Isovariant Étale Descent and Riemann-Roch for Algebraic Stacks

§1 Motivation

Recall Thomason's theorem from the early 1980s:

•l a prime invertible in the schemes X we consider, $\nu >> 0$

• $G/l^{\nu}(X)$: the mod$-l^{\nu}$ G-theory spectrum of X

• $G^{top}/l^{\nu}(X)$: the mod$-l^{\nu}$ topological G-theory spectrum of X

• base scheme: k an algebraically closed field, $char(k) \geq 0$

Thomason's Theorem: X of finite type over k. Then the augmentation:

$$G^{top}/l^{\nu}(X) \to \mathbb{H}_{et}(X, \mathbf{G}^{top}/l^{\nu}(\quad))$$

is a weak-equivalence. Therefore, there is a spectral sequence:

$$E_2^{s,t} = H^s_{et}(X, \pi_t \mathbf{G}^{top}/l^{\nu}(\quad)/l^{\nu}) \Rightarrow \pi_{t-s}(G^{top}/l^{\nu}(X$$

Among the *applications*:

• a general Riemann-Roch theorem, i.e. the square

$$\begin{array}{ccc} G/l^{\nu}(X) & \longrightarrow & G^{top}/l^{\nu}(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*^{top}} \\ G/l^{\nu}(Y) & \longrightarrow & G^{top}/l^{\nu}(Y) \end{array}$$

homotopy commutes for any proper map $f : X \to Y$.

• (trivial application) The E_2-terms of the above spectral sequence provide a definition of étale cohomology of X when X is smooth.

Goal of the Talk: sketch an outline of how to extend Thomason's theorem to algebraic stacks. We begin with

An elementary counterexample. k, X as before. G a finite constant group scheme acting on X. Then Thomason's theorem is *false* as stated for the quotient stack $[X/G]$!

Recall: $ob[X/G](T) = \{$principal G bundles $\mathcal{E}$ on T along with a G-equivariant map $f : \mathcal{E} \to X\}$

A morphism $(\mathcal{E} \to T)$ to $(\mathcal{E}' \to T')$ given by a commutative square

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\psi} & \mathcal{E}' \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & T' \end{array}$$

inducing an isomorphism $\mathcal{E} \xrightarrow{\alpha} \phi^*(\mathcal{E}')$ and where $f = f' \circ \phi$.

Sketch of a proof

- (Coherent sheaves on $[X/G]$) $\simeq$ (Coherent sheaves on X with descent data) $\simeq$ (G-equivariant sheaves on X). Therefore $G^{top}/l^\nu(X)$ $\simeq G^{top}(X,G)/l^\nu = G(X,G)/l^\nu[\beta^{-1}]$.

- $X \to [X/G]$ étale over of $[X/G]$. Hence any

$U \to X$ étale belongs to $[X/G]_{et}$. Therefore the presheaf $\mathbf{G}^{top}/l^{\nu}(\quad)$ on $[X/G]_{et}$ identifies with $U \to G^{top}/l^{\nu}(U)$.

• $EG \underset{G}{\times} X = cosk_0^{[X/G]} X$ so that

$\mathbb{H}_{et}([X/G], \mathbf{G}^{top}/l^{\nu}(\quad))$

$\simeq \mathbb{H}_{et}(EG \underset{G}{\times} X, \mathbf{G}^{top}/l^{\nu}(\quad)) \simeq G^{top}/l^{\nu}(EG \underset{G}{\times} X)$

• Therefore, it suffices to show $G^{top}(X, G)/l^{\nu} \to G^{top}(EG \underset{G}{\times} X)/l^{\nu}$ is not a weak-equivalence. This follows from classical results of Atiyah-Segal if $k = \mathbb{C}$.

Remarks

1. Similarly $G(\quad)_{\mathbf{Q}}$ does not have étale descent for algebraic stacks.

2. Most cohomology theories for stacks defined on the étale (or smooth) sites: for quotient stacks $[X/G]$ these are Borel style equivariant cohomology theories of X.

3. The above two are the main difficulties with the Riemann-Roch for algebraic stacks.

§2 Main Results

Theorem 1. Assume $\mathcal{S}$ is a gerbe over its coarse moduli space $\mathfrak{M}$ which exists as an algebraic space. Then $\mathcal{S}_{iso.et} \simeq \mathfrak{M}_{et}$

Theorem 2. Let $\mathcal{S}$ be an algebraic stacks. Then there exists a finite filtration $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \ldots \subseteq \mathcal{S}_n = \mathcal{S}$ by locally closed algebraic substacks with each $\mathcal{S}_i - \mathcal{S}_{i-1}$ a gerbe over its coarse moduli space.

The isovariant étale topos of $\mathcal{S}$ is obtained by gluing the isovariant étale topos of each $\mathcal{S}_i - \mathcal{S}_{i-1}$. This topos has enough points and they correspond to the geometric points of the coarse moduli spaces associated to each $\mathcal{S}_i - \mathcal{S}_{i-1}$.

Theorem 3. Let $\mathcal{S}$ be any algebraic stack. Then:

- the augmentation

$$G^{top}/l^{\nu}(\mathcal{S}) \to \mathbb{H}_{iso.et}(\mathcal{S}, \mathbf{G}^{top}/l^{\nu}(\quad))$$

is a weak-equivalence.

