Modules over convolution algebras from equivariant derived categories-I

Roy Joshua

Abstract. In this paper, we provide a general (functorial) construction of modules over convolution algebras (i.e. where the multiplication is provided by a convolution operation) starting with an appropriate equivariant derived category. The construction is sufficiently general to be applicable to different situations. One of the main applications is to the construction of modules over the graded Hecke algebras associated to complex reductive groups starting with equivariant complexes on the unipotent variety. It also applies to the affine quantum enveloping algebras of type A_n. As is already known, in each case the algebra can be realized as a convolution algebra. Our construction *turns suitable equivariant derived categories into an abundant source of modules over such algebras*; most of these are new, in that, so far the only modules have been provided by suitable Borel-Moore homology or cohomology with respect to a constant sheaf (or by an appropriate K-theoretic variant.) In a sequel to this paper we will apply these constructions to equivariant perverse sheaves and also obtain a general multiplicity formula for the simple modules in the composition series of the modules constructed here.

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0. **Introduction**

This paper has its origins in our effort to construct modules over Hecke algebras associated to a complex reductive group from equivariant complexes on its unipotent variety. As it became clear that our constructions apply in more general contexts, we decided to state our constructions in as broad a context as possible. The result is the present paper.

We begin with a review of equivariant derived categories in section 1. Section 2 is a continuation of section 1 where we establish some of the key properties of equivariant hypercohomology. We conclude section 2 by quoting an equivariant Riemann-Roch theorem from ([J-7](4.2), see also [T-2]) that suffices for the needs of the paper. Following Ginzburg (also Kazhdan and Lusztig), we consider convolution algebras in detail in the third section. We consider such algebras in two distinct contexts: those defined using a convolution operation in equivariant K-homology and those defined using a convolution operation in equivariant homology. The equivariant Riemann-Roch provides the compatibility of these algebras. We also consider the associativity of convolution in detail, as this forms the basis of the construction in section 4. Section 4 forms the heart of the paper, where we provide a general construction of modules over convolution algebras starting with an equivariant derived category. The key to this is a new interpretation of the convolution product (see lemma (4.3) and the discussion following it) in the setting of equivariant derived categories and also a thorough understanding of the associativity of the convolution. The main result of the paper is the following.

Let G denote a complex linear algebraic group acting on G-quasi-projective varieties (in the sense of (1.3.2)) $\overset{o}{U}$ and U and let $f : \overset{o}{U} \rightarrow U$ denote a G-equivariant *proper* map. Assume further that $\overset{o}{U}$ is *smooth.* Now one defines the structure of an associative (but not in general commutative) algebra on the equivariant homology $H_*^G(\overset{oo}{U}; \mathbb{Q})$ using a *convolution operation* as in (3.2.7). (Here $\overset{oo}{U} = \overset{o}{U} \underset{U}{\times} \overset{o}{U}$.) Let $D_b^{c,G}(\overset{oo}{U}; \mathbb{Q})$ denote the equivariant derived category associated to $\overset{oo}{U}$ defined as in section 1.

Theorem (See (4.6) and (4.7).) If $K^{\cdot} \in D_b^{c,G}(\overset{oo}{U}; \mathbb{Q})$, $\mathbb{H}_G^*(\overset{oo}{U}; K^{\cdot})$ has the structure of a left-module over the convolution algebra $H_*^G(\overset{oo}{U}; \mathbb{Q})$.

(The above theorem uses the pairing $\underline{\mathbb{Q}} \otimes K \rightarrow K$. If, instead, the pairing $K \otimes \underline{\mathbb{Q}} \rightarrow K$ is used, one obtains the structure of a right-module on $\mathbb{H}_G^*(\overset{oo}{U}; K)$.) This construction makes the equivariant derived category $D_b^{c,G}(\overset{oo}{U}; \mathbb{Q})$ an abundant source of modules over the convolution algebra. The fifth section considers various applications. Here we apply our results to produce an abundant supply of modules over Hecke algebras and over affine quantum enveloping algebras of type A_n. The final section considers the effect of the natural anti-involution (of the convolution algebra) on the module structures we have constructed. In sequels to this paper we will show that variants of the above construction provide the standard and co-standard modules as well as simple and self-dual modules over graded Hecke-algebras (with and without parameters.)

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Many of the constructions of this paper generalize in the context of generalized Verdier duality (as in [J-2], [J-4]) to provide a construction of modules over the *integral affine Hecke algebra* from equivariant presheaves of (equivariant) K-theory spectra. These will appear in a subsequent paper. An earlier version of the present paper in fact used some of these machinery; we have however removed all references to generalized Verdier duality in the present version, making the paper self-contained and dependent only on the theory of equivariant derived categories and on an equivariant form of Riemann-Roch as in (2.12).

We would also like to point out that weaker forms of the above theorem may be obtained by simpler techniques. For example assume the complex $K = Ri^!(\underline{\mathbb{Q}} \boxtimes L)$, $L \,\epsilon\, D_b^{c,G}(\overset{o}{U}; \mathbb{Q})$ (where $\overset{o}{U}$ is viewed as the second factor in $\overset{o}{U} \times \overset{o}{U}$, $\underline{\mathbb{Q}}$ is the obvious constant sheaf on the first factor and $i : \overset{oo}{U} \to \overset{o}{U} \times \overset{o}{U}$ is obvious closed immersion). Now it is possible to show that $\mathbb{H}^*_G(\overset{oo}{U}; K)$ has the structure of a left-module over the convolution algebra $H^G_*(\overset{oo}{U}; \mathbb{Q})$ in a simpler manner. However, one can see from (5.3.1), that there are many interesting cases that are not obtainable this way but to which our theorem applies; theorem (6.5.3) that shows how to interchange the left and right module-structures also will not apply to the above case.

Conventions. (0.1) Throughout the paper we will restrict to schemes of finite type over the complex numbers, which are also *quasi-projective or projective.* (We may assume without loss of generality that they are reduced, but we often need to consider schemes that are *not* irreducible.) We will refer to these as *quasi-projective (or projective) varieties.* In order to allow compact Lie group actions on these varieties, it is often convenient to enlarge the above situation. The generic term *space* will refer to either a quasi-projective complex variety as above or a locally-compact Hausdorff space with finite cohomological dimension. The first case will be called *the algebraic case* and the second *the topological. pt* will denote *point* = the terminal object in the category of spaces.

1. The equivariant derived category: a review In this section, we recall the basic results on equivariant derived categories from [J-5], section 6 (where the are stated in the l-adic setting). We have chosen to stick to the simplicial construction adopted in [J-5] of equivariant derived categories throughout. However one may equally well utilize the alternate approaches to equivariant derived categories in [B-L] and [B-G].

(1.1.0) We will assume the basic situation of (0.1). Accordingly we are in one of the following two situations: (i) X denotes a locally-compact Hausdorff space with finite cohomological dimension and provided with the (*left*) action by a compact Lie-group G *or* (ii) X denotes a complex quasi-projective variety provided with the (*left*) action by a complex linear algebraic group G. The first (the second) case will be referred to as the *topological case* (the *algebraic case*, respectively). *For the most part we will only need to consider the algebraic case. Therefore we discuss explicitly only this case.*

(1.1.1) In this situation one first forms the simplicial spaces $EG\underset{G}{\times}X$ in the usual manner (See [Fr]p. 4 for example.) Observe that $(EG\underset{G}{\times}X)_n = G^n \times X$ with the usual structure maps. Each of the face maps $d_i : (EG\underset{G}{\times}X)_n \to (EG\underset{G}{\times}X)_{n-1}$ is induced by the group-action $\mu : G \times X \to X$ and the projection $\pi_2 : G \times X \to X$ and hence is smooth in the algebraic case.

(1.1.2) Now assume $X.$ is a simplicial space (i.e. a simplicial object in the category of spaces), for example, one of the simplicial spaces obtained as above. One puts a *Grothendieck topology* on $X.$, denoted $Top(X.)$, by defining the objects to be maps $u : U \to X_n$, where u is the inclusion of an open set in X_n for some n. Given two such objects $u : U \to X_n$ and $v : V \to X_m$ for some n, m a morphism $\alpha : u \to v$ is given by a map $\alpha : U \to V$ that lies over a structure map $\alpha' : X_n \to X_m$ of the simplicial space $X.$. *If X_n is a complex algebraic variety for each n, we will always consider only the transcendental topology on X_n.*

(1.2.1). Let E denote a Noetherian ring. For the cases of interest we may assume E is either $\mathbb{Q}$, $\mathbb{C}$, or a field of characteristic 0. A sheaf of left (right) E-modules M on *a space* is *constructible* if it is constructible in the usual sense. If $X.$ *denotes a simplicial space*, a sheaf F of E-modules on $Top(X.)$ consists of a collection $\{F_n|n\}$, where each F_n is sheaf of E-modules on X_n, provided with a collection of maps $\phi(\alpha) : \alpha^*(F_n) \to F_m$ associated to each structure map $\alpha : X_m \to X_n$ of the simplicial space $X.$. These are required to satisfy certain obvious compatibility conditions as in ([Fr] p.14, for example). The category of such sheaves will be denoted by $Sh_E(X.)$. We will let $D_{b,l}(X.;\, E)$ ($D_{b,r}(X.;\, E)$) denote the derived category of all complexes of sheaves of left E-modules (right E-modules, respectively) with bounded cohomology sheaves. A sheaf F on $X.$ has *descent property* provided the map $\phi(\alpha) : \alpha^*(F_n) \to F_m$ (as above) associated to each structure map α, is an isomorphism. A complex $K^\cdot$ in $Sh_E(X.)$ or in $Sh_E(X.)$ *has descent* if all its cohomology sheaves $\{H^i(K_n)|n\}$ have descent. *Throughout the paper a complex will mean one whose differentials are of degree* $+1$. A sheaf $F = \{F_n|n\}$ of E-modules on $Top(X.)$ is *constructible* if each F_n on $Top(X_n)$ is constructible in the above sense; one may observe that if F is a sheaf on $Top(X.)$ with descent, F is constructible if and only if F_0 is constructible. Finally we say a sheaf $F = \{F_n|n\}$ on $Top(X.)$ is *locally constant* if F_0 is locally constant on $Top(X_0)$ and F is a sheaf with descent.

In the special case $X.$ is the simplicial space $EG\underset{G}{\times}X$ as in (1.1.1) and $F = \{F_n|n\}$ is a sheaf on $EG\underset{G}{\times}X$ with descent, we will say F *is a G-equivariant sheaf on $EG\underset{G}{\times}X$* and that F_0 *is a G-equivariant sheaf on X*. Now F is a lift of F_0 to $EG\underset{G}{\times}X$. (Conversely any sheaf K on X is equivariant if there is a sheaf $F = \{F_n|n\}$ on $EG\underset{G}{\times}X$ with descent so that $K = F_0$.)

(1.2.2) Assume the above situation. Observe that the subcategory of sheaves with descent is an abelian sub-category of $Sh_E(X.)$ closed under extensions. Therefore one defines the category $D^{des}_{b,l}(X.;\, E)$ ($D^{des}_{b,r}(X.;\, E)$) to be the full subcategory of the derived category $D_{b,l}(X.;\, E)$ ($D_{b,r}(X.;\, E)$, respectively) of complexes of sheaves having descent. $D^{c,des}_{b,l}(X.;\, E)$ ($D^{c,des}_{b,r}(X.;\, E)$, respectively) will denote the corresponding full subcategory of complexes with constructible cohomology sheaves. (*The subscripts l (and r) will be omitted to indicate any one of the above categories generically, the default being l.*)

(1.2.3) If $X. = EG\underset{G}{\times}X$ = the simplicial space associated to the action of a group G on a space X (as in (1.1.1)), we will denote the category $D^{des}_b(X.;\, E)$ ($D^{c,des}_b(X.;\, E)$) by $D^G_b(X;\, E)$ ($D^{c,G}_b(X;\, E)$, respectively).

(1.2.4) For later applications will need to generalize the set-up in (1.2.1) as follows. (See (2.P.4) and the remarks following it.) Let $\bar{G}$ denote a *discrete* group acting on the *right* on a simplicial space $X.$. (Observe that this action may be identified with the following data: for each $\bar{g} \in \bar{G}$ one is given a map of simplicial spaces $T_{\bar{g}} : X. \rightarrow X.$ such that if $\bar{g}_1, \bar{g}_2 \in \bar{G}$, the composite map $X. \xrightarrow{T_{\bar{g}_1}} X. \xrightarrow{T_{\bar{g}_2}} X.$ equals the map $X. \xrightarrow{T_{\bar{g}_1 . \bar{g}_2}} X.$) Now a sheaf F (of left (right) modules on $X.$) *provided with a $\bar{G}$-action* is a sheaf F on $X.$ provided with the following data: for each $\bar{g} \in \bar{G}$, there is given a map $F \rightarrow T_{\bar{g}*}F$ (of left (right) modules on $X.$) so that if $\bar{g}_1, \bar{g}_2 \in \bar{G}$, the composite map $F \rightarrow T_{\bar{g}_2 *}(F) \rightarrow T_{\bar{g}_2 *}T_{\bar{g}_1 *}F$ equals the given map $F \rightarrow (T_{\bar{g}_1 . \bar{g}_2})_* F$. This category of sheaves with $\bar{G}$-action will be denoted $Sh_E^{\bar{G}}(X.)$; this category is known to be an abelian category with enough injectives - see [Groth] chapter 5. Therefore, under the above hypotheses, one may define $D_b^{\bar{G}}(X.; E)$ to be the homotopy category of all complexes in $Sh_E^{\bar{G}}(X.)$ with bounded cohomology sheaves localized by inverting maps that are quasi-isomorphisms ($\cong$ the homotopy category of complexes of injectives in $Sh_E^{\bar{G}}(X.)$ with bounded cohomology sheaves). One defines the derived category $D_b^{des,\bar{G}}(X.; E)$ similarly.

(1.2.5) Observe that if $\bar{G}$ acts trivially on the simplicial space $X.$, a sheaf F on $X.$ with a $\bar{G}$-action corresponds to a sheaf F on $X.$ provided with a representation of the group $\bar{G}$ on F (i.e. at each stalk). If X is a space provided with the left-action of a group G along with the right-action of the discrete group $\bar{G}$ so that the two actions commute, one observes readily that the $\bar{G}$ action on X extends to an action on the simplicial space $EG \underset{G}{\times} X$. Now one may define the derived category $D_b^{G,\bar{G}}(X; E)$ to be $D_b^{des,\bar{G}}(EG \underset{G}{\times} X; E)$.

Equivariant derived functors

(1.3.1) Let $f : X. \rightarrow Y.$ denote a map between simplicial spaces $X.$ and $Y.$. Now f induces a map of sites : $V \rightarrow V \underset{Y_n}{\times} X_n$, $Top(Y.) \rightarrow Top(X.)$. One may define the functors $Rf_* : D_b^{c,des}(X.; E) \rightarrow D_b^{c,des}(Y.; E)$ in the obvious manner; if each $f : X_n \rightarrow Y_n$ is also *proper* one may let $Rf_! = Rf_*$. In general, if f admits a factorization $f = \bar{f} \circ j$ where $j : X. \rightarrow \bar{X}.$ is an open imbedding (in each degree) into a simplicial space $\bar{X}.$ and each $\bar{f}_n$ is a proper map, one may define $Rf_! = R\bar{f}_* j_!$.

If f is a G-equivariant map between spaces X and Y provided with the action by a group G, f induces a map $f^G : EG \underset{G}{\times} X \rightarrow EG \underset{G}{\times} Y$ of simplicial spaces. The induced functor $Rf_*^G : D_b^{c,des}(EG \underset{G}{\times} X; E) = D_b^{c,G}(X; E) \rightarrow D_b^{c,des}(EG \underset{G}{\times} Y; E) = D_b^{c,G}(Y; E)$ is given by $Rf_*^G = \{Rf_{n,*}^G : D_b^c((EG \underset{G}{\times} X)_n; E) \rightarrow D_b^c((EG \underset{G}{\times} Y)_n; E) | n\}$.

(1.3.2) If G denotes a complex linear algebraic group, we will consider mostly *G-quasi-projective varieties*. Recall that a G-variety X is G-quasi-projective, if there exists a G-equivariant locally-closed immersion of X into a projective space $\mathbb{P}^n$ provided with a linear action by G. A theorem of Sumihiro (see [Sum], Theorem 1) shows that if G is connected any normal quasi-projective variety is G-quasi-projective. (It is shown in [J-7](1.9), that the hypothesis that G be connected may be dropped; however this is not important for the present paper since the groups here will be usually connected.) If $f : X \rightarrow Y$ is a G-equivariant map between two G-quasi-projective varieties, one obtains a G-equivariant locally-closed immersion $X \rightarrow Y \times \mathbb{P}^n$, for some $n >> 0$. (G acts diagonally on the latter.) Let $\bar{X}$ = the closure of X in $Y \times \mathbb{P}^n$. Now one may factor f as $\bar{f} \circ j$, where $j : X \rightarrow \bar{X}$ is the obvious G-equivariant open immersion and $\bar{f} : \bar{X} \rightarrow Y \times \mathbb{P}^n \rightarrow Y$ is the obvious projection. In case f is a G-equivariant map as above, we will let the induced functor $Rf_!^G : D_b^{c,des}(EG\underset{G}{\times}X;\, E) = D_b^{c,G}(X;\, E) \to D_b^{c,des}(EG\underset{G}{\times}Y;\, E) = D_b^{c,G}(Y;\, E)$ be given by $R\bar{f}_*^G \circ j_!^G = \{R\bar{f}_{n,*}^G \circ j_{n,!}^G : D_b^c((EG\underset{G}{\times}X)_n;\, E) \to D_b^c((EG\underset{G}{\times}Y)_n;\, E)|n\}$. The above description of the functor $Rf_!$ shows readily that it is independent of the factorization $f = \bar{f} \circ j$.

(1.3.3) Let $K^{\cdot} = \{K_n|n\} \,\epsilon\, D_b^{c,G}(X;\, E)$. (One may view $K^{\cdot}$ as a differential graded object in the category $Sh_E(EG\underset{G}{\times}X)$.) Now $\{\Gamma((EG\underset{G}{\times}X)_n;\, K_n)|n\}$ is a cosimplicial object in the category of differential graded objects in the category of E-modules. Let $N(\{\Gamma((EG\underset{G}{\times}X)_n;\, K_n)|n\})$ denote the double complex given by applying the normalization functor. If $K \overset{\simeq}{\to} \mathcal{G}K$ is a quasi-isomorphism into a complex of flabby sheaves in $D_b^{c,G}(X;\, E)$, we let $\mathbb{H}_G(X;\, K)$ = the differential graded object given by $TOTN(\{\Gamma((EG\underset{G}{\times}X)_n;\, \mathcal{G}K_n)|n\})$, where TOT denotes the total complex of the double complex. We call this *the G-equivariant hypercohomology spectrum* of X with respect to K. *The equivariant hypercohomology* of X with respect to K are the cohomology groups of this differential graded object; these will be denoted $\mathbb{H}_G^*(X;\, K)$.

In this situation there exists a spectral sequence:

(1.3.4) $E_2^{s,t} = H_G^s(X; \mathcal{H}^t(F^{\cdot})) \Rightarrow \mathbb{H}_G^{s+t}(X;\, F^{\cdot})$

which *converges strongly* since $F^{\cdot}$ is a complex of sheaves of modules over E with bounded cohomology sheaves.

(1.3.5) *The equivariant hypercohomology of X with compact supports* may now be defined to be $\mathbb{H}_{G,c}^*(X;\, K) = \mathbb{H}_G^*(\bar{X};\, j_!K)$, where $j : X \to \bar{X}$ is a G-equivariant closed imbedding into a space $\bar{X}$ that is proper over pt (= Spec $\mathbb{C}$, in the algebraic case.)

Equivariant Verdier duality: local form. (See [J-5] section 6 for more details.)

(1.4.1) As discussed in (1.3.2) we will mostly consider G-quasi-projective varieties. If $BG.$ denotes the classifying simplicial space for G, we define the dualizing complex for the category $D_b^{c,des}(BG;\, E)$ to be the constant sheaf $\underline{E}$. If $\pi. : EG\underset{G}{\times}X \to BG.$ is the obvious map, we let $\mathbb{D}_E = R\pi.^!(\underline{E})$ be the dualizing sheaf for the category $D_b^{c,des}(EG\underset{G}{\times}X;\, E)$. If $K \,\epsilon\, D_b^{c,des}(EG\underset{G}{\times}X;\, E)$, we define $\mathbb{D}_E(K) = \mathcal{RH}om_E(K, \mathbb{D}_E)$. One may readily verify (observe that $(\mathbb{D}_E)_n$ = the dualizing complex for the category $D_b^c((EG\underset{G}{\times}X)_n;\, E)$) that the natural map $K \to \mathbb{D}_E(\mathbb{D}_E(K))$ is a quasi-isomorphism.

(1.4.2) Next assume that $f : X \to Y$ is a G-equivariant map of G-quasi-projective varieties. Now one defines $R(f^G)^! : D_b^{c,G}(Y;\, E) \to D_b^{c,G}(X;\, E)$ to be right adjoint to $R(f^G)_!$. (The existence of such a right adjoint follows readily from the description of $R(f^G)_!$ as $\{R\bar{f}_{n,*}^G \circ j_{n,!}^G : D_b^c((EG\underset{G}{\times}X)_n;\, E) \to D_b^c((EG\underset{G}{\times}Y)_n;\, E)|n\}$. We summarize the main results:

(1.4.3) **Proposition**. (*Local form of Verdier duality*). Assume the above situation.

(i) Now $Rf_!^G$ is independent of the factorization $f = \bar{f} \circ j$.

(ii) There exist natural quasi-isomorphisms: $Rf_!^G(L^{\cdot}) \simeq D_Y \circ Rf_*^G \circ D_X(L^{\cdot})$, $L^{\cdot} \,\epsilon\, D_b^{c,G}(X;\, E)$ and $Rf^{G^!}(K^{\cdot}) \simeq D_X \circ f^{G^*} \circ D_Y(K^{\cdot})$. $K^{\cdot} \,\epsilon\, D_b^{c,G}(Y;\, E)$.

(iii). There exists a natural map (= *the trace-map*) $tr(f^G, K^{\cdot}) : Rf_!^G \circ Rf^{G^!}K \to K$, *natural* in $K \,\epsilon\, D_b^{c,G}(Y;\, E)$.

(iv) $f : X \to Y$ denote a G-equivariant *smooth* map between G-quasi-projective varieties of relative dimension d. Now there exists a *natural isomorphism*

$P : f^*[2d] \to Rf^!$ as functors $D_b^{c,G}(Y;\, E) \to D_b^{c,G}(X;\, E)$.

(Since the Tate-twist remains constant for fixed X, Y and G, we have ignored it.)

(v) If in addition to the hypothesis in (i), f is proper, one obtains a natural identification of $R(f^G)_!$ with $R(f^G)_*$.

Proof. All of these may be readily established by observing $Rf_!^G = \{Rf_{n,!}^G : D_b^{c,des}((EG\underset{G}{\times}X)_n;\, E) \to D_b^{c,des}((EG\underset{G}{\times}Y)_n;\, E)|n\}$ and $Rf^{G^!} = \{Rf^{G^!}_n : D_b^{c,des}((EG\underset{G}{\times}Y)_n;\, E) \to D_b^{c,des}((EG\underset{G}{\times}X)_n;\, E)|n\}$. □

(1.5.1) *Proper or smooth base-change.* Let E

$$\begin{array}{ccc} X' & \xrightarrow[g']{} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

denote a pull-back square of schemes as in (1.1.0) (i.e. all the schemes here are acted on by a group G and all the maps are G-equivariant) so that f is also *proper* or g is *smooth.* If $K \;\epsilon\; D_b^{c,G}(X; E)$, we obtain natural quasi-isomorphisms:

$Rf'^G_* R(g'^G)^! K \simeq Rg^{G^!}(Rf^G{}_*(K))$, $Rf'^G{}_* g'^{G^*} K \simeq g^{G^*} Rf^G{}_* K$

Proof. This also follows from the same observations as in the proof of (1.4.3).

(1.5.2) *Naturality of the trace-map with respect to base change.* Assume the above situation.

Now $Rg^{G^!}(tr(f^G, K)) : Rg^{G^!} Rf^G{}_! Rf^{G^!} K \to Rg^{G^!} K$ identifies with the map

$tr(f'^G, Rg^{G^!}K) : Rf'^G_! Rg'^{G^!} Rf^{G^!}(K) \cong Rf'^G{}_! Rf'^{G^!}(Rg^{G^!}K) \to Rg^{G^!}(K)$

If g is smooth one obtains a similar result with g^{G^*} replacing $Rg^{G^!}$.

