COHOMOLOGY OPERATIONS IN MOTIVIC, ÉTALE AND DE RHAM-WITT COHOMOLOGY

PATRICK BROSNAN AND ROY JOSHUA

Abstract. In this note we explore the relationships between motivic cohomology operations and cohomology operations defined on mod-l étale cohomology as well as cohomology with respect to the De Rham-Witt sheaves. We show that the cohomology operations on mod-l motivic cohomology with l different from the characteristic of the ground field transform to the (classical) cohomology operations on mod-l étale cohomology upon inverting the motivic Bott element. We also show that when $l = p =$ the characteristic of the ground field, the classical operations in mod-p motivic cohomology transform to cohomology operations in cohomology with respect to the De Rham-Witt sheaves. We also introduce variants of the above cohomology operations that are covariant with respect to proper maps and consider several examples of push-forward formulae involving them.

Table of contents

1. Introduction

The main goals of this paper are to explore the relationships between the various cohomology operations: the motivic and classical operations in motivic cohomology, the cohomology operations in étale cohomology and the induced operations in cohomology with respect to the De Rham-Witt sheaves. We also consider variants of these cohomology operations that are covariant for proper maps and study various examples of push-forward formulae involving them.

Throughout the paper k will denote a fixed perfect field of characteristic $p \geq 0$ and we will restrict to the category of sooth schemes of finite type over k. X will denote such a scheme. $H^n_{\mathcal{M}}(X, \mathbb{Z}(r))$ will denote the motivic cohomology with *degree* n and *weight* r; $H^n_{\mathcal{M}}(X, \mathbb{Z}/l(r))$ will denote the corresponding mod-l-variant. Similarly $H^n_{et}(X, \mathbb{Z}/l(r))$ will denote the mod-l étale cohomology of X if l is different from $char(k) = p$ and $H^n(X, \nu(r))$ will denote the (Zariski or étale) cohomology with respect to the De Rham-Witt sheaves $\{\nu(r)|r \geq 0\}$.

In the first four sections we study comparisons between the motivic cohomology operations as in [Voev], the classical operations in motivic cohomology as in [J1] and operations in mod$-l$ étale cohomology with l different from the characteristic of the base field. In this situation we will also assume that k has a primitive l-th root of unity. $H^n_{\mathcal{M}}(X, \mathbb{Z}(r))$ will denote the motivic cohomology with *degree* n and *weight* r; $H^n_{\mathcal{M}}(X, \mathbb{Z}/l(r))$ will denote the corresponding mod-l-variant. Similarly $H^n_{et}(X, \mathbb{Z}/l(r))$ will denote the mod-l étale cohomology of X. We will restrict to smooth schemes of finite type over k.

Let $\beta \varepsilon H^0_{\mathcal{M}}(Spec \quad k, \mathbb{Z}/l(1))$ denote the *Motivic Bott element*: see section 3 for more details. In this situation, let $P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$ and $\beta P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$ denote the motivic cohomology operations defined in [Voev] and recalled below in the next section. As shown in section 3, these operations induce operations on mod-l-étale cohomology which we identify with the mod-l-motivic cohomology with the Bott element inverted: these will be denoted by the same symbols. By the results of Theorem 1.2 and section 6 of [J1] the complex $\mathcal{A} = R\Gamma(X_{et}, \mu_l)$ is an E_∞-algebra over an E_∞-operad.

The second author was supported by the IHES, the MPI (Bonn) and a grant from the NSA.

Therefore one obtains certain (*classical*) cohomology operations $Q^r : H^{2q}_{et}(X, \mu_l(q)) \to H^{2q+2r(l-1)}_{et}(X, \mu_l(q.l))$ and $\beta Q^r : H^{2q}_{et}(X, \mu_l(q)) \to H^{2q+2r(l-1)+1}_{et}(X, \mu_l(q.l))$

Then the main result of these sections is the following: this is discussed in section 4.

Theorem 1.1. *Assume the above situation. Then we obtain the following relations between the classical operations and the operations on étale cohomology induced by the motivic operations:*

$$Q^r = B^{(q-r).(l-1)}.P^r, \quad \beta Q^r = B^{(q-r).(l-1)}.\beta P^r$$

In the fifth section we show that the classical cohomology operations defined in mod$-p$ motivic cohomology as in [J1] provide cohomology operations in cohomology with respect to the De Rham-Witt sheaves. We state this as the following theorem:

Theorem 1.2. *There exist operations* $Q^s : H^q(X_{Zar}, \nu(t)) \to H^{q+(2s-t)(p-1)}(X_{Zar}, \nu(p.t))$ *and*

$\beta Q^s : H^q(X_{Zar}, \nu(t)) \to H^{q+(2s-t)(p-1)+1}(X_{Zar}, \nu(p.t))$.

These operations satisfy the properties as discussed in Theorem 5.1

The sixth section is devoted to cohomology operations that commute with respect to proper push-forwards and various examples of push-forward formulae involving these. The following is one such example and a few more are considered in that section.

Example 1.3. Let $dim(X) = 3$ and $l = 2$. Then we obtain an operation

$Q_1 : H^4(X, \mathbb{Z}/2(2)) \to H^6(X, \mathbb{Z}/2(4))$.

so that the composition with the push-forward $\pi_* : H^6(X, \mathbb{Z}/2(4)) \to H^0(Spec \quad k, \mathbb{Z}/2(1))$ is zero (assuming the validity of the Beilinson-Soulé vanishing conjecture.)

Here is an outline of the paper. In the next section, we recall the motivic cohomology operations from [Voev]. In the third section we recall the motivic Bott element. In the fourth section we first recall a well-known result that the operadic construction of classical cohomology operations leads to the cohomology operations Q^r and βQ^r that may be defined using equivariant cohomology. Then we complete the proof of Theorem 1.1. In the fifth section we consider cohomology operations when $l = p = char(k)$ and interpret these as operations in cohomology with respect to the De Rham-Witt sheaves. In the final section we consider variants of all of the above operations that commute with push-forwards by proper maps. We conclude by discussing several examples of such operations.

2. The motivic cohomology operations (after Voevodsky)

The basic reference for this section is [Voev]. We begin with the computation of the motivic cohomology of $B\pi$ where $\pi = \mathbb{Z}/l$ and $\pi = \Sigma_l$ (the symmetric group on l letters) where l is a fixed prime different from the characteristic ($= p$) of the ground field k.

