COMPARISON OF MOTIVIC AND CLASSICAL OPERATIONS IN MOD-l-MOTIVIC AND ÉTALE COHOMOLOGY

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Abstract. In this paper we explore the relationships between the motivic and classical cohomology operations defined on mod-l motivic cohomology. We also explore similar relationships in étale cohomology and conclude by considering certain operations that commute with proper push-forwards.

1. Introduction

Throughout the paper k will denote a fixed perfect field of characteristic $p \geq 0$. We will restrict to the category, Sm/k, of smooth separated schemes of finite type over k. If X denotes such a scheme, $H^n_{\mathcal{M}}(X, \mathbb{Z}(r))$ will now denote the motivic cohomology with *degree* n and *weight* r and $H^n_{\mathcal{M}}(X, \mathbb{Z}/l(r))$ will denote the corresponding mod-l-variant. Cohomology operations on motivic cohomology were defined and studied in [Voev1] and also just for Chow-groups in [Bros]. We call these *motivic operations* and these are characterized by the property that these are operations which are defined when $l \neq p$ and have the form:

$$P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1))) \text{ and}$$
$$\beta P^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1))).$$

When $i = 2j$, the above motivic cohomology groups identify with the usual mod-l Chow groups and then these operations were also defined in [Bros].

These operations are defined by making use of a geometric model for the classifying spaces of finite groups. By using a simplicial model for these classifying spaces, one obtains certain other operations in mod-l motivic cohomology, which we call *classical* and which are defined even if $l = p$. These operations now have the form:

$$Q^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)}_{\mathcal{M}}(X, \mathbb{Z}/l(jl)) \text{ and } \beta Q^r : H^i_{\mathcal{M}}(X, \mathbb{Z}/l(j)) \rightarrow H^{i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(jl)).$$

One of the main goals of this paper is to explore the relationship between these two types of operations in mod-l motivic cohomology. These elaborate on the results in an earlier preprint by the authors studying these relations after inverting the Bott-element, which is considerably easier.

The comparison of the total power operations, yields straightforward comparison between the motivic and classical operations first for classes with degree equal to twice the weight and then for classes with degree $\leq$ twice the weight by observing that both the classical and motivic operations are stable with respect to suspension in the degree. Since the classical operations are not stable with respect to suspension in the weight, the case when the degree $>$ twice the weight, is more involved and makes use of the Cartan formulae. The main comparison results may be summarized in the following theorem.

Theorem 1.1. *(See section 6 for more details.) Assume the base field has a primitive l-th root of unity and let $B \varepsilon H^0_{\mathcal{M}}(Spec\, k, \mathbb{Z}/l(1))$ denote the motivic Bott element. Let F denote a pointed simplicial sheaf on* $\mathrm{Sm/k}_{Nis}$.

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(i) *Let* $\alpha \varepsilon \tilde{H}^i_{\mathcal{M}}(F, \mathbb{Z}/(q)$ *for any* $i \le 2q$. *Then* $Q^r(\alpha) = B^{(q-r)(l-1)} P^r(\alpha)$, $\beta Q^r(\alpha) = B^{(q-r)(l-1)} \beta P^r(\alpha)$.

(ii) *For the remaining two cases let* $\alpha \varepsilon \tilde{H}^{2q+t}_{\mathcal{M}}(F, \mathbb{Z}/l(q))$, *with* $t > 0$.

If $t = 2t'$ *for some integer* t', *then*

$$B^{t'l} Q^r(x) = B^{(q+t'-r)(l-1)}.B^{t'} P^r(x) \quad \textit{and} \quad B^{t'l} \beta Q^r(x) = B^{(q+t'-r)(l-1)}.B^{t'} \beta P^r(x), \quad 0 \le r \le q+t'.$$

If $t = 2t'+1$ *for some integer* t', *then*

$$B^{(t'+1)l} Q^r(x) = B^{(q+t'+1-r)(l-1)}.B^{t'+1} P^r(x) \ \textit{ and}$$
$$B^{(t'+1)l} \beta Q^r(x) = B^{(q+t'+1-r)(l-1)}.B^{t'+1} \beta P^r(x), \quad 0 \le r \le q+t'+1.$$

(iii) *Both the motivic and classical operations extend to operations on étale cohomology with respect to the sheaf* μ_l. *If* F *denotes a pointed simplicial sheaf on* $\mathrm{Sm/k}_{et}$ *and* $\alpha \varepsilon \tilde{H}^i_{et}(F, \mu_l(q))$, *for any* $i \ge 0$, *then*

$$Q^r(\alpha) = B^{(q-r)(l-1)} P^r(\alpha), \quad \beta Q^r(\alpha) = B^{(q-r)(l-1)} \beta P^r(\alpha).$$

One may consult Examples 6.5 for various examples of the above relations. We conclude by constructing classical operations in motivic cohomology that commute with respect to proper pushforwards and applying them to several examples.

The following is an *outline* of the paper. We begin section 2 by reviewing quickly the cohomology of the classifying spaces of finite groups, using both the geometric and simplicial models for the classifying spaces. We discuss the total power operations in detail in the next two sections. First we recall the total power operations defined by Voevodsky for defining the motivic operations. We show that this may be modified to define total power operations for the classical operations, at least for classes whose degree is less than or equal to twice their weight. Both of these are first defined for algebraic cycles whose degree is twice their weight. Since the motivic operations are stable with respect to suspension in both the degree and the weight this suffices to define the motivic operations for all classes. However, since the classical operations are stable with respect to suspension in only the degree, the above total power operations do not define classical operations except for classes with degree less than or equal to twice their weight. Therefore we define total power operations in a different manner to be able to define classical operations without the above restriction and then show that these new total power operations agree with the ones defined above for classes with degree equal to twice their weight. This is carried out in detail in section 4.

At this point, the usual relations among the classical operations, like the Cartan formulae and Adem relations are by no means obvious. The quickest approach to establishing these for the classical operations is to show that the classical operations defined here identify with the operations defined operadically as in [J1, section 5] (making use of [May]), where such relations are known to hold. We prove this in section 5. The next section contains the key comparison theorem relating the motivic operations with the classical ones. We explore some applications of the above results in the last section. Here we construct classical cohomology operations that commute with proper push-forwards and work out several examples of such push-forward formulae.

Conventions. We restrict to smooth separated schemes of finite type over a field k and l will be a fixed prime. Usually this will be assumed to be different from the characteristic of k and k will be assumed to be provided with a primitive l-th root of unity, though such hypotheses are not required to define the classical operations. We will also consider simplicial schemes $X_\bullet$ over such a base field, where each X_n will be assumed to be a smooth scheme of finite type over k. $\mathrm{Sm/k}_{Zar}$ ($\mathrm{Sm/k}_{Nis}$, $\mathrm{Sm/k}_{et}$) will denote the category Sm/k provided with the big Zariski (Nisnevich or étale topology, respectively). If C denotes any one of these categories, $SSh(\mathrm{C})_+$ will denote the category of all pointed simplicial sheaves on C. $\mathcal{H}$C will denote the corresponding homotopy category obtained by inverting all $\mathbb{A}^1$-weak-equivalences. Any pointed simplicial scheme $X_\bullet$, with each $X_n \varepsilon(\mathrm{Sm/k})$ as well as any pointed simplicial set will be viewed as an object in each of the categories $SSh(\mathrm{Sm/k}_{Zar})_+$, $SSh(\mathrm{Sm/k}_{Nis})_+$ and $SSh(\mathrm{Sm/k}_{et})_+$ in the obvious manner.

The mod$-l$ motivic complex will be denoted $\mathbb{Z}/l$. This should be distinguished from the integers mod$-l$, which will be denoted Z/l. $H^*_{\mathcal{M}}$ will denote cohomology with respect to the motivic or mod$-l$ motivic complex computed on the Nisnevich or Zariski site: H^*_{et} will denote cohomology computed on the étale site. Often, when certain computations hold in any of these cases, we will simply use H^* to denote cohomology computed on any of these sites.

2. **Cohomology of the classifying space for a finite group**

We begin by recalling briefly the construction of the *geometric classifying space of a linear algebraic group*: originally this is due to Totaro - see [Tot]. Let G be a linear algebraic group over $S = Spec \quad k$ i.e. a closed subgroup-scheme in GL_n over S for some n. For a (closed) embedding $i : G \rightarrow GL_n$ *the geometric classifying space* $B_{gm}(G; i)$ of G with respect to i is defined as follows. For $m \geq 1$ let U_m be the open sub-scheme of $\mathbb{A}^{nm}$ where the diagonal action of G determined by i is free. Let $V_m = U_m/G$ be the quotient S-algebraic space of the action of G on U_m induced by the (diagonal) action of G on $\mathbb{A}^{nm}$; the projection $U_m \rightarrow V_m$ defines V_m as the quotient algebraic space of U_m by the free action of G and V_m is thus smooth. We have closed embeddings $U_m \rightarrow U_{m+1}$ and $V_m \rightarrow V_{m+1}$ corresponding to the embeddings $Id \times \{0\} : \mathbb{A}^{nm} \rightarrow \mathbb{A}^{nm} \times \mathbb{A}^n$ and we set $EG^{gm} = \lim\limits_{m \rightarrow \infty} U_m$ and $BG^{gm} = \lim\limits_{m \rightarrow \infty} V_m$ where the colimit is taken in the category of sheaves on $(\mathrm{Sm/k})_{Nis}$ or on $(\mathrm{Sm/k})_{et}$. Observe that if $G = \Sigma_n$ (or a subgroup of it) acting on $\mathbb{A}^n$ by permuting the n-coordinates and acting on $\mathbb{A}^{nm}$ diagonally, we may take $U_m = \{(u_1, \cdots, u_n) | u_i \varepsilon \mathbb{A}^m, u_i \neq u_j, i \neq j\}$. Moreover, in this case, the U_m may be shown to be an affine scheme readily and then [MFK, Proposition 0.7] shows that V_m is also affine. (These observations will be rather important for the construction of the total power operations constructed below.)

The equivariant motivic (étale cohomology) of a scheme X with an action by Σ_n will be defined to be $H^*_{\mathcal{M}}(E\Sigma_n^{gm} \underset{\Sigma_n}{\times} X, \mathbb{Z}/l(\star))$ $(H^*_{et}(E\Sigma_n^{gm} \underset{\Sigma_n}{\times} X, \mathbb{Z}/l(\star))$, respectively). The results in [MV, section 4] show that, one may also define equivariant étale cohomology using the simplicial construction for $E\Sigma_n$.

We recall the computation of the reduced equivariant motivic cohomology of F, where F is any pointed simplicial sheaf on $(\mathrm{Sm/k})_{Nis}$. (For example, $F = X_+$ where X is a given scheme with trivial action by Σ_l). (See [Voev1, Section 6].)

2.1. $\tilde{H}^*_{\mathcal{M}}(F \wedge (B\Sigma_l^{gm})_+; \mathbb{Z}/l(\star))$ is a free module over $\tilde{H}^*_{\mathcal{M}}(F; \mathbb{Z}/l(\star))$with a basis $\{c\bar{d}^i, d^i | i \geq 0\}$ where $\bar{d}$ is a class in $\tilde{H}^{2l-2}_{\mathcal{M}}(F \wedge (B\Sigma_l^{gm})_+; \mathbb{Z}/l(l-1))$ which is the mod-l reduction of a class $d \varepsilon \tilde{H}^{2l-2}_{\mathcal{M}}(F \wedge (B\Sigma_l^{gm})_+; \mathbb{Z}(l-1))$ and c is a class in $\tilde{H}^{2l-3}_{\mathcal{M}}(F \wedge (B\Sigma_l^{gm})_+; \mathbb{Z}/l(l-1))$ so that $\delta(c) = \bar{d}$.

