheses in (v) which also trivializes* p*, then* $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}E$ *is also cellular.*

(vii) If $p\colon E\to B$ *is an algebraic vector bundle such that* B *has a Zariski open cover satisfying the hypotheses as in (vi), then the Thom space* $\mathrm{Th}(p)$ *which is defined as the suspension spectrum associated to the quotient* $E/(E-B)$ *is also cellular.*

(viii) If $p: E\to B$ *is a principal* G*-bundle, for an algebraic group* G*, where both* $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}E$ *and* $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}G_+$ *are cellular, then so is* $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}B$*.*

(ix) $\Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}GL_n$ *is cellular for all* n*. The* $\mathbb{P}^1$*-suspension spectrum of the Stiefel variety* $V_k(\mathbb{A}^n)$ *= the set of all* k*-tuples of linearly independent vectors in* $\mathbb{A}^n$ *and the* $\mathbb{P}^1$*-suspension spectrum of the Grassmannian variety* $Gr_k(\mathbb{A}^n)$ *of* k*-planes in* $\mathbb{A}^n$ *are also cellular.*

Definition 4.4. (Linear and cellular schemes) Let k_1 denote a finite Galois extension of k_0. Then a scheme X over k_1 is *linear* if it has a finite filtration $F_0\subseteq F_1\subseteq\cdots F_{n-1}\subseteq F_n=X$ by closed sub-schemes so that there exists a sequence $m_0,m_1,\dots,m_n$, $a_0,a_1,\dots,a_l$ of non-negative integers with each $F_i-F_{i-1}=\sqcup\mathbb{A}^{a_i}_{k_1}\times\mathbb{G}^{m_i}_{m,k_1}$ which is a finite disjoint union of products of affine spaces and split tori isomorphic to $\mathbb{A}^{a_i}_{k_1}\times\mathbb{G}^{m_i}_{m,k_1}$. Moreover we require that each of the above summands is a connected component of F_i-F_{i-1}. We call $\{F_i|i\}$ a linear filtration.

A linear scheme where the strata F_i-F_{i-1} are all isomorphic to disjoint unions of affine spaces will be called a *cellular scheme*. For the most part, we will only consider linear schemes when the field $k_1=k_0$.

Remark 4.5. Observe in view of the definition that the if $f:k_0\to k_0'$ is any map of fields, the pulled-back scheme $X'=X\underset{Spec\,k_0}{\times}Spec\,k_0'$ is also cellular, with the i-th term of the induced filtration $F_i'=F_i\underset{Spec\,k_0}{\times}Spec\,k_0'$.

Examples 4.6.
- (i) Common examples of cellular schemes are schemes of the form G/B, G/P where G is a *split* reductive group over k_0 and B (P) is a Borel subgroup (parabolic subgroup). The existence of the cell-filtration follows readily from the Bruhat decomposition. Since G is assumed to be split over k_0, the Bruhat decomposition holds over k_0: see [Sp59, Theorem 16.1.3] and therefore, the Bruhat cells in the decomposition of G/B are affine spaces over k_0.
- (ii) Next assume that G is a reductive group defined over k_0 and that it is *not* necessarily split over k_0. It will be split over some finite Galois extension k_1 of k_0. Now the flag variety $G/B\underset{Spec\,k_0}{\times}Spec\,k_1$ has a Bruhat decomposition with the Bruhat-cells all affine spaces over k_1.
- (iii) Next let T denote a fixed maximal torus of G and let B denote a Borel subgroup containing T. Since G/T maps in the obvious manner to G/B with fiber being the unipotent radical of G, it follows G/T is also a cellular scheme. If G is split over k_0, then the cell structure of G/T will be defined by affine spaces defined over k_0; otherwise, the cell-structure will be defined by affine spaces defined over some finite Galois extension of k_0.

Proposition 4.7. *The* $\mathbb{P}^1$ *-motivic suspension spectrum of any smooth linear scheme (of finite type over* k_0*) is finite cellular. In particular, the* $\mathbb{P}^1$ *suspension spectra of cellular schemes are indeed finite cellular objects in* $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0,\mathrm{Gal})$*.*

Proof. Let $\phi=F_{-1}\subseteq F_0\subseteq F_1\subseteq\cdots\subseteq F_{n-1}\subseteq F_n=X$ denote the given filtration so that each $F_i-F_{i-1}=\sqcup\mathbb{A}^{m_i}\times\mathbb{G}_m^{n_i}$ which is a finite sum of products of affine spaces and split tori. We will show by descending induction on k that the suspension spectra $\Sigma_{\mathbb{P}^1}(X-F_k)$ are all finite cellular. When $k=n-1$, this is clear since $X=F_n$ and F_n-F_{n-1} is a finite sum of products of affine-spaces and split tori. Therefore, we may assume this is true for

$0 < k+1 \le n-1$ and we will show that this implies the corresponding statement for k. For this one observes the stable co-fiber sequence in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal})$:

$$\Sigma_{\mathbb{P}^1}(X - F_{k+1})_+ \rightarrow \Sigma_{\mathbb{P}^1}(X - F_k)_+ \rightarrow \Sigma_{\mathbb{P}^1}(X - F_k)/\Sigma_{\mathbb{P}^1}(X - F_{k+1}) \simeq \Sigma_{\mathbb{P}^1}(N/N - (F_{k+1} - F_k)) \tag{4.0.2}$$

where N is the normal bundle to the closed immersion $F_{k+1} - F_k \rightarrow X - F_k$. Observe that since X is smooth, $X - F_k$ which is open in X is also smooth and that the hypotheses imply that $F_{k+1} - F_k$ is also smooth. Therefore, the normal cone associated to the closed immersion $F_{k+1} - F_k \rightarrow X - F_k$ is in fact a normal bundle. The last weak-equivalence is provided by what is called the homotopy purity theorem: see [MV, section 3, Theorem 2.23]. This shows that one has cofiber-sequence of the simplicial presheaves forming the spectra above in the $\mathbb{A}^1$-localized category Psh_{mot}. Depending on which finite Galois extension k_1 of k_0 the above schemes are defined, we obtain compatible actions $Gal(k_1/k_0)$ on the above simplicial presheaves. On taking the corresponding $\mathbb{P}^1$-motivic suspension spectra, we also obtain compatible actions of Gal on the suspension coordinates. These observations prove that the the stable cofiber-sequence above lives in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal})$.

At this point one needs to observe that $F_{k+1} - F_k$ is a disjoint union of products of affine spaces and split-tori where each such summand is in fact a connected component in $F_{k+1} - F_k$. Therefore, the above normal bundle is in fact trivial: this uses the statement that vector bundles on affine spaces and tori are trivial. The statement for vector bundles over affine spaces is the Serre-conjecture, proven by Quillen and Suslin. Its extension to vector bundles over tori is proven in [Lam, Corollary 4.9, p. 146]. Therefore, it follows that $\Sigma_{\mathbb{P}^1}(N/N - (F_{k+1} - F_k))_+$ is finite cellular and so is $\Sigma_{\mathbb{P}^1}(X - F_{k+1})_+$ by the inductive hypothesis. Therefore, $\Sigma_{\mathbb{P}^1}(X - F_k)_+$ is also finite cellular. □

We proceed to obtain the étale analogue of the stable cofiber sequence in 4.0.2. At present, we obtain this only at the level of the étale topological types completed away from the residue characteristics, which nevertheless suffices for later applications in this paper.

Lemma 4.8. *Let S denote a pointed simplicial set, viewed as a constant pointed simplicial presheaf on Sm/k_0 with trivial action by Gal. Let $E \,\varepsilon\, \mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{Z/l_\infty}\Sigma_{\overline{\mathbf{T}}_{\mathrm{et}}})$, where $\overline{\mathbf{T}} = \mathbb{P}^1_{\bar{k}}$. Then*

$$\Gamma(U, \mathcal{H}om_{Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}), Gal/H_{Gal}}(Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}S), E^H) = \mathcal{H}om_{Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}), Gal/H_{Gal}}(Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}S), \Gamma(U, E^H))$$

where both $Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}S$ and $\Gamma(U, E^H)$ are viewed as constant presheaves of spectra.

Proof. For each fixed $U \,\varepsilon\, \mathrm{Sm}/k_0$, let Sm/U denote the sub-category of objects and morphisms over U. If $p_U : U \rightarrow Spec\, k_0$ denotes the structure map, then the above identification follows from the adjunction between p^* and p_*. (Here p^* denotes the restriction of presheaves from Sm/k_0 to Sm/U and p_* is the corresponding push-forward. □

Proposition 4.9. *Let X denote a smooth linear scheme over an algebraically closed field $\bar{k}$. Let $\phi = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = X$ denote the given filtration so that each $F_i - F_{i-1} = \sqcup \mathbb{A}^{m_i} \times \mathbb{G}_m^{n_i}$ which is a finite sum of products of affine spaces and tori.*

(i) Then one obtains the cofiber-sequence of objects of $\mathrm{pro} - \mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{Z/l_\infty}\Sigma_{\overline{\mathbf{T}}_{\mathrm{et}}})$:

$$\begin{aligned} Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}(X - F_{k+1})_{et+}) \rightarrow Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}(X - F_k)_{et+}) \rightarrow Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}(X - F_k)_{et+}/\Sigma_{\overline{\mathbf{T}}_{et}}(X - F_{k+1})_{et+}) \\ \simeq Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}(N/N - (F_{k+1} - F_k))_{et}) \end{aligned} \tag{4.0.3}$$

where $\overline{\mathbf{T}} = \mathbb{P}^1_{\bar{k}}$ and where N denotes the normal bundle associated to the closed immersion $F_k \rightarrow F_{k+1}$ and et denotes the étale topological type. Moreover the terms in the above diagram are viewed as the obvious constant presheaves of spectra with the action by the Galois group Gal only on the suspension coordinates.

(ii) Therefore, if E denotes any object in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k_0}, \mathrm{Gal}, \Sigma_{\overline{\mathbf{T}}_{\mathrm{et}}})$, one obtains the cofiber sequence

$$\mathcal{H}om_{Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}), Gal/H_{Gal}}(Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}(N/N - (F_{k+1} - F_k))_{et}), Z/l_n(E^{H_{Gal}}))$$

$$\rightarrow \mathcal{H}om_{Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}), Gal/H_{Gal}}(Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}(X - F_k)_{et}), Z/l_n(E^{H_{Gal}}))$$

$$\rightarrow Hom_{Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}), Gal/H_{Gal}}(Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}(X - F_{k+1})_{et}), Z/l_n(E^{H_{Gal}}))$$

Moreover, the first term above identifies with $\mathcal{H}om_{Z/l_\infty(\Sigma_{\overline{\mathbf{T}}_{et}}), Gal/H_{Gal}}(Z/l_\infty(\Sigma^{2c,c}_{\overline{\mathbf{T}}_{et}}(F_{k+1} - F_k)_+), Z/l_n(E^{H_{Gal}}))$ where c is the codimension of $F_{k+1} - F_k$ in F_{k+1}.

Proof. (i) uses the theory of étale tubular neighborhoods as worked out in [Fr, Proposition 15.6] and also [Fr, Theorem 15.7]. Now the first assertion in (ii) follows readily. As shown in the last proposition, the normal bundle N associated to the closed immersion of $F_{k+1} - F_k$ in F_{k+1} is trivial. Therefore, the remaining conclusions follow from the last lemma. □

Remark 4.10. On the other hand it may be important to point out that even if one has a stable cofiber sequence $\Sigma_{\mathbb{P}^1} U_+ \rightarrow \Sigma_{\mathbb{P}^1} X_+ \rightarrow \Sigma_{\mathbb{P}^1}(X/U)$ where U and X are schemes over k_0 where $U \subseteq X$ is open in X and both the suspension spectra $\Sigma_{\mathbb{P}^1} U_+$ and $\Sigma_{\mathbb{P}^1} Y_+$ are finite cellular, (where Y is the complement of U in X), there is no reason apriori for $\Sigma_{\mathbb{P}^1} X_+$ to be finite cellular. This is because, first of all one needs to be able to make use of the fact that $\Sigma_{\mathbb{P}^1} Y_+$ is finite cellular. For this, one needs X and Y to be smooth and also the normal bundle to Y in X to be trivial (or at least its Thom-space to be just a suspension spectrum associated to Y_+). This does not seem possible in general; however, if one considers generalized cohomology theories defined by spectra in $\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal})$ which have Thom-isomorphism, then the cohomology of Y with respect to this spectrum identifies with the cohomology of the Thom-space with respect to the same spectrum. Amplifying on these ideas, one will then be able to prove certain finiteness properties of generalized cohomology and homology with respect to spectra that have Thom-isomorphisms. These observations motivate the following definitions.