We begin by recalling briefly the construction of the *geometric classifying space of a linear algebraic group*: originally this is due to Totaro and Edidin-Graham - see [Tot] discussed in [J1, section 4]. Let G be a linear algebraic group over $S = Spec \quad k$ i.e. a closed subgroup-scheme in GL_n over S for some n. For a (closed) embedding $i : G \to GL_n$ *the geometric classifying space* $B_{gm}(G; i)$ of G with respect to i is defined as follows. For $m \geq 1$ let U_m be the open sub-scheme of $\mathbb{A}^{nm}_S$ where the diagonal action of G determined by i is free. Let $V_m = U_m/G$ be the quotient S-algebraic space of the action of G on U_m induced by the (diagonal) action of G on $\mathbb{A}^{nm}_S$; the projection $U_m \to V_m$ defines V_m as the quotient algebraic space of U_m by the free action of G and V_m is thus smooth. We have closed embeddings $U_m \to U_{m+1}$ and $V_m \to V_{m+1}$ corresponding to the embeddings $Id \times \{0\} : \mathbb{A}^{nm} \to \mathbb{A}^{nm} \times \mathbb{A}^n$ and we set $EG^{gm} = \lim_{m\to\infty} U_m$ and $BG^{gm} = \lim_{m\to\infty} V_m$ where the colimit is taken in the category of sheaves on $(sschms/S)_{Nis}$ or on $(sschms/S)_{et}$. Observe that if $G = \Sigma_n$ (or a subgroup of it) acting on $\mathbb{A}^n$ by permuting the n-coordinates, we may take $U_m = \{(x(1)_1, \cdots, x(1)_n, \cdots, x(m)_1, \cdots x(m)_n) | x(i)_j \neq x(i)_k, \quad for \quad all \quad i, \quad j \neq k\}$. (Moreover, in this case, the quotients V_m are in fact schemes.)

2.0.1. The following are proven in [Voev, section 6]:

- the map $i_m : U_m/G \to U_{m+1}/G$ defines an isomorphism on motivic cohomology of weight less than m.

• One has $H^*_{\mathcal{M}}(BG^{gm}, \mathbb{Z}(r)) = \lim_{\infty \leftarrow m} H^*_{\mathcal{M}}(U_m/G, \mathbb{Z}(r))$ where $r \geq 0$ is any weight.

• Let μ_l denote the group scheme of l-th roots of unity $\mu_l := ker(\mathbf{G}_m \xrightarrow{z^l} \mathbf{G}_m)$. (Observe that since the field k is assumed to have a primitive root of unity, one may identify μ_l with the constant sheaf $\pi = \mathbb{Z}/l$.) Then one has the identification:

$$B\mu_l = \mathcal{O}(-l)_{\mathbf{P}^\infty} - z(\mathbf{P}^\infty) \tag{2.0.2}$$

We have $U_m = \mathbf{A}^m - \{0\}$.

• Therefore, one has a cofibration sequence of the form

$$X_+ \wedge (B\mu_l)_+ \to X_+ \wedge (\mathcal{O}(-l)_{\mathbf{P}^\infty})_+ \to X_+ \wedge Th(\mathcal{O}(-l)) \tag{2.0.3}$$

• $e(\mathcal{O}(-l)) = l\sigma$ where $\sigma \in H^2_{\mathcal{M}}(\mathbf{P}^\infty; \mathbb{Z}(1))$ is the class of the first Chern class of $\mathcal{O}(-1)$ in motivic cohomology. Here X is any smooth scheme. Therefore, the long exact sequence defined by (2.0.3) is of the form

$$\ldots \to H^{*-2}_{\mathcal{M}}(X, \mathbb{Z}(\star - 1)[[\sigma]] \xrightarrow{l\sigma} H^*_{\mathcal{M}}(X, \mathbb{Z}(\star - 1))[[\sigma]] \to H^*_{\mathcal{M}}(B\mu_l, \mathbb{Z}(\star)) \to H^{*-1}(X, \mathbb{Z}(\star - 1)[[\sigma]] \to \ldots \tag{2.0.4}$$

(In the above long-exact-sequence and elsewhere, $*$ ($\star$) will denote the degree (the weight, respectively) in motivic cohomology.)

The short exact sequence of abelian groups $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/l \to 0$ defines a homomorphism $\delta : \tilde{H}^*_{\mathcal{M}}(-, \mathbf{Z}/l(\star)) \to \tilde{H}^{*+1}_{\mathcal{M}}(-, \mathbf{Z}(\star))$. Let v be Euler class of the line bundle on $X \times B\mu_l$ corresponding to the tautological representation of μ_l. There exists a unique element $u \in H^1(X \times B\mu_l, \mathbf{Z}/l(1))$ such that the restriction of u to $*$ is zero and $\delta(u) = v$. (Here $*$ denotes any k-rational point of $B\mu_l$ that lifts to a k-rational point of one of the U_m appearing in the definition of the $E\mu_l$.)

• We will denote by $\bar{v}$ the image of the class v in $H^2_{\mathcal{M}}(X \times B\mu_l, \mathbb{Z}/l(1))$. Now the elements $\bar{v}^i$ and $u\bar{v}^i$, $i \geq 0$ form a basis of $H^*_{\mathcal{M}}(X \times (B\mu_l), \mathbb{Z}/l(\star))$ over $H^*_{\mathcal{M}}(X, \mathbb{Z}/l(\star))$.

The *next key observation* is that the same arguments also hold for the mod-l étale cohomology of $B\mu_l$, so that we may conclude:

Let *cycl* denote the cycle map from mod-l motivic to mod-l étale cohomology. Let $c(\bar{v})$ denote the Euler class of of the same bundle on $X \times B\mu_l$ in $H^2_{et}(X \times B\mu_l, \mathbb{Z}/l(1))$. Then $c(\bar{v}) = cycl(\bar{v})$, and there exists a unique class $c(u) \varepsilon H^1_{et}(X \times B\mu_l, \mathbb{Z}/(1))$ so that $\delta(c(u)) = c(\bar{v})$ and $c(u) = cycl(u)$. Then the elements $c(\bar{v})^i$ and $c(u)c(\bar{v})^i$, $i \geq 0$ form basis of $H^*_{et}(X \times B\mu_l, \mathbb{Z}/l(\star))$ over $H^*_{et}(X, \mathbb{Z}/l(\star))$.