Let *cycl* denote the cycle map from motivic cohomology to étale cohomology. (By identifying the motivic cohomology with the higher Chow groups, these cycle maps identify with those defined in [Bl].) Now one may observe that the same computation as above holds in étale cohomology with the classes c and d replaced by their images under the above cycle map.

One may replace $E\Sigma_l^{gm}$ by *the simplicial model* $E\Sigma_l$ (which is given in degree m, by Σ_l^{m+1} and provided with the obvious structure maps) and $B\Sigma_l^{gm}$ by the corresponding *simplicial model* $B\Sigma_l$ in the above computation to obtain:

2.2. $\tilde{H}^*_{\mathcal{M}}(F_+ \wedge B\Sigma_l; \mathbb{Z}/l(\star))$ is a free module over $\tilde{H}^*_{\mathcal{M}}(F_+; \mathbb{Z}/l(\star))$ with a basis $\{x\bar{y}^i, \bar{y}^i | i \geq 0\}$ where $\bar{y}$ is a class in $\tilde{H}^{2l-2}_{\mathcal{M}}(F_+ \wedge B\Sigma_l; \mathbb{Z}/l(0))$ which is the mod-l reduction of a class $y \varepsilon \tilde{H}^{2l-2}_{\mathcal{M}}(F_+ \wedge B\Sigma_l; \mathbb{Z}(0))$ and x is a class in $\tilde{H}^{2l-3}_{\mathcal{M}}(F_+ \wedge B\Sigma_l; \mathbb{Z}/l(0))$ so that $\delta(x) = \bar{y}$.

Remark 2.1. One may observe that the main difference between the computations in 2.1 and in 2.2 is that the classes x, y and $\bar{y}$ have weight 0.

One may replace F above with a pointed simplicial sheaf on $(\mathrm{Sm})_{et}$ and the above cohomology with H^*_{et} to obtain:

2.3. $\tilde{H}^*_{et}(F_+ \wedge B\Sigma_l; \mathbb{Z}/l(\star))$ is a free module over $\tilde{H}^*_{et}(F_+; \mathbb{Z}/l(\star))$ with a basis $\{x\bar{y}^i, \bar{y}^i | i \geq 0\}$ where $\bar{y}$ is a class in $\tilde{H}^{2l-2}_{et}(F_+ \wedge B\Sigma_l; \mathbb{Z}/l(0))$ and x is a class in $\tilde{H}^{2l-3}_{et}(F_+ \wedge B\Sigma_l; \mathbb{Z}/l(0))$ so that $\delta(x) = \bar{y}$.

3. The total power operations:I

A key step in the comparison between the motivic and classical cohomology operations is a thorough understanding of the total power operation. We proceed to discuss this in detail.

If one only considers the case $i = 2j$, then $H^{2j}(X, \mathbb{Z}/l(j))$ for a smooth scheme X identifies with the mod-l usual Chow groups of X. Then the total power operation simply sends a class

$$\alpha \varepsilon H^{2j}(X, \mathbb{Z}/l(j)) \mapsto \alpha^l$$

which defines a class in $H^{2jl}(B\Sigma_l^{gm} \times X, \mathbb{Z}/l(jl))$.

In order to be able to extend this total power operation as a natural transformation

$$\tilde{\mathcal{P}}_l : H^{2j}(\quad ; \mathbb{Z}/l(j)) \to H^{2jl}(\quad \times B\Sigma_l^{gm}, \mathbb{Z}/l(jl))$$

defined on all simplicial sheaves on the big Zariski, Nisnevich or étale site over k, one needs to adopt the construction in [Voev1, section 5]. We will adopt this suitably modified to also define total power operations when $B\Sigma_l^{gm}$ is replaced by the simplicial model $B\Sigma_l$.

Next recall the following. An augmented simplicial object $X_\bullet$ in a category C consists of a simplicial object $Y_\bullet$ in C with $Y_i = X_i, i \geq 0$ together with an object $X_{-1} \varepsilon \mathrm{C}$ and an augmentation $\epsilon : Y_0 \to X_{-1}$ so that $d_0 \circ \epsilon = d_1 \circ \epsilon$, $i = 0, 1$.

Let $X_\bullet$ denote an augmented simplicial scheme. Let $k[X_\bullet] = \{k[X_n] | n\}$ denote the corresponding co-ordinate ring. A *finitely generated* $k[X_\bullet]$*-module* is given by a collection $\{M_n | n\}$ where each M_n is a finitely generated $k[X_n]$-module and provided with a compatible collection of maps $\{\alpha^*(M_n) \to M_m\}$ for each structure map $\alpha : X_m \to X_n$ of $X_\bullet$. $M_\bullet$ will be called *finitely generated projective (finitely generated free)* if each M_m is a finitely generated projective (free, respectively) $k[X_m]$-module.

Proposition 3.1. *Let $X_\bullet$ denote an augmented simplicial scheme so that X_{-1} is affine. If $M_\bullet = \{M_m | m\}$ is a finitely generated module on $X_\bullet$ which is the pull-back of a finitely generated $k[X_{-1}]$-module, then there exists a finitely generated free module $F_\bullet$ on $X_\bullet$ and a map $\phi : F_\bullet \to M_\bullet$ which is an epimorphism in each degree. In case $M_\bullet$ is the pull-back of a finitely generated projective $k[X_{-1}]$-module, one may also find a finitely generated projective $k[X_\bullet]$-module $N_\bullet$ so that $M_m \oplus N_n \cong F_n$ for all n and where the last isomorphism is compatible with the structure maps of the augmented simplicial scheme.*

Proof. Since $M_\bullet$ is the pull-back of a finitely generated $k[X_{-1}]$-module, it suffices to prove the first statement when the augmented simplicial scheme $X_\bullet$ has been replaced by the affine scheme X_{-1}. This is then clear since X_{-1} is affine. If N_{-1} is the kernel of the surjection, then $M_{-1} \oplus N_{-1} \cong F_{-1}$. This isomorphism pulls-back to a similar isomorphism $M_n \oplus N_n \cong F_n$ for each n and compatible with the structure maps of the augmented simplicial scheme $X_\bullet$. □

The following results relate the geometric classifying space $B\Sigma_l^{gm}$ with the simplicial classifying space $B\Sigma_l$.

Proposition 3.2. *Let U_N denote the open subscheme*

$$\{(u_1, \cdots, u_l) | u_i \varepsilon \mathbb{A}^N, u_j \neq u_k, \quad j \neq k\}$$

of $\mathbb{A}^{Nl}$. (As observed above this scheme and the quotient scheme $V_N = U_N/\Sigma_l$ are affine schemes.)

For any fixed integer $N > 0$, let $gs_N : E\Sigma_l \underset{\Sigma_l}{\times} U_N \rightarrow E\Sigma_l \underset{\Sigma_l}{\times} Spec\, k = B\Sigma_l$ denote the obvious map of simplicial schemes. This map induces a weak-equivalence in $\mathcal{H}SSh\mathrm{Sm/k}_{Nis}$ (and also in $\mathcal{H}SSh\mathrm{Sm/k}_{Zar}$ and $\mathcal{H}SSh\mathrm{Sm/k}_{et}$) on taking the colimit as $N \rightarrow \infty$.

Proof. This is immediate from the fact that the U_N get more and more connected as $N \rightarrow \infty$: see [MV, section 4.2]. $\square$

In view of the above proposition, we may approximate $B\Sigma_l$ by $E\Sigma_l \underset{\Sigma_l}{\times} U_N$ by taking N high enough. We will let U denote U_N and V denote V_N for a large N.

Observe that one has an obvious augmentation

$$E\Sigma_l \underset{\Sigma_l}{\times} U_N \rightarrow U_N/\Sigma_l \tag{3.0.1}$$

One may view this diagram as an augmented simplicial scheme. Observe that the scheme $V_N = U_N/\Sigma_l$ is affine so that Proposition 3.1 applies.

3.0.2. Recall from [Voev1, Theorem 2.1] that the functor $X \rightarrow H^{2n}(X, \mathbb{Z}/(n))$ is represented by the sheaf $U \mapsto Z/l_{tr}(\mathbb{A}^n)(U)/Z/l_{tr}(\mathbb{A}^n - \{0\})(U), U \varepsilon (\mathrm{Sm/k})_{Nis}$.

Let X denote a scheme in (Sm/k) and E, L vector bundles on X provided with an isomorphism $\phi : E \times_X L \rightarrow \mathbb{A}^N_X$ which is the N-dimensional trivial bundle on X. Given a cycle Z on E with coefficients in Z/l and equi-dimensional and finite over X, we consider the cycle on $L \times_X E \times_X L$ whose fiber over a point (x, l) of L is (Z_x, l), where Z_x denotes the fiber of Z over $x \varepsilon X$. It is observed in [Voev1, Construction 5.1] that this is an equi-dimensional cycle finite over L and that by identifying $E \times_X L$ with $\mathbb{A}^N_X$ using the isomorphism ϕ, one obtains a map of pointed sheaves $Th(L) \rightarrow Z/l_{tr}(\mathbb{A}^N)/Z/l_{tr}(\mathbb{A}^N - \{0\})$ (where $Th(L)$ denotes the Thom-space of L), i.e. a class in $\tilde{H}^{2N}(Th(L), \mathbb{Z}/l(N))$. (In fact this map identifies with the class $\Delta_*(q^*(Z)$, where $q : E \underset{X}{\times} L \rightarrow E$ is the obvious projection and $\Delta : E \underset{X}{\times} L \rightarrow L \underset{X}{\times} E \underset{X}{\times} L$ is the diagonal.) Moreover, making use of the Thom-isomorphism, $\tilde{H}^{2dim(E)}(X_+, \mathbb{Z}/l(dim(E))) \cong \tilde{H}^{2N}(Th(L), \mathbb{Z}/l(N))$, one observes that this defines a class in $\tilde{H}^{2dim(E)}(X_+, \mathbb{Z}/l(dim(E)))$ which is denoted $a(Z)$ and shown to be independent of the isomorphism ϕ: see [Voev1, Construction 5.1].

In view of 3.0.2, a class in $\tilde{H}^{2i}(X_+, \mathbb{Z}/l(i))$ may be represented by a cycle Z on $X \times \mathbb{A}^i$ equi-dimensional and finite over X. Let Z^l denote the l-th external power of Z: this is now a cycle on $(X \times \mathbb{A}^i)^l$. We will let $p^*(Z^l)$ denote its pull-back to $(X \times \mathbb{A}^i)^l \times U$, where $U = U_N$ for some suitably large N. Since Σ_l acts freely on U, one may observe that the cycle $p^*(Z^l)$ descends to a unique cycle Z' on $((X \times \mathbb{A}^i)^l \times U)/\Sigma_l$ equi-dimensional and finite over $(X^l \times U)/\Sigma_l$. On pulling back by the diagonal one obtains the cycle Z'' on $X \times (\mathbb{A}^{il} \times U)/\Sigma_l$.

3.0.3. Now $\bar{E} = (\mathbb{A}^l \times U)/\Sigma_l$ is a vector bundle on $V = U/\Sigma_l$ and the latter is affine. Moreover, recall that we have the augmented simplicial schemes: $E\Sigma_l \underset{\Sigma_l}{\times} (\mathbb{A}^l \times U) \to (\mathbb{A}^l \times U)/\Sigma_l$ and $E\Sigma_l \underset{\Sigma_l}{\times} U \to U/\Sigma_l$.