Proof. Observe that $Rg^{G^!} tr(f^G, K)$ is given by $Dg^{G^*} Rf^G_* f^{G^*} DK \overset{\simeq}{\Rightarrow} Dg^{G^*} DDRf^G{}_* DDf^{G^*} DK (= Rg^{G^!} Rf^G{}_! Rf^{G^!} K) \to Dg^{G^*} DK$. This is the dual of the natural map $g^{G^*} DK \to g^{G^*} Rf^G{}_* f^{G^*} DK$. On the other hand $tr(f'^G, Rg^{G^!}K)$ may be identified with the map $DRf'^G_* DDf'^{G^*} DDg^{G^*} DK \to Dg^{G^*} DK$, which may be identified with the dual of the natural map $g^{G^*} DK \to Rf'^G_* f'^{G^*} g^{G^*} DK$. Therefore it suffices to show the square

$$\begin{array}{ccc} g^{G^*} DK & \longrightarrow & g^{G^*} Rf^G_* f^{G^*} DK \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \simeq} \\ g^{G^*} DK & \longrightarrow & Rf'^G_* f'^{G^*} g^{G^*} DK \end{array}$$

or equivalently

$$\begin{array}{ccc} DK & \longrightarrow & Rf^G_* f^{G^*} DK \\ {\scriptstyle id}\downarrow & & \downarrow \\ DK & \longrightarrow & Rg^G_* Rf'^G{}_* f'^{G^*} g^{G^*} DK = Rf^G_* Rg'^G_* g'^{G^*} f^{G^*} DK \end{array}$$

commutes. This is clear. (Use the adjunction between Rf^G_* and f^{G^*} to convert the last diagram to:

$$\begin{array}{ccc} f^{G^*}DK & \xrightarrow{id} & f^{G^*}DK \\ {\scriptstyle id}\downarrow & & \downarrow \\ f^{G^*}DK & \longrightarrow & Rg'^G_* g'^{G^*} f^{G^*}DK \end{array} \quad). \quad \Box$$

(1.6) We conclude this section by recalling the definition of the equivariant intersection cohomology complexes. (See [Bryl-1] or [J-3]). Here E will denote either $\mathbb{Q}$ or $\mathbb{C}$. Assume $\{S\}$ is a G-invariant stratification of X into finitely many locally-closed smooth sub-schemes in the algebraic case, called *strata*, all which are of *pure dimension* (disjoint unions of manifolds of the same dimension, in the topological case, respectively). One may now form the corresponding filtration $\mathcal{U} = U_1 \subseteq U_2 \subseteq ... \subseteq U_n \subseteq U_{n+1} = X$. (Here n is the complex dimension of X and each successive difference $U_i - U_{i-1}$ is a disjoint union of strata of the same dimension $= n - i + 1$.) Since the strata are G-*invariant* this induces a corresponding filtration of the simplicial space $EG\underset{G}{\times}X$:

$$EG\underset{G}{\times}U_1 \to EG\underset{G}{\times}U_2 \to ... \to EG\underset{G}{\times}U_{n+1} = EG\underset{G}{\times}X$$

Let $j^G_i : EG\underset{G}{\times}U_i \to EG\underset{G}{\times}U_{i+1}$ denote the map induced by $j : U_i \to U_{i+1}$.

(1.6.1) Let $\mathcal{L}$ denote a G-equivariant locally constant sheaf on U_1. $\mathcal{L}$ defines a sheaf on the simplicial space $EG\underset{G}{\times}U_1$ as above. If p denotes a fixed perversity, this sheaf is extended to a complex of sheaves $IC^G_p(\mathcal{L})$ on $EG\underset{G}{\times}X$ called *the equivariant intersection cohomology complex with perversity p*. (The hypercohomology of $EG\underset{G}{\times}X$ with respect to this complex is called the *equivariant intersection cohomology of X with respect to* $\mathcal{L}$ and is denoted $IH^{G,*}_p(X;\mathcal{L})$. Such equivariant intersection cohomology complexes (for irreducible representations $\mathcal{L}$ of $\pi_1(U_1)$) form the simple objects in the category of equivariant perverse sheaves; these form the heart of the equivariant derived category $D^{c,G}_b(X;\,\mathbb{Q})$. (See [J-3] and [J-5] for more details.)

2.**Basic properties of equivariant hypercohomology and K-theory**

Assume the situation of (1.1.0). We begin by defining equivariant homology; this will always be with locally compact supports, so we will often call this merely equivariant homology.

(2.1)**Definition**. We define *the equivariant homology* of X relative to G to be

$$H^G_*(X;\,E) = \mathbb{H}^*_G(X;\,\mathbb{D}_E\ )$$

where $\mathbb{D}_E$ is the dualizing complex for the category $D^{c,G}_b(EG\underset{G}{\times}X;\,E)$ as in (1.4.1).

Let $X \to \tilde{X}$ denote a closed G-equivariant immersion of X into a smooth G-scheme $\tilde{X}$. Now one defines $K^G_0(X) = K^0_{G,X}(\tilde{X}) \cong$ the Grothendieck group of G-equivariant coherent sheaves on $\tilde{X}$ with supports contained in X. Throughout our discussion, M will denote a maximal compact subgroup of G. Now one lets $K^M_{A.S,0}(X) = K^{A.S,0}_{M,X}(\tilde{X})$ = the M-equivariant Atiyah-Segal K-theory of $\tilde{X}$ with supports in X and $K^M_{top,0}(X) = K^{top,0}_{M,X}(\tilde{X})$ = the M-equivariant topological K-theory of $\tilde{X}$ with supports in X. (See [J-7] sections 1 and 2 or [T-2] section 5 for a thorough discussion of these theories. There it is shown that $K^M_{A.S,0}(X)$ is independent of the chosen closed immersion $X \to \tilde{X}$.) Recall that if X is smooth, one obtains a Poincaré-duality isomorphism: $K^G_0(X) \cong K^0_G(X)$ where the former is the Grothendieck group of G-equivariant *locally free* coherent sheaves. Similar isomorphisms exist for equivariant Atiyah-Segal K-theory and equivariant topological K-theory.

Next we will list *a few of the properties of equivariant hypercohomology.* Throughout we will assume that $K \epsilon D^{c,G}_b(X; E)$, where E is either $\mathbb{Q}$ or $\mathbb{C}$.

(2.P.1) If $K \epsilon D^{c,G}_{b,l}(X; \mathbb{Q})$, $\mathbb{H}^*_G(X; K)$ is a finitely generated left-module over $\mathbb{H}^*_G(X; \mathbb{Q})$ and hence over $H^*(BG; \mathbb{Q})$. (Observe that $R(G) \cong R(M)$ where M is a maximal compact subgroup of G. Now [A-S] shows that if I_G denotes the augmentation ideal of $R(G)$ (i.e. the kernel of the rank-function: $R(G) \to \mathbb{Z}$) then, $H^*(BG; \mathbb{Q})$ is isomorphic to the completion of $R(G) \otimes \mathbb{Q}$ at the ideal I_G.)

(2.P.1') $K^G_0(X)$, $K^M_{A.S,0}(X)$ and $K^M_{top,0}(X)$ are modules over the representation ring $R(G) \cong R(M)$.

(2.P.2) *Restriction.* If $H \subseteq G$ is a closed subgroup, then there exists a natural restriction homomorphism (of $H^*(BG; E)$-modules) : $\mathbb{H}^*_G(X; K) \to \mathbb{H}^*_H(X; Res_H(K))$, where $Res_H(K)$ $(= i^*K$ as in (A.2)) is the obvious object in $D^{c,H}_b(X; E)$. Similarly, one obtains restriction homomorphisms (of $R(G)$-modules): $K^G_0(X) \to K^H_0(X)$, $K^M_{A.S,0}(X) \to K^{M'}_{A.S,0}(X)$ and $K^M_{top,0}(X) \to K^{M'}_{top,0}(X)$ if M' denotes $M \cap H$.

Remark. *The next two properties (2.P.3) and (2.P.4) find essential application in [J-6].*

(2.P.3) Let H denote a closed (and not necessarily connected) subgroup of G. Let $\bar{i} : EH \underset{H}{\times} X \to EG \underset{G}{\times} (G \underset{H}{\times} X)$ denote the obvious map (of simplicial spaces) induced by the map $X \to G \underset{H}{\times} X$ sending x to (e, x). Clearly this is a closed immersion (closed imbedding, in the topological case) in each degree. Let $K \epsilon D^{c,G}_b(G \underset{H}{\times} X; E)$. Clearly $\bar{i}^*(K)$ is a complex of sheaves on $EH \underset{H}{\times} X$. Now $\bar{i}^*$ induces an isomorphism:

(2.P.3*) $\mathbb{H}^*_H(X; \bar{i}^*(K)) \xrightarrow{\simeq} \mathbb{H}^*_G(G \underset{H}{\times} X; K)$

This is immediate from Theorem (A.1) in the appendix which shows that the functor $\bar{i}^*$ is fully-faithful. In particular if $L \epsilon D^{c,G}_b(X; E)$, one obtains the isomorphism $\mathbb{H}^*_H(X; \bar{i}^*(\pi^*(L))) \xrightarrow{\simeq} \mathbb{H}^*_G(G \underset{H}{\times} X; \pi^*(L))$ where $\pi : EG \underset{G}{\times} (G \underset{H}{\times} X) \to EG \underset{G}{\times} X$ is the obvious map induced by the projection $G \underset{H}{\times} X \to X$. (In this situation, the composite map $\pi \circ \bar{i} : EH \underset{H}{\times} X \to EG \underset{G}{\times} X$ will be denoted i as in (A.2).)

(2.P.3)' Let X denote a projective complex algebraic variety acted on by a complex linear algebraic group H. We will assume that the action is on the *right*. Let $H \to \bar{H}$ denote the imbedding of H as a closed algebraic subgroup. (We may also alternatively assume that both H and $\bar{H}$ are compact Lie groups.) Let $\bar{X} = (X \times \bar{H})/H$, where H acts on the right on $X \times \bar{H}$ by $(x, \bar{h}).h = (x.h, h^{-1}.\bar{h})$, $h \epsilon H$ and $\bar{h} \epsilon \bar{H}$. Now $\bar{H}$ acts on the right on $\bar{X}$ by: $(x, \bar{h}') \circ \bar{h} = (x, \bar{h}'.\bar{h})$, $x \epsilon X$, $\bar{h}'$, $\bar{h}$ $\epsilon \bar{H}$. Let $K \epsilon D^{c,\bar{H}}_b(\bar{X}; E)$. Let $i_X : X \to \bar{X}$ denote the obvious map (- this is clearly a closed imbedding); this is compatible with the right actions of H on X and $\bar{H}$ on $\bar{X}$. The corresponding induced map $EH \underset{H}{\times} X \to E\bar{H} \underset{\bar{H}}{\times} \bar{X}$ will also be denoted by i_X. Now i_X induces a *natural* isomorphism:

$\mathbb{H}^*_H(X;\, i_X^*(K)) \simeq \mathbb{H}^*_{\bar{H}}(\bar{X};\, K)$

To see this one may proceed as follows. One first observes readily that the fibers of the obvious map $p : \bar{X} = (X \times \bar{H})/H \to \bar{H}/H$ are isomorphic to X. Let $\bar{x}$ denote a fixed point of $\bar{H}/H$ and let $i_{\bar{x}} : \bar{x} \to \bar{H}/H$ denote the corresponding inclusion. One now obtains the pull-back square:

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \bar{X} = (X \times \bar{H})/H \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \bar{x} & \xrightarrow{i_{\bar{x}}} & (\bar{H})/H \end{array}$$

The left-most column is H-equivariant (where H acts trivially on $\bar{x}$) while the right-most column is $\bar{H}$-equivariant. The above diagram induces a map of the hypercohomology spectral sequences:

$E_2^{s,t}(1) = H^s(BH;\, R^t\bar{p}_* i_X^*(K)) \cong H^s(BH;\, i_{\bar{x}}^* R^t p_*(K)) \Rightarrow \mathbb{H}_H^{s+t}(X;\, i_X^* K)$ and

$E_2^{s,t}(2) = H^s(E\bar{H} \underset{\bar{H}}{\times} (\bar{H}/H);\, R^t p_*(K)) \Rightarrow \mathbb{H}_{\bar{H}}^{s+t}(\bar{X};\, K)$.

(Since X is projective one may use proper-base-change to obtain the $\cong$ on the first line.) In view of (2.P.3*) with $G = \bar{H}$, $X = \bar{x}$, we observe that the two E_2-terms above are naturally isomorphic. Clearly this is induced by a map of the above spectral sequences and therefore, we obtain an isomorphism at the abutments as well. (Recall that the above spectral sequences are strongly-convergent.) This completes the proof of (2.P.3)'.

(2.P.4). Assume in addition to the assumptions of (2.P.3) that H is also *normal* in G. Now we obtain a *right-action* by $\bar{G} = G/H$ on the space $G \underset{H}{\times} X$ that commutes with the left-action by G. If $g \,\epsilon\, G$ and $(g_0, x) \,\epsilon\, G \times X$ we let $(g_0, x).g = (g_0.g, g^{-1}.x)$. Recall that if $h \,\epsilon\, H$, (g_0, x) and $(g_0.h, h^{-1}.x)$ represent the same point in $G \underset{H}{\times} X$. Since H is normal in G, $h.g = g.h'$, for some $h' \,\epsilon\, H$. Therefore $(g_0.h, h^{-1}.x).g = (g_0.h.g, g^{-1}.h^{-1}.x) = (g_0.g.h', {h'}^{-1}.g^{-1}.x)$ which is identified with $(g_0.g, g^{-1}.x)$. It follows that we obtain a well-defined right-action of $\bar{G}$ on $G \underset{H}{\times} X$. One may readily see that this commutes with the left-action of G on the same space; therefore we obtain a right-action of $\bar{G}$ on the simplicial space $EG \underset{G}{\times} (G \underset{H}{\times} X)$. For each $\bar{g} \,\epsilon\, \bar{G}$, let $T_{\bar{g}} : EG \underset{G}{\times} (G \underset{H}{\times} X) \to EG \underset{G}{\times} (G \underset{H}{\times} X)$ denote the corresponding map. The simplicial space $EG \underset{G}{\times} X$ has the trivial action by $\bar{G} = G/H$ and the maps $\pi_n : (EG \underset{G}{\times} (G \underset{H}{\times} X))_n \to (EG \underset{G}{\times} X)_n$ are now G/H-equivariant for each n. One may now consider the derived categories $D_b^{G,G/H}(X;\, E)$ and $D_b^{G,G/H}(EG \underset{G}{\times} (G \underset{H}{\times} X);\, E)$ as in (1.2.5).

If follows that if $K \,\epsilon\, D_b^{c,G,G/H}(X;\, E)$, $\pi^*(K) \,\epsilon\, D^{c,G,G/H}(EG \underset{G}{\times} (G \underset{H}{\times} X);\, E)$ and that, therefore, there exists a natural map $\pi^*(K) \to (T_{\bar{g}})_* \pi^*(K) = \pi^*((T_{\bar{g}})_* K)$ of complexes on $EG \underset{G}{\times} (G \underset{H}{\times} X)$, for every $\bar{g} \,\epsilon\, \bar{G} = G/H$. On taking the hypercohomology, for each $\bar{g} \,\epsilon\, \bar{G}$, one obtains a map

(2.P.4.1) $T_{\bar{g}} : \mathbb{H}^*_H(X;\, i^*K) \cong \mathbb{H}^*_G(G \underset{H}{\times} X;\, \pi^*(K)) \to \mathbb{H}^*_G(G \underset{H}{\times} X;\, \pi^*(K)) \cong \mathbb{H}^*_H(X;\, i^*K)$

(Here the isomorphisms indicated by $\cong$ are from (2.P.3*).) One may now readily verify that this *provides an action of* $\bar{G} = G/H$ *on the hypercohomology groups* $\mathbb{H}^*_H(X;\, i^*K)$.

Moreover one obtains a 'fibration'

$$EH\underset{H}{\times}X \xrightarrow{i} EG\underset{G}{\times}X \xrightarrow{p.} B(G/H)$$

If K is a complex as above, one now obtains a Leray (or Leray-Hochshild-Serre) spectral sequence:

$$E_2^{s,t} = H^s(B(G/H);\, R^t p_*(K)) \Rightarrow \mathbb{H}_G^{s+t}\ (X;K)$$

where the functor p_* is the direct image functor for a slightly different site (called the simplicial site in [J-5](A.4)). We will assume in addition that G/H is *projective, for example, is finite.* In this case each of the maps p_n is *proper*. It follows as in ([J-5] (A.8)) that one may identify the stalks of $R^t p_*(K)$ with $\mathbb{H}_H^t(X;\, i^*(K))$. In particular if G/H is finite and $K \,\epsilon\, D_b^{c,G,G/H}(X;\, E)$ is as above, one obtains the identification:

(2.P.4.2) $\mathbb{H}_G^*(X;K) \simeq (\mathbb{H}_H^*(X;i^*(K)))^{(G/H)}$

Remarks. Now we will consider an example of the above situation. Let G be *not necessarily connected* and let $H = G^o =$ its connected component containing the identity. Now let $F = \{F_n|n\}$ denote any constructible sheaf of E-modules on BG; F_0 is clearly constant on $(BG)_0$. Therefore F is G-equivariant (or equivalently has descent) if and only if it is locally-constant on BG. (See the definition towards the end of (1.2.1).) Now locally constant sheaves on BG are classified by the monodromy representations of $\pi_1(BG,\, *) \cong G/G^0$ (at each stalk). Therefore F has an action of G/G^0 on each stalk. Since G/G^o acts trivially on BG, it follows that F belongs to $D_b^{c,G,G/G^o}(pt;\, E) = D_b^{c,des,G/G^o}(BG;\, E)$. (See (1.2.5).) If G acts on a space X and $\pi : EG\underset{G}{\times}X \rightarrow BG$. is the obvious map, it is now clear that $K = \pi^*(\mathcal{L}) \,\epsilon\, D^{c,G,G/G^o}(X;\, E)$, whenever $\mathcal{L}$ is a G-equivariant constructible sheaf of E-modules on BG. Moreover, by (2.P.4), G/G^o acts on $\mathbb{H}_{G^o}^*(X;\, i^*\pi^*\mathcal{L})$. (In [J-6] we show $D^{c,G}(pt;\, E) \simeq D_b^{c,G,G/G^o}(pt;\, E)$.)

(2.P.5). Let $f : X \rightarrow Y$ denote a G-equivariant map between two G-spaces as in (1.1.0). If $K \,\epsilon\, D_b^{c,G}(Y;\, E)$, the natural map $K \rightarrow R(f^G)_*(f^G)^*(K)$ (see (1.3.1)) induces a map $f^* : \mathbb{H}_G^*(Y;K) \rightarrow \mathbb{H}_G^*(X;(f^G)^*K)$ for each n. (Here $f^G : EG\underset{G}{\times}X \rightarrow EG\underset{G}{\times}Y$ is the obvious map induced by f.) Similarly f induces maps $f^* : K_G^0(Y) \rightarrow K_G^0(X)$; similar induced maps exist $f^* : K_M^{A.S,0}(Y) \rightarrow K_M^{A.S,0}(X)$ and $f^* : K_M^{top,0}(Y) \rightarrow K_M^{top,0}(X)$.

(2.P.6). Assume in addition to the situation in (2.P.5) that G is a complex linear algebraic group, X and Y are G-quasi-projective varieties and f is *proper*. Now the trace-map $tr(f^G, K) : R(f^G)_* R(f^G)^!(K) \simeq R(f^G)_! R(f^G)^!(K) \rightarrow K$ (see (1.4.1) and (1.4.3)(ii)) induces a map

$$f_* : \mathbb{H}_G^*(X; R(f^G)^!(K)) \rightarrow \mathbb{H}_G^*(Y;K).$$

If $K = \mathbb{D}_E$, the dualizing presheaf for the category $D_b^{c,G}(Y;\, E)$, one observes (in view of (2.1)) that the above map becomes

$$f_* : H_*^G(X;\, E) \rightarrow H_*^G(Y;\, E).$$

If, in addition, f is also *smooth*, one may identify $R(f^G)^!(K)$ with $(f^G)^*K[2d]$ and hence the above map may be identified with a map $f_* : \mathbb{H}_G^*(X;(f^G)^*(K)) \rightarrow \mathbb{H}_G^*(Y;K)$.

(2.P.6'). Under the same hypotheses, one obtains a map $f_* : K_0^G(X) \rightarrow K_0^G(Y)$. If $X \rightarrow \tilde{X}$ ($Y \rightarrow \tilde{Y}$) is a G-equivariant closed immersion into a smooth G-scheme $\tilde{X}$ ($\tilde{Y}$, respectively), one may define $f_* : K_{G,X}^0(\tilde{X}) \rightarrow K_{G,Y}^0(\tilde{Y})$ to be the composition: $K_{G,X}^0(\tilde{X}) \xrightarrow{\cong} K_0^G(X) \xrightarrow{f_*} K_0^G(Y) \xrightarrow{\cong} K_{G,Y}^0(\tilde{Y})$.

To define f_* in equivariant Atiyah-Segal and topological K-theory one proceeds as follows. Observe first that, under our hypotheses, the map f factors as a G-equivariant closed immersion $X \xrightarrow{i} Y \times \mathbb{P}^n$ and the projection $\pi : Y \times \mathbb{P}^n \to Y$. (Here G acts linearly on $\mathbb{P}^n$.) Now one may compute (see [Seg -2] or [T-2])

$$K^M_{A.S,0}(Y \times \mathbb{P}^n) = \underset{0 \le q \le n}{\oplus} K^M_{A.S,0}(Y)[-\mathcal{O}_{\mathbb{P}^n}(-q)]$$

Therefore one may define π_* to be projection to the summand indexed by $\mathcal{O}_{\mathbb{P}^n}(0)$. One defines $\pi_* : K^{A.S,0}_{M,Y\times\mathbb{P}^n}(\tilde{Y} \times \mathbb{P}^n) \to K^{A.S,0}_{M,Y}(\tilde{Y})$ to be the composition $\pi_* : K^{A.S,0}_{M,Y\times\mathbb{P}^n}(\tilde{Y} \times \mathbb{P}^n) \xrightarrow{\cong} K^M_{A.S,0}(Y \times \mathbb{P}^n) \xrightarrow{\pi_*} K^M_{A.S,0}(Y) \xrightarrow{\cong} K^{A.S,0}_{M,Y}(\tilde{Y})$. Clearly one obtains an induced map $i_* : K^{A.S,0}_{M,X}(\tilde{X}) \to K^{A.S,0}_{M,Y\times\mathbb{P}^n}(\tilde{Y} \times \mathbb{P}^n)$. Now one defines $f_* = \pi_* \circ i_*$. One defines f_* in equivariant topological K-theory similarly. (One may consult [J-7] for additional details.)

(2.P.7). Assume in addition to the situation in (1.5.2) that the map f is *proper* and the map g is *smooth*. Let $K \epsilon D^{c,G}_b(Y; E)$.