Next one may compute the mod-l motivic cohomology and mod-l étale cohomology of the symmetric group Σ_l similarly. We recall this from [Voev]:

2.1. $H^*_{\mathcal{M}}(X \times B\Sigma_l; \mathbb{Z}/l(\star))$ is a free module over $H^*_{\mathcal{M}}(X; \mathbb{Z}/l(\star))$with a basis $\{c\bar{d}^i, d^i | i \geq 0\}$ where $\bar{d}$ is a class in $H^{2l-2}_{\mathcal{M}}(X \times B\Sigma_n; \mathbb{Z}/l(l-1))$ which is the mod-l reduction of a class $d \varepsilon H^{2l-2}_{\mathcal{M}}(X \times B\Sigma_n; \mathbb{Z}(l-1))$ and c is a class in $H^{2l-3}_{\mathcal{M}}(X \times B\Sigma_n; \mathbb{Z}/l(l-1))$ so that $\delta(c) = \bar{d}$.

Going over the computation, one observes as in the case of $B\mu_l$, that the same computation carries over to mod-l étale cohomology.

Next we recall the definition of the cohomology operations of Voevodsky. Let X denote a smooth scheme over k.

Now the symmetric group Σ_l acts on $X^{\times^l}$ by permutations. In this context, one has the total power operation:

$$\tilde{\mathcal{P}}_l : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \to H^{il}_{\mathcal{M}}(E\Sigma_l \underset{\Sigma_l}{\times} X^{\times^l}, \mathbb{Z}/l(jl)) \tag{2.1.1}$$

Next one uses the pull-back by the diagonal $\Delta^* : H^{il}_{\mathcal{M}}(E\Sigma_l \underset{\Sigma_l}{\times} X^{\times^l}, \mathbb{Z}/l(jl)) \to H^{il}_{\mathcal{M}}(B\Sigma_l \times X, \mathbb{Z}/l(jl))$. We will denote the composition $\Delta^* \circ \tilde{\mathcal{P}}_l$ by $\mathcal{P}_l$. By the above results, $\underset{i,j}{\oplus} H^{in}_{\mathcal{M}}(B\Sigma_l \times X, \mathbb{Z}/l(jl))$ is a free module over $H^*_{\mathcal{M}}(X, \mathbb{Z}/(\star))$

with basis given by the elements $\bar{d}^r$ and $c\bar{d}^r$, $r \geq 0$. The operation P^r (βP^r) is defined by the formula:

$$\mathcal{P}_l(w) = \Sigma_{r\geq 0} P^r(w)\bar{d}^{d-r} + \beta P^r(w) c\bar{d}^{d-r-1}, \quad w \varepsilon H^{2d}(\quad, \mathbb{Z}/l(d)) \tag{2.1.2}$$

(A crucial observation is that, since the motivic cohomology operations are stable with respect to shifting degrees by 1, and also both degrees and weights by 1, this defines the operations P^r and βP^r on all $H^i_{\mathcal{M}}(\quad, \mathbb{Z}/l(j))$.)

Observe that so defined $P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \to H^{i+2r(l-1)}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$ and $\beta P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \to H^{i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$.

In view of the observations above, exactly the same definitions will define the cohomology operations in mod-l étale cohomology as well. We will denote the operations $P^r : H^i_{et}(X, \mathbb{Z}/l(j)) \to H^{i+2r(l-1)}_{et}(X, \mathbb{Z}/l(j + r(l-1)))$ ($\beta P^r : H^i_{et}(X, \mathbb{Z}/l(j)) \to H^{i+2r(l-1)+1}_{et}(X, \mathbb{Z}/l(j + r(l-1))))$ by P^r_{et} (βP^r_{et}, respectively). Therefore, we obtain the following result:

Theorem 2.1. *Denoting the cycle map from motivic cohomology to étale cohomology by cycl, we obtain:* $cycl \circ P^r = P^r_{et} \circ cycl$ *and* $cycl \circ \beta P^r = \beta P^r_{et} \circ cycl$.

3. Inverting the Motivic Bott element

Recall that if k is a field as above, we have:

$$\begin{aligned} H^p_{\mathcal{M}}(Spec \quad k, \mathbb{Z}(1)) &= 0, p \neq 1 \\ &= k^*, p = 1 \end{aligned} \tag{3.0.3}$$

Now the universal coefficient sequence associated to the short exact sequence $0 \to \mathbb{Z}(1) \xrightarrow{\times l} \mathbb{Z}(1) \to \mathbb{Z}/l(1) \to 0$ of motivic complexes, provides the isomorphism

$$H^0_{\mathcal{M}}(Spec \quad k, \mathbb{Z}/l(1)) \cong \mu_l(k) \tag{3.0.4}$$

The *Motivic Bott element* is the class in $H^0_{\mathcal{M}}(Spec \quad k, \mathbb{Z}/l(1))$ corresponding under the above isomorphism to the primitive l-th root of unity ζ. We will denote this element by B. Since $cycl(B) = \zeta$ in $H^*_{et}(\quad, \mu_l(*))$, multiplication by the class $cycl(B)$ induces an isomorphism: $H^*_{et}(\quad, \mu_l(r)) \to H^*_{et}(\quad, \mu_l(r+1))$. It follows that the cycle map *cycl* induces a map of cohomology functors:

$$cycl(B^{-1}) : H^*_{\mathcal{M}}(\quad, \mathbb{Z}/l(\star))[B^{-1}] \to H^*_{et}(\quad, \mu_l(\star)) \tag{3.0.5}$$

It is shown in [Lev] that this map is an isomorphism on smooth schemes.

3.1. Observe that by the multiplicative properties of the operations and the observation that $P^r(B) = 0$ if $r \geq 1$ ([Voev, Lemma 9.8]):

$$P^r(B^j\alpha) = B^j P^r(\alpha), \tag{3.1.1}$$

$$\beta P^r(B^j\alpha) = B^j \beta P^r(\alpha) \tag{3.1.2}$$

The above relations show that the motivic cohomology operations above induce operations on $H^*(\quad, \mathbb{Z}/l(\star))[B^{-1}]$ in the obvious manner: we define $P^r(\alpha.B^{-1}) = P^r(\alpha).B^{-1}$ and $\beta P^r(\alpha.B^{-1}) = \beta P^r(\alpha).B^{-1}$. Since we have already observed that the cohomology operations commute with the cycle map, it follows that the induced operations on $H^*(\quad, \mathbb{Z}/l(\star))[B^{-1}]$ may be identified with the cohomology operations on mod-l étale cohomology.