The pull-back of the vector bundle $\bar{E}$ to the simplicial scheme $E\Sigma_l \underset{\Sigma_l}{\times} U$ defines a vector bundle we will denote by $\hat{E}$. By invoking Proposition 3.1, one may find a vector bundle $\bar{L}$ on $V = U/\Sigma_l$ so that $\bar{E} \times_V \bar{L} \cong \mathbb{A}^N_V$ for some N. Therefore, the pull-back $\hat{L}$ of $\bar{L}$ to $E\Sigma_l \underset{\Sigma_l}{\times} U$ also has the property that $\hat{E} \times_{E\Sigma_l \underset{\Sigma_l}{\times} U} \hat{L}$ is a trivial bundle of rank N.

Next we let $\tilde{E} = X \times \bar{E}$, $\tilde{L} = X \times \bar{L}$ denote the pull-backs of $\bar{E}$ and $\bar{L}$ to $X \times V$. We also let $E = X \times \hat{E}$, $L = X \times \hat{L}$ denote the corresponding vector bundles on the simplicial scheme $X \times E\Sigma_l \underset{\Sigma_l}{\times} U$ obtained by pull-back from $X \times V$. Then $E \times_{X \times E\Sigma_l \underset{\Sigma_l}{\times} U} L$ is a trivial bundle of rank N on the simplicial scheme $X \times E\Sigma_l \underset{\Sigma_l}{\times} U$ and $\tilde{E} \times_X \tilde{L}$ is a trivial bundle of rank N on $X \times V$. Moreover, $E^{\oplus i}$ ($L^{\oplus i}$) will be the pull-back of the vector bundle $(\bar{E})^{\oplus i}$ ($(\bar{L})^{\oplus i}$, respectively). Observe that $Th(L^{\oplus i}) = X_+ \wedge Th((\hat{L})^{\oplus i})$, $Th((\tilde{L})^{\oplus i}) = X_+ \wedge Th((\bar{L})^{\oplus i})$ and that there is a natural map $Th(L^{\oplus i}) \to Th(\tilde{L}^{\oplus i})$.

In this context, the same arguments as above show that a cycle Z on $X \times \mathbb{A}^i$, equi-dimensional and finite over X defines (pointed) maps

$$\begin{aligned} \mathcal{P}_{gm}(Z) &: Th(\tilde{L}^{\oplus i}) \to Z/l_{tr}(\mathbb{A}^{iN})/Z_{tr}(\mathbb{A}^{iN} - 0) \quad \textit{and} \\ \mathcal{P}_{s}(Z) &: Th(L^{\oplus i}) \to Z/l_{tr}(\mathbb{A}^{iN})/Z_{tr}(\mathbb{A}^{iN} - \{0\}) \end{aligned} \tag{3.0.4}$$

with the latter being obtained by pre-composing the first map with the obvious map $Th(L^{\oplus i}) \to Th(\tilde{L}^{\oplus i})$. Clearly these correspond to classes in

$$\tilde{H}^{2iN}(Th(\tilde{L}^{\oplus i}), \mathbb{Z}/l(iN)) \text{ and } \quad \tilde{H}^{2iN}(Th(L^{\oplus i}), \mathbb{Z}/l(iN))$$

so that the second is the pull-back of the first class by the obvious map $Th(L^{\oplus i}) \to Th((\tilde{L})^{\oplus i})$. Making use of Thom-isomorphisms, these correspond to classes in

$$\tilde{H}^{2idim(E)}(X_+ \wedge (U/\Sigma_l)_+, \mathbb{Z}/l(idim(E))) \quad \text{and} \quad \tilde{H}^{2idim(E)}(X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_+), \mathbb{Z}/l(idim(E)))$$

so that the second is the pull-back of the first class by the obvious map $E\Sigma_l \underset{\Sigma_l}{\times} U \to U/\Sigma_l$. More precisely, we have defined maps

$$X_+ \wedge (U/\Sigma_l)_+ \to Z/l_{tr}(\mathbb{A}^{idim(E)})/Z_{tr}(\mathbb{A}^{idim(E)} - \{0\}) \text{ and}$$

$$X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_+) \to Z/l_{tr}(\mathbb{A}^{idim(E)})/Z_{tr}(\mathbb{A}^{idim(E)} - \{0\}).$$

so that the latter is obtained from the former by pre-composing with the augmentation $X_+ \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_+) \to X_+ \wedge (U/\Sigma_l)_+$. Moreover, the assignment $Z \to \mathcal{P}_{gm}(Z)$ and $Z \to \mathcal{P}_s(Z)$ are contravariantly functorial in X. Therefore, the same relations extend on taking a resolution of the given simplicial sheaf F by a simplicial scheme $X_\bullet$, with each X_n a smooth scheme. (Observe also that $rank(E) = l$ here.)

The contravariant functoriality of the above constructions in X shows that by taking a resolution of any simplicial sheaf on $(\mathrm{Sm/k})_{Nis}$ (and also in $SSh\mathrm{Sm/k}_{Zar}$ and $SSh\mathrm{Sm/k}_{et}$) by representables, one obtains the same maps as above when X is replaced by any simplicial sheaf on $(\mathrm{Sm/k})_{Nis}$. In particular, one may replace X by the pointed simplicial sheaf $Z/l_{tr}(\mathbb{A}^i)/Z_{tr}(\mathbb{A}^i - \{0\})$ to obtain maps

$$Z/l_{tr}(\mathbb{A}^i)/Z_{tr}(\mathbb{A}^i - \{0\}) \wedge (U/\Sigma_l)_+ \to Z/l_{tr}(\mathbb{A}^{idim(E)})/Z_{tr}(\mathbb{A}^{idim(E)} - \{0\}) \text{ and}$$
$$Z/l_{tr}(\mathbb{A}^i)/Z_{tr}(\mathbb{A}^i - \{0\}) \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_+) \to Z/l_{tr}(\mathbb{A}^{idim(E)})/Z_{tr}(\mathbb{A}^{idim(E)} - \{0\})$$

so that the latter is obtained from the former by pre-composing with the augmentation $E\Sigma_l \underset{\Sigma_l}{\times} U \to (U/\Sigma_l)$. Restating these as classes in cohomology, we observe that there exist natural transformations (defined on the categories $\mathcal{H}SSh(\mathrm{Sm}/\mathrm{k}_{Zar})_+$, $\mathcal{H}SSh(\mathrm{Sm}/\mathrm{k}_{Nis})_+$ and $\mathcal{H}SSh(\mathrm{Sm}/\mathrm{k}_{et})_+$):

$$P'_{gm} : \tilde{H}^{2i}(\quad, \mathbb{Z}/l(i)) \to \tilde{H}^{2il}(\quad \wedge (U_N/\Sigma_l)_+, \mathbb{Z}/l(il)) \text{ and}$$
$$P'_s : \tilde{H}^{2i}(\quad, \mathbb{Z}/l(i)) \to \tilde{H}^{2il}(\quad \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_N)_+, \mathbb{Z}/l(il))$$

so that the latter is obtained from the former by composing with the augmentation $E\Sigma_l \underset{\Sigma_l}{\times} U_N \to U_N/\Sigma_l$. Moreover, these are compatible as $N \to \infty$ which, together with Proposition 3.2 provides the following result.

Proposition 3.3. *We obtain natural transformations:*

$$\mathcal{P}_{gm} : \underset{N\to\infty}{lim} \tilde{H}^{2i}(\quad, \mathbb{Z}/l(i)) \to \underset{N\to\infty}{lim} \tilde{H}^{2il}(\quad \wedge (U_N/\Sigma_l)_+, \mathbb{Z}/l(il)) \text{ and}$$
$$\mathcal{P}_s : \underset{N\to\infty}{lim} \tilde{H}^{2i}(\quad, \mathbb{Z}/l(i)) \to \underset{N\to\infty}{lim} \tilde{H}^{2il}(\quad \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U_{N+}), \mathbb{Z}/l(i)) \cong \tilde{H}^{2il}(\quad \wedge B\Sigma_l, \mathbb{Z}/l(il))$$

on $\mathcal{H}SSh(\mathrm{Sm}/\mathrm{k}_{Nis})_+$ (and also on $\mathcal{H}SSh(\mathrm{Sm}/\mathrm{k}_{Zar})_+$ and $\mathcal{H}SSh(\mathrm{Sm}/\mathrm{k}_{et})_+$) so that the latter is obtained from the former by composing with the augmentation $E\Sigma_l \underset{\Sigma_l}{\times} U_N \to U_N/\Sigma_l$.

Definition 3.4. The natural transformation $\mathcal{P}_{gm}$ ($\mathcal{P}_s$) will be called the geometric total power operation (the simplicial total power operation, respectively).

3.1. Motivic operations. Next we recall the definition of the cohomology operations of Voevodsky. Let F denote a pointed simplicial sheaf on $(\mathrm{Sm}/\mathrm{k})_{Nis}$ (or on $(\mathrm{Sm}/\mathrm{k})_{Zar}$).

One starts with the total power operation :

$$\mathcal{P}_{gm} : \tilde{H}^{2i}_{\mathcal{M}}(F, \mathbb{Z}/l(i)) \to \tilde{H}^{2il}_{\mathcal{M}}(F \wedge (U/\Sigma_l)_+, \mathbb{Z}/l(il)) \tag{3.1.1}$$

By the results in 2.1, $\underset{i,j}{\oplus} \tilde{H}^{il}_{\mathcal{M}}(F \wedge (U/\Sigma_l)_+, \mathbb{Z}/l(jl))$ is a free module over $\tilde{H}^{*}_{\mathcal{M}}(F, \mathbb{Z}/(\star))$ with basis given by the elements $\bar{d}^r$ and $c\bar{d}^r$, $r \geq 0$. The operations P^r and βP^r are defined by the formula:

$$\mathcal{P}_{gm}(w) = \Sigma_{r\geq 0} P^r(w)\bar{d}^{i-r} + \beta P^r(w) c\bar{d}^{i-r-1}, \quad w \varepsilon \tilde{H}^{2i}(F, \mathbb{Z}/l(i)) \tag{3.1.2}$$

Observe that so defined $P^r : \tilde{H}^{2i}_{\mathcal{M}}(F, \mathbb{Z}/l(i)) \to \tilde{H}^{2i+2r(l-1)}_{\mathcal{M}}(F, \mathbb{Z}/l(i + r(l-1)))$ and $\beta P^r : \tilde{H}^{2i}_{\mathcal{M}}(F, \mathbb{Z}/l(j)) \to \tilde{H}^{2i+2r(l-1)+1}_{\mathcal{M}}(X, \mathbb{Z}/l(j + r(l-1)))$.

Behavior under suspension: A key observation is that, since the motivic cohomology operations are stable with respect to shifting degrees by 1, and also both degrees and weights by 1, this defines the operations P^r and βP^r on all $\tilde{H}^{i}_{\mathcal{M}}(F, \mathbb{Z}/l(j))$.

The classical operations are *not* stable with respect to suspension of weights, and therefore, one cannot define classical operations in general using the total power operations considered above. For this, we define a new total power operation when the simplicial model is used for the classifying spaces of finite groups. We also show that, when applied to classes with degree = twice their weight, these total power operations identify with the ones considered above. All of these are discussed in detail in the next section.

4. The total power operations:II

We proceed to define total power operations in a somewhat different manner so as to be able to define the classical operations on all classes. Let Σ_l denote the symmetric group on l-letters and let π denote a subgroup of Σ_l. Let $B\pi$ denote the simplicial classifying space of π with $E\pi \rightarrow B\pi$ denoting the associated principal π-fibration. We let $Z/l(E\pi)$ denote the chain complex obtained by taking the free Z/l-vector space in each simplicial degree and viewing that as a chain-complex in the usual manner using the alternating sums of the face maps as the differential. We let $Z/l(E\pi)^{\vee} = Hom(Z/l(E\pi), Z/l)$ which now forms a co-chain complex trivial in negative degrees.