(i) Now one obtains a commutative square:

$$\begin{array}{ccc} \mathbb{H}^*_G(X; Rf^{G!}K) & \xrightarrow{g'^*} & \mathbb{H}^*_G(X'; Rf'^{G!}g^*K) \\ \downarrow{f_*} & & \downarrow{f'_*} \\ \mathbb{H}^*_G(Y; K) & \xrightarrow{g^*} & \mathbb{H}^*_G(Y'; g^{G*}K) \end{array}$$

(ii) Let H denote a closed sub-group of G and let $i : EH \underset{H}{\times} X \to EG \underset{G}{\times} X$ denote the corresponding map. Now one obtains the commutative square (where the horizontal maps are the restriction homomorphisms):

$$\begin{array}{ccc} \mathbb{H}^*_G(X; Rf^{G!}K) & \longrightarrow & \mathbb{H}^*_H(X; Rf^{G!}i^*K) \\ \downarrow{f_*} & & \downarrow{f_*} \\ \mathbb{H}^*_G(Y; K) & \longrightarrow & \mathbb{H}^*_H(Y; i^*K) \end{array}$$

Proof. In view of the hypotheses, $Rg^{G!}$ may be identified with g^{G*} (modulo an even dimensional shift) and hence (1.5.2) provides the commutative square:

$$\begin{array}{ccccc} Rf^G_*Rf^{G!}(K) & \longrightarrow & Rg^G_*g^{G*}Rf^G_*Rf^{G!}K & \xrightarrow{\simeq} & Rg^G_*Rf'^G_*Rf'^{G!}g^{G*}K \\ \downarrow{tr(f^G,K)} & & \downarrow{Rg^G_*g^{G*}tr(f^G,K)} & & \downarrow{Rg^G_*tr(f'^G,g^{G*}K)} \\ K & \longrightarrow & Rg^G_*g^{G*}K & \xrightarrow{id} & Rg^G_*g^{G*}K \end{array}$$

Taking equivariant hypercohomology on Y one obtains the commutative square in (2.P.7)(i). It is clear that the left-vertical map (the right vertical map, the bottom map) in the above square induces the corresponding map in the square in (2.P.7)(i). To see that the top map in the above square induces the top row in the square (2.P.7)(i) observe the natural identification $Rg^G_*Rf'^G_*Rf'^{G!}g^{G*}K \simeq Rf^G_*Rg'^G_*g'^{G*}Rf^{G!}K$. (Now take equivariant hypercohomology on Y.) This proves (i). The same proof applies in (ii) where g^G is replaced by the map i, since by (A.2), $Ri^!$ may be identified with i^*.

(2.P.7)' Under the same hypotheses on f and g, one obtains a commutative square:

$$\begin{array}{ccc} K_0{}^G(X) & \xrightarrow{g'^*} & K_0^G(X') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ K_0^G(Y) & \xrightarrow{g^*} & K_0^G(Y') \end{array}$$

as well as similar commutative squares in equivariant Atiyah-Segal K-homology and in equivariant topological K-homology (with respect to the action of a maximal compact subgroup of G). These may be established by factoring the map f as the composition $X \xrightarrow{i} Y \times \mathbb{P}^n \xrightarrow{\pi} Y$ - the details are skipped.

(2.P.8) Next assume that G is a complex linear algebraic group acting on a complex algebraic variety X. If the action is *trivial* one may observe readily that $EG\underset{G}{\times}X \cong BG \times X$ and that therefore $H^*_G(X;K) \simeq H^*(BG;\mathbb{Q}) \otimes H^*(X;K_0)$ where $K = \{K_n|n\} \in D_b^{c,G}(X;\mathbb{Q})$. It follows that in this case $H^*_G(X;K)$ is a free $H^*(BG;\mathbb{Q}) \cong R(G)\otimes\mathbb{Q}\hat{}$-module which is also finitely generated if K is *constructible* - see (1.2.1). (Here the completion is at the augmentation ideal - see [A-S]).

(2.P.9). *Homotopy Property.* We will assume that G is a complex linear algebraic group acting on a complex quasi-projective variety X. Let $\mathcal{L}$ denote a locally-constant G-equivariant sheaf of $\mathbb{Q}$-vector spaces on X. If $\pi : X \to Y$ is a G-equivariant algebraic vector-bundle on X, then π^* induces an isomorphism:

$$\mathbb{H}^*_G(X;\pi^*(\mathcal{L})) \cong \mathbb{H}^*_G(Y;\mathcal{L}).$$

This follows readily from the Leray-spectral sequence:

$$E_2^{s,t} = \mathbb{H}^s_G(Y;R^t\pi_*(\pi^*\mathcal{L})) \Rightarrow \mathbb{H}^{s+t}_G(X;\mathcal{L})$$

since $R^t\pi_*(\pi^*\mathcal{L}) \cong 0$ unless $t=0$ and $\cong \mathcal{L}$ if $t=0$.

(2.P.10) **Proposition.**(*Projection formulae*).

Let X, Y denote complex quasi-projective varieties provided with the action of a complex linear algebraic group G and let $f : X \to Y$ denote a G-equivariant map. Let P, L, $L' \in D_b^{c,G}(Y;E)$ be provided with a pairing $P\otimes L \to L'$.

(i) Now we obtain the commutative square provided f is *proper*:

$$\begin{array}{ccccc} \mathbb{H}^*_G(X;(Rf^G)^!P)\otimes\mathbb{H}^*_G(Y;L) & \xrightarrow{id\otimes f^*} & \mathbb{H}^*_G(X;(Rf^G)^!P))\otimes\mathbb{H}^*_G(X;(f^G)^*L) & \longrightarrow & \mathbb{H}^*_G(X;(Rf^G)^!L') \\ {\scriptstyle f_*\otimes id}\downarrow & & & & \downarrow{\scriptstyle f_*} \\ \mathbb{H}^*_G(Y;P)\otimes\mathbb{H}^*_G(Y;L) & & \longrightarrow & & \mathbb{H}^*_G(Y;L') \end{array}$$

(i)' Suppose in addition $Y_1 \xrightarrow{i_1} Y$, $Y_2 \xrightarrow{i_2} Y$ and $Y_3 = Y_1 \cap Y_2 \xrightarrow{i_3} Y$ are three closed G-stable subspaces of Y and $P_0 \in D_b^{c,G}(Y_1;E)$ ($L_0 \in D_b^{c,G}(Y_2;E)$, $L'_0 \in D_b^{c,G}(Y_1\cap Y_2;E)$) so that $P = i_{1*}P_0$ ($L = i_{2*}L_0$, $L' = i_{3*}L'_0$, respectively). Let $X_1 \subseteq f^{-1}(Y_1)$ be a closed G-stable subspace so that the induced map $X_1 \to Y_1$ is *proper*, but not necessarily f. Now one obtains a commutative diagram (where $X_2 = f^{-1}(Y_2)$

and the induced map $X_i \to Y_i$, $i = 1,2$, is denoted f_i while the induced map $X_1 \cap X_2 \to Y_1 \cap Y_2$ is denoted $f_{1,2}$):

$$\begin{array}{ccc} \mathbb{H}^*_G(X_1;(Rf^G)^!P_0) \otimes \mathbb{H}^*_G(Y_2;L_0) \xrightarrow{id\otimes f_2^*} \mathbb{H}^*_G(X_1;(Rf_1^G)^!P_0)) \otimes \mathbb{H}^*_G(X_2;(f_2^G)^*L_0) & \longrightarrow & \mathbb{H}^*_G(X_1 \cap X_2;(Rf_{1,2}^G)^!L'_0) \\ \downarrow{\scriptstyle f_{1*}\otimes id} & & \downarrow{\scriptstyle f_{1,2*}} \\ \mathbb{H}^*_G(Y_1;P_0) \otimes \mathbb{H}^*_G(Y_2;L_0) & \longrightarrow & \mathbb{H}^*_G(Y_1 \cap Y_2;L'_0) \end{array}$$

(ii) Under the same hypotheses as in (i) one also obtains the following commutative square:

$$\begin{array}{ccc} \mathbb{H}^*_G(Y;P) \otimes \mathbb{H}^*_G(X;(Rf^G)^!L) \xrightarrow{f^*\otimes id} \mathbb{H}^*_G(X;(f^G)^*P) \otimes \mathbb{H}^*_G(X;(Rf^G)^!L) & \longrightarrow & \mathbb{H}^*_G(X;(Rf^G)^!L') \\ \downarrow{\scriptstyle id\otimes f_*} & & \downarrow{\scriptstyle f_*} \\ \mathbb{H}^*_G(Y;P) \otimes \mathbb{H}^*_G(Y;L) & \longrightarrow & \mathbb{H}^*_G(Y;L') \end{array}$$

(ii)' Suppose in addition to the hypotheses as in (i), $Y_1 \xrightarrow{i_1} Y$, $Y_2 \xrightarrow{i_2} Y$ and $Y_3 = Y_1 \cap Y_2 \xrightarrow{i_3} Y$ are three closed G-stable subspaces of Y and $P_0 \,\epsilon\, D_b^{c,G}(Y_1;\, E)$ ($L_0 \,\epsilon\, D_b^{c,G}(Y_2;\, E)$, $L'_0 \,\epsilon\, D_b^{c,G}(Y_1 \cap Y_2;\, E)$) so that $P = i_{1*}P_0$ ($L = i_{2*}L_0$, $L' = i_{3*}L'_0$, respectively). Let $X_2 \subseteq f^{-1}(Y_2)$ be a closed G-stable subspace so that the induced map $f_2 : X_2 \to Y_2$ is *proper*, but not necessarily f. Now one obtains the commutative diagram (where $X_1 = f^{-1}(Y_1)$ and the induced map $X_i \to Y_i$, $i = 1,2$, is denoted f_i while the induced map $X_1 \cap X_2 \to Y_1 \cap Y_2$ is denoted $f_{1,2}$):

$$\begin{array}{ccc} \mathbb{H}^*_G(Y_1;P_0) \otimes \mathbb{H}^*_G(X_2;(Rf^G)^!(L_0)) \xrightarrow{f_1^*\otimes id} \mathbb{H}^*_G(X_1;(f_2^G)^*P_0)) \otimes \mathbb{H}^*_G(X_2;(Rf^G)^!(L_0)) & \longrightarrow & \mathbb{H}^*_G(X_1 \cap X_2;(Rf_{1,2}^G)^!L'_0) \\ \downarrow{\scriptstyle id\otimes f_{2*}} & & \downarrow{\scriptstyle f_{1,2*}} \\ \mathbb{H}^*_G(Y_1;P_0) \otimes \mathbb{H}^*_G(Y_2;L_0) & \longrightarrow & \mathbb{H}^*_G(Y_1 \cap Y_2;L'_0) \end{array}$$

Proof. We will first show in detail how to obtain the first commutative diagram. We begin with the cartesian square:

(2.P.10.1)

$$\begin{array}{ccc} EG\underset{G}{\times}X \xrightarrow{\Delta} EG\underset{G}{\times}(X \times X) & \xrightarrow{id\times f^G} & EG\underset{G}{\times}(X \times Y) \\ \downarrow{\scriptstyle f^G} & & \downarrow{\scriptstyle f^G\times id} \\ EG\underset{G}{\times}Y & \xrightarrow{\Delta} & EG\underset{G}{\times}(Y \times Y) \end{array}$$

The top row in the above diagram will be denoted $\Gamma(f^G)$.

We will make use of the following notation. Let U, V denote two spaces as before provided with a G-action. Let $\pi_1 : EG\underset{G}{\times}(U \times V) \to EG\underset{G}{\times}U$ ($\pi_2 : EG\underset{G}{\times}(U \times V) \to EG\underset{G}{\times}V$ denote the projection to the first (second, respectively) factor. If F (F') denotes an object in $D_b^{c,G}(EG\underset{G}{\times}U;\, \mathbb{Q})$ ($D_b^{c,G}(EG\underset{G}{\times}V;\, \mathbb{Q})$, respectively) then $F \boxtimes F'$ will denote the object in $D_b^{c,G}(EG\underset{G}{\times}(U \times V); \mathbb{Q})$ given by $\pi_1^*(F) \otimes \pi_2^*(F')$. If $U = V$, we will let $F \otimes F'$ denote $\Delta^*(F \boxtimes F')$, where $\Delta : EG\underset{G}{\times}U \rightarrow EG\underset{G}{\times}(U \times U)$ is the diagonal. Now observe the existence of pairings:

(2.P.10.2) $\Delta^*(P \boxtimes L) \rightarrow L'$ or equivalently $P \boxtimes L \rightarrow \Delta_*(L')$ and

(2.P.10.3) $\Delta^*((f^G)^*P \boxtimes (f^G)^*L) \rightarrow (f^G)^*L'$ or equivalently $(f^G)^*P \boxtimes (f^G)^*L \rightarrow \Delta_*((f^G)^*L')$

Next we show the existence of a natural transformation:

(2.P.10.4) $\Gamma(f^G)^* \circ R(f^G \times id)^! \rightarrow (Rf^G)^! \circ \Delta^*$

Clearly such a map is adjoint to a natural transformation $R(f^G)_* \circ \Gamma(f^G)^* \circ R(f^G \times id)^! \rightarrow \Delta^*$. Since f is proper, we may use proper-base change (see (1.5.1)) to identify the left-hand-side with $\Delta^* \circ R(f^G \times id)_* R(f^G \times id)^!$. Therefore its suffices to show the existence of a natural transformation form the latter to Δ^*. The latter clearly exists, since we may apply Δ^* to the natural transformation (see (1.4.3)(iii)) $R(f^G \times id)_* R(f^G \times id)^! \rightarrow id$. Therefore we now obtain the map:

(2.P.10.5) $R(f^G \times id)^!(P \boxtimes L) \rightarrow R\Gamma(f^G)_*\Gamma(f^G)^*R(f^G \times id)^!(P \boxtimes L)$

$\rightarrow R\Gamma(f^G)_*(Rf^G)^!(\Delta^*(P \boxtimes L)) \rightarrow R\Gamma(f^G)_*((Rf^G)^!L')$

where the last map is obtained from the pairing $\Delta^*(P \boxtimes L) = P \otimes L \rightarrow L'$. *This map will henceforth be denoted* Φ. Clearly the pairing in (2.P.10.2) induces the map forming the bottom rows of the diagram (i) and (ii). The map Φ in (2.P.10.5) induces a map

$$\mathbb{H}_G^*(X; (Rf^G)^!P) \otimes \mathbb{H}_G^*(Y; L)) \xrightarrow{id \otimes f^*} \mathbb{H}_G^*(X; (Rf^G)^!P) \otimes \mathbb{H}_G^*(X; (f^G)^*L) \rightarrow \mathbb{H}_G^*(X; (Rf^G)^!L'))$$

forming the top row of the first diagram. Observe that the map Φ is adjoint to the map

(2.P.10.5)' $\Gamma(f^G)^*R(f^G \times id)^!(P \boxtimes L) \xrightarrow{\Phi'} (Rf^G)^!L'$

which is the composition of the map in (2.P.10.4) (applied to $P \boxtimes L$) with the map $\Delta^*(P \boxtimes L) \rightarrow L'$.

Now we obtain the diagram of pairings that commutes upto natural quasi-isomorphism :

(2.P.10.6)

$$\begin{array}{ccc}
 & & \Delta^*R(f^G \times id)_*R(f^G \times id)^!(P \boxtimes L) \\
 & \swarrow & \downarrow \simeq \\
Rf^G_*((Rf^G)^!L') & \xleftarrow{Rf^G_*(\Phi')} & Rf^G_*\Gamma(f^G)^*R(f^G \times id)^!(P \boxtimes L) \\
\downarrow id & & \downarrow \\
Rf^G_*R(f^G)^!(L') & \longleftarrow & Rf^G_*R(f^G)^!(\Delta^*(P \boxtimes L)) \\
\downarrow tr(f^G) & & \downarrow tr(f^G) \\
L' & \longleftarrow & \Delta^*(P \boxtimes L)
\end{array}$$

The commutativity of the bottom square in the above diagram follows from the observation that the trace map $tr(f^G)$ is natural. To establish the commutativity of the top square, it suffices to show the square

$$\begin{array}{ccc} R\Gamma(f^G)_* Rf^{G^!}L' & \xleftarrow{\Phi} & R(f^G \times id)^!(P \boxtimes L) \\ {\scriptstyle id}\downarrow & & \downarrow \\ R\Gamma(f^G)_* Rf^{G^!}L' & \longleftarrow & R\Gamma(f^G)_* Rf^{G^!}\Delta^*(P \boxtimes L) \end{array}$$

commutes; this is clear from the definition of the map Φ. The fact that the square in (2.P.10.1) is cartesian implies that one has the natural quasi-isomorphisms:

$$\Delta^* \circ R(f^G \times id)_* \xrightarrow{\simeq} Rf^G_* \circ \Gamma(f^G)^*$$

by proper-base change. This proves the top-most vertical map in the above diagram is a quasi-isomorphism. The commutativity of the top triangle in the above diagram is clear since we may define the third side to be the composition of the other two maps in the triangle. Clearly the diagram (2.P.10.6) provides the commutative diagram:

$$\begin{array}{ccc} R\Delta_* Rf^G_* R(f^G)^!(L') & \longleftarrow & R(f^G \times id)_* R(f^G \times id)^!(P \boxtimes L) \\ \downarrow & & \downarrow \\ R\Delta_*(L') & \longleftarrow & P \boxtimes L \end{array}$$

where the top row factors as in the diagram (2.P.10.6). Clearly this diagram provides the diagram in (2.P.10)(i). (Recall that $R\Delta_* Rf^G_* R(f^G)^! = R(f^G \times id)_* R\Gamma(f^G)_* R(f^G)^!$. This along with (2.P.10.5) shows the map in the top row above induces the map in the top row of (2.P.10)(i).). This completes the proof of (i).

To obtain the commutativity of the square (i)' we proceed as follows. Let $k_i : Y_3 = Y_1 \cap Y_2 \to Y_i$, $i = 1, 2$, denote the obvious closed immersions. Now observe that the square in (i)' factors as the composition of the squares:

$$\begin{array}{ccc} \mathbb{H}^*_G(X_1; Rf_1^{G^!}P_0) \otimes \mathbb{H}^*_G(Y_2; L_0) & \longrightarrow & \mathbb{H}^*_G(X_1; Rf^{G^!}P_0) \otimes \mathbb{H}^*_G(Y_3; k_2^*(L_0)) \\ {\scriptstyle f_{1*}\otimes id}\downarrow & & \downarrow{\scriptstyle f_{1*}\otimes id} \\ \mathbb{H}^*_G(Y_1; P_0) \otimes \mathbb{H}^*_G(Y_2; L_0) & \longrightarrow & \mathbb{H}^*_G(Y_1; P_0) \otimes \mathbb{H}^*_G(Y_3; k_2^*(L_0)) \end{array}$$

and

$$\begin{array}{ccccc} \mathbb{H}^*_G(X_1; (Rf_1^G)^!P_0) \otimes \mathbb{H}^*_G(Y_3; k_2^*(L_0)) & \xrightarrow{id\otimes f_2^*} & \mathbb{H}^*_G(X_1; (Rf_1^G)^!P_0)) \otimes \mathbb{H}^*_G(X_1 \cap X_2; (f^G_{1,2})^*(k_2^*(L_0))) & \longrightarrow & \mathbb{H}^*_G(X_1 \cap X_2; (Rf^G_{1,2})^! L'_0) \\ {\scriptstyle f_{1*}\otimes id}\downarrow & & & & \downarrow{\scriptstyle f_{1,2*}} \\ \mathbb{H}^*_G(Y_1; P_0) \otimes \mathbb{H}^*_G(Y_3; k_2^*(L_0)) & & \longrightarrow & & \mathbb{H}^*_G(Y_3; L'_0) \end{array}$$

Therefore it suffices to obtain the commutativity of the bottom diagram; this follows from (i) by replacing f (Y, L) by the induced map $X_1 \to Y_1$ $(Y_1, k_{1*}k_2^*(L_0)$ respectively). Observe that the pairing $P \otimes L \to L'$ is now replaced by the pairing $P_0 \otimes k_{1*}k_2^*(L_0) \to k_{1*}(L'_0)$.

The diagrams in (ii) and (ii)' are obtained by an argument entirely similar to that for (i) by considering the pull-back square

$$\begin{array}{ccccc}
EG\underset{G}{\times}X & \xrightarrow{\Delta} & EG\underset{G}{\times}(X\times X) & \xrightarrow{f^G\times id} & EG\underset{G}{\times}(Y\times X) \\
\downarrow{\scriptstyle f^G} & & & & \downarrow{\scriptstyle id\times f^G} \\
EG\underset{G}{\times}Y & & \xrightarrow{\qquad\Delta\qquad} & & EG\underset{G}{\times}(Y\times Y)
\end{array}$$

□

(2.P.10)' *Projection formula in equivariant K-homology.* Assume that in addition to the hypotheses of (2.P.10)(i) that $X \to \tilde{X}$, $Y \to \tilde{Y}$ are G-equivariant closed immersions into *smooth* G-quasi-projective varieties and the map f extends to a G-equivariant map $\tilde{f} : \tilde{X} \to \tilde{Y}$. Let $\tilde{g} : \tilde{Y} \to \tilde{Z}$ denote a G-equivariant map between smooth G-quasi-projective varieties, $Z \subseteq \tilde{Z}$ a G stable closed sub-variety. Let $g : \tilde{g}^{-1}(Z) \to Z$ be the induced map. Now one obtains the commutative square:

$$\begin{array}{ccccc}
K^0_{G,X}(\tilde{X}) \otimes K^0_{G,Z}(\tilde{Z}) & \longrightarrow & K^0_{G,X}(\tilde{X}) \otimes K^0_{G,\tilde{f}^{-1}(\tilde{g}^{-1}(Z))}(\tilde{X}) & \longrightarrow & K^0_{G,X\cap\tilde{f}^{-1}(\tilde{g}^{-1}(Z))}(\tilde{X}) \\
\downarrow{\scriptstyle f_*\otimes id} & & & & \downarrow{\scriptstyle \hat{f}_*} \\
K^0_{G,Y}(\tilde{Y}) \otimes K^0_{G,Z}(\tilde{Z}) & & \xrightarrow{\qquad id\otimes g^* \qquad} & & K^0_{G,Y\cap\tilde{g}^{-1}(Z)}(\tilde{Y})
\end{array}$$

and similar commutative squares in equivariant Atiyah-Segal K-theory as well as in equivariant topological K-theory (with respect to a maximal compact subgroup of G). The map $\hat{f}_*$ is the obvious map induced by f. Here the first map in the top row is $id \otimes \tilde{f}^* \circ \tilde{g}^*$. (Since all the varieties are G-quasi-projective, the map $\tilde{f}$ factors as the composition $\tilde{X} \to \tilde{Y} \times \mathbb{P}^n$ (where $\mathbb{P}^n$ is a large projective space on which G acts linearly) and the projection $\tilde{Y} \times \mathbb{P}^n \to \tilde{Y}$. Therefore it is clear the map $\tilde{f}$ has finite tor dimension so that the map $\tilde{f}^*$ is defined. A similar conclusion holds for the map $\tilde{g}$.) The commutative square in equivariant algebraic K-theory is clear by the familiar projection formula. To obtain the corresponding results in the other K-theories, one may use the factorization of f into a G-equivariant closed immersion followed by a projection $\mathbb{P}^n \times Y$, where G acts linearly on $\mathbb{P}^n$ and by the diagonal action on the product $\mathbb{P}^n \times Y$. (Or one may view these as generalized (equivariant) homology theories and invoke the projection formulae in such settings.)

Alternate form: Finally observe that one also obtains a similar commutative diagram when the pairings: $K^0_{G,Z}(\tilde{Z}) \otimes K^0_{G,X}(\tilde{X}) \to K^0_{G,X\cap\tilde{f}^{-1}(\tilde{g}^{-1}(Z))}(\tilde{X})$ and $K^0_{G,Z}(\tilde{Z}) \otimes K^0_{G,Y}(\tilde{Y}) \to K^0_{G,Y\cap\tilde{g}^{-1}(Z)}(\tilde{Y})$ are used. The corresponding diagram provides the equality: $(g^* \otimes id) \circ (id \otimes f_*) = \hat{f}_* \circ ((\tilde{f}^* \circ \tilde{g}^*) \otimes id)$. □

(2.P.11) **Corollary**. Assume the hypotheses of (2.P.10)(i). Let $L \in D^{c,G}_b(Y; E)$.

(i). Now the map

$f_* : \mathbb{H}^*_G(X; (Rf^G)^! L) \to \mathbb{H}^*_G(Y; L)$

is a map of modules over $H^*_G(Y; E)$ and hence over $H^*(B(G); E)$.

(ii). Assume in addition to the hypotheses of (2.P.10)(i) that X is *smooth*. Now the map $f_* : \mathbb{H}^*_G(X; (Rf^G)^! L) \to \mathbb{H}^*_G(Y; L)$ has the property that $f_*(f^*(\alpha)) = f_*(1).\alpha$, $\alpha \in \mathbb{H}^j_G(Y; L)$, $f^*(\alpha) =$ the image of $f^*(\alpha) \in \mathbb{H}^*_G(X;\ (f^G)^*(L))$ in $\mathbb{H}^*_G(X;\ (Rf^G)^!(L))$ and $1 \in H^0_G(X;\ \underline{E})$ is a generator of $H^0_G(X;\ \underline{E})$.