3.2. These operations on mod$-l$ étale cohomology will be denoted P^r and βP^r.

4. Comparison of cohomology operations in étale cohomology

We will begin by defining classical cohomology operations in étale cohomology. For this we start with a smooth scheme X and let $\mathcal{A} = R\Gamma(X_{et}, \mu_l)$. We let $\{N\mathbb{Z}E\Sigma_n|n\}$ denote the simplicial Barratt-Eccles operad defined in [J1, Definition 4.1]. By the results in Theorem 1.1 and section 6 of [J1], this acts on the complex $\mathcal{A} = R\Gamma(X_{et}, \mu_l)$. We will let $\mathcal{H}om(K, \mathbb{Z}/l)$ be denoted by $K^\vee$, if K is a complex of $\mathbb{Z}/l$-sheaves on $(smt.schms)_{et}$ or a complex of $\mathbb{Z}/l$-vector spaces. Let π denote the cyclic group $\mathbb{Z}/l$ imbedded as a subgroup of the symmetric group Σ_l.

If H is any subgroup of the symmetric group Σ_l, we will define the *equivariant cohomology* of $\mathcal{A}^{\otimes^l}$ with respect to H as follows: $H^*(\mathcal{A}^{\otimes^l}, H; \mathbb{Z}/l)$ will be the cohomology of the complex $(N\mathbb{Z}E\Sigma_l)^\vee \underset{ZH}{\otimes} \mathcal{A}^{\otimes^l}$.

4.0.1. For our comparison purposes, it is important to realize that the equivariant cohomology defined above is nothing other than equivariant étale cohomology. We proceed to explain this identification. First of all let H also denote the obvious constant group-scheme defined over the field k and associated to the sub-group H of Σ_l. Now H acts as a group scheme on the scheme X^l; therefore we may form the simplicial scheme $EH \underset{H}{\times} X^l$ in the obvious manner. We define the H-equivariant mod-l étale cohomology of X^l to be the mod-l étale cohomology of the simplicial scheme $EH \underset{H}{\times} X^l$. This identifies with the equivariant cohomology $H^*(\mathcal{A}^{\otimes l}, H, \mathbb{Z}/l)$. (In fact one may identify the complex $R\Gamma(EH \underset{H}{\times} X^l, \mathbb{Z}/l)$, up-to quasi-isomorphism, with the complex $(N\mathbb{Z}E\Sigma_l)^\vee \underset{ZH}{\otimes} \mathcal{A}^{\otimes^l}$.)

4.0.2. We need to also compare the H-equivariant mod-l étale cohomology defined above with the equivariant étale cohomology obtained by inverting the Bott element in H-equivariant mod-l motivic cohomology. Recall that the definition of H-equivariant mod-l-motivic cohomology uses the *geometric model* for the classifying space for H as opposed to the simplicial model. However, it is shown in [MV] that the two variants give isomorphic mod-l étale cohomology, i.e. $H^*_{et}(BG^{gm}, \mathbb{Z}/l) \cong H^*_{et}(BG, \mathbb{Z}/l)$ where BG denotes the simplicial classifying space for G considered above.

Let $\Delta^* : H^*(\mathcal{A}^{\otimes^l}, \pi; \mathbb{Z}/l) \to H^*(\mathcal{A}, \pi; \mathbb{Z}/l) \cong H^*(B\pi; \mathbb{Z}/l) \otimes H^*(\mathcal{A})$ denote the obvious map induced by the diagonal $:X \to X^l$. One may also observe readily that the l-th power map defines a map $H^*(\mathcal{A}) \to H^*(\mathcal{A}^{\otimes^l}, \pi; \mathbb{Z}/l)$, $a \mapsto a^l$. Let $\{w^i, vw^i | i \geq 0\}$ denote a basis of the $\mathbb{Z}/l$-vector space $H^*(B\pi; \mathbb{Z}/l)$. Here v has degree 1 and w has degree two. Since the cohomology operations are assumed to be stable, they are stable with respect to suspension so that it suffices to define these on classes of even degree. One defines cohomology operations $Q^s, \beta Q^s$ on $H^{2q}(\mathcal{A})$ by the formula: if $l = 2$, we let:

$$\Delta^*(x^2) = \Sigma_s Q^s(x) w^{(q-s)} + \beta Q^s(x) v w^{q-s-1} \tag{4.0.3}$$

and if $l > 2$, we let:

$$\Delta^*(x^l) = \Sigma_s (-1)^{d-s} Q^s(x) w^{(q-s)(l-1)} + (-1)^{d-s} \beta Q^s(x) v w^{(q-s)(l-1)-1} \tag{4.0.4}$$

In [J1, Section 7.1] we provided the action of the operad $\{NE\Sigma_n | n\}$ on the motivic complex $\mathbb{Z}/l^{\nu}{}^{mot}_{et}$ which is the mod-l^ν motivic complex sheafified on the big étale site of smooth schemes. One may identify the complex $\mathbb{Z}/l^\nu(r)^{mot}_{et}$ with $\mu_{l^\nu}(r)[0]$ up-to quasi-isomorphism: see [MVW, Theorem 10.3]. These lead to a somewhat different definition of the *classical* cohomology operations on mod$-l^\nu$ -étale cohomology as discussed in [J1, Section 8]. We will explain in outline that these operations are in fact identical. (Since most of this is folklore, we will be brief.)

Proposition 4.1. *The cohomology operations defined above coincide with the classical cohomology operations defined on mod-l étale cohomology in* [J1, Section 8].

Proof. For the rest of this section we will denote $\mathbb{Z}/l^{\nu}{}^{mot}_{et}$ by $\mathcal{A}$. The above action of the operad $\{NE\Sigma_n | n\}$ on the above complex provides us maps

$$\theta_n : N\mathbb{Z}E\Sigma_n \otimes \mathcal{A}^{\otimes^n} \to \mathcal{A} \tag{4.0.5}$$

We will let $\mathcal{H}om(K, \mathbb{Z}/l)$ be denoted by $K^\vee$, if K is a complex of $\mathbb{Z}/l$-sheaves on $(smt.schms)_{et}$. From the above pairing we obtain

$\theta^*_n : N\mathbb{Z}E\Sigma_n \otimes \mathcal{A}^\vee \to (\mathcal{A}^\vee)^{\otimes^n}$

where we define $\theta^*_n(h, a^\vee)(a_1 \otimes \cdots \otimes a_n) = < a^\vee, \theta_n(h \otimes a_1 \otimes \cdots \otimes a_n) >$, $a_i \varepsilon \mathcal{A}$, $a^\vee \varepsilon \mathcal{A}^\vee$ and $h \varepsilon N\mathbb{Z}E\Sigma_n$. It is a standard result in this situation that the map θ^*_n is a chain map and is an *approximation to the diagonal map* (i.e. homotopic to the diagonal map) $\Delta : \mathcal{A}^\vee \to (\mathcal{A}^\vee)^{\otimes^n}$. (Here, as well as elsewhere in this section, we use the observation that for any vector space V over $\mathbb{Z}/l$, a vector $v \varepsilon V$ (a vector $v^\vee \varepsilon V^\vee$) is determined by its pairing $< v, w >$ with all vectors $w \varepsilon V^\vee$ (its pairing $< u, v^\vee >$ with all vectors $u \varepsilon V$, respectively.).)