There exists a spectral sequence

$$E_2^{s,t} = H^s_{iso.et}(\mathcal{S}, \pi_t(\mathbf{G}^{top}/l^{\nu}(\quad))$$

$$\Rightarrow \pi_{t-s}(G^{top}/l^{\nu}(\mathcal{S}))$$

Applications

• Riemann-Roch for algebraic stacks: If $f : \mathcal{S}' \to \mathcal{S}$ is proper and of finite cohomological dimension, then

$$\begin{array}{ccc} G/l^{\nu}(\mathcal{S}') & \longrightarrow & G^{top}/l^{\nu}(\mathcal{S}') \\ f_* \downarrow & & \downarrow f_*^{top} \\ G/l^{\nu}(\mathcal{S}) & \longrightarrow & G^{top}/l^{\nu}(\mathcal{S}) \end{array}$$

commutes upto homotopy.

• For smooth stacks, the E_2-terms define Bredon-style cohomology theories for algebraic stacks.

• Definition of other Bredon-style cohomology-homology theories for algebraic stacks made possible using this site.

§3 Algebraic Stacks: a quick review

(schemes) $\subseteq$ (alg.spaces) $\subseteq$ (orbifolds) $\subseteq$ (D-M stacks) $\subseteq$ (Artin stacks)

• (schemes) $\subseteq$ functors($(schemes)^{op} \to$ (sets)) by Yoneda

• lack of fine moduli spaces means one needs to consider

$$(lax.functors) : (schemes)^{op} \to (groupoids)$$

Examples

1. *The moduli stack of smooth curves of genus g:*

ob($M_g(T)$) = { smooth proper maps $p : C \to T$— fibers geometrically connected curves of genus $= g\}$

morphisms: $(C \to T) \to (C' \to T')$ are commutative squares

$$\begin{array}{ccc} C & \longrightarrow & C' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

inducing an isomorphism $C \simeq C' \underset{T'}{\times} T$. This is a $D-M$ stack.

2. $Vect_r^X$*: X complete*

$\mathrm{ob}(Vect_r^X(T) = \{$ vector bundles $\mathcal{E}$ of rank r on $X \times T\}$,

morphisms $(\mathcal{E} \to T) \to (\mathcal{E}' \to T')$ given by a map $f : T \to T'$ and an isomorphism $\alpha : \mathcal{E} \simeq (id \times f)^*(\mathcal{E}')$

3. *Quotient Stacks*: $[X/G]$

Definitions

Let $\{U_i \to U | i\}$ be a flat cover.

A *stack* $\mathcal{S}$ is a lax-functor $(schemes)^{op} \to (groupoids)$ so that the following hold:

- *Gluing of morphisms*: given X, $Y \in \mathcal{S}(U)$ and $\phi_i : X_{|U_i} \to Y_{|U_i}$ which are compatible, then there exists a unique morphism $\phi : X \to Y$ so that $\phi_{|U_i} = \phi_i$ for all i.

- *Gluing of objects*: if $X_i \in \mathcal{S}(U_i)$ and $\phi_{i,j} : X_j | U_i \underset{U}{\times} U_j \to X_i | U_i \underset{U}{\times} U_j$ satisfy an obvious co-cycle condition, there exists an object $X \in \mathcal{S}(U)$

and $\phi_i : X_{|U_i} \xrightarrow{\simeq} X_i$ so that $\phi_{j,i} \circ \phi_i | U_i \underset{U}{\times} U_j = \phi_j | U_i \underset{U}{\times} U_j$.

Morphisms of stacks: lax natural transformations.

• An *algebraic stack* is a stack $\mathcal{S}$ so that (i) $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable, quasi-compact and separated and (ii) there exists a smooth surjective map $x : X \to \mathcal{S}$ with X an algebraic space.

• $\mathcal{S}$ is Deligne-Mumford (or D-M) if there exists an $x : X \to \mathcal{S}$ étale with X an algebraic space.

The Inertia Stack is defined by the cartesian square:

$$\begin{array}{ccc} I_{\mathcal{S}} & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \Delta \\ \mathcal{S} & \underset{\Delta}{\longrightarrow} & \mathcal{S} \times \mathcal{S} \end{array}$$

$f : \mathcal{S}' \to \mathcal{S}$ is *isovariant* if the induced map $I_{\mathcal{S}'} \to I_{\mathcal{S}} \underset{\mathcal{S}}{\times} \mathcal{S}'$ is an isomorphism.

Example: $\mathcal{S} = [X/G]$. Then $\mathcal{S}' \to \mathcal{S}$ is isovariant if and only if $\mathcal{S}' = [Y/G]$ for $Y \to X$ isovariant, i.e. equivariant and induces an isomorphism on the isotropy subgroups.

The Isovariant Étale site: $\mathcal{S}_{iso.et}$. Objects of this site are isovariant étale maps $\mathcal{S}' \to \mathcal{S}$.

Morphisms are commutative triangles of such stacks.

Reference:

R. Joshua: *Riemann-Roch for algebraic stacks: I*, to appear in Compositio Math, (in press)

Also see

http://www.math.ohio-state.edu/~ joshua/pub.html

for a preprint in dvi downloadable format.