Definition 4.11. Let E denote a ring spectrum in $\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal})$ as in 2.1. Then $\mathbf{Spt}(\mathrm{k}_0, \mathrm{E}, \mathrm{Gal})$ will denote the sub-category of all (Galois-equivariant) E-module spectra and E-module maps. This is a symmetric monoidal category with E as the unit. The internal hom in this category will be denoted $\mathcal{H}om_{\mathrm{E}}$.

Let $(\mathrm{E} - \mathrm{Cells})$ denote the collection

$(\mathrm{Cells}) \wedge E = \{(Gal/H)_+ \wedge \Sigma^{n,m}_{\mathbb{P}^1} TV \wedge E | V =$ an affine space over k, which is a representation of Gal/K, K, H subgroups of finite index in Gal with K normal in Gal, $n, m \geq 0\}$.

Then *cellular-objects* in $\mathbf{Spt}(\mathrm{k}_0, \mathrm{E}, \mathrm{Gal})$ is the smallest class of objects in $\mathbf{Spt}(\mathrm{k}_0, \mathrm{E}, \mathrm{Gal})$ so that (i) it contains $(\mathrm{E} - \mathrm{Cells})$ (ii) if X is weakly-equivalent to an E-cellular object, then X is an E-cellular object and (iii) if $\{X_i | i \,\varepsilon\, I\}$ is a collection of E-cellular objects indexed by a small category I, then $\underset{I}{\mathrm{hocolim}} X_i$ is also E-cellular. *Henceforth we will restrict to the full subcategory of* $\mathbf{Spt}(\mathrm{k}_0, \mathrm{E}, \mathrm{Gal})$ *consisting of cellular objects: this subcategory itself will be denoted by* $Spt(k_0, \mathrm{E}, Gal)$.

In the above situation, the class of *finite* E*-spectra* is the smallest class of objects in $\mathbf{Spt}(\mathrm{k}_0, \mathrm{E}, \mathrm{Gal})$ containing $(\mathrm{E} - \mathrm{Cells})$ with the following properties: (i) it is closed under finite sums (ii) if X is weakly-equivalent to a finite E-cellular object, then X is a finite E-cellular object and (iii) it is closed under finite homotopy pushouts in $\mathbf{Spt}(\mathrm{k}_0, \mathrm{E}, \mathrm{Gal})$.

5. **Finite T-spectra, finite** E**-spectra, Generalized Cohomology and Homology**

We will begin by first defining generalized motivic cohomology.

Definition 5.1. (Generalized motivic and étale cohomology). Let $\mathbf{T} = \mathbb{P}^1$ and let $\mathrm{E}, \mathcal{X} \,\varepsilon\, \mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0, \mathrm{Gal}, \mathbf{T})$ or $E, \mathcal{X} \,\varepsilon\, \mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0, \mathrm{Gal}, \mathbf{T})$. Then we let $\mathcal{H}om(\mathcal{X}, \mathrm{E})$ denote the internal hom in the category of spectra defined in [CJ1, 3.1.6 Mapping spectra]. We also let $\mathcal{RH}om(\mathcal{X}, \mathrm{E}) = \mathcal{H}om(\mathcal{X}, G\mathrm{E})$ where $G\mathrm{E}$ denotes the fibrant replacement defined by the Godement resolution. Given $A, B \,\varepsilon\, \mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0, \mathrm{Gal}, \mathbf{T})$, $[A, B]$ denotes Hom in the corresponding homotopy category. A similar meaning holds for $[A, B]$ if $A, B \,\varepsilon\, \mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0, \mathrm{Gal}, \mathbf{T})$. Then we let

$$h^{\bullet}(\mathcal{X}, \mathrm{E}, H) = \mathcal{RH}om(\mathcal{X}, \mathrm{E})^H \text{ and } h^{s+2|T_V|, T_V}(\mathcal{X}, \mathrm{E}, H) = [\Sigma(T_V) \wedge S^s, \mathcal{RH}om(\mathcal{X}, \mathrm{E})^H]$$

for any subgroup H of finite index in Gal. $h(\mathcal{X}, \mathrm{E}, H)$ will be called the generalized cohomology spectrum with respect to E and H. Observe that $\mathcal{RH}om(\mathcal{X}, \mathrm{E})$ belongs to $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0, \mathrm{Gal}, \mathbf{T})$ ($\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0, \mathrm{Gal}, \mathbf{T})$, respectively). Next we consider the special case where $\mathcal{X} = \Sigma_{\mathbf{T}} \wedge X_+$ is the $\mathbf{T}$-suspension spectrum associated to a scheme $X \,\varepsilon\, \mathrm{Sm}/k_0$. In this case, assuming that H acts trivially on the suspension coordinates, $h^{s+2|T_V|, T_V}(\mathcal{X}, \mathrm{E}, H)$ identifies with $[\Sigma(X_+) \wedge T_V \wedge S^s, \mathrm{E}^H]$.

Lemma 5.2. *Let* $\mathcal{X}$ *denote a finite* $\mathbf{T}$*-spectrum. Then* $\mathcal{RH}om_{\mathrm{E}}(\mathcal{RH}om_{\mathbf{T}}(\Sigma^{s+2|T_V|, T_V}_{\mathbf{T}} \mathcal{X}, \mathrm{E}), \mathrm{E})$ *is a finite* E*-module spectrum.*

Proof. This is simply the following observation:

$$\mathcal{RH}om_{\mathrm{E}}(\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma^{s+2|T_V|, T_V}_{\mathbf{T}} \wedge \mathcal{X}_+, \mathrm{E}), \mathrm{E}) \simeq \mathcal{RH}om_{E}(\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma^{s+2|T_V|, T_V}_{\mathbf{T}} \wedge \mathcal{X}_+, \Sigma_{\mathbf{T}}) \wedge \mathrm{E}, \mathrm{E})$$

$\simeq \mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{s+2|T_V|,T_V} \wedge \mathcal{X}_+, \Sigma_{\mathbf{T}}), \Sigma_{\mathbf{T}}) \wedge \mathrm{E} \simeq \Sigma^{s+2|T_V|,T_V}\mathcal{X}_+ \wedge \mathrm{E}.$ □

Next we proceed to define mod-l motivic and étale homology in such a way so that Spanier-Whitehead duality makes sense in the motivic and étale context.

Definition 5.3. (Generalized homology). We define generalized homology with respect to $\mathrm{E}\,\varepsilon\,\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0, \mathrm{Gal})$ or $E\,\varepsilon\,\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0, \mathrm{Gal})$ as follows. Let $p : X \rightarrow Spec\, k_0$ denote the obvious structure map. One first takes $\mathcal{RH}om_E(Rp_*(\mathrm{E}_{|X}), \mathrm{E})$ where $\mathrm{E}_{|X}$ denotes the restriction of E to the corresponding big site of X. One may also observe that $\mathrm{E}_{|X} \simeq \mathcal{RH}om(\quad, \mathrm{E})$ which is the sheaf associated to the presheaf $U \mapsto \mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}} \wedge U_+, \mathrm{E})$, U in the appropriate site on X. Here $\mathcal{RH}om$ is the derived functor of the internal hom functor in the category of sheaves of $\mathbf{T}$-spectra and $\mathcal{RH}om_{\mathrm{E}}$ is the derived functor of the internal hom-functor $\mathcal{H}om_{\mathrm{E}}$ in the category of E-module spectra on the appropriate big site. Then we let

$$h_{\bullet}(\mathcal{X}, \mathrm{E}, H) = \mathcal{RH}om_{\mathrm{E}}(Rp_*(E_{|X}), \mathrm{E}))^H,$$

$$h_{s+2|T_V|,T_V}(X, \mathrm{E}, H) = [\Sigma_{T_V} \wedge S^s, \mathcal{RH}om_{\mathrm{E}}(Rp_*(E_{|X}), \mathrm{E})^H].$$

Observe that $\mathcal{RH}om_{\mathrm{E}}(Rp_*(E_{|X}), \mathrm{E}))$ belongs to $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0, \mathrm{Gal}, \mathbf{T})$ or $E\,\varepsilon\,\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0, \mathrm{Gal}, \mathbf{T})$ depending on the context. If $\mathrm{E} = H(\mathbb{Z}/l)$, we will denote the corresponding homology by $H_{s+2|T_V|,T_V}(X, Z/l)$. $h(\mathcal{X}, \mathrm{E}, H)$ is called the generalized homology spectrum with respect to E.

In case $\mathrm{E} = H(\mu_l)$, the corresponding étale homology will be denoted $H^{et}_{s+2|T_V|,T_V}(X, \mu_l)$. Generalized Étale cohomology and homology are also considered [J86] in a slightly different context.

Remark 5.4. It is important to observe that the generalized homology defined above is *not* a Borel-Moore type homology theory, i.e. we are *not* considering homology with proper supports.

Proposition 5.5. *(i) The generalized homology defined above is a covariant functor for arbitrary maps.*

(ii) If $\Sigma_{\mathbf{T}} \wedge X_+$ is a finite $\mathbf{T}^1$-spectrum, the natural map

$$\Sigma_{\mathbf{T}} \wedge X_+ \wedge E \rightarrow \mathcal{RH}om_E(Rp_*(E|X), E)$$

is a weak-equivalence, so that $h_{s+2|T_V|,T_V}(X, E, H) = [\Sigma T_V \wedge S^s, Q(C(\Sigma_{\mathbb{P}^1} \wedge X_+) \wedge E))^H]$ where $C(\Sigma_{\mathbf{T}} \wedge X_+) \rightarrow \Sigma_{\mathbf{T}} \wedge X_+$ is a cofibrant replacement and $C(\Sigma_{\mathbf{T}} \wedge X_+) \wedge E \rightarrow Q(C(\Sigma_{\mathbf{T}} \wedge X_+) \wedge E)$ is a fibrant replacement.

(iii) If X is smooth scheme of finite type over k_0 with $p : X \rightarrow Spec\, k_0$ the structure map, then one has a natural weak-equivalence $\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}} X_+, E) \simeq Rp_*(E_{|X})$ in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0, \mathrm{Gal})$ ($\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0, \mathrm{Gal})$, respectively).

Proof. (i) This follows readily since generalized cohomology is a contravariant functor for arbitrary maps and the above definition of generalized homology is as a suitable *dual* of generalized cohomology. More specifically, let $f : X \rightarrow Y$ denote a map between two smooth schemes and let $p_X : X \rightarrow Spec\, k_0$ and $p_Y : Y \rightarrow Spec\, k_0$ denote the obvious structure maps. Then $p_X = p_Y \circ f$ and therefore, $Rp_{X*}(E_{|X}) \simeq Rp_{Y*}Rf_*(E_{|X})$ and so that $f^{-1}(E_{|Y})$ maps naturally to $E_{|X}$. This provides the map $E_{|Y} \rightarrow Rf_*(E_{|X})$ and hence the map $\mathcal{RH}om_E(Rp_{X*}(E_{|X}), E_{|Spec\, k_0}) \rightarrow \mathcal{RH}om_E(Rp_{Y*}(E_{|Y}), E)$. This proves (i).

(ii) Observe that $\Gamma(U, Rp_*(E_{|X})) = \Gamma(U \underset{Spec\, k_0}{\times} X, E) \simeq \mathcal{RH}om(\Sigma_{\mathbf{T}} U_+ \wedge X_+, E)$ where we have used the conventions of [CJ1, 3.1.7] of taking sections over the terminal object $Spec\, k_0$ to obtain the last identification. Therefore, the map in (ii) corresponds to a map $X_+ \wedge E \rightarrow \mathcal{RH}om_E(\mathcal{RH}om(\Sigma_{\mathbf{T}}(\quad)_+ \wedge X_+, E), E)$ which corresponds by adjunction to a map $X_+ \wedge \mathcal{RH}om(\Sigma_{\mathbf{T}}(\quad)_+ \wedge X_+, E) \rightarrow \mathcal{RH}om(\Sigma_{\mathbf{T}}(\quad)_+, E) \simeq R\Gamma((\quad)_+, E)$. Here $(\quad)$ varies over $U\,\varepsilon\,\mathrm{Sm}/k_0$. If E is fibrant, the last identifies with $\Gamma((\quad)_+, E)$. Using the assumption that $\Sigma_{\mathbf{T}} \wedge X_+$ is a finite $\mathbf{T}$-spectrum, one may reduce to the case where $\Sigma_{\mathbf{T}} \wedge X_+ = \Sigma_{\mathbf{T}}$ or an appropriate suspension of $\Sigma_{\mathbf{T}}$, in which case the proposition is clear.