Let K denote a possibly unbounded co-chain complex. Now $K^{\otimes^l}$ is the l-fold tensor product of K:the symmetric group Σ_l acts in the obvious manner on $K^{\otimes^l}$. Therefore, one may now form the co-chain complex:

$$Z/l(E\pi)^{\vee} \underset{Z/l[\pi]}{\otimes} K^{\otimes^l}$$

where the differentials of the tensor-product are induced by the differentials of the two factors in the usual manner. (Strictly speaking one needs to take the homotopy inverse limit of the cosimplicial object of co-chain complexes obtained this way: see [J1]. However, one may identify this with a suitable total chain-complex as in [Brow, Appendix].) In particular, the differential, $((Z/l(E\pi))^{\vee})^0 \underset{Z/l[\pi]}{\otimes} K^{\otimes^l})^n \rightarrow (Z/l(E\pi)^{\vee} \underset{Z/l(\pi)}{\otimes} (K^{\otimes^l})^{n+1}$ is such that if $z \varepsilon K^n$ is a cycle, then its l-th power $z^{\otimes^l}$ defines a cycle of degree nl in the above total complex we denote by

$$\tilde{Q}(z) \varepsilon (Z/l(E\Sigma_l)^{\vee} \underset{Z/l(\Sigma_l)}{\otimes} K^{\otimes^l})^{nl} \tag{4.0.3}$$

We will choose the complex K as follows. First we allow three distinct contexts:

(i) We work throughout on the site $(\mathrm{Sm}/k)_{Zar}$ with H^* denoting cohomology on the Zariski site.

(ii) We work throughout on the site $(\mathrm{Sm}/k)_{Nis}$ with H^* denoting cohomology on the Nisnevich site.

(iii) We work throughout on the site $(\mathrm{Sm}/k)_{et}$ with H^* denoting cohomology on the étale site.

Next observe that the category of (possibly unbounded) co-chain complexes of abelian sheaves on any of the above two sites is a quasi-simplicial model category in the sense of [Fausk] and therefore it is closed under homotopy inverse limits. Let $\mathcal{H}om$ denote the internal hom in this category. Then, given co-chain complexes of abelian sheaves M, N, we let $\mathcal{RH}om(M, N) = \mathcal{H}om(M, GN)$ with G denoting the homotopy inverse limit of the cosimplicial object defined by the Godement resolution computed on the appropriate site. $RHom$ will denote the external hom.

Let $X \varepsilon (\mathrm{Sm/k})$. We let $K = \Gamma(X, \mathcal{RH}om(M, \mathcal{A}(i)))$ where M is any chain complex of abelian sheaves trivial in negative degrees, $\mathcal{A}(i) = \mathbb{Z}/l(i)$ is the mod$-l$ motivic complex of weight i. Moreover, now $K = RHom(M \otimes Z(X), \mathcal{A}(i))$, where $Z(X)$ denotes the co-chain complex associated to the simplicial abelian presheaf defined by $\Gamma(U, Z(X)) = Z(\Gamma(U, X))$.

In fact we may start with a pointed simplicial sheaf F in $(\mathrm{Sm/k})_{Nis}$ and let M denote the normalized co-chain complex obtained by taking the associated free simplicial sheaf $Z/l(F)$ of Z/l-vector spaces (with the base point of F identified with 0) and re-indexing so that we obtain a co-chain complex. Then we define $\mathcal{RH}om(F, \mathcal{A}(i)) = \mathcal{RH}om(M, \mathcal{A}(i)) = \mathcal{H}om(M, G\mathcal{A}(i))$. The above definition makes implicit use of the adjunction between the free Z/l-vector space functor and the underlying functor sending a Z/l-vector space to the underlying set. A useful observation in this context is that the natural map $Z/l(S)\underset{Z/l}{\otimes} Z/l(T) \rightarrow Z/l(S \bigwedge T)$ is a weak-equivalence for any pointed simplicial presheaves S and T. (One may prove this as follows. First this is clear if S is a presheaf of pointed sets, i.e. it is true if S is replaced by its 0-th skeleton. One may prove using ascending induction on n, that the above map is a weak-equivalence when S is replaced by its n-skeleton. Finally take the colimit as $n \rightarrow \infty$ over the n-skeleta of S.) This will enable one to obtain the weak-equivalence $\mathcal{RH}om(M' \otimes M'', \mathcal{A}(i)) \simeq \mathcal{RH}om(F' \bigwedge F'', \mathcal{A}(i))$, when $M' = Z/l(F')$ and $M'' = Z/l(F'')$.

Then $K^{\otimes^l} = \Gamma(X, \mathcal{RH}om(M, \mathcal{A}(i)))^{\otimes^l} = \Gamma(X^l, \mathcal{RH}om(M, \mathcal{A}(i))^{\boxtimes^l})$ maps to $\Gamma(X, \mathcal{RH}om(M, \mathcal{A}(il)))$ by pull-back by the diagonal $\Delta : X \rightarrow X^l$. (In fact this makes use of the diagonal map $Z/l(F) \rightarrow Z/l(F)^{\otimes^l}$ and the pairing $\mathcal{A}^{\otimes^l} \rightarrow \mathcal{A}$.) We proceed to show this pairing is compatible with the obvious action of Σ_l. First observe that M being the normalized chain complex obtained from the simplicial abelian sheaf $Z/l(F)$ (re-indexed so as to become a co-chain complex), has the structure of a co-algebra over the Barratt-Eccles operad as shown in [B-F, 2.1.1 Theorem]. $\mathcal{A}$ has the structure of an algebra over the same operad as shown in [J1, Theorem 1.1]. Therefore, one may readily show that these structures provide $\mathcal{RH}om(M, \mathcal{A})$ the structure of an algebra over the tensor product of the Barratt-Eccles operad and the Eilenberg-Zilber operad: see [J1, Proposition 6.4]. Therefore, the above pairing is compatible with the obvious action of Σ_l and one obtains the obvious map

$$Z/l(E\Sigma_l)^{\vee} \underset{Z/l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{RH}om(Z/l(F), \mathcal{A}(i)))^{\otimes l} \rightarrow Z/l(E\Sigma_l)^{\vee} \underset{Z/l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{RH}om(Z/l(F), \mathcal{A}(il))).$$

(See for example (5.0.7), which explains such pairings in more detail.) One may identify the last term with

$$\Gamma(X, \mathcal{RH}om(Z/l(F) \underset{Z/l(\Sigma_l)}{\otimes} Z/l(E\Sigma_l), \mathcal{A}(il)) = \Gamma(X, \mathcal{RH}om(Z/l(F) \otimes Z/l(B\Sigma_l), \mathcal{A}(il)).$$

We denote the above composition

$$(4.0.4) \quad Z/l(E\Sigma_l)^{\vee} \underset{Z/l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{RH}om(Z/l(F), \mathcal{A}(i)))^{\otimes l} \rightarrow \Gamma(X, \mathcal{RH}om(Z/l(F) \otimes Z/l(B\Sigma_l), \mathcal{A}(il)))$$

by $\bar{\mathcal{Q}}_s$. As observed above any cycle $z\varepsilon\Gamma(X, RHom(M, \mathcal{A}))$ in degree n provides a cycle in degree nl in the source of the last map. Therefore, $\bar{\mathcal{Q}}_s(\tilde{\mathcal{Q}}(z))$ defines a cycle in the target of the last map in degree nl. This provides the natural transformation

$$(4.0.5) \qquad \mathcal{Q}_s : H^j(\quad, \mathbb{Z}/l(i)) \rightarrow H^{jl}(\quad \wedge B\Sigma_l, \mathbb{Z}/l(il))$$

for all j and all $i \geq 0$ on the category $\mathcal{HSSh}(\mathrm{Sm/k}_{Nis})_+$ and $\mathcal{HSSh}(\mathrm{Sm/k}_{et})_+$. (We call this the (second) *simplicial total power operation.*)

4.1. The classical operations. These are defined similar to the motivic operations using the total power operation $\mathcal{Q}_s$ defined above. Let F denote a pointed simplicial sheaf. The main point to recall is the computation of $\tilde{H}^*_{\mathcal{M}}(F \wedge B\Sigma_l; \mathbb{Z}/l)$ ($\tilde{H}^*_{et}(F \wedge B\Sigma_l; \mathbb{Z}/l)$) in 2.2 (2.3) which shows it is a free module over $\tilde{H}^*_{\mathcal{M}}(F; \mathbb{Z}/l)$ ($\tilde{H}^*_{et}(F; \mathbb{Z}/l)$, respectively) with basis $\{x\bar{y}^i, \bar{y}^i | i \geq 0\}$. The operation Q^r (βQ^r) is defined by the formula:

$$(4.1.1) \qquad \mathcal{Q}_s(w) = \Sigma_{r \geq 0} Q^r(w)\bar{y}^{j/2-r} + \beta Q^r(w)x\bar{y}^{j/2-r-1}, \quad w\varepsilon\tilde{H}^j(F, \mathbb{Z}/l(i)), \text{j even}$$

and extended to all j by observing that the classical operations are stable with respect to degree suspension. Observe that, so defined,

$$Q^r : \tilde{H}^j(F, \mathbb{Z}/l(i)) \to \tilde{H}^{j+2r(l-1)}(F, \mathbb{Z}/l(il)) \text{ and } \beta Q^r : \tilde{H}^j(F, \mathbb{Z}/l(i)) \to \tilde{H}^{j+2r(l-1)+1}(X, \mathbb{Z}/l(il)).$$

Behavior under suspension. In contrast to the motivic operations, these operations are compatible with shifting the degree alone by 1. This will follow from the comparison theorem in the next section.

Next we proceed to show that, for classes with degree = twice the weight, the total power operation $\mathcal{Q}_s$ identifies with the total power operation $\mathcal{P}_s$ defined above in Proposition 3.3.

Proposition 4.1. *Let* $\alpha \varepsilon \tilde{H}^{2i}_{\mathcal{M}}(F, \mathbb{Z}/l(i))$ *denote a class. Then* $\mathcal{Q}_s(\alpha) = \mathcal{P}_s(\alpha)$.

Proof. First we observe from [Voev1, Theorem 2.1] that since we are only considering cycles whose degree equals twice their weight, one may replace the motivic $\mathbb{Z}/l(m)[2m]$ by the sheaf associated to the presheaf $V \mapsto Z/l_{tr}(\mathbb{A}^m(V)/\mathbb{Z}/l(\mathbb{A}^m - 0)(V)$, V belonging to the big Nisnevich site $(\mathrm{Sm/k})_{Nis}$. Henceforth we will use $\mathcal{A}(m)$ to denote this sheaf when considering such cycles; when considering general cycles, $\mathcal{A}(m)$ will still denote the motivic complex $\mathbb{Z}/l(m)$.