We now define

$$d : N\mathbb{Z}E\Sigma_n \otimes \mathcal{A}^\vee \to N\mathbb{Z}E\Sigma_n \otimes (\mathcal{A}^\vee)^{\otimes^n} \tag{4.0.6}$$

by the formula $d(h, a^\vee) = (h, \theta_n^*(h, a^\vee))$. This in turn defines a map

$$d^* : (N\mathbb{Z}E\Sigma_n)^\vee \otimes \mathcal{A}^{\otimes^n} \to (N\mathbb{Z}E\Sigma_n)^\vee \otimes \mathcal{A} \tag{4.0.7}$$

by the formula:

$$< d^*(h^\vee, a_1 \otimes \cdots \otimes a_n), h' \otimes a^\vee >=< d(h', a^v), h^\vee \otimes a_1 \otimes \cdots \otimes a_n >$$

$$=< \theta_n(h', a_1 \otimes \cdots a_n), a^\vee > \otimes < h', h^\vee >.$$

(Here $h' \varepsilon N\mathbb{Z}E\Sigma n$, $h^\vee \varepsilon (N\mathbb{Z}E\Sigma n)^\vee$, $a^\vee \varepsilon A^\vee$, $a_i \varepsilon \mathcal{A}$.) We now let $n = l$ and let π denote the cyclic subgroup $\mathbb{Z}/l$ of Σ_l. One may recall that the action of $\sigma \varepsilon \Sigma_n$ on $NE\Sigma_n$ and of σ^{-1} on $\mathcal{A}^{\otimes^n}$ cancel out. Tracing through these actions of Σ_n on the maps in the above steps, one concludes that the map d^* induces a map on the quotients:

$$d^* : (N\mathbb{Z}E\Sigma_n)^\vee \underset{Z\pi}{\otimes} \mathcal{A}^{\otimes^n} \to (N\mathbb{Z}E\Sigma_n)^\vee \underset{Z\pi}{\otimes} \mathcal{A} \tag{4.0.8}$$

Now the cohomology of the complex $(N\mathbb{Z}E\Sigma_n)^\vee \underset{N(\mathbb{Z}(\pi))}{\otimes} \mathcal{A}$ identifies with $H^*(B\pi; \mathbb{Z}/l) \otimes H^*(\mathcal{A})$ whereas the cohomology of the complex $(N\mathbb{Z}E\Sigma_n)^\vee \underset{Z\pi}{\otimes} \mathcal{A}^{\otimes^n}$ identifies with the equivariant cohomology: $H^*(\mathcal{A}^{\otimes^n}, \pi; \mathbb{Z}/l)$. Therefore, the map d^* defines a map

$$\bar{d}^* : H^*(\mathcal{A}^{\otimes^n}, \pi; \mathbb{Z}/l) \to H^*(B\pi; \mathbb{Z}/l) \otimes H^*(\mathcal{A}) \tag{4.0.9}$$

One may also observe readily that the l-th power map defines a map $H^*(\mathcal{A}) \to H^*(\mathcal{A}^{\otimes^l}, \Sigma_l; \mathbb{Z}/l)$, $a \mapsto a^l$. Let $\{e_i, fe_i | i \geq 0\}$ denote a basis of the $\mathbb{Z}/l$-vector space $H_*(B\Sigma_n; \mathbb{Z}/l)$ dual to the basis $\{w^i, vw^i | i \geq 0\}$ for $H^*(B\Sigma_n; \mathbb{Z}/l)$, i.e. $< e_i, w^j >= 0$, if $i \neq j$ and $= 1$ if $i = j$. Also $< fe_i, w^j >= 0$ for all i , j, $< fe_i, vw^j >= 0$ for $i \neq j$ and $= 1$ for $i = j$. Observe that now we have the following computation for a class $x \varepsilon H^q(\mathcal{A})$:

$$< \bar{d}^*(x^l), (-)^\vee \otimes e_i >=< \bar{\theta}_l^*(e_i, (-)^\vee), x^l >=< (-)^\vee, \bar{\theta}_l(e_i, x^l) > \text{ and}$$

$$< \bar{d}^*(x^l), (-)^\vee \otimes fe_i >=< \bar{\theta}_l^*(fe_i, (-)^\vee), x^l >=< (-)^\vee, \bar{\theta}_l(fe_i, x^l) >$$

where $(-)^\vee \varepsilon H^*(\mathcal{A})^\vee$ and $\bar{\theta}$ ($\bar{\theta}_l^*$) is the map induced by θ (θ_l^*, respectively) on taking homology of the corresponding complexes. Since the map θ_l^* was observed to be chain homotopic to the diagonal, it follows that $\bar{d}^* = \Delta^*$ where Δ is the obvious diagonal. Therefore, the coefficient of w^i (vw^i) in the expansion of $\bar{d}^*(x^l) \varepsilon H^*(B\Sigma_l; \mathbb{Z}/l) \otimes H^*(\mathcal{A})$ identifies with $\bar{\theta}_i(e_i, x^l)$ ($\bar{\theta}_i(fe_i, x^l)$, respectively). This completes the proof of the proposition □

The formulae in (4.0.3) and (4.0.4) are stated in terms of the cohomology of the cyclic groups. This has a reformulation in terms of the cohomology of the symmetric groups Σ_l which will be readily comparable to the formula in (2.1.2). First one may compute the cohomology of the symmetric group $H^*(B\Sigma_l, \mathbb{Z}/l)$ to be the $\mathbb{Z}/l$-vector space with basis given by $\{y^i, xy^i | i \geq 0\}$ where y is a class in $H^{2l-2}(B\Sigma_l; \mathbb{Z}/l)$ and x is a class in $H^{2l-3}(B\Sigma_l; \mathbb{Z}/l)$. In fact $y = \prod_{i=1}^{l-1} iw = (l-1)! w^{l-1} = -w^{l-1}$ where w is the class in $H^2(B\pi; \mathbb{Z}/l)$ considered in (4.0.4). Now $x = -vw^{l-2}$. Then the cohomology operation Q^r and βQ^r expressed in terms of the equivariant cohomology with respect to the symmetric group replacing the equivariant cohomology with respect to the cyclic group $\mathbb{Z}/l$ has the following form:

$$\Delta^*(w^l) = \Sigma_{r \geq 0} Q^r(w) y^{d-r} + \beta P^r(w) x y^{d-r-1}, \quad w \varepsilon H^{2d}_{et}(\quad, \mathbb{Z}/l(d)) \tag{4.0.10}$$

This uniquely defines the cohomology operations as they are stable with respect to suspension and hence extend uniquely to cohomology classes with odd degree.