(iii). Assume in addition to the hypotheses of (2.P.10)(i) that $Y = Spec\ \ \mathbb{C}$ and $X = \mathcal{B}$ where $\mathcal{B}$ is a flag-manifold (i.e.. the variety of Borel subgroups associated to a complex reductive group $\mathbf{G}$ or the variety of parabolic subgroups of a complex reductive group $\mathbf{G}$, all conjugate to a given parabolic subgroup) and G is a closed subgroup of $\mathbf{G}$. Assume further that $L \ \epsilon \ D_b^{c,G}(Y;\ \mathbb{Q})$. In this case let $\bar{f}_* : \mathbb{H}^*_G(X; R(f^G)^! L) \simeq \mathbb{H}^*_G(X; (f^G)^* L) \to \mathbb{H}^*_G(Y; L)$ be the map defined by $\bar{f}_*(\alpha) = f_*(Td_X \otimes \alpha)$. Now $\bar{f}_* \circ f^* =$ the identity.

Proof. (i) follows from (2.P.10)(ii) by taking $P = \underline{E}$ and $L' = L$. To obtain (ii) take $P = \mathbb{D}_E$ and $L' = L$ in (2.P.10)(i). Since X is smooth, $(Rf^G)^!(\mathbb{D}_E) \simeq \mathbb{D}_E \simeq \underline{E}[2d]$, where d is the complex dimension of X and $\underline{E}$ denotes the obvious constant sheaf on X. Therefore $\mathbb{H}^*_G(X;\ (Rf^G)^!(\mathbb{D}_E)) \cong \mathbb{H}^*_G(X;\ \underline{E})$. Finally (iii) follows from (ii) (for the map $\bar{f}_*$) by observing that in this case $\bar{f}_*(1) = f_*(Td_X \otimes 1) = 1$. Observe that the class 1 is the equivariant Chern-character of the trivial line bundle and that if $f_* : K_0(Mod^G_{coh}(X)) \to K_0(Mod^G_{coh}(Y))$ is the obvious map on the Grothendieck groups of equivariant coherent sheaves, f_*(the trivial line bundle) $= 1$ by Borel-Weil-Bott as observed for example in [T-2] p. 594. Now the equivariant Riemann-Roch theorem in (2.12) shows $\bar{f}_*(1) = 1$. This completes the proof of (iii). □

We end this section by quoting the equivariant Riemann-Roch theorem from [J-7] section 4.

Let G denote a complex linear algebraic group acting on a G-quasi-projective variety Z and let M denote a maximal compact subgroup of G. Let $Z \to \tilde{Z}$ denote a G-equivariant closed immersion into a smooth G-quasi-projective variety. It is shown in [J-7] section 4, that there exist natural transformations:

$$K^G(Z) \xrightarrow{\rho} K^M_{A.S}(Z) \text{ and } \pi_0(K^M_{A.S}(Z)) \xrightarrow{B} K^M_{top,0}(Z) \cong K^G_{top,0}(Z).$$

where $K^G(Z)$ is the spectrum of the symmetric monoidal category of G-equivariant coherent sheaves on Z and $K^M_{A.S}(Z)$ is the Atiyah-Segal K-homology spectrum defined in [J-7] section 3. $K^M_{top,*}(Z)$ ($K^G_{top,*}(Z)$) is the M-equivariant (G-equivariant) topological K-theory of Z. It is also shown in [J-6] (2.7) and section 3 that there exists a *G-equivariant Chern-character* $ch^G : K^G_{top,0}(Z) \to H^G_*(Z;\ \mathbb{Q})$ and a Todd-homomorphism $\tau^G : K^G_{top,0}(Z) \to H^G_*(Z;\ \mathbb{Q})$ defined by $\tau^G(\alpha) = (Td_{\tilde{Z}}|Z) \otimes ch^G(\alpha)$, where $\alpha \ \epsilon \ K^G_{top,0}(Z)$ and $Td_{\tilde{Z}}|Z$ is the restriction of the G-equivariant Todd-class of $\tilde{Z}$ to Z. (Recall that $H^G_*(Z;\ \mathbb{Q}) \cong H^M_*(Z;\ \mathbb{Q})$.)

(2.12) **Equivariant Riemann-Roch** (See [J-7] (4.2) or [T-2].) Assume that $f : X \to Y$ is a G-equivariant *proper* map between two G-quasi-projective varieties. In this case one obtains the commutative diagram

$$\begin{array}{ccccccc}
\pi_0(K^G(X)) & \xrightarrow{\pi_0(\rho)} & \pi_0(K^M_{A.S}(X)) & \xrightarrow{B} & K^G_{top,0}(X) & \xrightarrow{\tau^G} & H^G_*(X;\mathbb{Q}) \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
\pi_0(K^G(Y)) & \xrightarrow{\pi_0(\rho)} & \pi_0(K^M_{A.S}(Y)) & \xrightarrow{B} & K^G_{top,0}(Y) & \xrightarrow{\tau^G} & H^G_*(Y;\mathbb{Q})
\end{array}$$

Here the last vertical map is the one induced by the trace map $Rf^G_* Rf^{G^!}(\mathbb{D}_\mathbb{Q}) \to \mathbb{D}_\mathbb{Q}$ (see (2.P.6)) where $\mathbb{D}_\mathbb{Q}$ is the dualizing complex for the category $D_b^{c,G}(Y;\ \mathbb{Q})$.

3. Convolution algebras via equivariant K-homology and homology: associativity of convolution.

(3.1) Let G denote a complex reductive group and let $f : \overset{o}{U} \to U$ denote a *proper* G-equivariant map of G-quasi-projective varieties. Assume further that $\overset{o}{U}$ is *smooth*.

The following are typical examples of this set-up. The first two examples lead to Hecke algebras; the third seems to be an un-explored new situation and the fourth is closely related to affine quantum universal enveloping algebras of type A_n. (One may show readily that all varieties are in fact G-quasi-projective.)

i). $\overset{o}{U} = \mathcal{B} =$ the variety of all Borel subgroups (or the variety of all parabolic subgroups conjugate to a fixed parabolic subgroup P) of a complex reductive group $\mathbf{G}$, $U =$ Spec $\mathbb{C}$, $G =$ a closed subgroup of $\mathbf{G}$ and $f =$ the obvious map.

ii). $\mathbf{G} =$ a complex reductive group, $U =$ a $\mathbf{G}$-stable open sub-variety of $\mathcal{U}$ (=the unipotent variety of $\mathbf{G}$), $\mathcal{B} =$ the variety of all Borel subgroups of $\mathbf{G}$, $\overset{o}{U} = \Lambda_U = \{(u, B) | u \,\epsilon\, U \cap B, B \,\epsilon\, \mathcal{B}\}$, $G =$ (a closed subgroup of $\mathbf{G}$) $\times \mathbb{C}^*$ and $f = \mu : \Lambda_U \to U$ the map sending $(u, B) \to u$. The action of G on U and on $\overset{o}{U}$ is described in more detail in (5.1.2) and (5.1.3).

iii) $\mathbf{G} =$ a complex reductive group, P a fixed parabolic subgroup of $\mathbf{G}$, $\mathcal{U} =$ the unipotent variety of $\mathbf{G}$, $\tilde{\mathcal{U}}^P =$ the partial desingularization of $\mathcal{U}$ $=\{(x, P') | x \,\epsilon\, \mathcal{U} \cap P', P'=$ a parabolic subgroup conjugate to $P\}$. Now the variety Λ (= the Springer desingularization of $\mathcal{U}$) maps naturally onto $\tilde{\mathcal{U}}^P$. Let this map be denoted η. Let $U =$ a $\mathbf{G}$-stable open sub-variety of $\tilde{\mathcal{U}}^P$, $\overset{o}{U} = \eta^{-1}(U)$, $f =$ the obvious map induced by η and G=(a closed subgroup of $\mathbf{G}$) $\times \mathbb{C}^*$ with the actions as in (5.1.2) and (5.1.3).

iv). (See [G-V].) Let d denote an integer ≥ 1 and let $\mathfrak{F} =$ the set of all n-step flags in $\mathbb{C}^d$ of the form $F = (0 = F_0 \subseteq F_1 \subseteq ... \subseteq F_n = \mathbb{C}^d)$. Let $M = \{(F, x) \,\epsilon\, \mathfrak{F} \times gl_d(\mathbb{C}^d) | x(F_i) \subseteq F_{i-1},\ i = 1, 2, ..., n\}$ and N= the variety of all $\mathbb{C}$-linear maps $x : \mathbb{C}^d \to \mathbb{C}^d$ so that $x^n = 0$. The group $GL(\mathbb{C}^d) \times \mathbb{C}^*$ acts on N and M as in (5.1.2) and (5.1.3). We let f denote the obvious projection to the second factor, U = any $GL(\mathbb{C}^d) \times \mathbb{C}^*$-stable open sub-variety of N and $\overset{o}{U} = f^{-1}(U)$. (Observe that M is smooth and f is proper.) We may let $G =$ (any closed subgroup of $GL(\mathbb{C}^d)$) $\times \mathbb{C}^*$. (See [G-V] or section 5 for more details.)

One may generalize the above set-up in (3.1) as follows. Let $s \,\epsilon\, G$ denote a fixed semi-simple element so that the fixed-point schemes U^s and $(\overset{o}{U})^s$ are *non-empty*. Now one may replace $f : \overset{o}{U} \to U$ by the map $f^s : (\overset{o}{U})^s \to U^s$ and G by $Z_G(s)$. We will refer to these cases as $(3.1)(i)_s$, $(3.1)(ii)_s$, $(3.1)(iii)_s$ and $(3.1)(iv)_s$.

(3.2.1) Assume the situation in (3.1). Let $\overset{oo}{U} = \overset{o}{U} \underset{U}{\times} \overset{o}{U}$. This is a G-stable subscheme of the *smooth* G-variety $\overset{o}{U} \times \overset{o}{U}$ where G acts diagonally on the variety $\overset{o}{U} \times \overset{o}{U}$. Let $i : \overset{oo}{U} = \overset{o}{U} \underset{U}{\times} \overset{o}{U} \to \overset{o}{U} \times \overset{o}{U}$ denote the obvious closed immersion. Let $p_{i,j} : \overset{o}{U} \times \overset{o}{U} \times \overset{o}{U} \to \overset{o}{U} \times \overset{o}{U}$ denote the projection to the (i,j)-th factor for $(i,j) = (1,2)$, $(i,j) = (2,3)$ and $(i,j) = (1,3)$. Let $\tilde{p}_{i,j}$= the induced projection $p_{i,j}^{-1}(\overset{oo}{U}) \to \overset{oo}{U}$; we let $\alpha_{i,j} : \overset{ooo}{U} \to p_{i,j}^{-1}(\overset{oo}{U})$. We let $\hat{p}_{i,j} = \tilde{p}_{i,j} \circ \alpha_{i,j}$.

(3.2.2) Now observe that $p_{i,j}^{-1}(\overset{oo}{U}) \subseteq \overset{o}{U} \times \overset{o}{U} \times \overset{o}{U}$ is a G-stable closed subscheme when G acts diagonally on the varieties considered. Now $\overset{ooo}{U} = p_{1,2}^{-1}(\overset{oo}{U}) \cap p_{2,3}^{-1}(\overset{oo}{U}) = \overset{o}{U} \underset{U}{\times} \overset{o}{U} \underset{U}{\times} \overset{o}{U}$. The properness of f, implies, by base-change that the projection $\hat{p}_{1,3} : \overset{ooo}{U} \rightarrow \overset{oo}{U}$ is *also proper.*

We recall the following terminology: If Z is a G-variety with a closed G-equivariant immersion $Z \rightarrow \tilde{Z}$ into a smooth G-variety, we let $K_0^G(Z) = \pi_0(K^G(Z))$, $K_{G,Z}^0(\tilde{Z}) = \pi_0(K_{G,Z}(\tilde{Z}))$. Now one first defines a convolution-product

(3.2.3)' $* : K^0_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \otimes K^0_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \rightarrow K^0_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U})$

For this one begins with the pairing:

$$K^0_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \otimes K^0_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \xrightarrow{p_{1,2}^* \otimes p_{2,3}^*} K^0_{G,p_{1,2}^{-1}(\overset{oo}{U})}(\overset{o}{U} \times \overset{o}{U} \times \overset{o}{U}) \otimes K^0_{G,p_{2,3}^{-1}(\overset{oo}{U})}(\overset{o}{U} \times \overset{o}{U} \times \overset{o}{U})$$

$$\rightarrow K^0_{G,\overset{ooo}{U}}(\overset{o}{U} \times \overset{o}{U} \times \overset{o}{U})$$

The map $\hat{p}_{1,3*} : K_0^G(\overset{ooo}{U}) \rightarrow K_0^G(\overset{oo}{U})$ induces by Poincaré-Lefschetz-duality a map $\hat{p}_{1,3*} : K^0_{G,\overset{ooo}{U}}(\overset{o}{U} \times \overset{o}{U} \times \overset{o}{U}) \rightarrow K^0_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U})$. The composition of the above maps defines the convolution in (3.2.3)'.

Making use of Poincaré-Lefschetz-duality, the convolution in (3.2.3)' induces the convolution:

(3.2.3) $* : K_0^G(\overset{oo}{U}) \otimes K_0^G(\overset{oo}{U}) \rightarrow K_0^G(\overset{oo}{U})$.

(Compare [C-G] (4.2.8) and section 2.5.) *This algebra will be denoted* $\mathbb{H}_{\mathbb{Z}}$ *henceforth.*

Let M denote a maximal compact subgroup of G. If Z is a G-variety with a closed G-equivariant immersion $Z \rightarrow \tilde{Z}$ into a smooth G-variety, we let $K^M_{A.S,0}(Z) = \pi_0(K^M_{A.S}(Z))$ and $K^{A.S,0}_{M,Z}(\tilde{Z}) = \pi_0(K^{A.S}_{M,Z}(\tilde{Z}))$. $K^{top,0}_{M,Z}(\tilde{Z})$ will denote the M-equivariant topological K-theory of $\tilde{Z}$ with supports in Z. Now one may define convolution-products:

(3.2.4)' $* : K^{A.S,0}_{M,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \otimes K^{A.S,0}_{M,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \rightarrow K^{A.S,0}_{M,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U})$ and

(3.2.4) $* : K^M_{A.S,0}(\overset{oo}{U}) \otimes K^M_{A.S,0}(\overset{oo}{U}) \rightarrow K^M_{A.S,0}(\overset{oo}{U})$.

The convolution-algebra in (3.2.4) will be denoted $\mathbb{H}_{\mathbb{Z},A.S}$. Similarly one defines convolution-products

(3.2.5)' $* : K^{top,0}_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \otimes K^{top,0}_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \rightarrow K^{top,0}_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U})$ and

(3.2.5) $* : K^G_{top,0}(\overset{oo}{U}) \otimes K^G_{top,0}(\overset{oo}{U}) \rightarrow K^G_{top,0}(\overset{oo}{U})$.

The convolution-algebra in (3.2.5) will be denoted $\mathbb{H}_{\mathbb{Z},top}$.

Next we consider a similar operation in equivariant homology. For this we will define a map $\hat{p}_{1,3*}$: $H^*_{G,\overset{ooo}{U}}(\overset{o}{U} \times \overset{o}{U} \times \overset{o}{U}; \mathbb{Q}) \rightarrow H^*_{G,\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}; \mathbb{Q})$ so as to make the square

$$
(3.2.6)\quad \begin{array}{ccc} H^*_{G,\overset{ooo}{U}}(\overset{o}{U}\times\overset{o}{U}\times\overset{o}{U};\mathbb{Q}) & \xrightarrow{\hat{p}_{1,3*}} & H^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\mathbb{Q}) \\ {\scriptstyle P-L}\downarrow & & \downarrow{\scriptstyle P-L} \\ H^G_*(\overset{ooo}{U};\mathbb{Q}) & \xrightarrow{\hat{p}_{1,3*}} & H^G_*(\overset{oo}{U};\mathbb{Q}) \end{array}
$$

Now one defines a convolution

(3.2.7)' $* : H^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\ \mathbb{Q})\otimes H^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\ \mathbb{Q})\to H^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\ \mathbb{Q})$

as in (3.2.3)'. Making use of the Poincaré-Lefschetz-duality, this also defines a convolution:

(3.2.7) $* : H^G_*(\overset{oo}{U};\ \mathbb{Q})\otimes H^G_*(\overset{oo}{U};\ \mathbb{Q})\to H^G_*(\overset{oo}{U};\ \mathbb{Q})$

(One may also observe the isomorphism: $H^G_*(\overset{oo}{U};\ \mathbb{Q})\cong H^M_*(\overset{oo}{U};\ \mathbb{Q})$.) This convolution algebra will be denoted $\mathbb{H}_{\mathbb{Q},gr}$. If H is a closed subgroup of G, one may define a similar convolution product when $H^G_*(\overset{oo}{U};\ \mathbb{Q})$ is replaced by $H^G_*((G\underset{H}{\times}\overset{oo}{U});\ \mathbb{Q})$.

Next we will define natural transformations

(3.2.8)' $BR' : K^{top,0}_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U})\to H^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\ \mathbb{Q})$ and

(3.2.8) $BR : K^G_{top}(\overset{oo}{U})\to H^G_*(\overset{oo}{U};\ \mathbb{Q})$

following ([C-G] (4.10.13).) If $\alpha\ \epsilon\ H^*_G(\overset{o}{U};\ \mathbb{Q})$ and $\beta\ \epsilon\ H^*_G(\overset{o}{U};\ \mathbb{Q})$ we will let $\alpha\times\beta$ denote their product in $H^*_G(\overset{o}{U}\times\overset{o}{U};\ \mathbb{Q})$. We will also let Td^G_i, $i=1,2,3$, denote the G-equivariant Todd-classes of the i-th factor $\overset{o}{U}$ in $\overset{o}{U}\times\overset{o}{U}\times\overset{o}{U}$. We will let $ch^G : K^{top,0}_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U})\to H^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\ \mathbb{Q})$ denote the equivariant local chern-character. (See [J-7] for details.) Let $\alpha\ \epsilon\ K^{top,0}_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U})$ and $\beta\ \epsilon\ K^G_{top,0}(\overset{oo}{U})$. Now we will let

$BR'(\alpha)=(1\times Td^G_2)\cup ch^G(\alpha)$ and $BR(\beta)=P-L(BR'(P-L^{-1}(\beta)))$

where $P-L$ denotes the appropriate Poincaré-Lefschetz duality isomorphism.

(3.3) **Theorem**. *Bivariant form of equivariant Riemann-Roch* (i). The natural transformation $\rho\ (B)$ considered in (2.12) maps the convolution algebra in (3.2.3) to the convolution algebra in (3.2.4) (the convolution algebra in (3.2.4) to the convolution algebra in (3.2.5), respectively).

(ii) The maps BR' and BR preserve the convolution-product.

Proof. (i) These follow immediately from the equivariant Riemann-Roch theorem in (2.12) since the above natural transformations clearly commute with inverse-images. One could also consult [T-2] which has similar equivariant Riemann-Roch theorems. (Such equivariant Riemann-Roch theorems are implicitly assumed in the literature to show the convolution on affine Hecke algebras is compatible with the convolution on the corresponding graded Hecke algebras.)

(ii) Clearly it suffices to prove that BR' preserves the convolution-product. This follows along the lines of [C-G] (4.10.13) making use of the equivariant Riemann-Roch (2.12). However we will give an argument for the sake of completeness. Let $\alpha,\ \alpha'\ \epsilon\ K^{top,0}_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U})$. Now

$BR'(\alpha)*BR'(\alpha')=\hat{p}_{1,3*}(p^*_{1,2}(BR'(\alpha))\cup p^*_{2,3}(BR'(\alpha')))$

$$= \hat{p}_{1,3*}(p_{1,2}^*((1 \times Td_2^G) \cup ch^G(\alpha)) \cup p_{2,3}^*((1 \times Td_3^G) \cup ch^G(\alpha')))$$
$$= \hat{p}_{1,3*}((1 \times Td_2^G \times Td_3^G) \cup (p_{1,2}^*(ch^G(\alpha)) \cup p_{2,3}^*(ch^G(\alpha'))))$$
$$= \hat{p}_{1,3*}((1 \times Td_2^G \times Td_3^G) \cup (ch^G(p_{1,2}^*(\alpha) \cup p_{2,3}^*(\alpha'))))$$
$$= \hat{p}_{1,3*}((p_{1,3}^*(Td_1^G \times 1)^{-1}) \cup (Td_1^G \times Td_2^G \times Td_3^G) \cup ch^G(p_{1,2}^*(\alpha) \cup p_{2,3}^*(\alpha')))$$
$$= (Td_1^G \times 1)^{-1} \cup \hat{p}_{1,3*}((Td_1^G \times Td_2^G \times Td_3^G) \cup ch^G(p_{1,2}^*(\alpha) \otimes p_{2,3}^*(\alpha')))$$
$$= (Td_1^G \times 1)^{-1} \cup (Td_1^G \times Td_3^G) \cup ch^G(\hat{p}_{1,3*}(p_{1,2}^*(\alpha) \otimes p_{2,3}^*(\alpha')))$$
$$= (1 \times Td_3^G) \cup ch^G(\hat{p}_{1,3*}(p_{1,2}^*(\alpha) \otimes p_{2,3}^*(\alpha'))) = BR'(\alpha * \alpha').$$

The third equality from the end is by the equivariant Riemann-Roch and the one immediately before that is by the projection formula. The rest are clear. $\square$

Next we proceed to show the associativity of the convolution operation so that we, in fact, obtain associative algebras. We will consider in detail only the convolution operation in (3.2.3). (It is hoped that the proof of (3.4) will provide better understanding of the proof of Theorem (4.6) (*see especially step 4*) in the next section.)

(3.4) **Theorem**. Under the convolution product in (3.2.3), one obtains the commutativity of the square:

$$(3.4*) \qquad \begin{array}{ccc} (K_0^G(\overset{oo}{U}) \otimes K_0^G(\overset{oo}{U}) \otimes K_0^G(\overset{oo}{U}) & \longrightarrow & K_0^G(\overset{oo}{U}) \otimes K_0^G(\overset{oo}{U}) \\ & & \downarrow \\ \cong \downarrow & & K_0^G(\overset{oo}{U}) \\ & & \uparrow \\ K_0^G(\overset{oo}{U}) \otimes (K_0^G(\overset{oo}{U}) \otimes K_0^G(\overset{oo}{U}) & \longrightarrow & K_0^G(\overset{oo}{U}) \otimes K_0^G(\overset{oo}{U}) \end{array}$$

Proof. In addition to the terminology in (3.2.1) we will adopt the following. For each integer $i = 1, 2, 3, 4$, we let $\overset{o}{U}_i = \overset{o}{U}$ indexed by i. For each pair (i, j), $1 \le i < j \le 4$, we let $\overset{o}{U}_{i,j}$ denote the product of 2 factors each isomorphic to $\overset{o}{U}$ and with the factors indexed by i and j. We let $\overset{oo}{U}_{i,j} = \overset{o}{U}_i \underset{U}{\times} \overset{o}{U}_j$. Let $\overset{o}{U}_{ijk}$, $1 \le i < j < k \le 4$ denote the product of 3 factors each isomorphic to $\overset{o}{U}$, with each of the factors indexed by i, j and k. We will let $\overset{o}{U}_{1,2,3,4}$ denote the product of four factors indexed by 1, 2, 3 and 4 with each factor isomorphic to $\overset{o}{U}$. Let $p_{i,j} : \overset{o}{U}_{ijk} \to \overset{o}{U}_{i,j}$ denote the projection to the (i, j)-th factor.