The next key step to is to invoke the following result proved in Proposition 3.2: the map gs_N : $E\Sigma_l \underset{\Sigma_l}{\times} U_N \to E\Sigma_l \underset{\Sigma_l}{\times} Spec\, k = B\Sigma_l$ of simplicial schemes induces a weak-equivalence in $\mathcal{H}SSh\mathrm{Sm/k}_{Nis}$ (and also in $\mathcal{H}SSh\mathrm{Sm/k}_{Zar}$ and $\mathcal{H}SSh\mathrm{Sm/k}_{et}$) on taking the colimit as $N \to \infty$. Therefore, one may replace $B\Sigma_l$ in the above definition of the classical operations by $E\Sigma_l \underset{\Sigma_l}{\times} U$, where $U = U_N$, $N >> 0$. i.e. First the map in (4.0.4) may be replaced by the map

$$(4.1.2) \quad Z/l(E\Sigma_l)^\vee \underset{Z/l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{RH}om(Z/l(F) \otimes Z/l(U), \mathcal{A}(i)))^{\otimes l}$$
$$\to \Gamma(X, \mathcal{RH}om(Z/l(F) \otimes Z/l(E\Sigma_l) \underset{Z/l(\Sigma_l)}{\otimes} Z/l(U), \mathcal{A}(il)))$$

Therefore, the total power operation $\mathcal{Q}_s$ may be defined as a map

$$(4.1.3) \qquad \mathcal{Q}_s : \tilde{H}^j(\quad, \mathcal{A}(i)) \to \tilde{H}^{jl}(\quad \wedge (E\Sigma_l \underset{\Sigma_l}{\times} U), \mathcal{A}(il))$$

Next will consider the case when $F = X$ which is a smooth scheme. A class in $\tilde{H}^{2i}(F, \mathcal{A}(i))$ may be represented by a cycle Z on $X \times \mathbb{A}^i$ equi-dimensional and finite over X. One first pulls-back the cycle Z to $p^*(Z^l)$ on $X \times \mathbb{A}^{il} \times U$. This cycle is invariant under the obvious action of the symmetric group Σ_l on $\mathbb{A}^{il} \times U$ and therefore defines a cycle in

$$Z/l(E\Sigma_l)^\vee \underset{Z/l(\Sigma_l)}{\otimes} \Gamma(X, \mathcal{RH}om(Z/l(U), \mathcal{A}(il))) = \Gamma(X, \mathcal{RH}om(Z/l(E\Sigma_l) \underset{Z/l(\Sigma_l)}{\otimes} Z/l(U), \mathcal{A}(il))).$$

Observe that this is the total complex of the double complex defined by the cosimplicial co-chain complex: $\{\Gamma(X, \mathcal{RH}om(Z/l(\Sigma_l^{\times^n}) \otimes Z/l(U), \mathcal{A}(il)))|n\}$. In fact this defines the cycle $\bar{\mathcal{Q}}_s(\tilde{\mathcal{Q}}(z))$ in

$$\Gamma(X, \mathcal{RH}om(Z/l(U), \mathcal{A}(il))) = \Gamma(X, \mathcal{RH}om(Z/l(E\Sigma_l)_0 \underset{Z/l(\Sigma_l)}{\otimes} Z/l(U), \mathcal{A}(il))$$

so that the pull-backs by d_0^* and d_1^* to classes in $\Gamma(X, \mathcal{RH}om(Z/l(\Sigma_l) \otimes Z/l(U), \mathcal{A}(il)))$ are the same. This will be denoted $\hat{\mathcal{Q}}_s(z)$.

(4.1.4) A *key observation* is that the assignment $z \mapsto \hat{\mathcal{Q}}_s(z)$ is contravariantly functorial.

Next let $\bar{E}$ and $\bar{L}$ denote the vector bundles on $V = U/\Sigma_l$ defined as in 3.0.3. Recall $rank(\bar{E}) = l$. Recall also that the pull-back of $\bar{E}$ to U is trivial and that since $\bar{L}$ is chosen such that $\bar{E} \oplus \bar{L}$ is trivial, the pull-back of $\bar{L}$ to U is also trivial. Recall the pull-back of $\bar{E}$ ($\bar{L}$) to $E\Sigma_l \underset{\Sigma_l}{\times} U$ is $\hat{E}$ ($\hat{L}$, respectively). (See the discussion preceeding (3.0.4) above.) It follows that for each fixed degree $n \geq 0$, $\hat{E}_n \oplus \hat{L}_n$ is trivial and that therefore that $\hat{L}_n$ also is trivial. Therefore, the same cycle as above defines the cycle

$$\Delta_*(q^*(\hat{\mathcal{Q}}_s(z)))\varepsilon\Gamma(X, \mathcal{RH}om(Z/l(Th(\hat{L}_0^{\oplus i})), \mathcal{A}(i(l + rank(\hat{L}))))$$

$$= RHom(Z(X) \otimes Z/l(E\Sigma_l)_0 \underset{Z/l(\Sigma_l)}{\otimes} (Z/l(Th(\bar{L}_0^{\oplus i})), \mathcal{A}(i(l + rank(L)))),$$

where $q : \hat{E}^{\oplus i} \underset{X}{\times} \hat{L}^{\oplus i} \to \hat{E}^{\oplus i}$ is the obvious projection and $\Delta : \hat{E}^{\oplus i} \underset{X}{\times} \hat{L}^{\oplus i} \to \hat{L}^{\oplus i} \underset{X}{\times} \hat{E}^{\oplus i} \underset{X}{\times} \hat{L}^{\oplus i}$ is the diagonal. Moreover, the two pull-backs by d_0 and d_1 to classes in $\Gamma(X, \mathcal{RH}om(Z/l(Th(\hat{L}_0^{\oplus i})), \mathcal{A}(i(l + rank(\hat{L})))))$ identify since the pull-backs by d_0 and d_1 of the class $\hat{\mathcal{Q}}_s(z)$ identify. On taking cohomology, this defines a class in $H^{2iN}(X_+ \wedge E\Sigma_l \underset{\Sigma_l}{\wedge} Th(\bar{L}^{\oplus i}), \mathbb{Z}/l(iN)) = H^{2iN}(Th(\tilde{L}^{\oplus i}), \mathbb{Z}/l(iN))$, where $N = rank(E) + rank(L)$. (Recall $l = rank(E)$ as well.) In fact, the definition of the total power operation $\mathcal{P}_s(z)$ as in Proposition 3.3 shows that, this class identifies with the class denoted $\mathcal{P}_s(z)$ above. The main point to observe now is that under Thom-isomorphism this class corresponds to the class in $H^{2il}(X_+ \wedge E\Sigma_l \underset{\Sigma_l}{\wedge} U_+, \mathbb{Z}/l(li)$ represented by the class $\hat{\mathcal{Q}}(z)$. This is proved in the lemma below. Since this class represents the class $\mathcal{Q}_s(z)$, we observe that $\mathcal{Q}_s(z) = \mathcal{P}_s(z)$ in this case.

Next we consider the general case when F denotes any pointed simplicial sheaf. Now one chooses the *standard resolution* of F by a simplicial scheme $X_\bullet$ as in [Voev1, section 3], with each X_n a smooth scheme. Then a class z in $\tilde{H}^{2i}(F, \mathbb{Z}/l(i))$ is represented by a cycle Z_0 on $X_0 \times \mathbb{A}^i$ equi-dimensional and finite over X_0 so that the two pull-backs $d_i^*(Z_0)$ to cycles on $X_1 \times \mathbb{A}^i$ equi-dimensional and finite over X_1 are the same for $i = 0$ and $i = 1$. The observation (4.1.4) shows that then $d_0^*(\hat{\mathcal{Q}}_s(Z_0)) = d_1^*(\hat{\mathcal{Q}}_s(Z_0))$. It follows that this cycle therefore defines a cycle in

$$Tot\{RHom(Z/l(F) \otimes Z/l(\Sigma_l^n) \otimes Z/l(U), \mathcal{A}(il)|n\} = RHom(Z/l(F) \otimes Z/l(E\Sigma_l) \underset{Z/l(\Sigma_l)}{\otimes} Z/l(U), \mathcal{A}(il))$$

(Here Tot denotes a suitable total complex.) We will denote this cycle by $\hat{\mathcal{Q}}_s(Z_0)_F$. Next observe that the vector bundles $\bar{E}$ and $\bar{L}$ as in 3.0.2 as well as the associated vector bundles $\hat{E}$ and $\hat{L}$ are defined independently of X, so that these are the same for all X_j, $j = 0, 1, \ldots,$. Therefore, we may define the associated vector bundle $E = X_\bullet \times \hat{E}$ and $L = X_\bullet \times \hat{L}$. Moreover it follows similarly that the two cycles $\Delta_*(q^*(d_j^*(\hat{\mathcal{Q}}_s(Z_0))))\varepsilon RHom(Z(X) \otimes Z/l(E\Sigma_l)_0 \underset{Z/l(\Sigma_l)}{\otimes} Z/l(Th(\bar{L}_0^{\oplus i})), \mathcal{A}(iN))$ are the same for $j = 0, 1$. Therefore, on taking cohomology, this defines a class in $H^{2iN}(Th(L^{\oplus i}), \mathbb{Z}/l(iN)) = H^{2iN}(X_{\bullet,+} \wedge E\Sigma_l \underset{\Sigma_l}{\wedge} Th(\bar{L}), \mathbb{Z}/l(iN)) \cong H^{2iN}(F \wedge E\Sigma_l \underset{\Sigma_l}{\wedge} Th(\bar{L}), \mathbb{Z}/l(iN))$, where $N = rank(E) + rank(L)$. In fact this class identifies with the class denoted $\mathcal{P}_s(z)$ above. Once again the lemma below shows that under Thom-isomorphism this class corresponds to the class in $H^{2il}(F_+ \wedge E\Sigma_l \underset{\Sigma_l}{\wedge} U_+, \mathbb{Z}/l(il)$ represented by the class $\hat{\mathcal{Q}}_s(Z_0)_F$. Since this class represents the class $\mathcal{Q}_s(z)$, we observe that $\mathcal{Q}_s(z) = \mathcal{P}_s(z)$ in this case as well, thereby completing the proof of the proposition. □

Lemma 4.2. *Let $z\varepsilon\tilde{H}^{2u}(X_{\bullet,+}, \mathcal{A}(u))$ for a simplicial scheme $X_\bullet$ be represented by a cycle Z_0 on $X_0 \times \mathbb{A}^u$ equi-dimensional and finite over X_0 so that the pull-backs $d_0^*(Z_0)$ and $d_1^*(Z_0)$ to cycles on $X_1 \times \mathbb{A}^u$ are the same. Let V denote the trivial vector bundle $X_\bullet \times \mathbb{A}^v$. Then the Thom-isomorphism sends the class z to the cycle represented by $\Delta_*(q^*(Z_0))\varepsilon\tilde{H}^{2u+2v}(Th(V), \mathcal{A}(u+v))$, where $q : V_0 \to X_0$*

is the projection and $\Delta : V_0 \underset{X_0}{\times} \mathbb{A}^u \to V_0 \underset{X_0}{\times} \mathbb{A}^u \underset{X_0}{\times} V_0$ *is the obvious diagonal. (Here* V_0 *is the restriction of* V *to* X_0*.)*

Proof. The Thom class $[T]$ corresponding to the vector bundle V is the class defined by the diagonal Δ_{V_0} which is a cycle on $V_0 \times V_0$ equi-dimensional and finite over V_0. Now the cup-product of the class z represented by the cycle Z_0 with $[T]$ corresponds to the cycle represented by $\Delta_*(q^*(Z_0))$. □

5. Comparison with the operadic definition of classical cohomology operations: properties of classical operations

An E^{∞}-structure on the motivic complex $\mathbb{Z}/l = \oplus_i \mathbb{Z}/l(i))$ is shown to lead to a somewhat different definition of the *classical* cohomology operations on mod$-l$ motivic cohomology as discussed in [J1, Section 5] and based on the earlier work [May]. We will presently show that these operations are in fact identical to the classical operations defined above. Since the classical operations defined operadically readily inherit several well-known properties, we are thereby able to carry over such properties to the classical cohomological operations defined above. Some of these properties of the classical cohomology operations, for example, the Cartan formulae are used in an essential manner in the comparison results in the next section.

The only other way to establish such properties for the classical cohomology operations would be by a tedious step-by-step verification of these properties following the approach in [St-Ep]. Therefore we prefer the approach adopted here, which is far simpler.