(iii) Let $V\,\varepsilon\,(\mathrm{Sm}/\mathrm{k}_0)_?$. Then $\Gamma(V, \mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}} X_+, E)) = \Gamma(X \underset{Speck_0}{\times} V, \mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}(X \underset{Speck_0}{\times} V)_+, E)$ compatible the action of the Galois group. Clearly one may identify $\Gamma(V, Rp_*(E_{|X}))$ with the last term. □

Remark 5.6. The last proposition shows that at least for schemes X whose T suspension spectra are finite $\mathbf{T}$-motivic spectra, the generalized homology has the same form as in algebraic topology: see [Sw], for example. Moreover, it justifies the definition of homology we have adopted.

Corollary 5.7. *The generalized homology and homology of smooth linear schemes are represented by finite E-module spectra.*

Proof. The identification in (ii) of the last Proposition along with Proposition 4.7 and Lemma 5.2 readily prove this. □

The rest of this section holds only in the non-equivariant framework. Though there are extensions of Thom-isomorphism to the equivariant setting, this is rather cumbersome; besides Thom isomorphism does not seem to play a major role in this paper. Therefore, we have chosen just to make some remarks on Thom isomorphism in the non-equivariant setting. Observe that, generalized cohomology and homology are now indexed by a pair of integers.

In case (X, Y) is a smooth pair with $c =$ the codimension of Y in X, we let $h_Y^{2*,*}(X, \mathrm{E}) = h^{2*,*}(\Sigma_{\mathbf{T}}(X/X-Y), \mathrm{E})$, where $\mathbf{T} = \mathbb{P}^1$.

Definition 5.8. (*Thom isomorphism*) We say that a spectrum E in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0)$ or $\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0)$ is *orientable* or equivalently that *it has the Thom-isomorphism property* if for every smooth pair (X, Y) with $c =$ the codimension of Y in X, there exists a (canonical) class $[T] \,\varepsilon\, h_Y^{2c,c}(X; \mathrm{E})$ so that the cup-product $\cup[T] : h^{2*,*}(Y, \mathrm{E}) \rightarrow h_Y^{2*+2c,*+c}(X, \mathrm{E})$ is an isomorphism for all $*$.

In the presence of Thom-isomorphism, one can extend the earlier results in this section to smooth schemes that are stratified by strata whose $\mathbf{T}$-suspension spectra are finite $\mathbf{T}$- spectra. We proceed to discuss this extension.

Let (X, Y) denote a smooth pair (with $c =$ the codimension of Y in X) over the base-scheme $Spec\, k_0$: recall this means, the structure maps of X and Y over $Spec\, k_0$ are smooth and that Y is (regularly) imbedded in X as a closed sub-scheme over $Spec\, k_0$. Let $p_X : X \rightarrow Spec\, k_0$ and $p_Y : Y \rightarrow Spec\, k_0$ denote the given structure maps. Given a sheaf F on $(\mathrm{Sm}/Spec\, k_0)_{Nis}$, we let $F_{|X}$ denote its restriction to $(\mathrm{Sm}/X)_{Nis}$. Therefore, we obtain presheaves of $\mathbf{T}$-motivic spectra on $(\mathrm{Sm}/X)_{Nis}$: $\Sigma_{\mathbf{T}} X_{|X,+}$, $\Sigma_{\mathbf{T}} Y_{|X,+}$ and $\Sigma_{\mathbf{T}}(X_{|X}/X_{|X} - Y_{|X})$ defined by $\Gamma(U, \Sigma_{\mathbf{T}} X_{|X}) = \Sigma_{\mathbf{T}} U_+ = \Sigma_{\mathbf{T}}(X \underset{X}{\times} U)_+$, $\Gamma(U, \Sigma_{\mathbf{T}} Y_{|X,+}) = \Sigma_{\mathbf{T}}(Y \underset{X}{\times} U)_+$ and $\Gamma(U, \Sigma_{\mathbf{T}}(X_{|X}/(X_{|X} - Y_{|X}))) = \Sigma_{\mathbf{T}}(U/(U - Y \underset{X}{\times} U))$.

Lemma 5.9. *Let* E *denote a* $\mathbf{T}$*-spectrum that is orientable in the above sense. Then* $Rp_{X*}(\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{s,t}(X_{|X}/(X_{|X} - Y_{|X})), \mathrm{E}_{|X})) \simeq Rp_{Y*}(\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{s,t} Y_{|Y,+}, \mathrm{E}_{|Y}))$

Proof. Let $S = Spec\, k_0$. Then the left-hand-side identifies with the presheaf $U \rightarrow \mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{s,t} X \underset{S}{\times} U/(X \underset{S}{\times} U - Y \underset{S}{\times} U), \mathrm{E}_{|X})$. By Thom-isomorphism, this identifies with $U \rightarrow \mathcal{RH}om_{\mathbf{T}}(\Sigma_{\mathbf{T}}^{s,t} Y \underset{S}{\times} U_+, \mathrm{E}_{|Y})$ and this clearly identifies with the presheaf on the right-hand-side. □

The next proposition shows the use of orientability.

Proposition 5.10. *Let X denote a smooth scheme provided with a stratification (i.e. a decomposition into locally closed smooth subschemes) so that the* $\mathbf{T}$*-motivic suspension spectra of the strata are all finite* $\mathbf{T}$*-motivic spectra. Assume further that these are all defined over a finite Galois extension k' of k_0 and let* E *denote a ring spectrum in* $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0)$ *(or* $\mathbf{Spt}_{\mathrm{et}}(\mathrm{k}_0)$*) which is orientable. Then* $\mathcal{RH}om_{\mathrm{E}}(Rp_{X*}\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{t,s} \wedge X_{|X,+}, \mathrm{E}_{|X}), \mathrm{E}))$ *is a finite* E*-spectrum.*

Proof. Clearly any finite $\mathbf{T}$-spectrum smashed with E is a finite E-spectrum. The definition of finite E-spectra, shows the following: if one has a (stable) cofiber sequence $A \rightarrow B \rightarrow B/A \rightarrow \Sigma_{S^1} A$ with all maps being maps of E-module spectra, and if two of the three terms, A, B and B/A are finite E-spectra, then so is the third. Therefore, one reduces to showing the following: let (X, Y) denote a smooth pair, so that the $\mathbf{T}$-suspension spectra of Y and $X - Y$ are finite $\mathbf{T}$-spectra. Then $\mathcal{RH}om_{\mathrm{E}}(Rp_{X*}\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{t,s} \wedge X_{|X+}, \mathrm{E}_{|X}), \mathrm{E}))$ is a finite E-spectrum.

A key step here is the identification (provided by Thom-isomorphism) as in the lemma above:

(5.0.4) $\mathcal{RH}om_{\mathrm{E}}(Rp_{X*}\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{t,s} \wedge (X_{|X}/X_{|X} - Y_{|X}), \mathrm{E}_{|X}), \mathrm{E}) \simeq \mathcal{RH}om_{\mathrm{E}}(Rp_{Y*}\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{t,s} \wedge Y_{|Y,+}, \mathrm{E}_{|Y}), \mathrm{E})$

Since $\Sigma_{\mathbf{T}}^{t,s} Y_+$ is assumed to be a finite $\mathbf{T}$-spectrum, this identifies with $\Sigma_{\mathbf{T}}^{t,s} \wedge Y_+ \wedge \mathrm{E}$.

Clearly this is a finite E-spectrum. Now one observes the stable cofiber sequence:

$\mathcal{RH}om_{\mathrm{E}}(Rp_{X-Y*}\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{t,s} \wedge (X - Y)_{|X-Y,+}, E_{|X-Y}), E)$

$\to \mathcal{RH}om_{\mathrm{E}}(Rp_{X*}\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{t,s} \wedge X_{|X,+}, \mathrm{E}_{|X}), \mathrm{E})$

$\to \mathcal{RH}om_{\mathrm{E}}(Rp_{Y*}\mathcal{RH}om_{\Sigma_{\mathbf{T}}}(\Sigma_{\mathbf{T}}^{t,s} \wedge Y_{|Y,+}, \mathrm{E}_{|Y}), \mathrm{E})$

The last term is what it is, in view of (5.0.4). We may now assume that $\Sigma_{\mathbf{T}}(X - Y)_+$ is finite $\mathbf{T}$-spectrum so that the first term is a finite E-module spectrum clearly. Using ascending induction on the cell-filtration, we may assume that the last term is also a finite E-module spectrum. Therefore, the conclusion follows. □

6. Formalism of $\mathbb{P}^1$ and E-duality

Let $E\,\varepsilon\,\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal})$ denote a fixed ring spectrum. Let $\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$ denote the category of Galois-equivariant E-module spectra. Recall that this is a symmetric monoidal category with E as the unit. *Throughout this section we will restrict to the full subcategory of* $(E - Cells)$*, or E-cellular objects in* $\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$*.* The internal hom in this category will be denoted $\mathcal{H}om_{\mathrm{E}}$. The *dual* of an object $F\,\varepsilon\,\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$ will mean $\mathcal{RH}om_{\mathrm{E}}(F, \mathrm{E})$ which is a derived functor of $\mathcal{H}om_{\mathrm{E}}$ applied to F and will be denoted $D_{\mathrm{E}}(F)$. (Often $D_{\mathrm{E}}(F)$ will be called the E-*dual* of F.) Next we observe the following important property of finite E-spectra.

Definition 6.1. We define $\wedge_{\mathrm{E}} = \overset{L}{\underset{\mathrm{E}}{\wedge}}$ where $\underset{\mathrm{E}}{\wedge}$ is the monoidal structure on $\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$. We will define *duality maps* $\mu : F \wedge_{\mathrm{E}} \Sigma^{s+2|T_V|, T_V} D_{\mathrm{E}}(F) \to \Sigma_{\mathbb{P}^1}^{s+2|T_V|, T_V}\mathrm{E}$ in the obvious way, since $\Sigma^{s+2|T_V|, T_V} D_{\mathrm{E}}(F) \simeq \mathcal{RH}om_{\mathrm{E}}(\mathrm{E} \wedge_{\mathrm{E}} F, \Sigma^{s+2|T_V|, T_V}\mathrm{E})$.

We will next verify that the above duality maps satisfy the usual formalism of Spanier-Whitehead duality: see [Sw, Chapter 14]. We will assume henceforth that the ring spectra we consider are all *commutative* ring-spectra.