Now we consider that the composition of the maps in the bottom row and the bottom half of the right column. Here we will use the following convention: $p_{1,2} : \overset{o}{U}_{1,2,4} \to \overset{o}{U}_{1,2}$, $p_{2,3} : \overset{o}{U}_{2,3,4} \to \overset{o}{U}_{3,4}$ and $p_{3,4} : \overset{o}{U}_{2,3,4} \to \overset{o}{U}_{3,4}$ are projections to the appropriate factors. We let $\hat{p}_{2,4} : \overset{ooo}{U} \to \overset{oo}{U}$ denote the obvious map induced by the composition $\overset{ooo}{U} \to \overset{o}{U}_{2,3,4} \xrightarrow{p_{2,4}} \overset{o}{U}_{2,4}$. We let $p_{2,4} : \overset{o}{U}_{1,2,4} \to \overset{o}{U}_{2,4}$ denote the projection to the factors 2 and 4; recall $\tilde{p}_{2,4} : p_{2,4}^{-1}(\overset{oo}{U}) \to \overset{oo}{U}$ denotes the map induced by $p_{2,4}$. Now the composition of the maps in the bottom row and the bottom half of the right column is given by the composition of the following maps:

$$K^0_{G,\overset{oo}{U}_{1,2}}(\overset{o}{U}_{1,2}) \otimes (K^0_{G,\overset{oo}{U}_{2,3}}(\overset{o}{U}_{2,3}) \otimes K^0_{G,\overset{oo}{U}_{3,4}}(\overset{o}{U}_{3,4}))$$

$$\xrightarrow{id\otimes\tilde{p}^*_{2,3}\otimes\tilde{p}^*_{3,4}} K^0_{G,\overset{oo}{U}_{1,2}}(\overset{o}{U}_{1,2}) \otimes (K^0_{G,p^{-1}_{2,3}(\overset{oo}{U}_{2,3})}(\overset{o}{U}_{2,3,4}) \otimes K^0_{G,p^{-1}_{3,4}(\overset{oo}{U}_{3,4})}(\overset{o}{U}_{2,3,4}))$$

$$\to K^0_{G,\overset{oo}{U}}(\overset{o}{U}_{1,2}) \otimes K^0_{G,\overset{ooo}{U}_{2,3,4}}(\overset{o}{U}_{2,3,4}) \xrightarrow{id\otimes\hat{p}_{2,4*}} K^0_{G,\overset{oo}{U}_{1,2}}(\overset{o}{U}_{1,2}) \otimes K^0_{G,\overset{oo}{U}_{2,4}}(\overset{o}{U}_{2,4})$$

$$\xrightarrow{id\otimes\tilde{p}^*_{2,4}} K^0_{G,\overset{oo}{U}_{1,2}}(\overset{o}{U}_{1,2}) \otimes K^0_{G,\tilde{p}^{-1}_{2,4}(\overset{oo}{U}_{2,4})}(\overset{o}{U}_{1,2,4}) \xrightarrow{\tilde{p}^*_{1,2}\otimes id} K^0_{G,\tilde{p}^{-1}_{1,2}(\overset{oo}{U}_{1,2})}(\overset{o}{U}_{1,2,4}) \otimes K^0_{G,\tilde{p}^{-1}_{2,4}(\overset{oo}{U}_{2,4})}(\overset{o}{U}_{1,2,4})$$

$$\to K_{G,\overset{ooo}{U}_{1,2,4}}(\overset{o}{U}_{1,2,4}) \xrightarrow{\hat{p}_{1,4*}} K^0_{G,\overset{oo}{U}_{1,4}}(\overset{o}{U}_{1,4})$$

For $1 \le i < j < k \le 4$, let $p_{i,j,k} : \overset{o}{U}_{1,2,3,4} \to \overset{o}{U}_{i,j,k}$ denote the projection to the factors indexed by i, j and k. We let $\overset{ooo}{U}_{i,j,k}$ denote the obvious closed subscheme $\overset{o}{U}\underset{U}{\times}\overset{o}{U}\underset{U}{\times}\overset{o}{U}$ of $\overset{o}{U} \times \overset{o}{U} \times \overset{o}{U} = \overset{o}{U}_{i,j,k}$. We let $\tilde{p}_{i,j,k} : p^{-1}_{i,j,k}(\overset{ooo}{U}_{i,j,k}) \to \overset{ooo}{U}_{i,j,k}$ and $\hat{p}_{i,j,k}$ = the composition of $\overset{oooo}{U} = \overset{o}{U}\underset{U}{\times}\overset{o}{U}\underset{U}{\times}\overset{o}{U}\underset{U}{\times}\overset{o}{U} \to p^{-1}_{i,j,k}(\overset{ooo}{U}_{i,j,k})$ with $\tilde{p}_{i,j,k}$.

Now (2.P.7)' applied to the cartesian square

$$\begin{array}{ccc} p^{-1}_{2,3,4}(\overset{ooo}{U}) & \xrightarrow{\tilde{p}_{2,3,4}} & \overset{ooo}{U} \\ {\scriptstyle \check{p}_{1,2,4}}\downarrow & & \downarrow{\scriptstyle \hat{p}_{2,4}} \\ p^{-1}_{2,4}(\overset{oo}{U}) & \xrightarrow{\tilde{p}_{2,4}} & \overset{oo}{U} \end{array}$$

(with $f : X \to Y$ equal to the map $\hat{p}_{2,4}$ and $g : Y' \to Y$ equal to the map $\tilde{p}_{2,4}$) shows: $(id \otimes \tilde{p}^*_{2,4}) \circ (id \otimes \hat{p}_{2,4*}) = (id \otimes \check{p}_{1,2,4*}) \circ (id \otimes \tilde{p}^*_{2,3,4})$. (Observe that $\tilde{p}_{2,4}$ and $\tilde{p}_{2,3,4}$ are smooth while $\hat{p}_{2,4}$ and $\check{p}_{1,2,4}$ are proper.)

Next we apply (the alternate form of) (2.P.10)' with $f = \check{p}_{1,2,4} : p^{-1}_{2,3,4}(\overset{ooo}{U}) \to p^{-1}_{2,4}(\overset{oo}{U})$, $\tilde{f} = p_{1,2,4} : \overset{o}{U}_{1,2,3,4} \to \overset{o}{U}_{1,2,4}$, $g = \tilde{p}_{1,2} : p^{-1}_{1,2}(\overset{oo}{U}) \to \overset{oo}{U}$ and $\tilde{g} = p_{1,2} : \overset{o}{U}_{1,2,4} \to \overset{o}{U} \times \overset{o}{U}$ and the pairings: $K^0_{G,Z}(\tilde{Z}) \otimes K^0_{G,X}(\tilde{X}) \to K^0_{G,X\cap\tilde{f}^{-1}(\tilde{g}^{-1}(Z))}(\tilde{X})$ and $K^0_{G,Z}(\tilde{Z}) \otimes K^0_{G,Y}(\tilde{Y}) \to K^0_{G,Y\cap\tilde{g}^{-1}(Z)}(\tilde{Y})$ and the map in the left-most-column $id \otimes f_*$. This shows that the composition:

$$K^{G,0}_{\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \otimes K^{G,0}_{p^{-1}_{2,3,4}(\overset{ooo}{U})}(\overset{o}{U}_{1,2,3,4}) \otimes \xrightarrow{(\tilde{p}^*_{1,2}\otimes id)\circ(id\otimes\check{p}_{1,2,4*})} K^{G,0}_{\overset{ooo}{U}_{1,2,4}}(\overset{o}{U}_{1,2,4})$$

factors as

$$K^{G,0}_{\overset{oo}{U}}(\overset{o}{U} \times \overset{o}{U}) \otimes K^{G,0}_{p^{-1}_{2,3,4}(\overset{ooo}{U})}(\overset{o}{U}_{1,2,3,4}) \otimes \xrightarrow{\hat{p}_{1,2,4*}\circ(\tilde{p}^*_{1,2,4}\circ\tilde{p}^*_{1,2}\otimes id)} K^{G,0}_{\overset{ooo}{U}_{1,2,4}}(\overset{o}{U}_{1,2,4})$$

As a consequence one concludes that the composition of the maps in the top row and the top half of the right column of the square in (3.4) is given by the composition:

$$K^{G,0}_{\overset{oo}{U}_{1,2}}(\overset{o}{U}_{1,2}) \otimes K^{G,0}_{\overset{oo}{U}_{2,3}}(\overset{o}{U}_{2,3}) \otimes K^{G,0}_{\overset{oo}{U}_{3,4}}(\overset{o}{U}_{3,4}) \xrightarrow{\tilde{p}^*_{1,2,4}\circ\tilde{p}^*_{1,2}\otimes\tilde{p}^*_{2,3,4}\circ\tilde{p}^*_{2,3}\otimes\tilde{p}^*_{2,3,4}\otimes\tilde{p}^*_{3,4}}$$

$$\to K^{G,0}_{p^{-1}_{1,2,3}(p^{-1}_{1,2}(\overset{oo}{U}_{1,2}))}(\overset{o}{U}_{1,2,3,4}) \otimes K^{G,0}_{p^{-1}_{1,2,3}(p^{-1}_{2,3}(\overset{oo}{U}_{2,3}))}(\overset{o}{U}_{1,2,3,4}) \otimes K^{G,0}_{p^{-1}_{1,3,4}(p^{-1}_{3,4}(\overset{oo}{U}_{3,4}))}(\overset{o}{U}_{1,2,3,4})$$

$$\to K^{G,0}_{\overset{\circ\circ\circ\circ}{U}}(\overset{o}{U}_{1,2,3,4}) \xrightarrow{\hat{p}_{1,4*}\circ\hat{p}_{1,2,4*}} K^{G,0}_{\overset{\circ\circ}{U}_{1,4}}(\overset{o}{U}_{1,4})$$

A similar analysis shows that the composition of the bottom row and the right column of the square in (3.4) is given by the composition:

$$K^{G,0}_{\overset{\circ\circ}{U}_{1,2}}(\overset{o}{U}_{1,2}) \otimes K^{G,0}_{\overset{\circ\circ}{U}_{2,3}}(\overset{o}{U}_{2,3}) \otimes K^{G,0}_{\overset{\circ\circ}{U}_{3,4}}(\overset{o}{U}_{3,4}) \xrightarrow{\tilde{p}^*_{1,2,3}\circ\tilde{p}^*_{1,2}\otimes\tilde{p}^*_{1,2,3}\circ\tilde{p}^*_{2,3}\otimes\tilde{p}^*_{1,3,4}\otimes\tilde{p}^*_{3,4}}$$

$$K^{G,0}_{p^{-1}_{1,2,4}(p^{-1}_{1,2}(\overset{\circ\circ}{U}_{1,2}))}(\overset{o}{U}_{1,2,3,4}) \otimes K^{G,0}_{p^{-1}_{2,3,4}(p^{-1}_{2,3}(\overset{\circ\circ}{U}_{2,3}))}(\overset{o}{U}_{1,2,3,4}) \otimes K^{G,0}_{p^{-1}_{2,3,4}(p^{-1}_{3,4}(\overset{\circ\circ}{U}_{3,4}))}(\overset{o}{U}_{1,2,3,4})$$

$$\to \otimes K^{G,0}_{\overset{\circ\circ\circ\circ}{U}}(\overset{o}{U}_{1,2,3,4}) \xrightarrow{\hat{p}_{1,4*}\circ\hat{p}_{1,3,4*}} K^{G,0}_{\overset{\circ\circ}{U}_{1,4}}(\overset{o}{U}_{1,4})$$

Clearly these compositions are the same since $p_{1,2}\circ p_{1,2,3} = p_{1,2}\circ p_{1,2,4}$, $p_{2,3}\circ p_{1,2,3} = p_{2,3}\circ p_{2,3,4}$, $p_{3,4}\circ p_{1,3,4} = p_{3,4}\circ p_{2,3,4}$ and $p_{1,4}\circ p_{1,3,4} = p_{1,4}\circ p_{1,2,4}$. $\square$

4. Construction of modules over convolution algebras from equivariant derived categories.

In this section, which forms the heart of the paper, we provide a general construction of modules over convolution algebras (as in (3.2.7)) starting with an appropriate equivariant derived category. We begin with a few technical lemmas aimed at constructing certain pairings in equivariant hypercohomology. We will assume $E = \mathbb{Q}$ or $\mathbb{C}$ throughout the rest of the paper. We will let **G** denote a fixed linear algebraic group acting on the **G**-quasi-projective varieties considered in this section. G will denote a closed algebraic subgroup or a compact subgroup of **G**. *Throughout this section we will let the map* $EG\underset{G}{\times}X \to EG\underset{G}{\times}Y$ *induced by any G-equivariant map $f : X \to Y$, (for any two G-spaces X and Y), be denoted merely by f for the most part -i.e. we omit the superscript G.*

(4.1) **Lemma**. Let

$$\begin{array}{ccc} Z & \xrightarrow{i'_X} & Y \\ {\scriptstyle i'_Y}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X & \xrightarrow{i_X} & \tilde{X} \end{array}$$

denote a *cartesian* square of G-equivariant closed immersions of G-quasi-projective varieties. Let $i = i_X \circ i'_Y = i_Y \circ i'_X$.

(i) Let $F,\ L \,\epsilon\, D^{c,G}_b(\tilde{X};\, E)$. Now one obtains the pairing $i_{X*}Ri_X{}^!F \otimes i_{Y*}Ri_Y{}^!L \to i_*Ri^!(F\otimes L) \to F\otimes L$ which induces the pairing:

$$\mathbb{H}^*_{G,X}(\tilde{X};\, F) \otimes \mathbb{H}^*_{G,Y}(\tilde{X};\, L) \to \mathbb{H}^*_{G,Z}(\tilde{X};\, F\otimes L)$$

(ii) If $F \,\epsilon\, D^{c,G}_b(\tilde{X};\, E)$ and $L' \,\epsilon\, D^{c,G}_b(Y;\, E)$, one obtains the pairing $i_{X*}Ri_X{}^!F \otimes i_{Y*}L' \to i_*Ri^!(F\otimes i_{Y*}L')$ which induces the pairing:

$$\mathbb{H}^*_{G,X}(\tilde{X};\, F) \otimes \mathbb{H}^*_{G,Y}(\tilde{X};\, i_{Y*}L') \to \mathbb{H}^*_{G,Z}(\tilde{X};\, F\otimes i_{Y*}L')$$

(iii) Let $F' \,\epsilon\, D^{c,G}_b(X;\, E)$ and $L' \,\epsilon\, D^{c,G}_b(Y;\, E)$. Now one obtains the pairing:

$i_{X*}F' \otimes i_{Y*}L' \xrightarrow{\simeq} i_*Ri^!(i_{X*}F' \otimes i_{Y*}L')$.

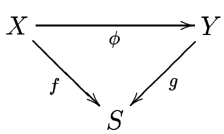
Proof. Observe that the support of $i_{X*}Ri_X{}^!F \otimes i_{Y*}Ri_Y{}^!L$ is contained in Z. Therefore the natural map $i_{X*}Ri_X{}^!F \otimes i_{Y*}Ri_Y{}^!L \to F \otimes L$ factors through $i_*Ri^!(F \otimes L)$. On taking equivariant hypercohomology, the above pairing of complexes of sheaves clearly induces the given pairing. This proves (i). The proofs of (ii) and (iii) are similar. $\square$

Take $F = L = \underline{E}$ = the obvious constant sheaf on $\tilde{X}$. *We will assume henceforth that $\tilde{X}$ is also smooth.* Now lemma (4.1) provides the pairing: $i_{X*}Ri_X^!(\underline{E}) \otimes i_{Y*}Ri_Y^!(\underline{E}) \to i_*Ri^!(\underline{E})$. Since $\tilde{X}$ is smooth, one may identify $\underline{E}$ with $\mathbb{D}_E^{\tilde{X}}$ = the dualizing complex for the category $D_b^{c,G}(\tilde{X}; E)$ modulo an even-dimensional shift. Therefore the above pairing may be identified with a pairing: $i_{X*}D_E^X \otimes i_{Y*}D_E^Y \to i_*D_E^Z$. This provides the pairing:

(4.2) $\mathbb{H}_G^*(X; D_E^X) \otimes \mathbb{H}_G^*(Y; D_E^Y) \to \mathbb{H}_G^*(Z; D_E^Z)$ (i.e. $\mathbb{H}_{G,X}^*(\tilde{X}; \underline{E}) \otimes \mathbb{H}_{G,Y}^*(\tilde{X}; \underline{E}) \to \mathbb{H}_{G,Z}^*(\tilde{X}; \underline{E})$)

on taking equivariant hypercohomology.

(4.3) **Lemma**. Consider a commutative triangle of G-equivariant maps between G-quasi-projective varieties:



where ϕ is an isomorphism. Now one obtains the following natural identifications:

(i) $Rg^! = \phi_*Rf^!$ as functors $D_b^{c,G}(S;E) \to D_b^{c,G}(X;E)$

(ii) $Rf_*Rf^! = Rg_*Rg^!$ as functors $D_b^{c,G}(S;E) \to D_b^{c,G}(S;E)$ and hence

(iii) $\mathbb{H}_G^*(X;Rf^!K) = \mathbb{H}_G^*(Y;Rg^!K)$, $K \in D_b^{c,G}(S;E)$.

(iv) If H is a closed subgroup of G, the above identifications are compatible with the restriction from G to H.

Proof. Since ϕ is an isomorphism, it is smooth and one may identify ϕ^* with $R\phi^!$. Moreover, $id = \phi_*\phi^* = \phi_*R\phi^!$. The commutativity of the above triangle shows $Rf^! = R\phi^! \circ Rg^!$. Now apply ϕ_* to both sides to obtain (i). Next observe $Rf_* = Rg_* \circ \phi_*$; therefore, on applying Rg_* to both sides of (i) we obtain $Rg_*Rg^! = Rg_*\phi_*Rf^! = Rf_*Rf^!$. This proves (ii). Finally take G-equivariant hypercohomology of S and the *canonical* identifications $\mathbb{H}_G^*(X;Rf^!K) = \mathbb{H}_G^*(S;Rf_*Rf^!K)$ and $\mathbb{H}_G^*(Y;Rg^!K) = \mathbb{H}_G^*(S;Rg_*Rg^!K)$ to obtain (iii). Let $i : EH\underset{H}{\times}S \to EG\underset{G}{\times}S$ denote the obvious map. Now, by (A.2) (in the appendix), $i^*Rf_*^GRf^{G^!}(K) \simeq Rf_*^HRf^{H^!}(i^*K)$, where $K \in D_b^{c,G}(S; E)$ and $i^*Rg_*^GRg^{G^!}(K) \simeq Rg_*^HRg^{H^!}(i^*K)$. This proves (iii). Now $i_*i^*Rf_*^GRf^{G^!}(K) \simeq i_*(Rf_*^HRf^{H^!}(i^*(K)))$ and $i_*i^*Rg_*^GRg^{G^!}(K) \simeq i_*Rg_*^HRg^{H^!}(i^*(K))$ by base-change. Take G-equivariant hypercohomology of S with respect to the above complexes to see that the identification in (iii) is compatible with restriction to the sub-group H. $\square$

We apply the above lemma as follows. For the remainder of this section we will assume the situation of (3.1) and the terminology in (3.2.1) and the proof of (3.4). For $(i,j) = (1,2)$, $= (2,3)$ or $= (1,3)$, let $\phi_{i,j} : \overset{o}{U}_{1,2,3} \to \overset{o}{U}_{1,2,3}$ denote the permutation that makes the following diagram commutative:

$$
(4.3.1)\qquad
\begin{array}{ccc}
\overset{ooo}{U} & \xrightarrow{\phi_{i,j}} & \overset{ooo}{U} \\
\Big\downarrow{\scriptstyle \alpha_{1,3}} & & \Big\downarrow{\scriptstyle \alpha_{i,j}} \\
p_{1,3}^{-1}(\overset{oo}{U}) & \xrightarrow{\phi_{i,j}} & p_{i,j}^{-1}(\overset{oo}{U}) \\
\Big\downarrow & & \Big\downarrow \\
\overset{o}{U}_{1,2,3} & \xrightarrow{\phi_{i,j}} & \overset{o}{U}_{1,2,3} \\
& {\scriptstyle p_{1,3}}\searrow \quad \swarrow{\scriptstyle p_{i,j}} & \\
& \overset{o}{U} \times \overset{o}{U} &
\end{array}
$$

The morphism $\phi_{i,j}$ induces isomorphisms: $p_{1,3}^{-1}(\overset{oo}{U}) \to p_{i,j}^{-1}(\overset{oo}{U})$ as well as $p_{1,2}^{-1}(\overset{oo}{U}) \cap p_{2,3}^{-1}(\overset{oo}{U}) \to p_{1,2}^{-1}(\overset{oo}{U}) \cap p_{2,3}^{-1}(\overset{oo}{U})$. (Observe that $p_{1,2}^{-1}(\overset{oo}{U}) \cap p_{2,3}^{-1}(\overset{oo}{U}) = \overset{ooo}{U}$.) These induced isomorphisms will also be denoted $\phi_{i,j}$. For example, if $(i,j) = (2,3)$, $\phi_{i,j}$ is the permutation that interchanges the first and second factor in $\overset{o}{U}_{1,2,3}$. Since $\phi_{i,j}$ (on $\overset{ooo}{U}$) is an automorphism, $R\phi_{i,j}^{!} \cong \phi_{i,j}^{*}$ is an automorphism of $D_b^{c,G}(\overset{ooo}{U};\, E)$ with inverse given by $\phi_{i,j*}$. It follows that

(4.3.2) $R\hat{p}_{1,3}^{!} = R\phi_{i,j}^{!} \circ R\hat{p}_{i,j}^{!}$ and $\phi_{i,j*}R\hat{p}_{1,3}^{!} = R\hat{p}_{i,j}^{!}$.

(4.3.3) Throughout the rest of this section we will let $\underline{E}$ denote the obvious constant sheaf on $\overset{o}{U} \times \overset{o}{U}$ and $\mathbb{D}_E = Ri^{!}(\underline{E})$. d will denote $dim_{\mathbb{C}}(\overset{o}{U})$.

(4.4)**Proposition**. Assume the above situation. Let $K \,\epsilon\, D_b^{c,G}(\overset{oo}{U};\, E)$ $(= D_{b,l}^{c,G}(\overset{oo}{U};\, E))$.

(i) Now there exists a canonical identification: $\mathbb{H}_G^*(\overset{ooo}{U};\, R\hat{p}_{i,j}^{!}K) = \mathbb{H}_G^*(\overset{ooo}{U};\, R\hat{p}_{1,3}^{!}K)$.

(ii) Under the same hypotheses, there exists a pairing:

$\mathbb{H}_G^*(\overset{oo}{U};\, \mathbb{D}_E) \otimes \mathbb{H}_G^*(\overset{oo}{U};\, K) \xrightarrow{\tilde{p}_{1,2}^* \otimes \tilde{p}_{2,3}^*} \mathbb{H}_G^*(p_{1,2}^{-1}(\overset{oo}{U});\, \tilde{p}_{1,2}^*\mathbb{D}_E) \otimes \mathbb{H}_G^*(p_{2,3}^{-1}(\overset{oo}{U});\, \tilde{p}_{2,3}^*(K))$

$\to \mathbb{H}_G^*(\overset{ooo}{U};\, R\hat{p}_{1,3}^{!}K) \xrightarrow{\hat{p}_{1,3*}} \mathbb{H}_G^*(\overset{oo}{U};\, K)$

(iii) In case $K \,\epsilon\, D_{b,r}^{c,G}(\overset{oo}{U};\, E)$, there exists a similar pairing:

$\mathbb{H}_G^*(\overset{oo}{U};\, K) \otimes \mathbb{H}_G^*(\overset{oo}{U};\, \mathbb{D}_E) \to \mathbb{H}_G^*(\overset{oo}{U};\, K)$

Proof. (i) follows immediately from (4.3). The proof of (iii) is entirely similar to that of (ii) and is skipped. To obtain (ii) we apply (4.1) with $\tilde{X} = \overset{o}{U}_{1,2,3}$, $X = \tilde{p}_{2,3}^{-1}(\overset{oo}{U})$, $Y = \tilde{p}_{1,2}^{-1}(\overset{oo}{U})$ and $Z = \overset{ooo}{U}$. Let $i_{i,j} : \overset{oo}{U}_{i,j} \to \overset{o}{U}_{i,j}$ denote the obvious closed immersion and let $\bar{i}_{i,j} : p_{i,j}^{-1}(\overset{oo}{U}) \to \overset{o}{U}_{1,2,3}$ denote the obvious induced closed immersions for $(i,j) = (1,2)$ and $= (2,3)$. Let $i_{1,2,3} : \overset{ooo}{U} \to \overset{o}{U}_{1,2,3}$ denote the composite closed immersion. Therefore lemma (4.1)(i) provides the pairing:

(4.4.1) $\mathbb{H}_G^*(\overset{oo}{U};\, \mathbb{D}_E) \otimes \mathbb{H}_G^*(\overset{oo}{U};\, K) \cong \mathbb{H}_{G,\overset{oo}{U}}^*(\overset{o}{U} \times \overset{o}{U};\, \underline{E}) \otimes \mathbb{H}_{G,\overset{oo}{U}}^*(\overset{o}{U} \times \overset{o}{U};\, i_*K)$

$\to \mathbb{H}_{G,p_{1,2}^{-1}(\overset{oo}{U})}^*(\overset{o}{U}_{1,2,3};\, p_{1,2}^*\underline{E}) \otimes \mathbb{H}_{G,p_{2,3}^{-1}(\overset{oo}{U})}^*(\overset{o}{U}_{1,2,3};\, p_{2,3}^*(i_*K))$

$$\to \mathbb{H}^*_{G,\overset{ooo}{U}}(\overset{o}{U};\, p^*_{1,2}(\underline{E}) \otimes p^*_{2,3}(i_*(K))) \cong \mathbb{H}^*_G(\overset{ooo}{U};\, Ri^!_{1,2,3}p^*_{2,3}(i_*(K)))$$

$$\cong \mathbb{H}^*_G(\overset{ooo}{U};\, Ri^!_{1,2,3}\bar{i}_{2,3*}\tilde{p}^*_{2,3}K) \cong \mathbb{H}^*_G(\overset{ooo}{U};\, R\alpha^!_{2,3}R\tilde{p}^!_{2,3}K[-2d])$$

$$\cong \mathbb{H}^*_G(\overset{ooo}{U};\, R\hat{p}^!_{2,3}K[-2d])$$

(Recall that $\tilde{p}_{1,2}$ is smooth and the map i denotes the closed immersion $\overset{oo}{U} \to \overset{o}{U} \times \overset{o}{U}$. Therefore $\tilde{p}^*_{2,3}K \simeq R\tilde{p}^!_{2,3}K[-2d]$. Recall also that $\alpha_{2,3} : \overset{ooo}{U} \to p^{-1}_{2,3}(\overset{oo}{U})$ is the obvious closed immersion and that $\hat{p}_{2,3} = \tilde{p}_{2,3} \circ \alpha_{2,3}$. This provides the last equality.) Now (i) shows the last term may be identified with $\mathbb{H}^*_G(\overset{ooo}{U};\, R\hat{p}^!_{1,3}K[-2d])$. Therefore we will compose the above pairing with the map $\hat{p}_{1,3*} : \mathbb{H}^*_G(\overset{ooo}{U};\, R\hat{p}^!_{1,3}K[-2d]) \to \mathbb{H}^*_G(\overset{oo}{U};\, K[-2d])$. Finally we may ignore the shift $[-2d]$ since $\mathbb{H}^*_G(\overset{oo}{U};\, K[-2d]) \cong \mathbb{H}^*_G(\overset{oo}{U};\, K)$. $\square$

(4.4)'**Definition**. The pairing in (4.4)(ii) ((4.4)(iii)) defines the *convolution product*:

$* : \mathbb{H}^*_G(\overset{oo}{U};\, \mathbb{D}_E) \otimes \mathbb{H}^*_G(\overset{oo}{U};\, K) \to \mathbb{H}^*_G(\overset{oo}{U};\, K)$ ($* : \mathbb{H}^*_G(\overset{oo}{U};\, K) \otimes \mathbb{H}^*_G(\overset{oo}{U};\, \mathbb{D}_E) \to \mathbb{H}^*_G(\overset{oo}{U};\, K)$, respectively)

For the rest of the paper we will consider explicitly only the first product; the details for the second one are entirely similar.