Proof of Theorem 1.1.

Here the observations in 4.0.1 and 4.0.2 are important. In addition, one needs to observe that weight-suspension in mod-l étale cohomology is defined by multiplication by the Bott element B: since B is a unit, the weight suspension is an isomorphism in mod-l étale cohomology.

Therefore, the main difference of the above formula with the one in 3.2 is that the classes y^i and xy^i have no weight, or equivalently have weight 0. Observe that the operations Q^r and βQ^r defined above are maps:

$Q^r : H^{2q}_{et}(X, \mu_l(q)) \rightarrow H^{2q+2r(l-1)}_{et}(X, \mu_l(q.l))$ and

$\beta Q^r : H^{2q}_{et}(X, \mu_l(q)) \rightarrow H^{2q+2r(l-1)+1}_{et}(X, \mu_l(q.l))$

Since the operations above raise a cohomology class in $H^{2q}_{et}(X, \mu_l(q))$ to the l-th power, and the classes y^i and xy^i have zero-weight, the Tate-twist $q.l$ appears in the target of these operations.

Therefore, we obtain the relation between the classical operations and the operations on étale cohomology induced by the motivic operations as follows:

$$Q^r = B^{(q-r).(l-1)}.P^r, \quad \beta Q^r = B^{(q-r).(l-1)}.\beta P^r$$

Remark 4.2. There are alternate procedures for obtaining cohomology operations on $\oplus_r H^*(X_{et}, \mu_l(r))$. One approach is to use [Ep] which applies in more generality to provide cohomology operations on cohomology with respect to any *strictly associative and commutative* sheaf of algebras which are torsion. Over a separably closed field, one may also use the the cohomology operations defined on the étale homotopy types.

5. **Operations in mod$-p$ motivic cohomology and cohomology with respect to the De Rham-Witt sheaves**

Recall from [J1, section 8] that we obtain classical cohomology operations

$Q^s : H^q(X, \mathbb{Z}/p(t)) \rightarrow H^{q+2s(p-1)}(X, \mathbb{Z}/p(p.t))$ and

$\beta Q^s : H^q(X, \mathbb{Z}/p(t)) \rightarrow H^{q+2s(p-1)+1}(X, \mathbb{Z}/p(p.t))$.

These operations satisfy the properties discussed in [J1, Theorem 8.2].

Let $\nu(r)$ be the sheaf that is kernel of $W^* - C : Z\Omega^r_{X/S} \rightarrow \Omega^r_{X^{(p)}/S}$. Here $X^{(p)}$ is the scheme obtained as the pull-back of $X \underset{S}{\times} S$ where the map $S \rightarrow S$ is the absolute Frobenius and $S = Spec\, k$ is the base field. Moreover W^* is defined as the adjoint to the obvious map $\Omega^r_{X/S} \rightarrow W_* \Omega^r_{X^{(p)}/S}$ and $Z\Omega^r_{X/S}$ denotes the kernel of the differential $d : \Omega^r_{X/S} \rightarrow \Omega^{r+1}_{X/S}$. (See [Ill, 2.4] for more details.) It is known that $\nu(0) = \mathbb{Z}/p$, $\nu(1) = dlog(\mathcal{O}^*_X)$ and that $\nu(r)$, viewed as a sheaf on X_{et} is generated locally by $dlog(x_1). \cdots dlog(x_r)$, $x_i \varepsilon \mathcal{O}^*_X$.

It is shown in [GL, Theorem 8.4] that if X is a smooth integral scheme over k and k is perfect, then one has the natural isomorphism

$$H^s(X_?, \nu(r)) \cong H^{s+r}(X_?, \mathbb{Z}/p(r)) \tag{5.0.11}$$

where the right-hand-side is the mod$-p$ motivic cohomology of the scheme X and $X_?$ denotes either X_{Zar} or X_{et}. Therefore, it is clear that the operations considered above define operations on $\oplus_r H^*(X_?, \nu(r))$ and we obtain the following theorem. Moreover, the above isomorphism suggests that these classical operations are likely to be the only operations that exist in $mod - p$ motivic cohomology: we obtain these as a direct consequence of the operad actions constructed in [J1]. See [J1, Theorem 8.2] for more details.

Theorem 5.1. *Assume the above situation. Let $X_?$ denote either X_{Zar} or X_{et}.*

(i) Operations on the Zariski and étale cohomology with respect to the sheaves $\nu(r)$s: One obtains operations $Q^s : H^q(X_?, \nu(t)) \rightarrow H^{q+(2s-t)(p-1)}(X_?, \nu(pt))$ and $\beta Q^s : H^q(X_?, \nu(t)) \rightarrow H^{q++(2s-t)(p-1)+1}(X_?, \nu(pt))$. These satisfy the following properties:

(ii) Contravariant functoriality: If $f : X \rightarrow Y$ is a map between smooth schemes over k, $f^* \circ Q^s = Q^s \circ f^*$

(iii) Let $x \varepsilon H^q(X_?, \nu(t))$. $Q^s(x) = 0$ if $2s - t > q$, $\beta Q^s(x) = 0$ if $2s - t \geq q$ and if $(2s - t = q)$, then $Q^s(x) = x^p$.

(iv) If β is the Bockstein, $\beta \circ Q^s = \beta Q^s$.

(v) Cartan formulae: For all primes l, $Q^s(x \otimes y) = \underset{i+j=s}{\Sigma} Q^i(x) \otimes Q^j(y)$ and

$\beta Q^s(x \otimes y) = \underset{i+j=s}{\Sigma} \beta Q^i(x) \otimes Q^j(y) + Q^i(x) \otimes \beta Q^j(y)$

(vi) Adem relations: With the same terminology as before:

If $a < lb$, and $\epsilon = 0, 1$ one has

$$\beta^\epsilon Q^a Q^b = \Sigma_i (-1)^{a+i}(a - pi, (l-1)b - a + i - 1)\beta^\epsilon Q^{a+b-i} Q^i \tag{5.0.12}$$

where $\beta^0 Q^s = Q^s$ while $\beta^1 Q^s = \beta Q^s$. One also has

$$\begin{aligned} \beta^\epsilon Q^a \beta Q^b = (1-\epsilon)\Sigma_i (-1)^{a+i}(a - pi, (l-1)b - a + i - 1)\beta Q^{a+b-i} Q^i \\ -\Sigma_i (-1)^{a+i}(a - pi - 1, (l-1)b - a + i)\beta^\epsilon Q^{a+b-i} \beta Q^i \end{aligned} \tag{5.0.13}$$

(vii) The operation Q^s and βQ^s commute with change of base fields.