Proposition 6.2. *Let* $F\,\varepsilon\,\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$ *denote a finite E-module spectrum. (i) Then the obvious natural map* $F \to D_{\mathrm{E}}(D_{\mathrm{E}}(F))$ *is a weak-equivalence. (ii) If* $K\,\varepsilon\,\mathbf{Spt}_{\mathrm{E,Gal}}$*, the natural map* $D_{\mathrm{E}}(F) \wedge_E K \to \mathcal{RH}om_{\mathrm{E}}(F, K)$ *is a weak-equivalence.*

Proof. In view of the definition of finite E-spectra, it suffices to prove both statements when $F = \mathrm{E}$ or an appropriate $\mathbb{P}^1$-suspension of E. Now the main observation is that E is the unit of the symmetric monoidal structure on $\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$, so that $\mathcal{RH}om_{\mathrm{E}}(\mathrm{E}, \mathrm{E}) \simeq \mathrm{E}$ and hence $\mathrm{E} \simeq \mathcal{RH}om_{\mathrm{E}}(\mathcal{RH}om_{\mathrm{E}}(\mathrm{E}, \mathrm{E}), \mathrm{E})$. This proves (i). $\mathcal{RH}om_{\mathrm{E}}(\mathrm{E}, K) \simeq K \simeq \mathcal{RH}om_{\mathrm{E}}(\mathrm{E}, \mathrm{E}) \wedge_E K$ which proves (ii) when $F = \mathrm{E}$. The general case follows from this readily by using the fact that F is a a finite E-module spectrum. □

Let $X, X^*\,\varepsilon\,\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$ and let $\mu : X^* \wedge_{\mathrm{E}} X \to \Sigma_{\mathbb{P}^1}^{s+2|T_V|, T_V}\mathrm{E}$ denote a map. Then we define homomorphisms:

(6.0.4) $$D_\mu : [P, Q \wedge_{\mathrm{E}} X^*] \to [P \wedge_{\mathrm{E}} X, Q] \quad \text{and}$$

(6.0.5) $${}_\mu D : [P, X \wedge_{\mathrm{E}} Q] \to [X^* \wedge_{\mathrm{E}} P, Q]$$

as follows for any $P, Q\,\varepsilon\,\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$. Here $[\quad , \quad]$ denotes Hom in the homotopy category associated to $\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$. (First observe that since E is assumed to be a commutative ring spectrum, one may use left and right module structures interchangeably.) Given a map $f : P \to Q \wedge_{\mathrm{E}} X^*$, $D_\mu(f)$ will be represented by the composite map:

$$P \wedge_{\mathrm{E}} X \overset{f \wedge_{\mathrm{E}} id}{\to} Q \wedge_{\mathrm{E}} X^* \wedge_{\mathrm{E}} X \overset{id \wedge_{\mathrm{E}} \mu}{\to} Q \wedge_{\mathrm{E}} \Sigma_{\mathbb{P}^1}^{s+2|T_V|, T_V} E \simeq \Sigma_{\mathbb{P}^1}^{s+2|T_V|, T_V} Q$$

If $g : P \to X \wedge_{\mathrm{E}} Q$ is a map, ${}_\mu D(g)$ is represented by

$$X^* \wedge_{\mathrm{E}} P \overset{id \wedge_{\mathrm{E}} g}{\to} X^* \wedge_{\mathrm{E}} X \wedge_{\mathrm{E}} Q \overset{\mu \wedge id}{\to} \Sigma_{\mathbb{P}^1}^{s, T_V} E \wedge_{\mathrm{E}} Q \simeq \Sigma_{\mathbb{P}^1}^{s+2|T_V|, T_V} Q.$$

Definition 6.3. We say that $\mu : X^* \wedge_{\mathrm{E}} X \to \Sigma_{\mathbb{P}^1}^{s+2|T_V|, T_V}\mathrm{E}$ is an E-*duality map* if and only if the two maps D_μ and ${}_\mu D$ are isomorphisms for all spectra $F\,\varepsilon\,\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathrm{E})$. When the spectrum E is $\Sigma_{\mathbb{P}^1}$, we call an E-duality map, a $\Sigma_{\mathbb{P}^1}$-duality map or a motivic Spanier-Whitehead duality map.

Proposition 6.4. μ *is an* E*-duality map if and only if* D_μ *and* ${}_\mu D$ *are both isomorphisms for* $P = \Sigma_{\mathbb{P}^1}^{s+2|T_V|, T_V} E$ *and* $Q = \Sigma_{\mathbb{P}^1}^{s'+2|T_W|, T_W} E$

Proof. This may be proven as in [Sw, 14.22 Proposition] using the layer filtration defined in [CJ1, 3.1.8 Layer filtrations on spectra] since we have already restricted to spectra that are E-cellular. □

Theorem 6.5. *(Formal properties of* E*-duality)*

(i) If $\mu : X^* \wedge_{\mathrm{E}} X \rightarrow \Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}\mathrm{E}$ *is an* E*-duality map, then so is* $\mu \circ \tau : X \wedge_{\mathrm{E}} X^* \rightarrow \Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}\mathrm{E}$

(ii) Suppose $\mu : X^* \wedge_{\mathrm{E}} X \rightarrow \Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}\mathrm{E}$ *and* $\nu : Y^* \wedge_{\mathrm{E}} Y \rightarrow \Sigma_{\mathbb{P}^1}^{s'+2|T_W|,T_W}\mathrm{E}$ *are duality maps. Then both maps*

$$[X, Y]\overset{D_\mu}{\rightarrow}[Y^* \wedge_{\mathrm{E}} X, \Sigma_{\mathbb{P}^1}^{s'+2|T_W|,T_W}\mathrm{E}] \text{ and } [Y^*, X^*]\overset{\nu D}{\rightarrow}[Y^* \wedge_{\mathrm{E}} X, \Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}\mathrm{E}]$$

are isomorphisms. Assume $s \geq s'$ *and that* $T_V = T_W \wedge T_U$*. Denoting by* $f^* : Y^* \rightarrow X^*$ *the induced map associated to a given map* $f : X \rightarrow Y$*,* f^* *is characterized by the homotopy-commutativity of*

$$\begin{tikzcd}
Y^* \wedge_{\mathrm{E}} X \arrow[r, "f^*\wedge_{\mathrm{E}} id"] \arrow[d, "id\wedge_{\mathrm{E}} f"'] & X^* \wedge_{\mathrm{E}} X \arrow[d, "\mu"] \\
Y^* \wedge_{\mathrm{E}} Y \arrow[r, "\Sigma_{\mathbb{P}^1}^{s-s'+2|T_U|,T_U}\nu"] & \Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}\mathrm{E}
\end{tikzcd}$$

(iii) if $\mu : X^* \wedge_{\mathrm{E}} X \rightarrow \Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}E$*,* $\nu : Y^* \wedge_{\mathrm{E}} Y \rightarrow \Sigma_{\mathbb{P}^1}^{s'+2|T_W|,T_W}\mathrm{E}$ *and* $\pi : Z^* \wedge Z \rightarrow \Sigma_{\mathbb{P}^1}^{s"+2|T_U|,T_U}\mathrm{E}$ *are duality maps, and* $f : X \rightarrow Y$ *and* $g : Y \rightarrow Z$ *are* E*-module maps, then (a)* $(f^*)^* \simeq f$ *(b)*$(g \circ f)^* \simeq f^* \circ g^*$ *and (c)* $id^* = id$*.*

(iv) If $\mu : X^* \wedge_{\mathrm{E}} X \rightarrow \Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}\mathrm{E}$ *and* $\hat{\mu} : \hat{X}^* \wedge_{\mathrm{E}} X \rightarrow \Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}\mathrm{E}$ *are both E-duality maps, then there is a weak-equivalence* $h : X^* \rightarrow \hat{X}^*$ *so that* $\hat{\mu} \circ (h \wedge id) \simeq \mu$ *and* h *is unique upto homotopy.*

(v) If $\mu : X^* \wedge_{\mathrm{E}} X \rightarrow \Sigma_{\mathbb{P}^1}^{s+2|T_V|,T_V}\mathrm{E}$ *and* $\nu : \hat{Y}^* \wedge_{\mathrm{E}} Y \rightarrow \Sigma_{\mathbb{P}^1}^{s'+2|T_W|,T_W}\mathrm{E}$ *are both E-duality maps, then so is the composite* $(\mu, \nu) : X^* \wedge_{\mathrm{E}} Y * \wedge_{\mathrm{E}} Y \wedge_{\mathrm{E}} X \rightarrow X^* \wedge_E \Sigma_{\mathbb{P}^1}^{s'+2|T_W|,T_W}\mathrm{E} \wedge_{\mathrm{E}} X \rightarrow \Sigma_{\mathbb{P}^1}^{s,t}\mathrm{E} \wedge_{\mathrm{E}} \Sigma_{\mathbb{P}^1}^{s'+2|T_W|,T_W}\mathrm{E} \cong \Sigma_{\mathbb{P}^1}^{s+s'+2|T_V|+2|T_W|,T_V\wedge T_W}\mathrm{E}$*.*

Proof. These all may be proved exactly as in [Sw, Chapter 14] making use of the earlier results in this section. Therefore we skip the details. □

7. Motivic stable homotopy and cohomotopy of $G/N(T)$

Throughout this section, G *will denote* GL_n *(for some positive integer* n*) or a connected reductive group that is defined and split over the field* k_0*.)*

The main goal of this section is to show that the stable motivic homotopy and cohomotopy of $G/N(T)$, when completed away from the characteristic of k_0 is finite as a completed $\mathbb{P}^1$-spectrum, in fact is the completion of the étale homotopy type of $\bar{G}/N(\bar{T})$ smashed with the $\mathbb{P}^1$-spectrum. Here $\bar{G}$ ($\bar{T}$) is the algebraic group G (its maximal torus) base-extended to the algebraic closure of k_0. Before we proceed to the rather technical proofs below, we hope the following detailed explanations will shed some light on the proofs.

The motivic stable homotopy and cohomotopy of the varieties G/B and G/T are fairly easy to compute using the Bruhat decomposition for G which in fact provides a nice cell decomposition for the above varieties. We adopt the following strategy to extend these results to $G/N(T)$.

We may define a left action of $N(T)$ on G/T as follows: $(n, gT) \mapsto gn^{-1}T$. To see this is a left action, observe that if $n, n' \,\varepsilon\, N$, then $(n', (n, gT)) \mapsto (n', gn^{-1}T) \mapsto gn^{-1}n'^{-1}T = g(n'n)^{-1}T$. This is the image of $(n'n, gT)$. Moreover $gn^{-1}T = gT$ if and only if $n^{-1}T = T$, i.e. if and only if $n \,\varepsilon\, T$. Therefore, the above action of $N(T)$ induces an action by $W = N(T)/T$, where W is the Weyl group of G. Moreover, the same arguments show this is also a free action. It follows that one has a fibration sequence : $W = N(T)/T \rightarrow G/T \rightarrow G/N(T)$. Since we have assumed that G is split over k_0, it is important to observe that $(N(T)/T)(k_0) \cong W$, i.e. every element in W is represented by a k_0-rational point of $N(T)$. See [Sp59, 16.1.3 Theorem].

It is tempting to conclude from this that, since the $\mathbb{P}^1$-suspension spectrum of G/T_+ is a finite spectrum, so is the $\mathbb{P}^1$-suspension spectrum of $G/N(T)_+$. However, the above fibration sequence is one *only unstably* and therefore such a conclusion does not follow, unless proven using other arguments. Presently we prove this using a comparison with the completed étale homotopy type of $\bar{G}/N(\bar{T})$. Finally since $\bar{G}$ lifts to a corresponding reductive group defined over the complex numbers, it is possible to compare the completed étale homotopy type of $\bar{G}/N(\bar{T})$ with the corresponding variety associated to the complex group $G_{\mathbb{C}}$. At this point one may replace $G_{\mathbb{C}}$ with its compact form K and eventually reduce to comparing with the completion of $K/N(T_K)$. $K/N(T_K)$ is well-known to be a finite complex.

Next we identify $G/N(T)$ with $EW \underset{W}{\times} G/T$: this involves looking at Henselizations (and strict Henselizations) of $G/N(T)$ at points on it and considering the corresponding inverse images under the map $EW \underset{W}{\times} G/T \to G/N(T)$. What seems quite important for the computations below and the above identification is the following:

Proposition 7.1. *The map $G/T \to G/N(T)$ is finite étale with fibers given by the constant étale group scheme $W_{k_0} = W \wedge Spec\, k_0 = \sqcup_{w \varepsilon W} Spec\, k_0$. Moreover, if k is a field extension of k_0, for each k-rational point of $G/N(T)$, there are exactly W, k-rational points of G/T mapping to it. In particular, if $\bar{g} \varepsilon G/N(T)(k)$ and $\mathcal{O}^h_{G/N(T),\bar{g}}$ denotes the Henselizations at $\bar{g}$, then $Spec \mathcal{O}^h_{G/N(T),\bar{g}} \underset{G/N(T)}{\times} G/T \cong \sqcup_{w \varepsilon W} Spec \mathcal{O}^h_{G/N(T),\bar{g}}$.*

Proof. The first statement essentially follows from the observations made earlier regarding the free-action of W on G/T. One may also want to observe that since G is smooth, so are G/T and $G/N(T)$. Therefore, the map $G/T \to G/N(T)$ is smooth of relative dimension 0, i.e. it is étale. $G/N(T)$ identifies with the variety of all maximal tori in G. Any such maximal torus remains stable on conjugation by any element of its normalizer and there are precisely W distinct maximal tori that are contained in the normalizer of a given maximal torus. This

Throughout this proof k will denote any extension field of k_0 appearing as a residue field in $G/N(T)$. One may readily observe that for each such k-rational point of $G/N(T)$, there are exactly W k-rational points of G/T lying over it. (Since G is split over k_0, this holds for the k_0 rational points. To see it for k-rational points, where k is any extension field of k_0, one may use the identification $G_k/H_k \cong (G/H) \underset{Spec\, k_0}{\times} Spec\, k$: here $H = T$, or $N(T)$ and $X_k = X \underset{Spec\, k_0}{\times} Spec\, k$ for any k_0-scheme X.)