Proposition 5.1. *The cohomology operations defined above coincide with the classical cohomology operations defined on mod-l motivic cohomology in* [J1, Section 5]*.*

Proof. Recall the simplicial Barratt-Eccles operad is the operad $\{NZ(E\Sigma_n)|n\}$ where $E\Sigma_n$ denotes the simplicial bar-resolution of the finite group Σ_n and $NZ(E\Sigma_n)$ denotes the normalized chain complex associated to the simplicial abelian group $Z(E\Sigma_n)$. The operad structure obtained this way is discussed in [J1]. We will assume that it is an action by the simplicial Barratt-Eccles operad on the motivic complex that provides its E_∞-structure. The above action of the operad $\{NZ(E\Sigma_n)|n\}$ on the complex $\mathcal{A} = \oplus_{n\ge 0}\mathbb{Z}/l(n)$ provides us maps

$$\theta_l : NZ(E\Sigma_l) \otimes \mathcal{A}^{\otimes^l} \to \mathcal{A} \tag{5.0.5}$$

Recall that $K^\vee$ denotes $\mathcal{H}om(K, Z/l)$, if K is any complex of Z/l-vector spaces. From the above pairing we obtain

$$\theta_l^* : NZ(E\Sigma_l) \otimes \mathcal{A}^\vee \to (\mathcal{A}^\vee)^{\otimes^l}$$

where we define $\theta_l^*(h, a^\vee)(a_1 \otimes \cdots \otimes a_l) =< \theta_l(h \otimes a_1 \otimes \cdots \otimes a_l), a^\vee >$, $a_i \varepsilon \mathcal{A}$, $a^\vee \varepsilon \mathcal{A}^\vee$ and $h \varepsilon NZ(E\Sigma_l)$. In fact these pairings provide the dual $\mathcal{A}^\vee$ with the structure of a co-algebra over the operad $\{NZ(E\Sigma_l|l\}$. It is a standard result in this situation that the map θ_l^* is a chain map and is an *approximation to the diagonal map* (i.e. homotopic to the diagonal map) $\Delta : \mathcal{A}^\vee \to (\mathcal{A}^\vee)^{\otimes^l}$. (Here, as well as elsewhere in this section, we use the observation that for any vector space V over Z/l, a vector $v \varepsilon V$ (a vector $v^\vee \varepsilon V^\vee$) is determined by its pairing $< v, w >$ with all vectors $w \varepsilon V^\vee$ (its pairing $< u, v^\vee >$ with all vectors $u \varepsilon V$, respectively.).)

We next take the dual of the pairing θ_l^* to define a chain-map:

$$((\mathcal{A}^\vee)^\vee)^{\otimes^l} \to NZ(E\Sigma_l)^\vee \otimes (\mathcal{A}^\vee)^\vee.$$

The formula defining the chain map θ_l^* shows that this map sends $\mathcal{A}^{\otimes^l} \subseteq ((\mathcal{A}^\vee)^\vee)^{\otimes^l}$ to $NZ(E\Sigma_l)^\vee \otimes \mathcal{A}$. Clearly there is a pairing $NZ(E\Sigma_l)^\vee \otimes NZ(E\Sigma_l)^\vee \to NZ(E\Sigma_l)^\vee$ induced by the diagonal $\Delta : E\Sigma_l \to$

$E\Sigma_l \times E\Sigma_l$. Tensoring the last map with $NZ(E\Sigma_l)^\vee$ and making use of this pairing provides us with the map:

(5.0.6) $$d : (NZ(E\Sigma_l))^\vee \otimes \mathcal{A}^{\otimes^l} \to (NZ(E\Sigma_l))^\vee \otimes \mathcal{A}$$

One may recall that the action of $\sigma \varepsilon \Sigma_l$ on $NZ(E\Sigma_l)$ and of σ^{-1} on $\mathcal{A}^{\otimes^l}$ cancel out. Tracing through these actions of Σ_l on the maps in the above steps, one concludes that the map d induces a map on the quotients:

(5.0.7) $$\bar{d} : (NZ(E\Sigma_l))^\vee \underset{Z\Sigma_l}{\otimes} \mathcal{A}^{\otimes^l} \to (NZ(E\Sigma_l))^\vee \underset{Z\Sigma_l}{\otimes} \mathcal{A}$$

Now the cohomology of the complex $(NZ(E\Sigma_l))^\vee \underset{N(Z(\Sigma_l))}{\otimes} \mathcal{A}$ identifies with $H^*(B\Sigma_l; Z/l) \otimes H^*(\mathcal{A})$ whereas the cohomology of the complex $(NZ(E\Sigma_l))^\vee \underset{Z\Sigma_l}{\otimes} \mathcal{A}^{\otimes^l}$ identifies with the equivariant cohomology: $H^*(\mathcal{A}^{\otimes^l}, \Sigma_l; Z/l)$. Therefore, the map $\bar{d}$ defines a map

(5.0.8) $$\bar{d}^* : H^*(\mathcal{A}^{\otimes^l}, \Sigma_l; Z/l) \to H^*(B\Sigma_l; Z/l) \otimes H^*(\mathcal{A})$$

The formula defining d also shows that the map $\bar{d}^*$ is a map of $H^*(B\Sigma_l, Z/l)$-modules. One may also observe readily that the l-th power map defines a map $H^*(\mathcal{A}) \to H^*(\mathcal{A}^{\otimes^l}, \Sigma_l; Z/l)$, $a \mapsto a^l$. Let $\{e_i, fe_i | i \geq 0\}$ denote a basis of the Z/l-vector space $H_*(B\Sigma_l; Z/l)$ dual to the basis $\{y^i, xy^i | i \geq 0\}$ for $H^*(B\Sigma_l; Z/l)$, i.e. $< e_i, y^j >= 0$, if $i \neq j$ and $= 1$ if $i = j$. Also $< fe_i, y^j >= 0$ for all i , j, $< fe_i, xy^j >= 0$ for $i \neq j$ and $= 1$ for $i = j$. Observe that now we have the following computation for a class $\alpha \varepsilon H^q(\mathcal{A})$:

$< \bar{d}^*(\alpha^l), e_i \otimes (-)^\vee >=< \bar{\theta}_l^*(e_i, (-)^\vee), \alpha^l >=< (-)^\vee, \bar{\theta}_l(e_i, \alpha^l) >$ and

$< \bar{d}^*(\alpha^l), fe_i \otimes (-)^\vee >=< \bar{\theta}_l^*(fe_i, (-)^\vee), \alpha^l >=< (-)^\vee, \bar{\theta}_l(fe_i, \alpha^l) >$

where $(-)^\vee \varepsilon H^*(\mathcal{A})^\vee$ and $\bar{\theta}_l^*$ is the map induced by θ_l^* on taking homology of the corresponding complexes. Since the map θ_l^* was observed to be chain homotopic to the diagonal, it follows that $\bar{d}^* = \Delta^*$ where Δ is the obvious diagonal. Therefore, the coefficient of y^i (xy^i) in the expansion of $\bar{d}^*(\alpha^l) \varepsilon H^*(B\Sigma_l; \mathbb{Z}/l) \otimes H^*(\mathcal{A})$ identifies with $\bar{\theta}_i(e_i, \alpha^l)$ ($\bar{\theta}_i(fe_i, \alpha^l)$, respectively). This completes the proof of the proposition □

The main point of the above comparison is to provide the following corollary where the corresponding results are shown to hold for the classical operations defined operadically in [J1, Theorem 5.3] invoking the results of [May].

Theorem 5.2. *Let F denote a pointed simplicial sheaf on $(\mathrm{Sm}/k)_{Nis}$ in which case H^* will denote cohomology computed on the Nisnevich site or on $(\mathrm{Sm}/k)_{et}$ in which case H^* will denote cohomology computed on the étale site.*

The classical cohomology operations $Q^s : \tilde{H}^q(F, \mathbb{Z}/l(t)) \to \tilde{H}^{q+2s(l-1)}(F, \mathbb{Z}/l(l.t))$ and

$\beta Q^s : \tilde{H}^q(F, \mathbb{Z}/l(t)) \to \tilde{H}^{q+2s(l-1)+1}(F, \mathbb{Z}/l(l.t))$.

satisfy the following properties:

(i) Contravariant functoriality: if $f : F' \to F$ is a map between simplicial sheaves, $f^* \circ Q^s = Q^s \circ f^*$

(ii) Let $x \varepsilon \tilde{H}^q(F, \mathbb{Z}/l(t))$. $Q^s(x) = 0$ if $2s > q$, $\beta Q^s(x) = 0$ if $2s \geq q$ and if $(q = 2s)$, then $Q^s(x) = x^l$.

(iii) If β is the Bockstein, $\beta \circ Q^s = \beta Q^s$.

(iv) Cartan formulae: For all primes l, $Q^s(x \otimes y) = \underset{i+j=s}{\Sigma} Q^i(x) \otimes Q^j(y)$ and

$\beta Q^s(x \otimes y) = \underset{i+j=s}{\Sigma} \beta Q^i(x) \otimes Q^j(y) + Q^i(x) \otimes \beta Q^j(y)$

(v) Adem relations For each pair of integers $i \geq 0, j \geq 0$, we let $(i,j) = \frac{(i+j)!}{i!j!}$ with the convention that $0! = 1$. We will also let $(i,j) = 0$ if $i < 0$ or $j < 0$. (See [May, p. 183]*.) With this terminology we obtain:*

If ($l > 2$, $a < lb$, and $\epsilon = 0, 1$) or if ($l = 2$, $a < lb$ and $\epsilon = 0$) one has

$$\beta^\epsilon Q^a Q^b = \Sigma_i (-1)^{a+i}(a - li, (l-1)b - a + i - 1)\beta^\epsilon Q^{a+b-i}Q^i \tag{5.0.9}$$

where $\beta^0 Q^s = Q^s$ while $\beta^1 Q^s = \beta Q^s$. If $l > 2$, $a \leq lb$ and $\epsilon = 0, 1$, one also has

$$\begin{aligned} \beta^\epsilon Q^a \beta Q^b = (1-\epsilon)\Sigma_i(-1)^{a+i}(a - li, (l-1)b - a + i - 1)\beta Q^{a+b-i}Q^i \\ -\Sigma_i(-1)^{a+i}(a - li - 1, (l-1)b - a + i)\beta^\epsilon Q^{a+b-i}\beta Q^i \end{aligned} \tag{5.0.10}$$

(vi) The operations Q^s commute with the simplicial suspension isomorphism in $H^n(F; \mathbb{Z}/l(r)) \cong H^{n+1}(S^1_s F; \mathbb{Z}/l(r))$.

(vii) The operation Q^s commutes with change of base fields and also with the higher cycle map into mod–l étale cohomology.

Remark 5.3. It is important to observe that Q^0 is not the identity. The property (ii) above shows that in general $Q^0(x) = x^l$, if $x \varepsilon H^0(X, \mathbb{Z}/l(t)) = \tilde{H}^0(X_+, \mathbb{Z}/l(t))$ for any smooth scheme X and any $t \geq 0$. This will play a major role in the comparison results in the next section.

6. Comparison between the motivic and classical operations

In view of the results established in the earlier sections we are able to provide a nearly complete comparison of the motivic and classical operations.