(4.4)" Assume that H is a closed subgroup of G. Now one may define similar convolution products: $\mathbb{H}^*_G((G\underset{H}{\times}\overset{oo}{U});\, \mathbb{D}_E) \otimes \mathbb{H}^*_G((G\underset{H}{\times}\overset{oo}{U});\, K) \to \mathbb{H}^*_G((G\underset{H}{\times}\overset{oo}{U});\, K)$, if $K \,\epsilon\, D^{c,G}_b((G\underset{H}{\times}\overset{oo}{U});\, E)$. For this, it suffices to replace the diagram (4.3.1) where each term α has been replaced by the corresponding term $G\underset{H}{\times}\alpha$. One may check readily that (4.4) carries over verbatim to the new setting.

(4.5) **Theorem**. The above pairing has the following properties:

(i) If H is a closed subgroup of G, the pairing in (4.4)' is compatible with the restriction map to the equivariant hypercohomology with respect to H. Similarly if

$$\begin{array}{ccc} H' & \longrightarrow & G' \\ \downarrow & & \downarrow \\ H & \longrightarrow & G \end{array}$$

is a commutative diagram where the maps are all inclusions of closed subgroups, the pairing in (4.4)" is compatible with the restriction $\mathbb{H}^*_G((G\underset{H}{\times}\overset{oo}{U});\quad) \to \mathbb{H}^*_{G'}((G'\underset{H'}{\times}\overset{oo}{U});\quad)$.

(ii) The pairing in (4.4)' is natural in K. i.e. if $K' \overset{\phi}{\to} K$ is a map in $D^{c,G}_b(\overset{oo}{U};\, E)$, the corresponding induced map $\mathbb{H}^*_G(\overset{oo}{U};\, K') \to \mathbb{H}^*_G(\overset{oo}{U};\, K)$ is compatible with the pairings in (4.4)'. (A similar conclusion holds for the pairing in (4.4)".)

(iii) Assume that H is a closed *normal* subgroup of G with *finite index* and that $K \,\epsilon\, D^{c,G,G/H}_b(\overset{oo}{U};\, E)$. Let $i : EH\underset{H}{\times}X \to EG\underset{G}{\times}X$ denote the obvious map. Now the action of G/H provided by (2.P.4) on $\mathbb{H}^*_H(\overset{oo}{U};\, i^*K)$ commutes with the pairings in (4.4)'.

Proof. To prove (i) ((ii)) we need to go back and show that each step in the definition of the convolution product is compatible with the restriction to a closed subgroup (the map $K' \to K$, respectively). One may observe readily that the pairing:

$$(4.5.1)\ \mathbb{H}^*_G(\overset{oo}{U};\, \mathbb{D}_E) \otimes \mathbb{H}^*_G(\overset{oo}{U};\, K) \xrightarrow{\tilde{p}^*_{1,2}\otimes\tilde{p}^*_{2,3}} \mathbb{H}^*_G(p^{-1}_{1,2}(\overset{oo}{U});\, \tilde{p}^*_{1,2}\mathbb{D}_E) \otimes \mathbb{H}^*_G(p^{-1}_{2,3}(\overset{oo}{U});\, \tilde{p}^*_{2,3}(K))$$

$$\to \mathbb{H}^*_G(\overset{ooo}{U};\, R\hat{p}^!_{2,3}K)$$

is compatible with restriction to a closed subgroup H. So is the identification

$$(4.5.2)\ \mathbb{H}^*_G(\overset{ooo}{U};\, R\hat{p}^!_{2,3}K) = \mathbb{H}^*_G(\overset{ooo}{U};\, R\hat{p}^!_{1,3}K)$$

by (4.3)(iv). Finally the map $\hat{p}_{1,3*} : \mathbb{H}^*_G(\overset{ooo}{U};\, R\hat{p}^!_{1,3}K) \to \mathbb{H}^*_G(\overset{oo}{U};\, K)$ is also compatible with the restriction to a subgroup H as (2.P.7)(iii) shows. This completes the proof of the first assertion in (i). The proof of the second assertion in (i) is entirely similar.

To prove (ii) one observes that the pairing in (4.5.1) and the identification in (4.5.2) are compatible with the map $K' \to K$. So is the map $\hat{p}_{1,3*} : \mathbb{H}^*_G(\overset{ooo}{U};\, R\hat{p}^!_{1,3}K) \to \mathbb{H}^*_G(\overset{oo}{U};\, K)$ by the naturality of the trace maps in (1.4.3)(iii). Now consider the last assertion. Let $EG\underset{G}{\times}(G\underset{H}{\times}\overset{oo}{U}) \xrightarrow{\pi} EG\underset{G}{\times}\overset{oo}{U}$ denote the obvious map. Let $\bar{g} \in \bar{G} = G/H$ denote a fixed element, let $T_{\bar{g}} : EG\underset{G}{\times}(G\underset{H}{\times}\overset{oo}{U}) \to EG\underset{G}{\times}(G\underset{H}{\times}\overset{oo}{U})$ denote the induced map and let $\pi^*K \to (T_{\bar{g}})_*(\pi^*K)$ be the induced map as in (2.P.4). Now (4.5)(ii) shows the pairing in (4.4)' commutes with the corresponding induced map

$$T_{\bar{g}} : \mathbb{H}^*_G(EG\underset{G}{\times}(G\underset{H}{\times}\overset{oo}{U});\, \pi^*(K)) \to \mathbb{H}^*_G(EG\underset{G}{\times}(G\underset{H}{\times}\overset{oo}{U});\, T_{\bar{g}*}\pi^*(K)) \cong \mathbb{H}^*_G(EG\underset{G}{\times}(G\underset{H}{\times}\overset{oo}{U});\, \pi^*(K))$$

One may now show, using the second assertion in (4.5)(i) applied to the inclusion of the pairs $(H \subseteq H) \to (H \subseteq G)$, that the identifications

$\mathbb{H}^*_H(\overset{oo}{U};\, i^*K) \simeq H^*(EG\underset{G}{\times}(G\underset{H}{\times}\overset{oo}{U});\, \pi^*K)$ and

$\mathbb{H}^*_H(\overset{oo}{U};\, i^*K) \simeq \mathbb{H}^*_H(\overset{oo}{U};\, i^*T_{\bar{g}*}K) \simeq H^*(EG\underset{G}{\times}(G\underset{H}{\times}\overset{oo}{U});\, T_{\bar{g}*}\pi^*(K))$

are provided by the corresponding restriction maps and therefore are compatible with the pairing in (4.4)'. □

(4.6) **Theorem**. The first pairing in (4.4)' provides a commutative diagram:

$$(4.6*)\quad \begin{array}{ccc}
(\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E) \otimes \mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E)) \otimes \mathbb{H}^*_G(\overset{oo}{U};K) & \longrightarrow & \mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E) \otimes \mathbb{H}^*_G(\overset{oo}{U};K) \\
\downarrow \cong & & \downarrow \\
 & & \mathbb{H}^*_G(\overset{oo}{U};K) \\
 & & \uparrow \\
\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E) \otimes (\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E) \otimes \mathbb{H}^*_G(\overset{oo}{U};K)) & \longrightarrow & \mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E) \otimes \mathbb{H}^*_G(\overset{oo}{U};K)
\end{array}$$

The second pairing in (4.4)' provides a similar commutative diagram with $\mathbb{H}^*_G(\overset{oo}{U};K)$ in the left-most position.

Proof. The proof of this theorem follows along the lines of (3.4) showing the associativity of the convolution product in (3.2.3). (See Step 4.) Throughout we will continue to follow the notation adopted in the proof of (3.4) and in the discussion above. *The reader is advised to first look at step 4 of the proof for an overall view of the proof before returning to the details in steps 1 through 3.*

Step 1 We begin by considering the composition of the maps in the bottom row and the bottom half of the right-most column. This is clearly the composition of the following maps:

$$\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E))\otimes\mathbb{H}^*_G(\overset{oo}{U};K)\cong\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\,\underline{E})\otimes\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\,\underline{E})\otimes\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\,i_*K)$$

$$(4.6.1)\ \xrightarrow{id\otimes\tilde{p}^*_{2,3}\otimes\tilde{p}^*_{3,4}}\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U}\ ;\,\underline{E})\otimes\mathbb{H}^*_{G,p^{-1}_{2,3}(\overset{oo}{U})}(\overset{o}{U}_{2,3,4}\ ;\,p^*_{2,3}\underline{E})\otimes\mathbb{H}^*_{G,p^{-1}_{3,4}(\overset{oo}{U})}(\overset{o}{U}_{2,3,4};\,p^*_{3,4}(i_{3,4*}(K)))$$

$$(4.6.2)\ \to\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U}\ ;\,\underline{E})\otimes\mathbb{H}^*_{G,\overset{ooo}{U}}(\overset{o}{U}_{2,3,4}\ ;\,p^*_{2,3}(\underline{E})\otimes p^*_{3,4}(i_{3,4*}K))$$

$$(4.6.3)\ \xrightarrow{\cong}\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U}\ ;\,\underline{E})\otimes\mathbb{H}^*_G(\overset{ooo}{U}_{2,3,4};\,R\hat{p}^!_{3,4}K[-2d])$$

$$\cong\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U}\ ;\,\underline{E})\otimes\mathbb{H}^*_G(\overset{ooo}{U}_{2,3,4};\,R\hat{p}^!_{2,4}K[-2d])$$

$$(4.6.4)\ \xrightarrow{id\otimes\hat{p}_{2,4*}}\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U}\ ;\,\underline{E})\otimes\mathbb{H}^*_G(\overset{oo}{U};K[-2d])\cong\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U}\ ;\,\underline{E})\otimes\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\,i_{2,4*}K[-2d])$$

$$(4.6.5)\ \xrightarrow{\tilde{p}^*_{1,2}\otimes\tilde{p}^*_{2,4}}\mathbb{H}^*_{G,p^{-1}_{1,2}(\overset{oo}{U})}(\overset{o}{U}_{1,2,4}\ ;\,p^*_{1,2}(\underline{E}))\otimes\mathbb{H}^*_{G,p^{-1}_{2,4}(\overset{oo}{U})}(\overset{o}{U}_{1,2,4};\,p^*_{2,4}(i_{2,4*}K[-2d]))$$

$$(4.6.6)\ \to\mathbb{H}^*_{G,\overset{ooo}{U}_{1,2,4}}(\overset{o}{U}_{1,2,4};\,p^*_{1,2}(\underline{E})\otimes p^*_{2,4}(i_{2,4*}K[-2d]))\cong\mathbb{H}^*_G(\overset{ooo}{U}_{1,2,4};\,R\hat{p}^!_{2,4}(K[-4d]))$$

$$\cong\mathbb{H}^*_G(\overset{ooo}{U}_{1,2,4};\,R\hat{p}^!_{1,4}(K[-4d]))$$

$$(4.6.7)\ \xrightarrow{\hat{p}_{1,4*}}\mathbb{H}^*_G(\overset{oo}{U};K[-4d]).$$

(Here $i_{i,j}:\overset{oo}{U}_{i,j}\to\overset{o}{U}_i\times\overset{o}{U}_j$ is the obvious closed immersion. $\overset{ooo}{U}_{i,j,k}=\overset{o}{U}\underset{U}{\times}\overset{o}{U}\underset{U}{\times}\overset{o}{U}$ with the factors $\overset{o}{U}$ indexed by i, j and k. The identifications $\mathbb{H}^*_G(\overset{ooo}{U}_{2,3,4};R\hat{p}^!_{3,4}K[-2d])=\mathbb{H}^*_G(\overset{ooo}{U}_{2,3,4};R\hat{p}^!_{2,4}K[-2d])$ in (4.6.3) and $\mathbb{H}^*_G(\overset{ooo}{U}_{1,2,4};R\hat{p}^!_{2,4}K[-4d])=\mathbb{H}^*_G(\overset{ooo}{U}_{1,2,4};R\hat{p}^!_{1,4}K[-4d])$ in (4.6.6) follow from an application of lemma (4.3). The first $\cong$ in (4.6.6) and the first $\cong$ in (4.6.3) follow from an identification as in (4.4.1).)

Step 2. Here we will show that the composition of the maps in (4.6.1) through (4.6.5) may be replaced by the following:

$$\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E))\otimes\mathbb{H}^*_G(\overset{oo}{U};K)$$

$$\cong\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\,\underline{E})\otimes\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\,\underline{E})\otimes\mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}\times\overset{o}{U};\,i_*K)$$

$$(4.6.1)'\ \xrightarrow{\tilde{p}^*_{1,2}\otimes\tilde{p}^*_{2,3}\otimes\tilde{p}^*_{3,4}}\mathbb{H}^*_{G,p^{-1}_{1,2}(\overset{oo}{U})}(\overset{o}{U}_{1,2,4};\,p^*_{1,2}(\underline{E}))\otimes\mathbb{H}^*_{G,p^{-1}_{2,3}(\overset{oo}{U})}(\overset{o}{U}_{2,3,4}\ ;\,p^*_{2,3}\underline{E})\otimes\mathbb{H}^*_{G,p^{-1}_{3,4}(\overset{oo}{U})}(\overset{o}{U}_{2,3,4}\ ;\,p^*_{3,4}(i_{3,4*}(K)))$$

$$\to\mathbb{H}^*_{G,p^{-1}_{1,2}(\overset{oo}{U})}(\overset{o}{U}_{1,2,4};\,p^*_{1,2}(\underline{E}))\otimes\mathbb{H}^*_{G,\overset{ooo}{U}_{2,3,4}}(\overset{o}{U}_{2,3,4};\,p^*_{2,3}(\underline{E})\otimes p^*_{3,4}(i_{3,4*}K))$$

$$(4.6.2)'\cong\mathbb{H}^*_{G,p^{-1}_{1,2}(\overset{oo}{U})}(\overset{o}{U}_{1,2,4};\,p^*_{1,2}(\underline{E}))\otimes\mathbb{H}^*_G(\overset{ooo}{U}_{2,3,4}\ ;\,R\hat{p}^!_{3,4}K[-2d])$$

$$(4.6.3)' \xrightarrow{id\otimes\tilde{p}^*_{2,3,4}} \mathbb{H}^*_{G,p_{1,2}^{-1}(\overset{\circ\circ}{U})}(\overset{\circ}{U}_{1,2,4};\, p^*_{1,2}(\underline{E})) \otimes \mathbb{H}^*_G(p^{-1}_{2,3,4}(\overset{\circ\circ\circ}{U}_{2,3,4});\, \tilde{p}^*_{2,3,4}(R\hat{p}^!_{3,4}K[-2d]))$$

$$(4.6.4)' = \mathbb{H}^*_{G,p_{1,2}^{-1}(\overset{\circ\circ}{U})}(\overset{\circ}{U}_{1,2,3};\, p^*_{1,2}(\underline{E})) \otimes \mathbb{H}^*_G(p^{-1}_{2,3,4}(\overset{\circ\circ\circ}{U}_{2,3,4});\, \tilde{p}^*_{2,3,4}(R\hat{p}^!_{2,4}K[-2d]))$$

$$(4.6.5)' = \mathbb{H}^*_{G,p_{1,2}^{-1}(\overset{\circ\circ}{U})}(\overset{\circ}{U}_{1,2,4};\, p^*_{1,2}(\underline{E})) \otimes \mathbb{H}^*_G(p^{-1}_{2,3,4}(\overset{\circ\circ\circ}{U}_{2,3,4});\, R\check{p}^!_{1,2,4}(\tilde{p}^*_{2,4}K[-2d]))$$

$$(4.6.6)' \xrightarrow{id\otimes\check{p}_{1,2,4*}} \mathbb{H}^*_{G,p_{1,2}^{-1}(\overset{\circ\circ}{U})}(\overset{\circ}{U}_{1,2,3};\, p^*_{1,2}(\underline{E})) \otimes \mathbb{H}^*_G(p^{-1}_{2,4}(\overset{\circ\circ}{U});\, (\tilde{p}^*_{2,4}K[-2d]))$$

$$= \mathbb{H}^*_{G,p_{1,2}^{-1}(\overset{\circ\circ}{U})}(\overset{\circ}{U}_{1,2,4};\, p^*_{1,2}(\underline{E})) \otimes \mathbb{H}^*_{G,p_{2,4}^{-1}(\overset{\circ\circ}{U})}(\overset{\circ}{U}_{1,2,4};\, p^*_{2,4}(i_{2,4*}(K[-2d])))$$

First we will show that the above maps are defined as stated. Observe the cartesian square

$$(4.6.8)\quad \begin{array}{ccc} p^{-1}_{2,3,4}(\overset{\circ\circ\circ}{U}) & \xrightarrow{\tilde{p}_{2,3,4}} & \overset{\circ\circ\circ}{U} \\ {\scriptstyle \check{p}_{1,2,4}}\downarrow & & \downarrow{\scriptstyle \hat{p}_{2,4}} \\ p^{-1}_{2,4}(\overset{\circ\circ}{U}) & \xrightarrow{\tilde{p}_{2,4}} & \overset{\circ\circ}{U} \end{array}$$

(4.6.9) Observe that the maps $\tilde{p}_{2,4}$ and hence $\tilde{p}_{2,3,4}$ are smooth and that $d =$ the dimension of the fibers of these maps. Now, it follows that, $\tilde{p}^*_{2,3,4}R\hat{p}^!_{2,4}(K) \simeq R\tilde{p}^!_{2,3,4}R\hat{p}^!_{2,4}(K)[-2d] = R\check{p}^!_{1,2,4}R\tilde{p}^!_{2,4}(K)[-2d] = R\check{p}^!_{1,2,4}\tilde{p}^*_{2,4}(K)$. One therefore obtains the identification in (4.6.5)'; it also shows that the last map $id \otimes \check{p}_{1,2,4*}$ is defined. The isomorphism in (4.6.2)' is provided by an application of (4.1)(i) as in the proof of (4.4.1). It remains to verify the identification in (4.6.4)'.

Now the assertion that the composition of the maps in (4.6.1) through (4.6.5) may be replaced by the composition of the maps in (4.6.1)' through (4.6.5)' is equivalent to the commutativity of the diagram (4.6.10) - see attached page 31.1.

The bottom square commutes by applying (2.P.7)(i) to the cartesian square in (4.6.8) with $f : X \to Y$ ($g : Y' \to Y$) given by the map $\hat{p}_{2,4}$ ($\tilde{p}_{2,4}$, respectively). Observe that $\tilde{p}_{2,4}$ and $\tilde{p}_{2,3,4}$ are smooth while $\hat{p}_{2,4}$ and $\check{p}_{1,2,4}$ are proper. (The same square provides the identification : $\tilde{p}^*_{2,3,4}R\hat{p}^!_{2,4}K = R\check{p}^!_{1,2,4}\tilde{p}^*_{2,4}(K)$ as discussed above. This defines the map $\check{p}_{1,2,4*}$ on the right.) To obtain the commutativity of the top square (and the identification in (4.6.4)') one argues as follows. One begins with the natural transformation $R\hat{p}^!_{3,4} \to R\tilde{p}_{2,3,4*}\tilde{p}^*_{2,3,4}R\hat{p}^!_{3,4}$. Now use the natural identification $R\hat{p}^!_{3,4} = \phi_{3,4*}R\hat{p}^!_{2,4}$ to identify this with: $\phi_{3,4*}R\hat{p}^!_{2,4} \to R\tilde{p}_{2,3,4*}\tilde{p}^*_{2,3,4}\phi_{3,4*}R\hat{p}^!_{3,4} \xrightarrow{\simeq} R\tilde{p}_{2,3,4*}\psi_{3,4*}\tilde{p}^*_{2,3,4}R\hat{p}^!_{3,4} = \phi_{3,4*}R\tilde{p}_{2,3,4*}\tilde{p}^*_{2,3,4}R\hat{p}^!_{2,4}$. (Here $\phi_{3,4}$ and $\psi_{3,4}$ are the isomorphisms that make the diagram commute:

$$\begin{array}{ccccc} p^{-1}_{2,3,4}(\overset{\circ\circ\circ}{U}_{2,3,4}) & & \xrightarrow{\psi_{3,4}} & & p^{-1}_{2,3,4}(\overset{\circ\circ\circ}{U}_{2,3,4}) \\ {\scriptstyle \tilde{p}_{2,3,4}}\downarrow & & & & \downarrow{\scriptstyle \tilde{p}_{2,3,4}} \\ \overset{\circ\circ\circ}{U}_{2,3,4} & & \xrightarrow{\phi_{3,4}} & & \overset{\circ\circ\circ}{U}_{2,3,4} \\ & {\scriptstyle \hat{p}_{2,4}}\searrow & & \swarrow{\scriptstyle \hat{p}_{3,4}} & \\ & & \overset{\circ\circ}{U} & & \end{array}$$

.)

Now apply $R\hat{p}_{3,4*}$ to both sides to obtain the natural transformation:

$R\hat{p}_{2,4*}R\hat{p}^!_{2,4} \to R\hat{p}_{2,4*}R\tilde{p}_{2,3,4*}\tilde{p}^*_{2,3,4}R\hat{p}^!_{2,4}$

In other words we have obtained the commutative square:

$$\begin{array}{ccc} R\hat{p}_{3,4*}R\hat{p}^!_{3,4}K & \longrightarrow & R\hat{p}_{3,4*}R\tilde{p}_{2,3,4*}\tilde{p}^*_{2,3,4}R\hat{p}^!_{3,4}K \\ \downarrow & & \downarrow \\ R\hat{p}_{3,4*}R\hat{p}^!_{3,4}K & \longrightarrow & R\hat{p}_{3,4*}R\tilde{p}_{2,3,4*}\tilde{p}^*_{2,3,4}R\hat{p}^!_{3,4}K \end{array}$$

(The two vertical maps are the identity -see (4.3)(ii).) Taking equivariant hypercohomology on $\overset{oo}{U}$, this provides the top square in (4.6.10).