Let $U \to G/N(T)$ denote an étale neighborhood of a k-rational point u_k of $G/N(T)$ together with a lift of the k-rational point on $G/N(T)$ to U, which we denote by $\tilde{u}_k$. (Recall that the Henselization of $G/N(T)$ at a given k-rational point is the *Spec* of the colimit of $O_{U,\tilde{u}_k}$ as U runs over such étale neighborhoods of u_k.) It is clear that if $V = U \underset{G/N(T)}{\times} G/T$, then there is a lift of $\tilde{u}_k$ to V. In fact there are W such lifts corresponding to the points in G/T lying over u. Therefore, it suffices to show that etale neighborhoods of the form $U \underset{G/N(T)}{\times} G/T$ are cofinal in the system of étale neighborhoods of the lifts of u_k to G/T which appear in the Henselization of G/T at these points.

For this, we make a couple of observations first. Let $X \to Y$ denote a finite étale map over k so that X is a W-torsor over Y and that $X \underset{Y}{\times} X \cong \underset{w \varepsilon W}{\sqcup} X$. Suppose moreover that over every point $y \varepsilon Y$, there are exactly W-points $x \varepsilon X$ with $k(x) \cong k(y)$. Then if $y \varepsilon Y$ is a point, the étale neighborhoods of y of the form U together with a lift of the $k(y)$-rational point which are obtained as follows are cofinal in the system of étale neighborhoods that appear in the Henselization of Y at y: let V denote an étale neighborhood of a point x in the fiber over y, with $k(x) = k(y)$ along with a lift of this $k(x)$-rational point, then let $U = V$ viewed now as an étale neighborhood of y. This is clear, since given any étale neighborhood W of y appearing in the system of neighborhoods used in the Henselization of Y at y, its inverse image $W \underset{Y}{\times} X$ is an étale neighborhood of any point in the fiber over y and the map $W \underset{Y}{\times} X \to Y$ factors through W.

Next let $v : V \to G/T$ be an étale map with a lift v_k of a k-rational point of G/T in the fiber over u_k. Then the composition $V \to G/T \to G/N(T)$ is an étale map with a lift of the k-rational point u_k to V. Observe that $G/T \to G/N(T)$ is a torsor that trivializes over G/T, i.e. $G/T \underset{G/N(T)}{\times} G/T \cong \underset{w \varepsilon W}{\sqcup} G/T$. Therefore, $V \underset{G/N(T)}{\times} G/T \cong \underset{w \varepsilon W}{\sqcup} V$ which clearly maps to $\underset{w \varepsilon W}{\sqcup} G/T$. Varying V over a cofinal system of étale neighborhoods of v_k, then $V \underset{G/N(T)}{\times} G/T$ is cofinal in the system of étale neighborhoods that appear in the Henselizations at all the lifts of u_k to G/T. The last statement now follows readily in view of these observations. □

The generalized cohomology of $G/N(T)$ then identifies with the generalized cohomology of the simplicial scheme $EW \underset{W}{\times} G/T$ which may be computed using the spectral sequence: $E_1^{s,t} = h^t(W^s \times G/T; E) \cong \oplus_{w \varepsilon W^s} h^t(G/B; E) \Rightarrow$

$h^{s+t}(EW \underset{W}{\times} G/T; E)$ where E is a spectrum in $\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathbb{P}^1)$. Such spectral sequences exist both on the Nisnevich and étale sites, after first replacing E by an Ω-spectrum and then after replacing this Ω-spectrum by the fibrant simplicial presheaf forming its 0-th term. One may compute the generalized cohomology $h^t(G/B; E)$ readily using the Bruhat decomposition on G/B and this computation works fine on both the Nisnevich and étale sites. These computations will show that apart from the mod-l motivic or étale cohomology of the base field k, the E_1-terms are essentially the same for the Nisnevich and étale sites and in fact also for the case when the base field is algebraically closed. Now a comparison of these spectral sequences will enable one to compute the completed motivic homotopy and cohomotopy of $G/N(T)$ as claimed. In order to carry out such a comparison, we will need to make use of the natural maps defined in (3.3.1) between the étale homotopy types and the motivic homotopy types.

Corollary 7.2. *Let $E \,\varepsilon\, \mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathbb{P}^1)$. Then the quotient map $EW \underset{W}{\times} G/T \to G/N(T) = (G/T)/W$ induces a weak-equivalence $h_{Nis}(G/N(T), E) \simeq h_{Nis}(EW \underset{W}{\times} G/T, E)$ where $h_{Nis}(\quad, E)$ denotes the generalized cohomology spectrum computed on the Nisnevich site with respect to E. If $\bar{G}$ ($\bar{T}$) denotes the base extension of G (T) to the algebraic closure of k_0, then the map $EW \underset{W}{\times} \bar{G}/\bar{T} \to \bar{G}/N(\bar{T})$ induces a weak-equivalence $h_{et}(\bar{G}/N(\bar{T}), E) \simeq h_{et}(EW \underset{W}{\times} \bar{G}/\bar{T}, E)$ where $h_{et}(\quad, E)$ denotes the generalized cohomology spectrum computed on the étale site with respect to E, provided the homotopy groups of E are l torsion or E is Z/l-complete.*

Proof. First we will consider the case where E is replaced by an abelian sheaf (or abelian presheaf). In this case, the Leray spectral sequence in [J02, Theorem 4.2] applies in view of Proposition 7.1 and provides an isomorphism of the cohomology of $G/N(T)$ with respect to E with the cohomology of $EW \underset{W}{\times} G/T$ with respect to E. The general case when $E \,\varepsilon\, \mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathbb{P}^1)$ follows from this using the Atiyah-Hirzebruch type spectral sequences whose E_2-terms are $H^s_{Nis}(G/N(T), \pi_{-t}(E))$ and $H^s_{Nis}(EW \underset{W}{\times} G/T, \pi_{-t}(E))$. These spectral sequences exist as argued above (by assuming E is an Ω-spectrum and then by replacing E by the fibrant simplicial presheaf forming its 0-term) and converge strongly in view of the fact that the Nisnevich sites of $G/N(T)$ and G/T have finite cohomological dimension. This proves the first statement. The second follows similarly under the stronger hypotheses. $\square$

In these computations it seems essential that the linear algebraic group G be *split* over k_0, i.e. a maximal torus is split over k_0, the root data and the Weyl group are all defined over the given field k_0. In this case the action of the Weyl-group on G/T is defined over k_0 and so is the Bruhat decomposition: see [Spr98, Chapter 16] or [B-T65, 5. 15 Théorème]. The hypothesis that the group G be reductive is used only to ensure that it lifts to characteristic 0. We will denote by $\bar{G}_{\mathbb{C}}$ ($\bar{T}_{\mathbb{C}}$, $N(\bar{T}_{\mathbb{C}})$) the lift of $\bar{G}$ ($\bar{T}$, $N(\bar{T})$) to characteristic 0, i.e. over $\mathbb{C}$.

One may want to recall that the action of the Galois group Gal is only on the suspension coordinates of the spectra below.

Theorem 7.3. *Let X denote a linear scheme over k_0 (see Definition 4.4) which is also smooth and where the strata are isomorphic to product of affine spaces and split tori over k_0. We may also let $X = G/N(T)$ where $N(T)$ is the normalizer of a maximal torus in G, with G a split connected reductive group G, split over k_0 and T (B) is a fixed maximal torus of G (B a Borel subgroup containing T, respectively). Let $\bar{X}$ denote the scheme $X \underset{Spec\, k_0}{\times} Spec\, k$. Let l denote a fixed prime different from $char(k)$ and let Z/l_∞ and $\widetilde{Z/l}_\infty$ denote the Z/l-completions in the sense of* [CJ1, section 4]*.*

(i) Then the natural map

$$\Sigma_{\bar{\mathbb{P}^1}_{et}} \bar{X}_{et,+} \overset{L}{\underset{\Sigma_{\bar{\mathbb{P}^1}_{et}}}{\wedge}} \Sigma_{\mathbb{P}^1} \to \Sigma^{\mathbb{P}^1}_\infty X_{mht,+}$$

induces weak-equivalences

$$h_{Nis}(X, Z/l_\infty(\Sigma_{\mathbb{P}^1}), H) \overset{\simeq}{\to} h_{et}(\bar{X}, Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}), H) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})^H}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})^H$$

where h_{Nis} (h_{et}) denotes the generalized cohomology spectrum computed on the Nisnevich site (étale site, respectively).

(ii) Next let X denote linear schemes of the following form: (a) G/B or (b) G/T where G is a split connected reductive classical group G, split over k_0 and T (B) is a fixed maximal torus of G (B a Borel subgroup containing

T, respectively). We may also let $X = G/N(T)$ where $N(T)$ is the normalizer of a maximal torus in G, with G as above. Then one obtains the weak-equivalences:

$$Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}\bar{X}_{et,+})^H \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})^H}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})^H \simeq h^{et}(\bar{X}, Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}), H) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})^H}{\wedge}} \widetilde{Z/l}_\infty(\Sigma^\infty_{\mathbb{P}^1})^H \simeq h^{Nis}(X, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})^H)$$

of spectra in $\mathbf{Spt}_{\mathrm{s}}(\mathrm{k}_0, \mathrm{Gal}, \mathbb{P}^1)$.

Here we have used $Z/l_\infty\Sigma_{\bar{\mathbb{P}}^1_{et}}\bar{X}_{et,+}$ to denote the homotopy limit of the obvious pro-object denoted by the same symbol. Also, h^{Nis} (h^{et}) denotes the generalized homology spectrum computed on the Nisnevich site (étale site, respectively).

(iii) If X is any one of the schemes as in (ii), then it admits a lifting to a linear scheme $\bar{X}_{\bar{\mathbb{C}}}$ defined over $\mathbb{C}$ and one also obtains the weak-equivalences:

$$Z/l_\infty(\Sigma_{S^2}\bar{X}_{\mathbb{C}})) \simeq Z/l_\infty(\Sigma_{S^2}\bar{X}_{\mathbb{C}})_{et})$$

Proof. Since the action of the Galois group *Gal* is only on the suspension coordinates of the spectrum, we will suppress the group H and prove the statements only when $H = e$. This suffices since the action of the Galois group on a spectrum belonging to $\mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathbb{P}^1)$ is only on the suspension coordinates which are of the form $\mathbb{P}^1_K = \underset{Gal/K}{\wedge}\mathbb{P}^1$ and there it is by permuting the factors in the $\wedge$. Therefore this action is compatible with the action of *Gal* on the other spectra that appear in the proof where the suspension coordinates are $\mathbb{P}^1_{et,Gal} = \underset{Gal/K}{\wedge}\mathbb{P}^1_{et}$ or $S^2_K = \underset{Gal/K}{\wedge}S^2$.