6.0.11. The Motivic Bott element. Throughout the rest of this section, we will assume that the field k has a primitive l-th root of unity. Recall that we have:

$$\begin{aligned} H^p_{\mathcal{M}}(Spec \quad k, \mathbb{Z}(1)) &= 0, p \neq 1 \\ &= k^*, p = 1 \end{aligned} \tag{6.0.12}$$

Now the universal coefficient sequence associated to the short exact sequence $0 \to \mathbb{Z}(1) \overset{\times l}{\to} \mathbb{Z}(1) \to \mathbb{Z}/l(1) \to 0$ of motivic complexes, provides the isomorphism

$$H^0_{\mathcal{M}}(Spec \quad k, \mathbb{Z}/l(1)) \cong \mu_l(k) \tag{6.0.13}$$

The *Motivic Bott element* is the class in $H^0_{\mathcal{M}}(Spec \quad k, \mathbb{Z}/l(1))$ corresponding under the above isomorphism to the primitive l-th root of unity ζ. We will denote this element by B. Since $cycl(B) = \zeta$ in $H^*_{et}(\quad, \mu_l(*))$, multiplication by the class $cycl(B)$ induces an isomorphism: $H^*_{et}(\quad, \mu_l(r)) \to H^*_{et}(\quad, \mu_l(r+1))$. It follows that the cycle map $cycl$ induces a map of cohomology functors:

$$cycl(B^{-1}) : H^*_{\mathcal{M}}(\quad, \mathbb{Z}/l(\star))[B^{-1}] \to H^*_{et}(\quad, \mu_l(\star)). \tag{6.0.14}$$

It is shown in [Lev] that this map is an isomorphism on smooth schemes.

As observed above, the cohomology $H^*(B\Sigma_l^{gm};\mathbb{Z}/l)$ maps naturally to $H^*(B\Sigma_l;\mathbb{Z}/l)$ under which the total power operation $\mathcal{P}_{gm}$ maps to the total power operation $\mathcal{P}_s$. Therefore, a simple comparison of the degrees and weights of the classes involved provides the following proposition.

Proposition 6.1. *Assume that the base field k has a primitive l-th root of unity. Let* $\alpha \varepsilon H^{2q}_{\mathcal{M}}(X,\mathbb{Z}/l(q)$ *for some* $q \geq 0$ *with* $X\varepsilon(\mathrm{Sm/k})$. *Then*

$$Q^r(\alpha) = B^{(q-r).(l-1)}.P^r(\alpha), \quad \beta Q^r(\alpha) = B^{(q-r).(l-1)}.\beta P^r(\alpha)$$

for $r \leq q$. *For* $r > q$, $Q^r(\alpha) = 0 = P^r(\alpha)$.

Corollary 6.2. *The same relation holds for any class* $\alpha \varepsilon \tilde{H}^i_{\mathcal{M}}(F,\mathbb{Z}/(q))$ *when* F *is any pointed simplicial sheaf on* $(\mathrm{Sm/k})_{Nis}$ *provided* $i \leq 2q$.

Proof. We will first observe that the relations hold when $i = 2q$ and F is any pointed simplicial sheaf on $(\mathrm{Sm/k})_{Nis}$. This follows readily in view of the observation that the two total power operations $\mathcal{P}_{gm}$ and $\mathcal{P}_s$ are compatible as natural transformations defined on the category of all pointed simplicial sheaves on $(\mathrm{Sm/k})$. Next we consider the statement when $i < 2q$. For example, if $i = 2q-1$, $\tilde{H}^{2q-1}_{\mathcal{M}}(F,\mathbb{Z}/l(q)) \cong \tilde{H}^{2q}_{\mathcal{M}}(\Sigma^1_s \wedge F,\mathbb{Z}/l(q))$. Now using the observation that both the motivic and classical operations are stable with respect to the suspension $\Sigma^1_s \wedge$, such a degree-suspension reduces this to the case when $i = 2q$, which has been proved already. Observe also that when $i \leq 2q$, one knows that $Q^r(\alpha) = 0 = P^r(\alpha)$ for $r > q$, (see [Voev1, Lemma 9.9] for a proof of the last equality) so that for the classes α for which Q^r is non-zero, the exponent $(q-r)(l-1)$ of B is ≥ 0. (If $i > 2q$ this may no longer be true apriori.)

In case F is in fact a scheme $X\varepsilon(\mathrm{Sm/k})$, the identification $H^i_{\mathcal{M}}(X,\mathbb{Z}/(q)) \cong CH^j(X, 2q-i; l)$ shows that these groups are trivial if $i > 2q$. Therefore, it suffices to consider the case when $i \leq 2q$ in case F is in fact a scheme $X\varepsilon(\mathrm{Sm/k})$. □

Next we will consider what may be said about the case $i > 2q$. First observe that the Bott element B defines a class in $H^0_{\mathcal{M}}(X,\mathbb{Z}/l(1))$ for any smooth scheme X by pull-back. Next consider a a pointed simplicial sheaf F. Then one finds a resolution of F by pointed simplicial schemes $X_{\bullet,+}$: see [Voev1, section 3]. The structure map $X_1 \to Spec\, k$ factors through the structure map $X_0 \to Spec\, k$, so that B pulls-back to define a class (still denoted) $B\varepsilon\tilde{H}^0_{\mathcal{M}}(F,\mathbb{Z}/l(1))$.

Lemma 6.3. *Let* F *denote a pointed simplicial sheaf on* $(\mathrm{Smt}/k)_{Nis}$. *Then*
(i) $Q^0(B) = B^l$.
(ii) *if* $x\varepsilon\tilde{H}^q_{\mathcal{M}}(F,\mathbb{Z}/l(t))$, *then* $Q^r(B.x) = B^l Q^r(x)$ *and* $\beta Q^r(B.x) = B^l \beta Q^r(x)$ *for all* $x\varepsilon\tilde{H}^q_{\mathcal{M}}(F,\mathbb{Z}/l(t))$.

Proof. (i) Take $x = B$ in Theorem 5.2(ii). Then $q = 0 = s$ there so that $Q^r(B) = 0$ for $r > 0$ and $Q^0(B) = B^l$. This proves (i). (ii) now follows from (i) making use of the Cartan formula in Theorem 5.2(iv). □

Our basic technique to handle the case where the degree > twice the weight (i.e. $i > 2q$) is to apply suitable weight and degree suspensions so as to reduce to the case where the degree = twice the weight. Then we handle this case by the comparison above. Both the motivic and classical operations commute with degree suspension, and the motivic operations commute with weight suspensions as well. The classical operations do not, however, commute with weight suspensions. But weight suspensions may effected by multiplying with the class B and the behavior of the classical operations with respect to tensoring with B is explained by the results above. Therefore, we obtain the extension of our comparison to classes of all degree and weight as explained below.

Proposition 6.4. *Suppose* $x\varepsilon\tilde{H}^{2q+t}_{\mathcal{M}}(F, \mathbb{Z}/l(q))$, *with* $t > 0$.

(i) If $t = 2t'$ *for some integer* t', *then*

$$B^{t'l}Q^r(x) = B^{(q+t'-r)(l-1)}.B^{t'}P^r(x) \text{ and } B^{t'l}\beta Q^r(x) = B^{(q+t'-r)(l-1)}.B^{t'}\beta P^r(x), \quad 0 \le r \le q+t'.$$

(ii) If $t = 2t'+1$,

$$B^{(t'+1)l}Q^r(x) = B^{(q+t'+1-r)(l-1)}.B^{t'+1}P^r(x) \text{ and}$$
$$B^{(t'+1)l}\beta Q^r(x) = B^{(q+t'+1-r)(l-1)}.B^{t'+1}\beta P^r(x), \quad 0 \le r \le q+t'+1.$$

Proof. To obtain (i), one first applies an iterated weight suspension t'-times: this is effected by multiplying x by $B^{t'}$. Now the class $B^{t'}x\varepsilon\tilde{H}^{2q+2t'}_{\mathcal{M}}(F, \mathbb{Z}/l(q+t'))$, so that one may apply the comparison in Proposition 6.1 to it and obtain:

$$Q^r(B^{t'}x) = B^{(q+t'-r)(l-1)}P^r(B^{t'}x) \text{ and } \beta Q^r(B^{t'}x) = B^{(q+t'-r)(l-1)}\beta P^r(B^{t'}x).$$

Making use of Lemma 6.3, we see that $Q^r(B^{t'}x)$ simplifies to $B^{t'l}Q^r(x)$ while $\beta Q^r(B^{t'}x)$ simplifies to $B^{t'l}\beta Q^r(x)$. $P^r(B^{t'}x) = B^{t'}P^r(X)$ and $\beta P^r(B^{t'}x) = B^{t'}\beta P^r(x)$. These prove (i). To obtain (ii), one needs to apply an iterated weight suspension $t'+1$-times followed by a degree suspension once. This produces the class $\Sigma^1_s B^{t'+1}x\varepsilon\tilde{H}^{2q+2t'+2}_{\mathcal{M}}(\Sigma^1_s F, \mathbb{Z}/l(q+t'+1))$. Now one applies the comparison in Proposition 6.1 to it. Then one makes use of Lemma 6.3 to pull-out the B from the left-hand-side. □

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Examples 6.5. (i) Take $t = 1$. In this case one obtains $B^lQ^r(x) = B^{(q+1-r)(l-1)}BP^r(x)$ and $B^l\beta Q^r(x) = B^{(q+1-r)(l-1)}B\beta P^r(x)$. One may now also take $r = q$ to obtain, $B^lQ^q(x) = B^lP^q(x)$ and $B^l\beta Q^q(x) = B^l\beta P^q(x)$. Since B is not invertible, multiplication by B need not be injective and therefore, one cannot conclude that therefore $Q^q(x) = P^q(X)$ or that $\beta Q^q(x) = \beta P^q(x)$.

(ii) Take $t = 2$. In this case one obtains $B^lQ^r(x) = B^{(q+1-r)(l-1)}BP^r(x)$ and $B^l\beta Q^r(x) = B^{(q+1-r)(l-1)}B\beta P^r(x)$.

(iii) Take $t = 3$. In this case one obtains $B^{2l}Q^r(x) = B^{(q+2-r)(l-1)}B^2P^r(x)$ and $B^{2l}\beta Q^r(x) = B^{(q+2-r)(l-1)}B^2\beta P^r(x)$. If, in addition, $r = q+1$, then this becomes $B^{2l}Q^r(x) = B^{l+1}P^r(x)$ and $B^{2l}\beta Q^r(x) = B^{l+1}\beta P^r(x)$. Once again, since B in not invertible, one cannot conclude that therefore $B^{l-1}Q^r(x) = P^r(x)$ or that $B^{l-1}\beta Q^r(x) = \beta P^r(x)$.

Observe that by the multiplicative properties of the operations and the observation that $P^r(B) = 0$ if $r \ge 1$ ([Voev1, Lemma 9.8]):

$$P^r(B^j\alpha) = B^jP^r(\alpha), \tag{6.0.15}$$

$$\beta P^r(B^j\alpha) = B^j\beta P^r(\alpha). \tag{6.0.16}$$

The above relations show that the motivic cohomology operations above induce operations on $H^*(\quad, \mathbb{Z}/l(\star))[B^{-1}]$ in the obvious manner: we define $P^r(\alpha.B^{-1}) = P^r(\alpha).B^{-1}$ and $\beta P^r(\alpha.B^{-1}) = \beta P^r(\alpha).B^{-1}$. Next we proceed to compare these induced motivic and classical operations on mod$-l$ étale cohomology,

Proposition 6.6. *(Comparison of operations in mod$-l$ étale cohomology.) Assume that the base field k has a primitive l-th root of unity. Let F denote a pointed simplicial sheaf on* $(\mathrm{Sm}/k)_{et}$. *Let* $\alpha\varepsilon H^i_{et}(F, \mu_l(q))$ *for some* $q \ge 0$. *Then*

$$Q^r(\alpha) = B^{(q-r).(l-1)}.P^r(\alpha), \quad \beta Q^r(\alpha) = B^{(q-r).(l-1)}.\beta P^r(\alpha)$$

for $r \le i/2$ *and all* $i \ge 0$. *For* $Q^r(\alpha) = 0$ *for* $r > i/2$ *and* $P^r(\alpha) = 0$ *for* $r \ge i/2$.