Proof. The assertion that the degree and the weight of the source and target is as indicated follows from the isomorphisms in (5.0.11). The existence of the operations on cohomology computed on the is clear again from the same isomorphism. The properties for the operations defined are induced by the properties of the corresponding operation on mod-p motivic cohomology considered in [J1, Theorem 8.4]. □

6. Cohomological operations that commute with proper push-forwards and Examples

The operations considered so far commute with pull-backs only and do not commute with push-forwards by proper maps. In this section we modify the above operations to obtain operations that commute with proper-push-forwards. The key to this is the following formula, which follows by a deformation to the normal cone argument as shown in [FL, Chapter VI]. We state this for the convenience of the reader. Recall that motivic cohomology is a contravariant functor on smooth schemes. By identifying motivic cohomology with higher Chow groups, one may show the former is also covariant for proper maps.

Proposition 6.1. *Let*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & W \arrow[d, "g"] \\ X' \arrow[r, "i'"] & W' \end{tikzcd}$$

denote a cartesian square with all schemes smooth and with the vertical maps either regular closed immersions or projections from a projective space. Let the normal bundle associated to i (i) be N (N', respectively). Then the square commutes:

$$\begin{tikzcd} H^*(X', \mathbb{Z}/l(\bullet)) \arrow[r, "{i'}_*"] \arrow[d, "e(N)f^*"] & H^*(W', \mathbb{Z}/l(\bullet)) \arrow[d, "g^*"] \\ H^*(X, \mathbb{Z}/l(\bullet)) \arrow[r, "i_*"] & H^*(W, \mathbb{Z}/l(\bullet)) \end{tikzcd}$$

where $N = f^*(N')/N$ is the excess normal bundle and $e(N)$ denotes the Euler-class of N. In case g and hence f are also closed immersions with co-normal sheaves N_g and N_f, respectively, then $N \cong N_{g|X}/N_f$. Moreover if a finite constant group scheme G acts on the above schemes, the corresponding assertions holds in the G-equivariant motivic cohomology defined below.

Definition 6.2. Let G denote a finite group acting on a scheme X. Then we let $\mathbb{H}_G(X, \mathbb{Z}/(r)) = \underset{\Delta}{\text{holim}}\, R\Gamma(EG \underset{G}{\times} X, \mathbb{Z}/l(r))$ following the terminology in [J2, Section 6]. We let $H^n_G(X, \mathbb{Z}/l(r)) = \pi_{-n}(\mathbb{H}_G(X, \mathbb{Z}/l(r))$.

Remarks 6.3. 1. One may now verify that if $G = \mathbb{Z}/l$, for a fixed prime l, then

$$H^*_G(Spec \quad k, \mathbb{Z}/l(\bullet)) \cong H^*(Spec \quad k, \mathbb{Z}/l(\bullet)) \otimes H^*_{sing}(BG, \mathbb{Z}/l)$$

where $H^*(Spec \quad k, \mathbb{Z}/l(\bullet))$ denotes the motivic cohomology of $Spec \quad k$ and $H^*_{sing}(BG, \mathbb{Z}/l)$ denotes the singular cohomology of the space BG with Z/l-coefficients. Recall that if $l = 2$, $H^*_{sing}(BG, \mathbb{Z}/l)$ is a polynomial ring in one variable and when $l > 2$, $H^*_{sing}(BG, \mathbb{Z}/l) = \mathbb{Z}/l[t] \otimes \Lambda[\nu]$ where $\beta t = \nu$ and $\Lambda[\nu]$ denotes an exterior algebra in one generator ν.

2. The situation where will apply the above proposition will be the following: X will denote a given smooth scheme and X' will denote $X^{\times^l}$. W will denote another smooth scheme provided with a closed immersion $X \to W$ and W' will denote $W^{\times^l}$. In this case the normal bundle associated to the diagonal imbedding of X in $X^{\times^l}$ is $T_{X^{\times^{l-1}}}$ (the normal bundle associated to the diagonal imbedding of W in $W^{\times^l}$ is $T_{W^{\times^{l-1}}}$,respectively). As equivariant vector bundles for the obvious permutation action of $\mathbb{Z}/l$ on $X^{\times^l}$ and $W^{\times^l}$ these identify with $R \otimes_k T_X$ and $R \otimes_k T_W$ where R is the representation of $\mathbb{Z}/l$ given by $(k[x]/(x^l - 1))/k$. For a line bundle $\mathcal{E}$, let $w(\mathcal{E}, t) = 1 + c_1(L)^{l-1} t$. One extends the definition of $w(\mathcal{E}, t)$ to all vector bundles $\mathcal{E}$ by making this class take short exact sequences to products. Then the Euler-class $e(R \otimes_k T_X) = t^{dim(X)} w(T_X, 1/t)$ and $e(R \otimes_k T_W) = t^{dim(W)} w(T_W, 1/t)$.

At this point, we may adopt the arguments as in [Br] to define cohomology operations that are compatible with push-forwards by proper maps between quasi-projective schemes. i.e. Let $Q^\bullet : H^*(X, \mathbb{Z}/l(\bullet)) \to H^*(X, \mathbb{Z}/l(\bullet))$ denote the total operation defined by $Q^\bullet = \Sigma_s Q^s$. Now we define the *covariantly functorial* operations Q_s by letting

$$Q_\bullet = \Sigma_s Q_s = Q^\bullet \cap w(T_X)^{-1} \tag{6.0.14}$$

(Recall that the class $w(T_X)$ is invertible.) If we re-index motivic cohomology homologically, (i.e. if X is proper and of pure dimension d, we let $H_n(X, \mathbb{Z}/l(r)) = H^{2d-n}(X, \mathbb{Z}/(d - r)))$ the operations Q_s map $H_n(X, \mathbb{Z}/l(t))$ to $H_{n-2s(l-1)}(X, \mathbb{Z}/l(tl - d(l - 1)))$.