Let X denote any of the schemes considered above. Then we obtain the identifications (upto weak-equivalences):

$$h_{et}(\bar{X}, Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \simeq \mathcal{H}om_{\Sigma_{\overline{\mathbb{P}^1}_{et}}}(\Sigma_{\overline{\mathbb{P}^1}_{et}}\bar{X}_{et,+}, Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \text{ and}$$

$$h_{Nis}(X, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})) \simeq \mathcal{H}om_{\Sigma^\infty_{\mathbb{P}^1}}(\Sigma_{\mathbb{P}^1}(X_{mht,+}), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})).$$

Moreover there is a natural map

$$\begin{aligned}(7.0.6)\quad h_{et}(\overline{X}, Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) &\simeq \mathcal{H}om_{\Sigma_{\overline{\mathbb{P}^1}_{et}}}(\Sigma_{\overline{\mathbb{P}^1}_{et}}\overline{X}_{et,+}, Z/l_\infty(\Sigma^\infty_{\overline{\mathbb{P}^1}_{et}})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \\ &\to \mathcal{H}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \overset{L}{\underset{Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})}{\wedge}} (Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})\overline{X}_{et,+}, \quad \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})))\end{aligned}$$

where $h_{et}(\bar{X}, Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}))$ ($h_{Nis}(X, Z/l_\infty(\Sigma_{\mathbb{P}^1}))$) is the generalized étale cohomology spectrum of $\bar{X}$ (generalized cohomology spectrum of X computed on the Nisnevich site, respectively). It is shown in Proposition 4.9, that the last term $\mathcal{H}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \overset{L}{\underset{Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})}{\wedge}} (Z/l_\infty(\Sigma^\infty_{\overline{\mathbb{P}^1}_{et}})\overline{X}_{et,+}, \quad \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})))$ provides a localization sequence when X is a linear scheme provided with the obvious stratification. There is also an obvious map

$$\begin{aligned}(7.0.7)\quad h_{Nis}(X, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})) &= \mathcal{H}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(\Sigma_{\mathbb{P}^1}(X_{mht,+}), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})) \\ &\to \mathcal{H}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \overset{L}{\underset{Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})}{\wedge}} (Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})\bar{X}_{et,+}, \quad \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})))\end{aligned}$$

induced by the map $\Sigma_{\overline{\mathbb{P}^1}_{et}}\bar{X}_{et,+} \overset{L}{\underset{\Sigma_{\overline{\mathbb{P}^1}_{et}}}{\wedge}} \Sigma_{\mathbb{P}^1} \to \Sigma_{\mathbb{P}^1}X_{mht,+}$.

We proceed to show that these maps induce weak-equivalences. Assume for the time being that X is a smooth linear scheme like G/T. Propositions 4.7 and 4.9 show that the terms on either side of (7.0.6) and (7.0.7) have localization sequences associated to the obvious stratification of such a linear scheme and that therefore we reduce to the case of a stratum which is the product of an affine space and a torus. Another application of the same localization sequences reduces to proving this when X is an affine space. Since the normal bundle to this stratum is trivial, we reduce to proving this when $\Sigma_{\mathbb{P}^1}X_{mht,+}$ ($\Sigma_{\bar{\mathbb{P}}^1_{et}}X_{et,+}$) is replaced by $\Sigma_{\mathbb{P}^1}[T_N]$ ($\Sigma_{\bar{\mathbb{P}}^1_{et}}[T_N]$, respectively) where N is the normal bundle to the stratum X and T_N is the corresponding Thom-space. Recall that the etale homotopy type of any affine space $\mathbb{A}^n_k$ completed at l is trivial, since k is algebraically closed. Therefore, in this case $h_{et}(\bar{X}, Z/l_\infty(\Sigma_{S^2})) \simeq h_{et}(Spec\, k, Z/l_\infty(\Sigma_{S^2}))$. Since, by definition $\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})$ is a fibrant object (i.e. an object that is $\mathbb{A}^1$-local) in the model category $\mathbf{Spt}_{\mathrm{s}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{Nis}, \mathbb{P}^1)$, a similar conclusion holds for $h_{Nis}(X, \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}))$. Therefore, both $h_{Nis}(\Sigma_{\mathbb{P}^1}[T_N], \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}))$ and $h_{et}(\Sigma_{\bar{\mathbb{P}}^1_{et}}[T_N], Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})$ identify with $\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})[-T_N]$. The definition of the smash-product of spectra next shows

$$\mathcal{H}om_{\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})}(\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \overset{L}{\underset{Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})}{\wedge}} (Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})\Sigma_{\bar{\mathbb{P}}^1_{et}}[T_N], \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})))$$

$$\simeq \mathcal{H}om_{Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})}((Z/l_\infty(\Sigma_{\overline{\mathbb{P}^1}_{et}})\Sigma_{\bar{\mathbb{P}}^1_{et}}[T_N], \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})))$$

Moreover making use of the properties of the completion functor, the last term also identifies with $\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})[-T_N]$. Therefore the map in question is a weak-equivalence in this case. The arguments above show that then the same holds for G/T and by the same arguments for any linear scheme that is smooth. These prove the statements in (i) for generalized cohomology for all varieties except $G/N(T)$.

Next recall $G/N(T)$ ($\bar{G}/N(\bar{T})$) identifies with the simplicial scheme $EW \underset{W}{\times} G/T$. ($EW \underset{W}{\times} \bar{G}/\bar{T}$, respectively). Therefore, the generalized cohomology of these objects are obtained as the homotopy limit $\underset{\Delta}{\mathrm{holim}}\{h_{et}(W^n \times \bar{G}/\bar{T}, Z/l_\infty(\Sigma_{S^2}))$ and $\underset{\Delta}{\mathrm{holim}}\{h_{Nis}(W^n \times G/T, \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}))$. Therefore, the maps in (7.0.6) and (7.0.7) will be weak-equivalences with $X = G/N(T)$ if all the maps in the diagram

$$(\underset{\Delta}{\mathrm{holim}}\{h_{et}(W^n \times \bar{G}/\bar{T}, Z/l_\infty\Sigma_{\bar{\mathbb{P}}^1_{et}})|n\}) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \to \underset{\Delta}{\mathrm{holim}}\{h_{et}(W^n \times \bar{G}/\bar{T}, Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})|n\}$$

$$\to \underset{\Delta}{\mathrm{holim}}\, h_{et}(W^n \times \bar{G}/\bar{T}, \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})) \leftarrow \underset{\Delta}{\mathrm{holim}}\, h_{Nis}(W^n \times G/T, \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}))$$

are also weak-equivalences. Since homotopy inverse limits preserve weak-equivalences between suitably fibrant objects, the last map is a weak-equivalence. The map before that may be easily seen to be a weak-equivalence, so that it reduces to showing the first map above is also a weak-equivalence. At this point, the hypotheses show that $\bar{G}/N(\bar{T}) = EW \underset{W}{\times} \bar{G}/\bar{T}$ lifts to characteristic 0, i.e. one may now assume the ground field is $\mathbb{C}$. Therefore, this map identifies with the map:

$$h(\underset{\Delta}{\mathrm{hocolim}}\{W^n \times \bar{G}/\bar{T}_{\mathbb{C}}|n\}, Z/l_\infty(\Sigma_{S^2})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \to h(\underset{\Delta}{\mathrm{hocolim}}\{W^n \times \bar{G}/\bar{T}_{\mathbb{C}}|n\}, \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}))$$

which in turn identifies with the map

$$\mathcal{H}om_{\Sigma^\infty_{S^2}}(\Sigma_{\bar{\mathbb{P}}^1_{et}}(EW \underset{W}{\times} (\bar{G}/\bar{T})_{\mathbb{C}}, Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \to \mathcal{H}om_{\Sigma_{\bar{\mathbb{P}}^1_{et}}}(\Sigma_{\bar{\mathbb{P}}^1_{et}}(EW \underset{W}{\times} (\bar{G}/\bar{T})_{\mathbb{C}}, \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})).$$

where $\mathcal{H}om_{\Sigma^\infty_{S^2}}$ denotes the internal hom in the category $\mathbf{Spt}_{\mathrm{s}}(\mathrm{k_0}, \mathrm{Gal}, \mathrm{S^2},)$. It is well-known that $\bar{G}/N(\bar{T})_{\mathbb{C}}$ has the homotopy type of a finite complex. Since both sides above are homotopy invariant, we may now assume $\bar{G}/N(\bar{T})_{\mathbb{C}}$ itself is a finite cell-complex. In this case an induction on the number cells will how that the last map above is a weak-equivalence.

These prove that the maps in (7.0.6) and (7.0.7) induces a weak-equivalence for $X = G/N(T)$ as well. These prove the statements in generalized cohomology for all the varieties considered.

Moreover, the proof of (ii) is clear in view of the arguments above and the definition of generalized homology: see 5.3. This also shows that $Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}\bar{X}_{et,+}))$ is a finite $Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})$-module spectrum: here we have used $Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}\bar{X}_{et,+}))$ to denote the homotopy limit of the obvious pro-object denoted by the same symbol when $X = G/N(T)$. Clearly the same conclusions hold when X is a linear scheme. In order to complete the proof of the theorem, we take the duals of $A = h_{et}(\bar{X}, Z/l_\infty(\Sigma_{\mathbb{P}^1_{et}})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})$ and $h_{Nis}(G/N(T), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}))$ by applying $\mathcal{H}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(\quad, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}))$. Then the identifications above show that

$$\mathcal{H}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(A, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})) \simeq Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}} \bar{X}_{et+}) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}).$$

By the definition of generalized homology as in Definition 5.3,

$$h^{Nis}(X, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})) = \mathcal{H}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(h_{Nis}(X, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})).$$

Therefore, we see that $h^{Nis}(X, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})) \simeq Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}} \bar{X}_{et+}) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})$ completing the proof of the theorem. □

Corollary 7.4. *Assume the hypothesis of the theorem. Then:*

$$\mathcal{H}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})^H}(h^{Nis}(G/N(T), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}), H), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})^H) \simeq h_{Nis}(G/N(T), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}), H) \text{ and}$$

$$\mathcal{H}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})^H}(h_{Nis}(G/N(T), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}), H), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})^H) \simeq h^{Nis}(G/N(T), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}), H).$$

Moreover, $h^{Nis}(G/N(T), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}), H) \simeq \widetilde{Z/l}_\infty((\Sigma_{\mathbb{P}^1} G/N(T)))^H \simeq (\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \overset{L}{\underset{\Sigma_{\mathbb{P}^1}}{\wedge}} \Sigma_{\mathbb{P}^1} G/N(T)_+)^H$.

Proof. Since the maps involved are all compatible with respect to the action of the Galois group, Gal, it suffices to prove the statements with $H = e$, the trivial group. In view of the hypotheses, one may lift the reductive group $\bar{G}$, and the root data to characteristic 0 so that $\bar{G}/N(\bar{T})$ also lifts to characteristic 0. It is well-known that over the complex numbers $G/N(T)$ has the homotopy type of a finite complex: for example, one may replace G by its compact form. Therefore a comparison of its completed homotopy type with the completed etale homotopy type of $\bar{G}/N(\bar{T})$ shows that that $Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}(\bar{G}/N(\bar{T}))_{et+}) \simeq Z/l_\infty \Sigma_{\bar{\mathbb{P}}^1_{et}}(G_{\mathbb{C}}/N(T_{\mathbb{C}}))$ is a finite cell spectrum over the completed sphere spectrum $Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})$. These prove the first two weak-equivalences. To see the last, one may first observe that the weak-equivalence $h^{Nis}(G/N(T), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})) \simeq Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}(\bar{G}/N(\bar{T})_{et+})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}^\infty)$ proven in Theorem 7.3 implies that $h^{Nis}(G/N(T), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})) \simeq \underset{\Delta}{\text{hocolim}}\{h^{Nis}((EW \underset{W}{\times} G/T)_n, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}))|n\}$. Since

$$h^{Nis}((EW \underset{W}{\times} G/T)_n, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})) \simeq \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}(EW \underset{W}{\times} G/T)_n) \simeq \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \overset{L}{\underset{\Sigma_{\mathbb{P}^1}}{\wedge}} (EW \underset{W}{\times} G/T)_n$$

for all $n \geq 0$, and $\underset{\Delta}{\text{hocolim}}$ commutes with $\overset{L}{\underset{\Sigma_{\mathbb{P}^1}}{\wedge}}$ we obtain the last weak-equivalence. □

8. Construction of Transfer maps

This is one of the few places in the paper, where we need to use both non-equivariant and equivariant spectra with respect to the action of the Galois group: we will therefore adopt the terminology in [CJ1, 3.1.5 Equivariant vs. non-equivariant spectra]. Accordingly $\Sigma_{\mathbb{P}^1, Gal}$ ($\Sigma_{\mathbb{P}^1}$) will denote the Galois-equivariant motivic $\mathbb{P}^1$-spectrum (the non-equivariant motivic $\mathbb{P}^1$-spectrum, respectively). We will let $\widetilde{\wedge}_{Gal} = \overset{L}{\underset{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1, Gal})}{\wedge}}$, $\wedge_{Gal} = \overset{L}{\underset{(\Sigma_{\mathbb{P}^1, Gal})}{\wedge}}$, $\widetilde{\wedge} = \overset{L}{\underset{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}{\wedge}}$, and $\wedge = \overset{L}{\underset{(\Sigma_{\mathbb{P}^1})}{\wedge}}$.