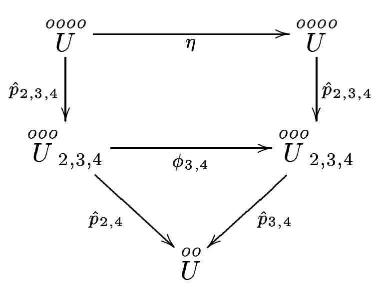
Step 3. Now we consider the composition of the maps (4.6.6) through (4.6.7). We will show the existence of a commutative diagram (4.6.11) (see attached page 31.2), where the composition of the maps in the bottom row and the bottom half of the right column will correspond to the composition of the maps in (4.6.6) and (4.6.7).

The commutativity of the bottom square will follow from the projection formula (2.P.10)(ii') applied to the following situation. In (2.P.10)(ii') take $Y = \overset{o}{U}_{1,2,4}$, $Y_1 = p^{-1}_{1,2}(\overset{oo}{U})$, $Y_2 = p^{-1}_{2,4}(\overset{oo}{U})$, $X = \overset{o}{U}_{1,2,3,4}$, $f = p_{1,2,4}$ and $X_2 = p^{-1}_{2,3,4}(\overset{ooo}{U}_{2,3,4})$. The cartesian square (4.6.8) shows f induces a map $f_2 = \check{p}_{1,2,4}$: $X_2 \to Y_2$; it is induced by base-change from the proper map $\hat{p}_{2,4}$ and hence is proper. $X_1 = f^{-1}(Y_1)$. Now $X_1 \cap X_2 = \overset{oooo}{U}$ while $Y_1 \cap Y_2 = \overset{ooo}{U}$ and $f_{1,2} : X_1 \cap X_2 \to Y_1 \cap Y_2$ is the map $\hat{p}_{1,2,4}$. Let $P = \bar{i}_{1,2*}\tilde{p}^*_{1,2}\underline{E}$, $L = \bar{i}_{2,4*}\tilde{p}^*_{2,4}K$ and $L' = i_{1,2,4*}R\hat{p}^!_{2,4}K[-4d]$. (One readily observes the existence of a pairing $P \otimes L \to L'$ on Y - see the arguments in (4.4).) The commutativity of the top-square follows from the arguments in step 2. The commutativity of the second square from the top follows from the commutative square (4.6.8) and the observation (4.6.9) in step 2. (The bottom row in this square is the map $\tilde{p}^*_{1,2,4} \otimes id$.) The identification in the bottom-part of the right-most column results from an application lemma (4.3). This completes the proof of the commutativity of the diagram in (4.6.11).

Next consider the composition:

$$\mathbb{H}^*_{G,p^{-1}_{1,2,4}(p^{-1}_{1,2}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\underline{E}) \otimes \mathbb{H}^*_{G,p^{-1}_{2,3,4}(\overset{ooo}{U})}(\overset{o}{U}_{1,2,3,4}; p^*_{2,3,4}(p^*_{3,4}(i_{3,4*}K)))$$
$$\cong \mathbb{H}^*_{G,p^{-1}_{1,2,4}(p^{-1}_{1,2}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\underline{E}) \otimes \mathbb{H}^*_G(p^{-1}_{2,3,4}(\overset{ooo}{U}_{2,3,4}); \tilde{p}^*_{2,3,4}(R\hat{p}^!_{3,4}K)[-2d])$$
$$\to \mathbb{H}^*_G(\overset{oooo}{U}; R\hat{p}^!_{2,3,4}R\hat{p}^!_{3,4}K[-4d]) = \mathbb{H}^*_G(\overset{oooo}{U}; R\hat{p}^!_{2,3,4}R\hat{p}^!_{2,4}K[-2d]) = \mathbb{H}^*_G(\overset{oooo}{U}; R\hat{p}^!_{1,2,4}R\hat{p}^!_{2,4}K[-2d])$$

The last equality follows from the equality of the maps $p_{2,4} \circ p_{2,3,4} = p_{2,4} \circ p_{1,2,4}$. The first equality follows from an application of lemma (4.3) to the commutative diagram



(where $\eta(u_1,u_2,u_3,u_4) = (u_1,u_3,u_2,u_4)$). It should be now clear (see the bottom part of step 2) that the composition of the top three maps in the right-most column of the diagram (4.6.11) may be identified with the composition of the above maps above. Now apply the map

$$\hat{p}_{1,2,4*} : \mathbb{H}^*_G(\overset{oooo}{U}_{1,2,3,4};\ R\hat{p}^!_{1,2,4}R\hat{p}^!_{2,4}K[-4d]) \to \mathbb{H}^*_G(\overset{ooo}{U}_{1,2,4};\ R\hat{p}^!_{2,4}K[-4d]).$$

Now another application of lemma (4.3) identifies the last term with

$$\mathbb{H}^*_G(\overset{ooo}{U}_{1,2,4};\ R\hat{p}^!_{1,4}K[-4d]).$$

Finally one applies the map

$$\hat{p}_{1,4*} : \mathbb{H}^*_G(\overset{ooo}{U}_{1,2,4};\ R\hat{p}^!_{1,4}K[-4d]) \to \mathbb{H}^*_G(\overset{oo}{U};\ K[-4d]).$$

Step 4. (*Compare this with the proof of (3.4).*) We will first summarize the results of the three steps above. These show that the composition of the maps forming the bottom row and the bottom half of the right-most column in (4.6.*) is obtained as the composition of the maps in (4.6.12) through (4.6.16).

$$(4.6.12)\ \mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{oo}{U};K) \xrightarrow{\tilde{p}^*_{1,2,4}\circ\tilde{p}^*_{1,2}\otimes\tilde{p}^*_{2,3,4}\circ\tilde{p}^*_{2,3}\otimes\tilde{p}^*_{2,3,4}\circ\tilde{p}^*_{3,4}}$$

$$\mathbb{H}^*_G(p^{-1}_{1,2,4}(p^{-1}_{1,2}(\overset{oo}{U}));\ \tilde{p}^*_{1,2,4}\tilde{p}^*_{1,2}\mathbb{D}_E)\otimes\mathbb{H}^*_G(p^{-1}_{2,3,4}(p^{-1}_{2,3}(\overset{oo}{U}));\ \tilde{p}^*_{2,3,4}\tilde{p}^*_{2,3}\mathbb{D}_E)\otimes\mathbb{H}^*_G(p^{-1}_{2,3,4}(p^{-1}_{3,4}(\overset{oo}{U}));\ \tilde{p}^*_{2,3,4}\tilde{p}^*_{3,4}K)$$

$$\simeq \mathbb{H}^*_G(\overset{o}{U}_{1,2,3,4};\ \bar{\bar{i}}_{1,2*}\tilde{p}^*_{1,2,4}\tilde{p}^*_{1,2}\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{o}{U}_{1,2,3,4};\ \bar{\bar{i}}_{2,3*}\tilde{p}^*_{2,3,4}\tilde{p}^*_{2,3}\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{o}{U}_{1,2,3,4};\ \bar{\bar{i}}_{3,4*}\tilde{p}^*_{2,3,4}\tilde{p}^*_{3,4}K)$$

For each triple $i \neq j \neq k$, let $\bar{\bar{i}}_{i,j} : p^{-1}_{i,j,k}(p^{-1}_{i,j}(\overset{oo}{U}_{i,j})) \to \overset{oooo}{U}_{1,2,3,4}$ denote the obvious closed immersion induced by $\bar{i}_{i,j}$. Now one may identify $\bar{\bar{i}}_{1,2*}\tilde{p}^*_{1,2,4}\tilde{p}^*_{1,2}(\mathbb{D}_E) = p^*_{1,2,4}\bar{i}_{1,2*}\tilde{p}^*_{1,2}Ri^!_{1,2}\underline{E} = p^*_{1,2,4}\bar{i}_{1,2*}R\bar{i}^!_{1,2}p^*_{1,2}(\underline{E}) = \bar{\bar{i}}_{1,2*}R\bar{\bar{i}}^!_{1,2}p^*_{1,2,4}p^*_{1,2}\underline{E}$. Similarly $\bar{\bar{i}}_{2,3*}\tilde{p}^*_{2,3,4}\tilde{p}^*_{2,3}(\mathbb{D}_E) = \bar{\bar{i}}_{2,3*}R\bar{\bar{i}}^!_{2,3}p^*_{2,3,4}p^*_{2,3}\underline{E}$. Moreover $p^*_{2,3,4}p^*_{3,4}(i_{3,4*}K) = \bar{\bar{i}}_{3,4*}\tilde{p}^*_{2,3,4}\tilde{p}^*_{3,4}K$. Therefore one may identify the last term with

$$(4.6.13)\ \mathbb{H}^*_{G,p^{-1}_{1,2,4}(p^{-1}_{1,2}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,2,4}p^*_{1,2}(\underline{E}))\otimes\mathbb{H}^*_{G,p^{-1}_{2,3,4}(p^{-1}_{2,3}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{2,3,4}p^*_{2,3}(\underline{E}))$$

$$\otimes\mathbb{H}^*_{G,p^{-1}_{2,3,4}(p^{-1}_{3,4}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{2,3,4}p^*_{3,4}(i_{3,4*}K))$$

Clearly the latter maps naturally to

$$(4.6.14)\ \mathbb{H}^*_{G,p^{-1}_{1,2,4}(p^{-1}_{1,2}(\overset{ooo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,2,4}(p^*_{1,2}(\underline{E})))\otimes\mathbb{H}^*_{G,p^{-1}_{2,3,4}(\overset{ooo}{U})}(\overset{o}{U}_{1,2,3,4};\ p^*_{2,3,4}(p^*_{2,3}(\underline{E})\otimes p^*_{3,4}(i_{3,4*}K)))$$

$$\simeq \mathbb{H}^*_{G,p^{-1}_{1,2,4}(p^{-1}_{1,2}(\overset{ooo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,2,4}(p^*_{1,2}(\underline{E})))\otimes\mathbb{H}^*_{G,p^{-1}_{2,3,4}(\overset{ooo}{U})}(\overset{o}{U}_{1,2,3,4};\ p^*_{2,3,4}(p^*_{3,4}(i_{3,4*}K)))$$

$$\to \mathbb{H}^*_{G,\overset{oooo}{U}}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,2,4}p^*_{1,2}(\underline{E})\otimes p^*_{2,3,4}(p^*_{3,4}(i_{3,4*}K))) = \mathbb{H}^*_{G,\overset{oooo}{U}}(\overset{o}{U}_{1,2,3,4};\ p^*_{2,3,4}(p^*_{3,4}(i_{3,4*}K)))$$

$$\cong \mathbb{H}^*_G(\overset{o}{U}_{1,2,3,4};\ \bar{i}_*R\bar{i}^!p^*_{2,3,4}(p^*_{3,4}(i_{3,4*}K))) = \mathbb{H}^*_G(\overset{oooo}{U};\ R\hat{p}^!_{2,3,4}R\hat{p}^!_{3,4}K[-4d])$$

The steps 1 through 3 show that one needs to apply the map $\hat{p}_{1,4*}\circ\hat{p}_{1,2,4*}$ to the last term in order to complete the definition of the product:

$$(4.6.15)\ \mathbb{H}^*_G(\overset{oooo}{U};\ R\hat{p}^!_{2,3,4}R\hat{p}^!_{3,4}K[-4d]) = \mathbb{H}^*_G(\overset{oooo}{U};\ R\hat{p}^!_{1,2,4}R\hat{p}^!_{2,4}K[-4d])$$

$$\xrightarrow{\hat{p}_{1,2,4*}} \mathbb{H}^*_G(\overset{ooo}{U};\ R\hat{p}^!_{2,4}K[-4d]) = \mathbb{H}^*_G(\overset{ooo}{U};\ R\hat{p}^!_{1,4}K[-4d]) \xrightarrow{\hat{p}_{1,4*}} \mathbb{H}^*_G(\overset{oo}{U};\ K[-4d])$$

If one considers the composition of the top row and the top half of the right-most column in (4.6.*), one may show that it is obtained as the composition of the following maps. (In the place of the projection-formula (2.P.10)(ii)' used in step 3, one will need to use (2.P.10)(i)' instead.)

$$\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{oo}{U};K)\xrightarrow{\tilde{p}^*_{1,2,3}\circ\tilde{p}^*_{1,2}\otimes\tilde{p}^*_{1,2,3}\circ\tilde{p}^*_{2,3}\otimes\tilde{p}^*_{1,3,4}\circ\tilde{p}^*_{3,4}}$$

(4.6.12)' $\mathbb{H}^*_G(p^{-1}_{1,2,3}(p^{-1}_{1,2}(\overset{oo}{U}));\ \tilde{p}^*_{1,2,3}\tilde{p}^*_{1,2}\mathbb{D}_E)\otimes\mathbb{H}^*_G(p^{-1}_{1,2,3}(p^{-1}_{2,3}(\overset{oo}{U}));\ \tilde{p}^*_{1,2,3}\tilde{p}^*_{2,3}\mathbb{D}_E)\otimes\mathbb{H}^*_G(p^{-1}_{1,3,4}(p^{-1}_{3,4}(\overset{oo}{U}));$ $\tilde{p}^*_{1,3,4}\tilde{p}^*_{3,4}K)$

$$\simeq\mathbb{H}^*_G(\overset{o}{U}_{1,2,3,4};\ \bar{\bar{i}}_{1,2*}\tilde{p}^*_{1,2,3}\tilde{p}^*_{1,2}\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{o}{U}_{1,2,3,4};\ \bar{\bar{i}}_{2,3*}\tilde{p}^*_{1,2,3}\tilde{p}^*_{2,3}\mathbb{D}_E)\otimes\mathbb{H}^*_G(\overset{o}{U}_{1,2,3,4};\ \bar{\bar{i}}_{3,4*}\tilde{p}^*_{1,3,4}\tilde{p}^*_{3,4}K)$$

(Here $\bar{\bar{i}}_{1,2}:p^{-1}_{1,2,3}(p^{-1}_{1,2}(\overset{oo}{U}))\to\overset{o}{U}_{1,2,3,4}$ and $\bar{\bar{i}}_{3,4}:p^{-1}_{1,3,4}(p^{-1}_{3,4}(\overset{oo}{U}))\to\overset{o}{U}_{1,2,3,4}$ are the obvious closed immersions.) One can identify $\bar{\bar{i}}_{1,2*}\tilde{p}^*_{1,2,3}\tilde{p}^*_{1,2}(\mathbb{D}_E)=p^*_{1,2,3}\bar{i}_{1,2*}\tilde{p}^*_{1,2}Ri^!_{1,2}\underline{E}=p^*_{1,2,3}\bar{i}_{1,2*}R\bar{i}^!_{1,2}p^*_{1,2}(\underline{E})=$ $\bar{\bar{i}}_{1,2*}R\bar{\bar{i}}^!_{1,2}p^*_{1,2,3}p^*_{1,2}\underline{E}$. Similarly $\bar{\bar{i}}_{2,3*}\tilde{p}^*_{1,2,3}\tilde{p}^*_{2,3}(\mathbb{D}_E)=\bar{\bar{i}}_{2,3*}R\bar{\bar{i}}^!_{2,3}p^*_{1,2,3}p^*_{2,3}\underline{E}$ and $p^*_{1,3,4}p^*_{3,4}(i_{3,4*}K)\simeq\bar{\bar{i}}_{3,4*}\tilde{p}^*_{1,3,4}\tilde{p}^*_{3,4}K$. Therefore one may identify the last term with

(4.6.13)' $\mathbb{H}^*_{G,p^{-1}_{1,2,3}(p^{-1}_{1,2}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,2,3}p^*_{1,2}(\underline{E}))\otimes\mathbb{H}^*_{G,p^{-1}_{1,2,3}(p^{-1}_{2,3}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,2,3}p^*_{2,3}(\underline{E}))$

$\otimes\mathbb{H}^*_{G,p^{-1}_{1,3,4}(p^{-1}_{3,4}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,3,4}p^*_{3,4}(i_{3,4*}K))$

Clearly the latter maps naturally to

(4.6.14)' $\mathbb{H}^*_{G,p^{-1}_{1,2,3}(\overset{ooo}{U})}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,2,3}(p^*_{1,2}(\underline{E})\otimes p^*_{2,3}(\underline{E})))\otimes\mathbb{H}^*_{G,p^{-1}_{1,3,4}(p^{-1}_{3,4}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,3,4}p^*_{3,4}(i_{3,4*}K))$

$$\simeq\mathbb{H}^*_{G,p^{-1}_{1,2,3}(\overset{ooo}{U})}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,2,3}(p^*_{2,3}(\underline{E})))\otimes\mathbb{H}^*_{G,p^{-1}_{1,3,4}(p^{-1}_{3,4}(\overset{oo}{U}))}(\overset{o}{U}_{1,2,3,4};\ p^*_{1,3,4}p^*_{3,4}(i_{3,4*}K))$$

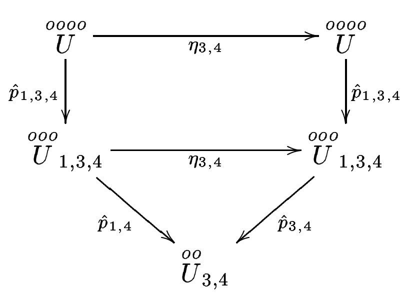
$$\to\mathbb{H}^*_{G,\overset{oooo}{U}}(\overset{o}{U}_{1,2,3,4};\ \underline{E}\otimes p^*_{1,3,4}p^*_{3,4}(i_{3,4*}K))=\mathbb{H}^*_G(\overset{o}{U}_{1,2,3,4};\ \bar{i}_*R\bar{i}^!(p^*_{1,3,4}p^*_{3,4}(i_{3,4*}K)))$$

$$=\mathbb{H}^*_G(\overset{oooo}{U};\ R\hat{p}^!_{1,3,4}R\hat{p}^!_{3,4}K[-4d])$$

If we apply an argument corresponding to those in steps 1 through 3 for the composition of the top row and the top half of the right column in (4.6.*), one sees that now it is necessary to apply the map

$$\hat{p}_{1,4*}\circ\hat{p}_{1,3,4*}:\mathbb{H}^*_G(\overset{oooo}{U};\ R\hat{p}^!_{1,3,4}R\hat{p}^!_{3,4}K[-4d])\to\mathbb{H}^*_G(\overset{oo}{U};\ K[-4d])$$

to complete the corresponding product. To see that this map is in fact defined, apply lemma (4.3) to the commutative diagram:



Here $\eta_{3,4} : \overset{ooo}{U} \rightarrow \overset{ooo}{U}$ is the permutation that interchanges the first and third factors and the map $\eta_{3,4} : \overset{oooo}{U} \rightarrow \overset{oooo}{U}$ is induced by the above map. This identifies the last term in (4.6.11) with $\mathbb{H}^*_G(\overset{oooo}{U}; R\hat{p}^!_{1,3,4} R\hat{p}^!_{1,4} K[-4d])$. Now one simply applies the map

(4.6.15)' $\hat{p}_{1,4*} \circ \hat{p}_{1,3,4*} : \mathbb{H}^*_G(\overset{oooo}{U}; R\hat{p}^!_{1,3,4} R\hat{p}^!_{1,4} K[-4d]) \rightarrow \mathbb{H}^*_G(\overset{oo}{U}; K[-4d])$.

To see that the maps in (4.6.12) through (4.6.14) are the same as the maps in (4.6.12)' through (4.6.14)' one may observe $p_{1,2} \circ p_{1,2,4} = p_{1,2} \circ p_{1,2,3}$, $p_{2,3} \circ p_{2,3,4} = p_{2,3} \circ p_{1,2,3}$, $p_{3,4} \circ p_{2,3,4} = p_{3,4} \circ p_{1,3,4}$ and use the associativity of the usual tensor-product-pairing in equivariant hypercohomology. To see that the map in (4.6.15) is identical to the map in (4.6.15)' one observes: $p_{1,4} \circ p_{1,2,4} = p_{1,4} \circ p_{1,3,4}$. This completes the proof of theorem (4.6). □

(4.7) **Corollary**. Assume the situation in (4.5). (i) If $K \in D^{c,G}_b(\overset{oo}{U}; \mathbb{Q})$, $\mathbb{H}^*_G(\overset{oo}{U}; K^\cdot)$ *has the structure of a left as well as right module* over the convolution-algebra $\mathbb{H}_{\mathbb{Q},gr}$ defined in (3.2.7).

(ii) The above structures are natural in $K \in D^{c,G}_b(\overset{oo}{U}; \mathbb{Q})$ and with respect to closed subgroups of G.

(iii) If H is a closed *normal* subgroup with finite index, and $K \in D^{c,G,G/H}_b(\overset{oo}{U}; \mathbb{Q})$ the group G/H has a natural action on $\mathbb{H}^*_H(\overset{oo}{U}; i^*K^\cdot)$ that is compatible with the structures in (i) and (ii). (Here $i : EH \underset{H}{\times} X \rightarrow EG \underset{G}{\times} X$ is the obvious map.)

Proof. (i) is clear from (4.6). One uses the pairing $\underline{\mathbb{Q}} \otimes K$ (the pairing $K \otimes \underline{\mathbb{Q}} \rightarrow K$) to obtain the left-module structure (the right-module structure, respectively). Now (ii) and (iii) are clear from (4.5). *Note: the above module structures do not provide a bi-module structure - see section 6.*

Next we consider questions of compatibility of the convolution with the ring structure on $K^0_G(U)$.

(4.8) Assume the situation of (4.4). Let $\pi : \overset{o}{U} \underset{U}{\times} \overset{o}{U} \rightarrow U$ denote the obvious map. Since $f : \overset{o}{U} \rightarrow U$ is proper, it follows by base-change, that π is also proper. Now π induces a map $K^0_G(U) \xrightarrow{\pi^*} K^0_G(\overset{o}{U} \underset{U}{\times} \overset{o}{U}) \xrightarrow{\phi} K^G_0(\overset{o}{U} \underset{U}{\times} \overset{o}{U})$. (Here ϕ is the obvious map from equivariant K-cohomology to K-homology sending a locally free coherent sheaf to itself viewed only as a coherent sheaf.)

(4.9)**Proposition**. If $\alpha \in K^0_G(U)$ and $\beta \in K^0_G(U)$

$\phi(\pi^*(\alpha)) * \phi(\pi^*(\beta)) = \phi(\pi^*(\alpha \cup \beta)).\hat{p}_{1,3*}(\phi(1))$

where $\phi(1) \in K^G_0(\overset{o}{U} \underset{U}{\times} \overset{o}{U})$ is the image of the class $1 \in K^0_G(\overset{o}{U} \underset{U}{\times} \overset{o}{U})$.

Proof. $\phi(\pi^*(\alpha)) * \phi(\pi^*(\beta)) = \hat{p}_{1,3*}(\tilde{p}^*_{1,2}\phi(\pi^*(\alpha)) \otimes \tilde{p}^*_{2,3}\phi(\pi^*(\beta)))$. Since $\tilde{p}^*_{1,2}\phi(\pi^*(\alpha)) \otimes \tilde{p}^*_{2,3}\phi(\pi^*(\beta))$ has supports in $\overset{ooo}{U}$, one may identify it (as a K-theory-class) with $\alpha^*_{1,2}\tilde{p}^*_{1,2}\phi(\pi^*(\alpha)) \otimes \alpha^*_{2,3}\tilde{p}^*_{2,3}\phi(\pi^*(\beta)) = \phi(\hat{p}^*_{1,2}(\pi^*(\alpha)) \otimes \hat{p}^*_{2,3}(\pi^*(\beta))) = \phi(\hat{p}^*_{1,3}(\pi^*(\alpha) \otimes \pi^*(\beta))$. (Recall $\alpha_{i,j} : \overset{ooo}{U} \rightarrow p^{-1}_{i,j}(\overset{oo}{U})$ is the obvious map.) The last identification results from the observation $\pi \circ \hat{p}_{i,j} = \pi \circ \hat{p}_{1,3}$. Therefore

$\phi(\pi^*(\alpha)) * \phi(\pi^*(\beta)) = \hat{p}_{1,3*}(\phi(\hat{p}^*_{1,3}(\pi^*(\alpha) \otimes \pi^*(\beta)))) = (\pi^*(\alpha \cup \beta)).\hat{p}_{1,3*}(\phi(1))$

by the projection formula. □

(4.10) **Corollary**. (i) If $\hat{p}_{1,3*}(\phi(1)) = \phi(1) \in K_0^G(\overset{o}{U} \underset{U}{\times} \overset{o}{U})$, it follows that the map π^* in (4.8) sends the cup-product on $K_G^0(U)$ to the convolution on $K_0^G(\overset{o}{U} \underset{U}{\times} \overset{o}{U})$. The above map is injective, if, in addition $\pi_*(\phi(1))$ is a unit in $K_0^G(U)$.