Proposition 6.4. *Let $f : X \to Y$ denote a proper map between quasi-projective schemes over Spec k. Then $Q_\bullet \circ f_* = f_* \circ Q_\bullet$.*

Proof. Since X and Y are quasi-projective, f may be factored as a closed immersion $i : X \to Y \times \mathbb{P}^n$ for some projective space $\mathbb{P}^n$ and the obvious projection $\pi : Y \times \mathbb{P}^n \to Y$. Therefore, it suffices to prove the assertion separately for $f = i$ and for $f = \pi$. The case $f = i$ is clear from the statements above. Next observe that $\mathbb{P}^n$ is a linear scheme and therefore the motivic cohomology of $X \times \mathbb{P}^n$ is given by an obvious Kunneth formula: see [AJ, Appendix] for example. Therefore the Cartan formula immediately implies the required assertion for the case $f = \pi$. □

We proceed to consider various examples.

6.1. **Examples.** The first example we consider is an operation $Q_s : H_q(X, \mathbb{Z}/l(t)) \to H_{q-2s(l-1)}(X, \mathbb{Z}/l(tl - d(l - 1)))$ on a projective smooth scheme X of dimension d so that the composition with the proper map $\pi_* : H_{q-2s(l-1)}(X, \mathbb{Z}/l(ldl)) \to H_{q-2s(l-1)}(Spec \quad k, \mathbb{Z}/l(tl - d(l - 1)))$ is in fact zero. In view of the fact that the operation Q_s commutes with proper push-forward, it suffices to take $q > 2t$ so that the group $H_q(Spec \quad k, \mathbb{Z}/l(t)) = 0$.

For example, one may take $dim(X) = 3$, $q = 3$, $t = 1$, $s = 1$ and $l = 2$. Now we have the operation

$$Q_1 : H_2(X, \mathbb{Z}/2(1)) \to H_0(X, \mathbb{Z}/2(-1)).$$

In cohomology notation this identifies with an operation $Q_1 : H^4(X, \mathbb{Z}/2(2)) \to H^6(X, \mathbb{Z}/2(4))$. The projection to $Spec \quad k$ sends the source to the group $H_2(Spec \quad k, \mathbb{Z}/2(1)) \cong H^{-2}(Spec \quad k, \mathbb{Z}/2(-1))$ which is zero assuming the Beilinson-Soulé vanishing conjecture. It follows that $\pi_* \circ Q_1 = 0$ assuming the validity of this conjecture. Recall that $H^4(X, \mathbb{Z}/2(2))$ identifies with $CH^2(X, \mathbb{Z}/2)$. Therefore any closed integral sub-scheme of X of codimension 2 defines a class in this group. If α is such a class, our conclusion is that $\pi_*(Q_1(\alpha)) = 0$.

So far we did not put any restriction on the prime l. If we require $l = p$, the last operation takes on the form $Q_1 : H^2(X, \nu(2)) \to H^2(X, \nu(4))$ where cohomology denotes cohomology computed either on the Zariski or étale sites.

As another example, we may assume again $dim(X) = 3$, $l = 3$. Now we obtain the operation $Q_1 : H^3(X, \mathbb{Z}/3(2)) \rightarrow H^7(X, \mathbb{Z}/3(6))$. Re-indexing homologically this identifies with $Q_1 : H_3(X, \mathbb{Z}/3(1)) \rightarrow H_{-1}(X, \mathbb{Z}/3(-3))$. Once again one may prove that the composition $\pi_* \circ Q_1 = 0$. In case $l = p$, this operation now takes on the form $Q_1 : H^1(X, \nu(2)) \rightarrow H^1(X, \nu(6))$.

As yet another example, we will presently show that the only classical operations that send the usual mod-l Chow groups to the usual mod-l Chow groups are the power operations. Recall that the usual mod$-l$ Chow groups are given by the mod$-l$ motivic cohomology groups $H^{2n}(X, \mathbb{Z}/l(n))$. Now let $Q^s : H^{2t}(X, \mathbb{Z}/l(t)) \rightarrow H^{2t+2s(l-1)}(X, \mathbb{Z}/l(lt))$ be given so that the $2t + 2s(l-1) = 2lt$. Then $2s(l-1) = 2t(l-1)$ so that $s = t$. Therefore we see from [J1, Theorem 8.2(ii)] that the given operation is none other than the l-th power operation.

References

[AJ] Reza Akhtar and Roy Joshua: *Kunneth decomposition for quotient varieties*, Indag. Math., **17**(3), 319-344, (2006)

[Br] P. Brosnan: *Cohomology operations in Chow theory*, Transactions of the AMS, 355, (2003), no. 5, 1869-1903.

[Ep] D. B. A. Epstein, *Steenrod operations in Homological Algebra*, Invent. Math, **1**, 152-208, (1966).

[FL] W. Fulton and S. Lang, *Riemann-Roch algebra*, Grund. Math. **277**, Springer, (1985).

[GL] T. Geisser and M. Levine, *K-theory of fields in characteristic p*, Invent. Math., **139**, no.3, (2000), 459-493

[HH] C. Haesemeyer and J. Hornbostel :*Motives and Étale motives with finite coefficients*, K-theory, (2005), 34:195-207.

[Ill] L. Illusie, *Complexe de De Rham-Witt et Cohomologie cristalline*, Ann. Scient. Ec. Norm. Sup., **12**, (1979), 501-661

[J1] Roy Joshua : *Motivic E_∞-algebras and the motivic E_∞-dga*, preprint, (2006)

[J2] Roy Joshua : *The intersection theory on algebraic stacks*, K-theory, **27** no.2, 134-195, (2002)

[Lev] Marc Levine: *Inverting the motivic Bott element*, K-theory, 19, (2000), no.1, 1-28.

[MV] F. Morel and V. Voevodsky, $\mathbb{A}^1$*-homotopy theory of schemes*, Inst. Hautes tudes Sci. Publ. Math. No. 90 (1999), 45-143 (2001).

[MVW] C. Mazza, V. Voevodsky and C. Weibel, *Notes on motivic cohomology*, Clay Mathematics Monographs, **2**, AMS, (2006)

[Tot] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), 249-281, Proc. Symposia in Pure Math, **67**, AMS, Providence, (1999)

[Voev] Vladimir Voevodsky: *Reduced Power Operations in motivic cohomology*, Publ. Mathematiques d'IHES, 98, (2003), 1-57.

[W] C. Weibel, *A road map of motivic homotopy and homology theory* in Axiomatic, enriched and motivic homotopy theory, 385–392, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer Acad. Publ., Dordrecht, 2004.

Department of Mathematics, University of British Columbia, Vancouver, Canada
E-mail address: pbrosnan@math.ubc.ca

Department of Mathematics, Ohio State University, Columbus, Ohio, 43210, USA.
E-mail address: joshua@math.ohio-state.edu