Associated to any self-map $f : G/N(T) \to G/N(T)$ over the base field k, we will presently define a *Gal*-equivariant pre-transfer map

$$\text{(8.0.8)} \quad \text{tr}(f)'_{Gal} : \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1, Gal}) \simeq \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1, Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1}(Spec\, k)_+ \to \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1, Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1, Gal}(G/N(T))_+$$

following the approach taken by Becker-Gottlieb (see [BG76] and also [DP]). In fact, the discussion in [DP] puts these constructions in a sufficiently general context that seems to apply to the situation at hand as well.

Given an $F \,\varepsilon\, \mathbf{Spt}(\mathrm{k}_0, \mathrm{Gal}, \mathbb{P}^1)$, we let $D^{Gal}(F) = \mathcal{RH}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal})}(F, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}))$. The non-equivariant version of this dual will be $D(F) = \mathcal{RH}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(F, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}))$. For the discussion below, we will let $F = \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}(G/N(T)_+)$.

We start with the obvious evaluation map μ (which we called the duality map) (defined in 6.1). Then we take its dual followed by a suitable suspension to obtain a map

$$\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}) \to D^{Gal}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}(G/N(T)_+)) \widetilde{\wedge}_{Gal} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(G/N(T)_+).$$

It is possible to take any non-negative integral values for s and $|T_V|$, in particular both may be 0. We will denote this map by ϕ_{Gal} from now on. This map is Gal-equivariant, when Gal acts diagonally on the right. (Observe that the Gal action on $\mathcal{RH}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal})}(F, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}))$ is by $(g, f) \mapsto g \circ f \circ g^{-1}$, $f \,\varepsilon\, \mathcal{RH}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(F, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}))$ and $g \,\varepsilon\, Gal$.)

Let $\phi : \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}^{s+2|T_V|,|T_V|}) \to D(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1}(G/N(T)_+)) \widetilde{\wedge} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1}^{s+2|T_V|,|T_V|}(G/N(T)_+)$

denote the corresponding non-equivariant map.

Definition 8.1. We define the *Gal-equivariant pre-transfer*, $tr(f)'_{Gal}$, to be the composition

$(\mu \wedge id) \circ (id \wedge \Delta(f)) \circ \phi_{Gal} : \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}) \to D^{Gal}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal} G/N(T)_+) \widetilde{\wedge}_{Gal} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,|T_V|}(G/N(T)_+)$

$\to D^{Gal}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,|T_V|} G/N(T)_+) \widetilde{\wedge}_{Gal} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}(G/N(T)_+) \widetilde{\wedge}_{Gal} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}(G/N(T)_+)$

$\to \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}) \wedge_{Gal} (\Sigma_{\mathbb{P}^1,Gal} G/N(T)_+)$.

Here $\Delta(f) : \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1}(G/N(T)_+) \to \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}(G/N(T)_+) \widetilde{\wedge}_{Gal} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}(G/N(T)_+)$ is the map $(\widetilde{Z/l}_\infty(id) \wedge_{Gal} f \wedge id) \circ \Delta$. We define the *(non-equivariant) pre-transfer*, $tr(f)'$, to be the corresponding composition involving non-equivariant spectra.

Proposition 8.2. *Assume the above situation. Then we obtain the Gal-equivariant weak-equivalence of Gal-equivariant spectra:*

$$\mathcal{RH}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal})}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}(G/N(T)_+), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}))$$

$$\simeq \mathcal{RH}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1} G/N(T)_+, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}))$$

Proof. We first obtain the identification:

(8.0.9) $$\mathcal{RH}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal})}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}(G/N(T)_+), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}))$$
$$\simeq \mathcal{RH}om_{\Sigma_{\mathbb{P}^1,Gal}}(\Sigma_{\mathbb{P}^1,Gal} G/N(T)_+, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}))$$ and

(8.0.10)
$$\mathcal{RH}om_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \wedge_{Gal} \Sigma_{\mathbb{P}^1}(G/N(T)_+), \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal})) \simeq \mathcal{RH}om_{\Sigma_{\mathbb{P}^1}}(\Sigma_{\mathbb{P}^1} G/N(T)_+, \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}))$$

Next recall from [CJ1, 3.1.5 Equivariant vs. non-equivariant spectra] that the spectrum $\Sigma_{\mathbb{P}^1,Gal}$ is a module spectrum over $\Sigma_{\mathbb{P}^1}$ and that the Galois group Gal has no action on $G/N(T)$. Therefore,

$$\Sigma_{\mathbb{P}^1,Gal} G/N(T)_+ \simeq \Sigma_{\mathbb{P}^1,Gal} \overset{L}{\underset{\Sigma_{\mathbb{P}^1}}{\wedge}} \Sigma_{\mathbb{P}^1} G/N(T)_+.$$

This then induces a weak-equivalence between the right-hand-sides of (8.0.9) and (8.0.10). Therefore, it suffices to show that this weak-equivalence preserves the Gal-action. If $f \,\varepsilon\, \mathcal{RH}om_{\Sigma_{\mathbb{P}^1}}(\Sigma_{\mathbb{P}^1} G/N(T)_+, Z/l(\Sigma_{\mathbb{P}^1,Gal}))$, let its image under the above map be $\bar f \,\varepsilon\, \mathcal{RH}om_{\Sigma_{\mathbb{P}^1,Gal}}(\Sigma_{\mathbb{P}^1,Gal} G/N(T)_+, Z/l(\Sigma_{\mathbb{P}^1,Gal}))$. Then $(gf)(m) = g(f(m))$, $m \,\varepsilon\, \Sigma_{\mathbb{P}^1} G/N(T)_+$, $g \,\varepsilon\, Gal$ and $\bar f(s.m) = sf(m)$, if m is as above and $s \,\varepsilon\, \Sigma_{\mathbb{P}^1,Gal}$. Therefore, $(g.\bar f)(s.m) = g\bar f(g^{-1}s.m) = g.(g^{-1}s.f(m)) = s.gf(m)$. The last equality follows from the observation that the Gal action on $\Sigma_{\mathbb{P}^1,Gal}$ is compatible with its ring structure, the first equality is because Gal acts on $\bar f$ by $(g\bar f)(s.m) = g.\bar f(g^{-1}s.m)$ and the next equality follows from the definition of $\bar f$. Now $\overline{(gf)}(s.m) = s.gf(m) = g\bar f(s.m)$. This shows that the map $f \mapsto \bar f$ is compatible with the Gal-action. □

Corollary 8.3. *Assume the above situation. Then the Galois-equivariant pre-transfer $tr(f)'_{Gal}$ identifies with $tr(f)' \overset{L}{\underset{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})}{\wedge}} id_{\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal})}$.*

Proof. The proof is clear in view of the last proposition. □

Now the following result is the key to the use of the pre-transfer:

Proposition 8.4. *The composite map*

$$\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}^{s+2|T_V|,|T_V|}) \overset{tr(f)'}{\to} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \wedge \Sigma s+2|T_V|,|T_V|_{\mathbb{P}^1}(G/N(T)_+) \overset{pr}{\to} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}^{s+2|T_V|,|T_V|})$$

where pr denotes the obvious collapse map has degree Λ_f = the Lefschetz-number of f.

Proof. The key observation that we make is the following: since

$\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1}(G/N(T)_+) \simeq Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et,+}} \overline{G/N(T)}_{et+}) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et,+}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})$, it follows that

$D(\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1} G/N(T)_+) \simeq \mathcal{RH}om_{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}(Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}(\overline{G/N(T)}_{et+})), Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et,+}})}{\wedge}} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1})$.

Therefore, the required property follows from the corresponding transfer constructed in the context of étale homotopy for the variety $\overline{G/N(T)}$ and in fact reduces to the corresponding property of the classical Becker-Gottlieb transfer defined for the lifting of $\overline{G/N(T)}$ to characteristic 0. □

Corollary 8.5. *The composite map*

$$pr_{Gal} \circ tr(f)'_{Gal} : \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}) \overset{tr(f)'_{Gal}}{\to} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,|T_V|} G/N(T)_+ \overset{pr}{\to} \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V})$$

is a unit in $[Z/l(\Sigma_{\mathbb{P}^1,Gal}), Z/l(\Sigma_{\mathbb{P}^1,Gal})]$ where $[A,B]$ denotes Hom in the homotopy category $\mathbf{HSpt}_{mot}(k_0, Gal)$ and/or $\mathbf{HSpt}_{et}(k_0, Gal)$ and where $pr : \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}(G/N(T)_+) \to \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1_Gal})$ is the obvious collapse map.

Proof. The above results show that the composition $pr_{Gal} \circ tr(f)'_{Gal}$ identifies with $(pr \circ tr(f)') \overset{L}{\underset{\Sigma_{\mathbb{P}^1}}{\wedge}} id_{\Sigma_{\mathbb{P}^1,Gal}}$. The corollary follows. □

Definition 8.6. (The transfer.) Next we proceed to construct the transfer map $tr(f)_{Gal} : \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(BG_+) \to \widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(EG\underset{G}{\times} G/N(T)_+)$.

The key observations are the identifications:

$\Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(EG\underset{G}{\times} G/N(T)_+) \simeq EG \ltimes_G \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(G/N(T)_+)$,

$\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(EG\underset{G}{\times} G/N(T)_+) \simeq EG \ltimes_G (\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(G/N(T)_+))$,

$\Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(EG\underset{G}{\times}(Spec\, k)_+) \simeq EG \ltimes_G \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}$, and

$\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge_{Gal} \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(EG\underset{G}{\times}(Spec\, k)_+) \simeq EG \ltimes_G (\widetilde{Z/l}_\infty(\Sigma_{\mathbb{P}^1,Gal}) \wedge \Sigma_{\mathbb{P}^1,Gal}^{s+2|T_V|,T_V}(Spec\, k)_+)$.

(Here $X \ltimes Y = X_+ \wedge Y$ when Y is a pointed simplicial presheaf and X is an unpointed simplicial presheaf.)

8.1. **Motivic cohomology of linear schemes and of $G/N(T)$.** Though the following computation is not used elsewhere in the paper, we include it here since it will put the results of the last theorem in better perspective. $H^*(\quad ;\mathbb{Z}/l)$ will denote mod-l-motivic cohomology and $H^*_{et}(\quad ;\mu_l)$ will denote étale cohomology with respect to μ_l.

Proposition 8.7. *Let X denote one of the schemes G/B, G/T or $G/N(T)$ defined over the field k_0 and let k denote a fixed algebraic closure of k_0. Then*

(i) $H^*_{\mathcal{M}}(X,\mathbb{Z}/l) \cong CH^*(X;Z/l)\underset{Z/l}{\otimes} H^*_{\mathcal{M}}(Spec\, k_0;\mathbb{Z}/l)$ if $X = G/B$ or $X = G/T$ and in these cases $CH^*(X;Z/l) \cong H^*_{et}(\bar{X},\mu_l)$ where $\bar{X} = X \underset{Spec\, k_0}{\times} Spec\, k$

(ii) If $X = G/N(T)$, $H^*_{\mathcal{M}}(X,\mathbb{Z}/l) \cong H^*_{et}(\bar{X},\mu_l)\underset{Z/l}{\otimes} H^*_{\mathcal{M}}(Spec\, k_0;\mathbb{Z}/l)$.

(iii) In all of the above cases $H^*_{\mathcal{M}}(X;\mathbb{Z}/l)$ is a free module over $H^*_{\mathcal{M}}(Spec\, k_0;\mathbb{Z}/l)$. Moreover, in all of the above cases, on also obtains the identification $H^*_{\mathcal{M}}(X;\mathbb{Z}/l) \cong H^*(RHom(M(X)/l,Z/l))\underset{Z/l}{\otimes} H^*_{\mathcal{M}}(Spec\, k_0;\mathbb{Z}/l)$ where $M(X)/l$ denotes the mod$-l$ motive of X: recall this is a complex of sheaves.

Proof. First we consider the case where $X = G/B$. In this case, the Bruhat decomposition provides decomposition of G/B into affine spaces. The arguments apply more generally to any projective smooth scheme X provided with the action of the multiplicative group $\mathbb{G}_m$ and the resulting Bialynicki-Birula decomposition: see [delBano]. Though these were originally stated for the case where the ground field is algebraically closed, they have now been extended to the case where the ground field is any field: see [Bros]. The basic idea is that the long-exact localization sequences in higher Chow groups break-up into short exact sequences. We will provide some details of this argument, mainly for the sake of completeness.