Proof. For the case $r \leq q$ this follows from Proposition 6.1. For the other cases it follows by expanding the exponents of B on both sides of the formulae in Proposition 6.4 and canceling out all the powers of B on the left-hand-side. □

7. Cohomological operations that commute with proper push-forwards and Examples

The operations considered so far commute with pull-backs only and do not commute with push-forwards by proper maps. In this section we modify the above operations to obtain operations that commute with proper-push-forwards. The goal of this discussion is to consider the analogues of degree formulae in mod-p motivic cohomology: such degree formulae have played a major role in some of the applications of motivic cohomology operations. The key to this is the following formula, which follows by a deformation to the normal cone argument as shown in [FL, Chapter VI]. We state this for the convenience of the reader. Recall that motivic cohomology is a contravariant functor on smooth schemes. By identifying motivic cohomology with higher Chow groups, one may show the former is also covariant for proper maps.

Proposition 7.1. *Let*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & W \arrow[d, "g"] \\ X' \arrow[r, "i'"] & W' \end{tikzcd}$$

denote a cartesian square with all schemes smooth and with the vertical maps either regular closed immersions or projections from a projective space. Let the normal bundle associated to i (i) be N (N', respectively). Then the square commutes:

$$\begin{tikzcd} H^*(X', \mathbb{Z}/l(\bullet)) \arrow[r, "{i'}_*"] \arrow[d, "e(N)f^*"] & H^*(W', \mathbb{Z}/l(\bullet)) \arrow[d, "g^*"] \\ H^*(X, \mathbb{Z}/l(\bullet)) \arrow[r, "i_*"] & H^*(W, \mathbb{Z}/l(\bullet)) \end{tikzcd}$$

where $N = f^*(N')/N$ is the excess normal bundle and $e(N)$ denotes the Euler-class of N. In case g and hence f are also closed immersions with normal bundles N_g and N_f, respectively, then $N \cong N_{g|X}/N_f$. Moreover if a finite constant group scheme G acts on the above schemes, the corresponding assertions holds in the G-equivariant motivic cohomology defined below.

Definition 7.2. Let G denote a finite group acting on a scheme X. Then we let $\mathcal{H}_G(X, \mathbb{Z}/(r)) = \underset{\Delta}{\mathrm{holim}}\, R\Gamma(EG \underset{G}{\times} X, \mathbb{Z}/l(r))$ following the terminology in [J2, Section 6]. We let $H^n_G(X, \mathbb{Z}/l(r)) = \pi_{-n}(\mathcal{H}_G(X, \mathbb{Z}/l(r))$.

Remarks 7.3. 1. One may now verify that if $G = Z/l$, for a fixed prime l, then

$$H^*_G(Spec\, k, \mathbb{Z}/l(\bullet)) \cong H^*(Spec \quad k, \mathbb{Z}/l(\bullet)) \otimes H^*_{sing}(BG, Z/l)$$

where $H^*(Spec\, k, \mathbb{Z}/l(\bullet))$ denotes the motivic cohomology of $Spec\, k$ and $H^*_{sing}(BG, Z/l)$ denotes the singular cohomology of the space BG with Z/l-coefficients. Recall that if $l = 2$, $H^*_{sing}(BG, Z/l)$ is a polynomial ring in one variable and when $l > 2$, $H^*_{sing}(BG, Z/l) = Z/l[t] \otimes \Lambda[\nu]$ where $\beta t = \nu$ and $\Lambda[\nu]$ denotes an exterior algebra in one generator ν.

2. The situation where will apply the above proposition will be the following: X will denote a given smooth scheme and X' will denote $X^{\times^l}$. W will denote another smooth scheme provided with a

closed immersion $X \to W$ and W' will denote $W^{\times^l}$. In this case the normal bundle associated to the diagonal imbedding of X in $X^{\times^l}$ is $T_{X^{\times^{l-1}}}$ (the normal bundle associated to the diagonal imbedding of W in $W^{\times^l}$ is $T_{W^{\times^{l-1}}}$,respectively). As equivariant vector bundles for the obvious permutation action of $\mathbb{Z}/l$ on $X^{\times^l}$ and $W^{\times^l}$ these identify with $R\otimes_k T_X$ and $R\otimes_k T_W$ where R is the representation of $\mathbb{Z}/l$ given by $(k[x]/(x^l-1))/k$. For a line bundle $\mathcal{E}$, let $w(\mathcal{E},t) = 1+c_1(L)^{l-1}t$. One extends the definition of $w(\mathcal{E},t)$ to all vector bundles $\mathcal{E}$ by making this class take short exact sequences to products. Then the Euler-class $e(R\otimes_k T_X) = t^{dim(X)}w(T_X, 1/t)$ and $e(R\otimes_k T_W) = t^{dim(W)}w(T_W, 1/t)$.

At this point, we may adopt the arguments as in [Bros] to define cohomology operations that are compatible with push-forwards by proper maps between quasi-projective schemes. i.e. Let $Q^\bullet$: $H^*(X,\mathbb{Z}/l(\bullet)) \to H^*(X,\mathbb{Z}/l(\bullet))$ denote the total operation defined by $Q^\bullet = \Sigma_s Q^s$. Now we define the *covariantly functorial* operations Q_s by letting

(7.0.17) $$Q_\bullet = \Sigma_s Q_s = Q^\bullet \cap w(T_X)^{-1}$$

(Recall that the class $w(T_X)$ is invertible.) If we re-index motivic cohomology homologically, (i.e. if X is proper and of pure dimension d, we let $H_n(X,\mathbb{Z}/l(r)) = H^{2d-n}(X,\mathbb{Z}/(d-r)))$ the operations Q_s map $H_n(X,\mathbb{Z}/l(t))$ to $H_{n-2s(l-1)}(X,\mathbb{Z}/l(tl-d(l-1)))$.

Proposition 7.4. *Let $f : X \to Y$ denote a proper map between quasi-projective schemes over $Spec \quad k$. Then $Q_\bullet \circ f_* = f_* \circ Q_\bullet$.*

Proof. Since X and Y are quasi-projective, f may be factored as a closed immersion $i : X \to Y \times \mathbb{P}^n$ for some projective space $\mathbb{P}^n$ and the obvious projection $\pi : Y \times \mathbb{P}^n \to Y$. Therefore, it suffices to prove the assertion separately for $f = i$ and for $f = \pi$. The case $f = i$ is clear from the statements above. Next observe that $\mathbb{P}^n$ is a linear scheme and therefore the motivic cohomology of $X \times \mathbb{P}^n$ is given by an obvious Kunneth formula: see [AJ, Appendix] for example. Therefore the Cartan formula immediately implies the required assertion for the case $f = \pi$. □

We proceed to consider various examples.

7.1. Examples. The first example we consider is an operation

$$Q_s : H_q(X,\mathbb{Z}/l(t)) \to H_{q-2s(l-1)}(X,\mathbb{Z}/l(tl-d(l-1)))$$

on a projective smooth scheme X of dimension d so that the composition with the proper map $\pi_* : H_{q-2s(l-1)}(X,\mathbb{Z}/l(tl-d(l-1))) \to H_{q-2s(l-1)}(Spec \quad k,\mathbb{Z}/l(tl-d(l-1)))$ is in fact zero.

For example, one may take $dim(X) = 3$, $q = 2$, $t = 1$, $s = 1$ and $l = 2$. Now we have the operation

$$Q_1 : H_2(X,\mathbb{Z}/2(1)) \to H_0(X,\mathbb{Z}/2(-1)).$$

In cohomology notation this identifies with an operation $Q_1 : H^4(X,\mathbb{Z}/2(2)) \to H^6(X,\mathbb{Z}/2(4))$. The projection to $Spec \quad k$ sends the source to the group $H_2(Spec \quad k,\mathbb{Z}/2(1)) \cong H^{-2}(Spec \quad k,\mathbb{Z}/2(-1)) \cong CH^{-1}(Speck\, k,\mathbb{Z}/2) = 0$. It follows that $\pi_* \circ Q_1 = 0$. Recall that $H^4(X,\mathbb{Z}/2(2))$ identifies with $CH^2(X,\mathbb{Z}/2)$. Therefore any closed integral sub-scheme of X of codimension 2 defines a class in this group. If α is such a class, our conclusion is that $\pi_*(Q_1(\alpha)) = 0$.

So far we did not put any restriction on the prime l. Next we assume $l = p$. Let $\nu(r)$ be the sheaf that is kernel of $W^* - C : Z\Omega^r_{X/S} \to \Omega^r_{X^{(p)}/S}$. Here $X^{(p)}$ is the scheme obtained as the pull-back of $X \underset{S}{\times} S$ where the map $S \to S$ is the absolute Frobenius and $S = Spec\, k$ is the base field. Moreover W^* is defined as the adjoint to the obvious map $\Omega^r_{X/S} \to W_*\Omega^r_{X^{(p)}/S}$ and $Z\Omega^r_{X/S}$ denotes the kernel of the differential $d : \Omega^r_{X/S} \to \Omega^{r+1}_{X/S}$. (See [Ill, 2.4] for more details.) It is known that $\nu(0)$ = the constant sheaf Z/p, $\nu(1) = dlog(\mathcal{O}^*_X)$ and that $\nu(r)$, viewed as a sheaf on X_{et} is generated locally by $dlog(x_1). \cdots dlog(x_r)$, $x_i \varepsilon \mathcal{O}^*_X$.

It is shown in [GL, Theorem 8.4] that if X is a smooth integral scheme over k and k is perfect, then one has the natural isomorphism (induced by a quasi-isomorphism $\nu(r)[-r] \simeq \mathbb{Z}/p(r)$) $H^s(X, \nu(r)) \cong H^{s+r}(X, \mathbb{Z}/p(r))$, where cohomology denotes cohomology computed either on the Zariski or étale sites.

Therefore, if we require $l = p$ and the field k is perfect, the last operation takes on the form

$$Q_1 : H^2(X, \nu(2)) \to H^2(X, \nu(4))$$

where cohomology denotes cohomology computed either on the Zariski or étale sites.

As another example, we may assume $dim(X) = 4$, $q = 3$, $t = 1$, $s = 1$ and $l = 3$. Now we obtain the operation $Q_1 : CH^3(X, \mathbb{Z}/3, 1) \cong H^5(X, \mathbb{Z}/3(3)) \to H^9(X, \mathbb{Z}/3(9))$. Re-indexing homologically this identifies with $Q_1 : H_3(X, \mathbb{Z}/3(1)) \to H_{-1}(X, \mathbb{Z}/3(-5))$. Now $\pi_* \circ Q_1 = Q_1 \circ \pi_*$ and π_* maps the group $H_3(X, \mathbb{Z}/3(1))$ to $H_3(Spec\, k, \mathbb{Z}/3(1)) \cong H^{-3}(Spec\, k, \mathbb{Z}/3(-1)) \cong CH^{-1}(Spec\, k, \mathbb{Z}/3, 1) = 0$ since the higher Chow groups indexed by the codimension are trivial for negative codimension. Therefore, the composition $\pi_* \circ Q_1 = 0$. In case $l = p$, this operation now takes on the form

$$Q_1 : H^2(X, \nu(3)) \to H^0(X, \nu(9)).$$

As yet another example, we will presently show that the only classical operations that send the usual mod-l Chow groups to the usual mod-l Chow groups are the power operations. Recall that the usual mod$-l$ Chow groups are given by the mod$-l$ motivic cohomology groups $H^{2n}(X, \mathbb{Z}/l(n))$. Now let $Q^s : H^{2t}(X, \mathbb{Z}/l(t)) \to H^{2t+2s(l-1)}(X, \mathbb{Z}/l(lt))$ be given so that the $2t + 2s(l-1) = 2lt$. Then $2s(l-1) = 2t(l-1)$ so that $s = t$. Therefore we see from Theorem 5.2(ii) that the given operation is none other than the l-th power operation.

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