First one observes that $X^{\mathbb{G}_m}$ is smooth and projective and breaks up into finitely many components, X_α. $\alpha = 0,\cdots,n$. (When $X = G/B$, these components are just k-rational points on X.) Then X itself breaks up as the disjoint union of locally closed smooth sub-schemes X^+_α, so that (i) each X^+_α is provided with a $\mathbb{G}_m$-equivariant map $\pi_\alpha : X^+_\alpha \to X_\alpha$, (ii) $(X^+_\alpha)^{\mathbb{G}_m} = X_\alpha$. Moreover, there exists a descending filtration $X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n \supseteq Y_{n+1} = \phi$ by closed $\mathbb{G}_m$-stable sub-schemes Y_i, with $X^+_\alpha = Y_\alpha - Y_{\alpha+1} = X^+_\alpha$.

Let $d_\alpha =$ the codimension of X^+_α in X and let Γ_{π_α} denote the closure of the graph of π_α in $X_\alpha \times X$. This defines a class, m_α, in $CH^{dim(X_\alpha)+d_\alpha}(X_\alpha \times X)$. Composition with this class defines maps: $CH^{i-d_\alpha}(X_\alpha, j; Z/l)\overset{p^*_\alpha}{\to} CH^{i-d_\alpha}(X_\alpha \times X, j; Z/l)\overset{\cap m_\alpha}{\to} CH^{i+dim(X_\alpha)}(X_\alpha \times X, j; Z/l)\overset{p_{2*}}{\to} CH^i(X, j; Z/l)$ where p_i denotes the projection to the i-th factor. It is shown in [delBano, Theorem 2.4] that these maps provide a decomposition:

$$\underset{\alpha}{\oplus} CH^{i-d_\alpha}(X_\alpha, j; Z/l)\overset{\cong}{\to} CH^i(X, j; Z/l) \tag{8.1.1}$$

In our case $X = G/B$ and each X_α is a k-rational point. Moreover, taking $j = 0$, shows that $CH^*(X, Z/l) \cong \oplus_{w\,\epsilon\, W} Z/l$, where the summands correspond to the various Schubert-cells indexed by the Weyl group W. Therefore, after identifying the higher Chow-groups of G/B with its motivic-cohomology, we obtain the isomorphism:

$$H^*_{\mathcal{M}}(G/B;\mathbb{Z}/l) \cong CH^*(G/B, Z/l)\underset{Z/l}{\otimes} H^*_{\mathcal{M}}(Spec\, k;\mathbb{Z}/l) \tag{8.1.2}$$

Moreover, in this case it is clear that $H^*_{et}(\overline{G/B};\mu_l) \cong \oplus_{w\,\epsilon\, W} Z/l$. Therefore, these statements prove all the assertions in the theorem, except for the last part of (iii) for the case $X = G/B$.

Next we consider the last assertion in (iii) for $X = G/B$. observe first that $M(X)/l$ is a resolution of the constant sheaf $\underline{Z}/l_X$ so that, $R\Gamma(X,\mathbb{Z}/l) \simeq RHom(M(X)/l,\mathbb{Z}/l)$. One has a natural map $RHom(M(X)/l,\mathbb{Z}/l)\overset{L}{\underset{\mathbb{Z}/l}{\otimes}} Z/l \to RHom(M(X)/l, Z/l)$. Using the cell-filtration on X, one may show that this map is a quasi-isomorphism. This proves the last assertion in (iii).

When $X = G/T$, observe that there is a natural smooth map $G/T \to G/B$ whose fibers are all $R_u(B)$ and hence affine spaces. The homotopy property for motivic cohomology shows therefore, that the above statements for $X = G/B$ carry over to $X = G/T$.

Next we consider the case $X = G/N(T)$. Recall that we already showed we may identify the simplicial scheme $EW \underset{W}{\times} G/T$ with $G/N(T)$. Therefore the corresponding assertions for $G/N(T)$ follow from the ones for G/T by observing that the cohomology spectra of a simplicial scheme is defined as the homotopy limit of the cohomology spectra of the schemes in each simplicial degree. □

9. Motivic Atiyah duality

Recall Atiyah duality is the statement that the Thom-space of the normal bundle to imbedding a compact manifold in a large sphere is Spanier-Whitehead dual to the stable homotopy type of the manifold. This was extended to projective algebraic varieties over algebraically closed fields in the context of étale homotopy theory in [J86]. Recall that in [Voev, Proposition 2.7], it is shown that for every projective smooth variety X, there exists a vector bundle $\mathcal{N}$ together with a Thom-Pontrjagin collapse map: $TP : T^{d+n} \to Th(\mathcal{N})$. Here T^{d+n} is a $\mathbf{T}$-sphere of dimension $d+n$ (and identified with the projective space $\mathbb{P}^{d+n}_{k_0}$) and $Th(\mathcal{N})$ is the Thom-space of the vector bundle $\mathcal{N}$. As another application of the theory developed in this paper, we are able to provide an independent proof that this collapse map provides a version of *Atiyah-duality* (see [At]) in the motivic context when X is assumed to be both linear and smooth projective. This is independent of [Voev] which contains a (rather difficult to follow) proof for general smooth projective schemes.)

The starting point of this in all contexts is the construction of a Thom-Pontrjagin collapse map from a large sphere to a suitable Thom-space. In [Voev, Proposition 2.7], Voevodsky obtained a version of this which we will employ. It is shown in [Voev, Proposition 2.7] that there exists a vector bundle N on $\mathbb{P}^d$ along with a map

(9.0.3) $$TP_1 : T^{d+n} \to Th_{\mathbb{P}^d}(N)$$

where T^{d+n} is the sphere $S^{d+n} \wedge \mathbb{G}_m^{\wedge^n}$. This map is *degree one* in the sense that $TP^*(a) = t$ where a denotes the top class in the motivic cohomology of $Th_{\mathbb{P}^d}(N)$ and t denotes the canonical generator of $H^{2d+2n}(T^{d+n}, \mathbb{Z}(d+n))$

Next assume X is a projective smooth scheme over k_0 imbedded in the projective space $\mathbb{P}^d$ with normal bundle V. Then the immersion $N - \mathbb{P}^d \to N - X$ induces a second Thom-Pontrjagin collapse map

(9.0.4) $$TP_2 : Th_{\mathbb{P}^d}(N) \to Th_X(N) \simeq Th_X(N_{|X} \oplus V).$$

This is also degree one in the sense that the TP_2^* pulls-back the top class of $Th_X(N_{|X} \oplus V)$ in motivic cohomology to the top class of $Th_{\mathbb{P}^d}(N)$. Composing the two Thom-Pontrjagin collapse maps, we obtain

(9.0.5) $$TP : T^{d+n} \to Th_X(N_{|X} \oplus V) \text{ and } TP_{et} : \bar{T}^{d+n}_{et} \to Th_{\bar{X}}(\bar{N}_{|\bar{X}} \oplus \bar{V})_{et}$$

where the subscript et denotes the étale topological type and the $^-$ denotes the corresponding schemes defined over the algebraic closure of k_0. Next we may compose TP with the diagonal map $\Delta : Th_X(N_{|X} \oplus V) \to X_+ \wedge Th_X(N_{|X} \oplus V)$. Taking the corresponding $\mathbb{P}^1$-suspension spectra, we obtain the map $\mu : \Sigma_{\mathbb{P}^1} T^{d+n} \to \Sigma_{\mathbb{P}^1} X_+ \wedge \Sigma_{\mathbb{P}^1} Th_X(N_{|X} \oplus V)$.

Theorem 9.1. *Assume in addition that X is a linear scheme. Then the following hold:*

(i) Both $\Sigma_{\mathbb{P}^1}(X)$ and $\Sigma_{\mathbb{P}^1} \wedge Th_X(N_{|X} \oplus V)$ are finite $\Sigma_{\mathbb{P}^1}$-module spectra. In fact $\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1}(X) \simeq \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \overset{L}{\underset{Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}} Z/l_\infty(\Sigma_{\bar{\mathbb{P}}^1_{et}}(X_{et}))$ and $\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \wedge \Sigma_{\mathbb{P}^1}(Th_X(N_{|X} \oplus V)) \simeq \widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1}) \overset{L}{\underset{\Sigma_{\bar{\mathbb{P}}^1_{et}}}{\wedge}} \Sigma_{\bar{\mathbb{P}}^1_{et}}(Th_X(N_{|X} \oplus V)_{et})$

(ii) The dual of the map $id_{\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})} \wedge \mu$ is a duality map in the sense of Definition 6.1.

Proof. The finiteness of $\Sigma_{\mathbb{P}^1}(X)$ is already established in Proposition 4.7. We will consider the given filtration $\phi = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = X$ so that each $F_i - F_{i-1} = \sqcup \mathbb{A}^{m_i} \times \mathbb{G}_m^{n_i}$ which is a finite sum of products of affine spaces and split tori. We will show by descending induction on k that the suspension spectra $\Sigma_{\mathbb{P}^1} Th_{X-F_k}(N_{|X-F_k} \oplus V_{|X-F_k})$ are all finite cellular. When $k = n-1$, this is clear since $X = F_n$ and $F_n - F_{n-1}$ is a finite sum of products of affine-spaces and split tori. Therefore any vector bundle over $F_n - F_{n-1}$ is trivial and we may assume $\Sigma_{\mathbb{P}^1} Th_{X-F_k}(N_{|X-F_k} \oplus V_{|X-F_k})$ is finite cellular for $1 \le k+1 \le n$ and we will show that this implies the corresponding statement for $k = k-1$. For this one observes the stable co-fiber sequence in $\mathbf{Spt}_{\mathrm{mot}}(\mathrm{k}_0)$:

$$\Sigma_{\mathbb{P}^1}(Th_{X-F_{k+1}}(N_{|X-F_{k+1}} \oplus V_{|X-F_{k+1}})) \to \Sigma_{\mathbb{P}^1}(Th_{X-F_k}(N_{|X-F_k} \oplus V_{|X-F_k})) \to \Sigma_{\mathbb{P}^1}\Sigma^N_{\mathbb{P}^1}(F_{k+1}-F_k)_+$$

where $N = rank(N) + rank(V) + codim_{F_{k+1}-F_k}(X - F_k)$. The existence of this stable cofiber sequence follows from the homotopy purity theorem as in [MV, section3, Theorem 2.23]. The identification of the last term of the above stable cofiber sequence as claimed follows from the observation that the stratum $F_{k+1} - F_k$ is assumed to be a disjoint union of affine spaces and tori and hence that every vector bundle over it is trivial as observed in the proof of Proposition 4.7. This proves that finiteness of the $\mathbb{P}^1$-suspension spectra in the first statement. The comparison with completed suspension spectra associated to the étale homotopy types follows as in the proof of Theorem 7.3.

In view of (i), the map $id_{\widetilde{Z/l_\infty}(\Sigma_{\mathbb{P}^1})} \wedge \mu$ may be identified with the map $id \wedge Z/l_\infty(\bar{\mu}_{et})$, where $\wedge$ denotes $\overset{L}{\underset{Z/l(\Sigma_{\bar{\mathbb{P}}^1_{et}})}{\wedge}}$ and id denotes the identity map of $\widetilde{Z/l_\infty}\Sigma_{\mathbb{P}^1}$ and $\bar{\mu}_{et}$ denotes the induced map on the étale topological types of the corresponding objects defined over the algebraic closure of k_0. Therefore, it suffices to show that the map $Z/l_\infty(\mu_{et})$ is a duality map. At this point one may invoke [J86] to complete the proof of (ii). (The basic technique there is to show that slant-product with the class $\bar{\mu}_{et}\hat{(}id_{d+n})$ induces a map of Atiyah-Hirzebruch spectral sequences defining generalized étale homology and cohomology with respect to any constant presheaf of spectra whose homotopy groups are l-primary torsion. This reduces to showing that the above slant-product induces an isomorphism at the E_2-terms of these spectral sequences which are étale homology and cohomology. Then one makes of use a triangle involving Thom-isomorphism, Poincaré duality and the slant-product with the class $\bar{\mu}_{et}(id_{d+n})$ as in [Sw, 14.42 Proposition].) □

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