(ii) If, in addition, the obvious map $R(G) \to K_G^0(U)$ is also injective, one obtains an imbedding of the representation ring $R(G)$ into the convolution algebra $K_0^G(\overset{o}{U} \underset{U}{\times} \overset{o}{U})$.

(iii) If, in addition to the hypotheses in (i), the natural transformations $K_G^0(U) \to \pi_0(K_M^{A.S}(U))$ and $K_0^G(\overset{oo}{U}) \to \pi_0(K^M_{A.S}(\overset{oo}{U}))$ (see (2.12)) are isomorphisms, the map π^* sends the cup-product on $H_G^*(U;\, \mathbb{Q})$ to the convolution on $H_*^G(\overset{oo}{U};\, \mathbb{Q})$.

Proofs of all the assertions except the second one in (i) and (iii) are clear. The second assertion in (i) follows readily from the projection formula applied to the map π. Observe that the maps $\pi_0(K_M^{A.S}(U)) \otimes \mathbb{Q} \to \pi_0(K_M^{top}(U)) \otimes \mathbb{Q} \xrightarrow{ch^M} \mathbb{H}_G^*(U;\, \mathbb{Q})$ and the corresponding ones for $\overset{oo}{U}$ are isomorphisms on completing the first term at the augmentation ideal. (Recall $K_G^0(U) \otimes \mathbb{Q}$ is a commutative sub-algebra of the convolution algebra $K_0^G(\overset{oo}{U}) \otimes \mathbb{Q}$ and that the latter is also finitely generated as a module over $K_G^0(U) \otimes \mathbb{Q}$. Therefore the completion of $K_0^G(\overset{oo}{U}) \otimes \mathbb{Q}$ may be identified with the tensor-product $K_0^G(\overset{oo}{U}) \otimes \mathbb{Q} \underset{K_G^0(U) \otimes \mathbb{Q}}{\otimes} (K_G^0(U) \otimes \mathbb{Q})_{I_G}\,\hat{}\,$.) This proves (iii). □

5. Applications to Hecke algebras and to affine quantum groups of type A_n..

In this section we will consider the examples in (3.1)(i)-(iv) in more detail.

(5.1.1) Let $\mathbf{G}$ denote a connected reductive algebraic group over $\mathbb{C}$. We will let $\mathcal{B}$ denote the variety of all Borel-subgroups of $\mathbf{G}$. Let $\mathcal{U}$ denote the variety of all unipotent elements in G. Making use of the exponential mapping from the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ to $\mathbf{G}$, one may observe that $\mathcal{U}$ is isomorphic to the variety $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$. (The inverse of this isomorphism from $\mathcal{U}$ to $\mathcal{N}$ will be denoted by log.) Let $T^*\mathcal{B}$ denote the cotangent bundle to $\mathcal{B}$; observe that one may identify $T^*\mathcal{B}$ with the desingularization of $\mathcal{U}$ (see [Stein-2]) given by $\Lambda = \{(u, B) | u \in \mathcal{U},\ B \in \mathcal{B} \text{ and } u \in B\}$. (One may now identify the obvious map $\mu : \Lambda \to \mathcal{U}$ given by $(u, B) \to u$ with the moment-map $T^*\mathcal{B} \to \mathcal{N}$.)

(5.1.2) The group $G = \mathbf{G} \times \mathbb{C}^*$ acts on Λ on the *right* by $(u, B).(g, q) = (g^{-1}.u^q.g,\ g^{-1}.B.g)$. (Here the exponent u^q is defined as follows: $log(u^q)$ is the element $qlog(u)$. To see that this in-fact defines a right-action one may consider the corresponding action on $\mathcal{N}$.)

(5.1.3) Moreover one may define a right-action of $G \times \mathbb{C}^*$ on the unipotent variety $\mathcal{U}$ by $u.(g, q) = g^{-1}.u^q.g,\ u \in \mathcal{U}$. The stabilizer at u will be denoted $M(u)$. Now observe that the unipotent conjugacy-class $\mathcal{C}_u$ containing a given unipotent element u and its closure are stable under this action; moreover the map $\mu : \Lambda \to \mathcal{U}$ is also $G \times \mathbb{C}^*$-equivariant.

(5.1.4). One may also observe $\mathcal{U} \simeq \mathcal{N}$ is a cone and that for the induced action of the torus $\mathbb{C}^* = 1 \times \mathbb{C}^* \subseteq \mathbf{G} \times \mathbb{C}^*$ on $\mathcal{U}$, the only fixed point is the origin; this defines a *contraction* of $\mathcal{U}$ to its origin.

(5.2)**Proposition**. (a) The map $\pi^* : K^0_G(U) \to K^G_0(\overset{oo}{U})$ sends the tensor-product to the convolution in the following cases: (3.1)(i), (3.1)(ii) and $(3.1)(i)_{(s,1)}$, $(3.1)(ii)_{(s,1)}$ where $(s,1) \, \epsilon \, \mathbf{G} \times \mathbb{C}^*$ is a fixed semi-simple element, $G = Z_{\mathbf{G}}(s) \times \mathbb{C}^*$ and U $(\overset{o}{U})$ is replaced by $U^{(s,1)}$ (by $(\overset{o}{U})^{(s,1)}$, respectively). The same holds in the case of (3.1)(iv) and $(3.1)(iv)_{(s,1)}$ where $(s,1) \, \epsilon \, GL(\mathbb{C}^d) \times \mathbb{C}^*$ is a fixed semi-simple element and $G = Z_{GL(\mathbb{C}^d)}(s) \times \mathbb{C}^*$

(b) The map $\pi^* : H^*_G(U;\, \mathbb{Q}) \to H^G_*(\overset{oo}{U};\, \mathbb{Q})$ sends the cup-product to the convolution in the following cases: (3.1)(i), (3.1)(ii). The same holds in the cases $(3.1)(\mathrm{i})_{(s,1)}$ and $(3.1)(\mathrm{ii})_{(s,1)}$ where $(s,1) \, \epsilon \, \mathbf{G} \times \mathbb{C}^*$ is a fixed semi-simple element, $G = Z_{\mathbf{G}}(s) \times \mathbb{C}^*$ and U, $\overset{o}{U}$ are replaced by $U^{(s,1)}$, $\overset{o}{U}^{(s,1)}$ respectively .

Proof. (a) We will first consider the case of (i) in (3.1). Now $U = Spec \ \mathbb{C}$ and $\overset{o}{U} = \mathcal{B}$ and $f : \mathcal{B} \to Spec \ \mathbb{C}$ is the obvious map. Now the Borel-Weil-Bott theorem shows $Rf_*(\mathcal{O}_{\mathcal{B}}) = Rf_*(f^*(\underline{\mathbb{C}})) \cong \underline{\mathbb{C}}$. It follows that $Rp_{1,3*}(\mathcal{O}_{\mathcal{B}\times\mathcal{B}\times\mathcal{B}}) \cong \mathcal{O}_{\mathcal{B}\times\mathcal{B}}$. Since $\mathcal{O}_{\mathcal{B}\times\mathcal{B}}$ represents the class $1 \, \epsilon \, K^0_G(\mathcal{B}\times\mathcal{B}) \cong K^G_0(\mathcal{B}\times\mathcal{B})$, it follows by an argument as in (4.9) and (4.10) that one obtains an injection of $R(G) = K^0_G(U)$ into the convolution algebra $K^G_0(\mathcal{B}\times\mathcal{B})$ defined in (3.2.3). It is shown in [J-7] (5.3.3) that the map $K^G_0(\mathcal{B}\times\mathcal{B}) \to \pi_0(K^M_{A.S}(\mathcal{B}\times\mathcal{B}))$ is an isomorphism. Therefore the equivariant Riemann-Roch theorem in (2.12) now proves that the obvious map $H^*(BG;\, Q) = H^*_G(U;\, \mathbb{Q}) \to \mathbb{H}^*_G(\mathcal{B}\times\mathcal{B};\, \mathbb{Q})$ is an injection sending the cup-product to the convolution. An entirely similar argument applies to the case $\mathcal{B}$ = the variety of all parabolics conjugate to a given one. (Observe that the Borel-Weil-Bott theorem holds for G/P also; in fact this follows from the Borel-Weil-Bott theorem for all G/B.)

Next we will consider (ii) in (3.1). Now one obtains the commutative squares:

$$\begin{array}{ccc} K^0_G(U) & \longrightarrow & K^G_0(\overset{oo}{U}) \\ \downarrow & & \downarrow \\ K^0_T(U) & \longrightarrow & K^T_0(\overset{oo}{U}) \end{array} , \qquad \begin{array}{ccc} H^*_G(U;\mathbb{Q}) & \longrightarrow & \mathbb{H}^*_G(\overset{oo}{U};\mathbb{Q}) \\ \downarrow & & \downarrow \\ H^*_T(U;\mathbb{Q}) & \longrightarrow & \mathbb{H}^*_T(\overset{oo}{U};\mathbb{Q}) \end{array}$$

where the two vertical maps are split injections (as shown in [J-7] (6.2) or [T-2] Theorem (1.13)) and where T denotes a maximal torus of G. Recall $G = \mathbf{G} \times \mathbb{C}^*$ and hence $T = \mathbf{T} \times \mathbb{C}^*$, for a maximal torus $\mathbf{T}$ of $\mathbf{G}$. Moreover the vertical maps on the right-hand-sides preserve the convolutions defined in (3.2.3) and (3.2.7) as shown by Theorem (4.5). Therefore it suffices to prove the proposition with the group G replaced by T. i.e. it suffices to show that the maps $K^0_T(U) \to K^0_T(\overset{oo}{U})$ and $H^*_T(U;\mathbb{Q}) \to \mathbb{H}^*_T(\overset{oo}{U};\mathbb{Q})$ are injections sending the cup-product on the first term to the convolution on the last. Let p denote the prime ideal in $R(T)$ corresponding to the sub-torus $\mathbb{C}^* = 1 \times \mathbb{C}^*$ in T. Localizing at p, now one obtains the isomorphisms

$$K^0_T(U)_p \simeq K^0_T(U^{\mathbb{C}^*}) \simeq K^0_T(pt)_p = R(T)_p \text{ and } K^T_0(\overset{oo}{U})_p \simeq K^T_0((\overset{oo}{U})^{\mathbb{C}^*})_p \simeq K^T_0(\mathcal{B}\times\mathcal{B})_p.$$

and similar ones in equivariant cohomology. Let ϕ denote the map from equivariant K-cohomology to equivariant K-homology. If π_* ($\hat{p}_{1,3*}$) denotes the map induced in equivariant K-homology by π ($\hat{p}_{1,3}$, respectively), one may now observe

$$\pi_*(\phi(1)) = \phi(1) \text{ in } R(T)_p \text{ and } \hat{p}_{1,3*}(\phi(1)) = \phi(1) \text{ in } K^T_0(\mathcal{B}\times\mathcal{B})_p.$$

Therefore the conclusions of (4.10) (i) and (ii) are true after localization at p. To see that the second holds before localization at p, it suffices to observe that $K^T_0(\overset{oo}{U})$ is a projective module over $R(T)$.

(See [C-G] (6.5.13).). (The first also very likely holds without localizing $K^0_T(U)$ at p; this will involve showing $K^0_T(U)$ is a torsion free.) This completes the proof of (a) for (3.1)(ii) and the proofs of (3.1)(iv), $(3.1)(\mathrm{i})_{(s,1)}$, $(3.1)(\mathrm{ii})_{(s,1)}$ and $(3.1)(\mathrm{iv})_{(s,1)}$ are similar. (Obsere that one may easily reduce the case of $(3.1)(i)_{(s,1)}$, $(3.1)(ii)_{(s,1)}$, $(3.1)(\mathrm{iv})_{(s,1)}$ to the case of (3.1)(i), (3.1)(ii) and (3.1)(iii) respectively .)

(b) The hypotheses of (4.10)(iii) are shown to be satisfied in [J-7](5.5) for (3.1)(i) and (3.1)(ii). (See also [C-G].) One may readily reduce $(3.1)(\mathrm{i})_{(s,1)}$ and $(3.1)(\mathrm{ii})_{(s,1)}$ to the case of (3.1)(i) and (3.1)(ii) respectively . □

(5.3.1). Assume the situation of (3.1)(i). Now theorem (4.6) shows there exists a functor:

$D^{c,G}_b(\mathcal{B}\times\mathcal{B};\mathbb{Q}) \rightarrow$(left- (and right) modules over the convolution algebra $\mathbb{H}^*_G(\mathcal{B}\times\mathcal{B};\mathbb{D}_{\mathbb{Q}})$)

given by $K^{\cdot} \rightarrow \mathbb{H}^*_G(\mathcal{B}\times\mathcal{B};K^{\cdot})$. As an example of such a complex one may take $K^{\cdot}$ to be the equivariant intersection cohomology complex on the closure of a G-orbit for the diagonal action of G on $\mathcal{B}\times\mathcal{B}$. Or one may start with a complex $L^{\cdot}\,\epsilon\, D^{c,G}_b(Spec\ \mathbb{C};\mathbb{Q})$ and let $K^{\cdot}=\pi^*(L^{\cdot})$ where $\pi: EG\underset{G}{\times}(\mathcal{B}\times\mathcal{B})\rightarrow BG$ is the obvious map.

(5.3.2) In the remaining cases of (3.1) we define two functors

J^* and $J^!: D^{c,G}_b(U;E)\rightarrow$ (modules over the convolution algebra $\mathbb{H}^*_G(\overset{oo}{U};\mathbb{D}_E)$)

Recall $f:\overset{o}{U}\rightarrow U$, $\Delta:\overset{o}{U}\rightarrow \overset{oo}{U}=\overset{o}{U}\underset{U}{\times}\overset{o}{U}$ are the obvious maps. Let $L^{\cdot}\,\epsilon\, D^{c,G}_b(U;E)$. Now $\Delta_*(f^*(L^{\cdot}))$ and $\Delta_*(Rf^!(L^{\cdot}))\,\epsilon\, D^{c,G}_b(\overset{oo}{U};E)$. Therefore we let

$J^*(L^{\cdot})=\mathbb{H}^*_G(\overset{oo}{U};\Delta_*(f^*(L^{\cdot})))$ and $J^!(L^{\cdot})=\mathbb{H}^*_G(\overset{oo}{U};\Delta_*(Rf^!(L^{\cdot})))$.

In the case of (3.1)(ii), recall that $U=$ a $G\times\mathbb{C}^*$-stable open sub-variety of the unipotent variety of $\mathbf{G}$ and $f:\overset{o}{U}\rightarrow U$ is the map induced by $\mu:\Lambda\rightarrow\mathcal{U}$. Therefore, one may take $L^{\cdot}$ to be the equivariant intersection cohomology complex on the closure of a unipotent conjugacy-class in U. The convolution algebra (with $U=\mathcal{U}$), in this case, is the graded Hecke algebra associated to $\mathbf{G}$; the bivariant form of the equivariant Riemann-Roch theorem in (3.3) now shows the modules provided by the functors in (5.3.2) are also modules over the corresponding affine Hecke algebra. Let $E=\mathbb{Q}$. Taking $L^{\cdot}=j_!(\underline{\mathbb{Q}})$, where $j:\mathcal{C}_u\rightarrow U$ is the closed immersion of a unipotent conjugacy class, one may now verify that

$J^*(L^{\cdot})\cong\mathbb{H}^*_{G\times\mathbb{C}^*}(\Lambda_{\mathcal{C}_u};\mu^*_{\mathcal{C}_u}(\underline{\mathbb{Q}}))\cong H^*_{M(u)}(\mathcal{B}_u;\mathbb{Q})$ and

$J^!(L^{\cdot})\cong\mathbb{H}^*_{G\times\mathbb{C}^*}(\Lambda_{\mathcal{C}_u};R\mu^!_{\mathcal{C}_u}(\underline{\mathbb{Q}})\cong H^{M(u)}_*(\mathcal{B}_u;\mathbb{Q})$.

where $\mu_{\mathcal{C}_u}:\Lambda_{\mathcal{C}_u}\rightarrow\mathcal{C}_u$ is the map induced by μ. The first isomorphisms on either line follow from the observation that j is a closed immersion. The second isomorphisms on either line follow from (2.P.3)' with $\bar{H}=G\times\mathbb{C}^*$, $H=M(u)$, $X=\mathcal{B}_u$ and $\bar{X}=(\mathcal{B}_u\times G\times\mathbb{C}^*)/M(u)\cong\Lambda_{\mathcal{C}_u}$. If $(s,q)\,\epsilon\, M(u)\subseteq G\times\mathbb{C}^*$ denotes a fixed semi-simple element, one obtains corresponding functors defined on $D^{c,Z_{G\times\mathbb{C}^*}(s,q)}_b(U^{(s,q)};\mathbb{Q})$ that provide modules over the graded Hecke-algebra $H^*(BZ_{G\times\mathbb{C}^*}(s,q);\mathbb{Q})\underset{H^*(BG\times\mathbb{C}^*;\mathbb{Q})}{\otimes}H^*(BG\times\mathbb{C}^*;\mathbb{Q})$. If $L^{\cdot}=j_!(\underline{\mathbb{Q}})$, where $j:\mathcal{C}^{(s,q)}_u\rightarrow U^{(s,q)}$ is the closed immersion associated to a unipotent conjugacy class, then

$J^*(L^{\cdot})\cong\mathbb{H}^*_{Z_{G\times\mathbb{C}^*}(s,q)}(\Lambda_{\mathcal{C}^{(s,q)}_u};\mu^*_{\mathcal{C}^{(s,q)}_u}(\underline{\mathbb{Q}}))\cong H^*_{M(u)\cap Z_{G\times\mathbb{C}^*}(s,q)}(\mathcal{B}^s_u;\mathbb{Q})$ and

$$J^!(L^\cdot) \cong \mathbb{H}^*_{Z_{G\times\mathbb{C}^*}(s,q)}(\Lambda_{\mathcal{C}_u^{(s,q)}};\, R\mu^!_{\mathcal{C}_u^{(s,q)}}(\underline{\mathbb{Q}})) \cong H_*^{M(u)\cap Z_{G\times\mathbb{C}^*}(s,q)}(\mathcal{B}_u^s;\,\mathbb{Q}).$$

where $\mu_{\mathcal{C}_u^{(s,q)}} : \Lambda_{\mathcal{C}_u^{(s,q)}} \to \mathcal{C}_u^{(s,q)}$ is the obvious map induced by μ.

Let $\mathbf{G}$ be the complex dual group associated to a p-adic reductive group G_p. Now the correspondence between modules over the affine Hecke algebra and p-adic representations of G_p shows our constructions make the equivariant derived category on the unipotent variety of $\mathbf{G}$ an abundant source of such p-adic representations.

The case of (3.1)(iii) seems to be somewhat less explored as of now. Somewhat related to it is the case of graded Hecke algebras with unequal parameters and this is considered in detail in [L-1], [L-2] and also in [J-8].

Next we consider the case of (3.1)(iv). First we recall the following results from ([G-V] sections 7 and 8). Let $U_q(\widehat{sl_n})$ denote the quantized enveloping algebra associated to the affine Cartan matrix of the root system R_{af} (see [Dr]). Let $\mathbf{U}$ denote the quotient of $U_q(\widehat{sl_n})$ obtained by specializing the central elements C, C^{-1} to 1. Now it is shown in ([G-V] section 8) that there exists a sequence of algebra homomorphisms:

$$U_q(\widehat{sl_n}) \to \mathbf{U} \to K_0^{GL_d\times\mathbb{C}^*}(\overset{oo}{U})$$

where the last one has the structure of the convolution algebra defined as in (3.2.3). Now one may compose this with the obvious map (as in (3.3)) into the convolution algebra $\mathbb{H}^*_{GL_d(\mathbb{C}^d)\times\mathbb{C}^*}(\overset{oo}{U};\,\mathbb{D}_{\mathbb{Q}})$ defined as in (3.2.7). Therefore the functors considered in (5.3.2) now provide functorial constructions of modules over the algebra $U_q(\widehat{sl_n})$ starting with the equivariant derived category $D_b^{c,GL_d(\mathbb{C}^d)\times\mathbb{C}^*}(U;\,\mathbb{Q})$. Observe that $U = \{x \,\epsilon\, End_{\mathbb{C}}(\mathbb{C}^d) | x^n = 0\}$; clearly this is a sub-variety of the variety of all nilpotent elements in the lie-algebra $gl_d(\mathbb{C}^d)$. If $(s,q) \,\epsilon\, GL_g(\mathbb{C}^d)\times\mathbb{C}^*$ is a fixed semi-simple element, one may define a similar construction of modules starting with the equivariant derived category $D_b^{c,Z_{GL_d(\mathbb{C}^d)\times\mathbb{C}^*}}(U^{(s,q)};\,\mathbb{Q})$.

6. The effect of the natural anti-involution on the module structures

In this final section we will consider the natural anti-involution on the convolution algebra $H_*^G(\overset{oo}{U};\,\mathbb{Q})$ and its effect on the module structures we have constructed in section 4.

Assume the situation of (3.1). Now the map $\tau : \overset{oo}{U} \to \overset{oo}{U}$ interchanging the two factors induces an anti-involution τ^* on $H_*^G(\overset{oo}{U};\,\mathbb{Q})$ in the following sense. If $\tau^*(\alpha)$, $\tau^*(\alpha') \,\epsilon\, H_*^G(\overset{oo}{U};\,\mathbb{Q})$ are homogeneous of degree n and n' respectively,

(6.1) $\tau^*(\alpha * \alpha') = (-1)^{n.n'}(\tau^*(\alpha') * \tau^*(\alpha))$

(See [C-G] (3.5.9) for the case of the affine Hecke algebra. This will also follow from our more general result (6.3) below. In case $H_n^G(\overset{oo}{U};\,\mathbb{Q}) = 0$ for all *odd* n as in many of the examples, there is no need for the factor $(-1)^{n.n'}$.)

(6.2) We will view the map $\tau^* : \overset{oo}{U} \to \overset{oo}{U}$ as an action of the finite group $\mathbb{Z}/2.\mathbb{Z}$ on the space $\overset{oo}{U}$ commuting with the given diagonal action of G on $\overset{oo}{U}$. Therefore the discussion (1.2.4) applies; we therefore obtain an induced action of $\mathbb{Z}/2.\mathbb{Z}$ on the simplicial space $EG\underset{G}{\times}\overset{oo}{U}$ and the corresponding derived category $D_b^{G,\mathbb{Z}/2.\mathbb{Z}}(\overset{oo}{U};\ \mathbb{Q})$. Next assume that $K \epsilon D_b^{G,\mathbb{Z}/2.\mathbb{Z}}(\overset{oo}{U};\ \mathbb{Q})$. Now the results of (2.P.4) apply to define an action of $\mathbb{Z}/2.\mathbb{Z}$ on the equivariant homology group $H_*^G(\overset{oo}{U};\ \mathbb{Q})$ as well as on $\mathbb{H}_G^*(\overset{oo}{U};\ K)$. We will denote this by τ^* as well.

(6.3)**Proposition**. Assume the situation in (6.2). Let $\alpha \epsilon H_*^G(\overset{oo}{U};\ \mathbb{Q}) \cong \mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}{}^2;\ \underline{\mathbb{Q}})$ and $\beta \epsilon \mathbb{H}_G^*(\overset{oo}{U};\ K) = \mathbb{H}^*_{G,\overset{oo}{U}}(\overset{o}{U}{}^2;\ i_*(K))$. Assume that $\tau^*(\alpha)$ is homogeneous of degree n and that $\tau^*(\beta)$ is homogeneous of degree m. Now

$$\tau^*(\alpha * \beta) = (-1)^{n.m}(\tau^*(\beta) * \tau^*(\alpha))$$

Proof. We will use the terminology adopted in sections 3 and 4. Observe (from (4.4)): $\tau(\alpha * \beta) = \tau_{1,3}^*(\hat{p}_{1,3*}(\Delta^*(\tilde{p}_{1,2}^*(\alpha) \times \tilde{p}_{2,3}^*(\beta))))$. Here $\tau_{1,3} : \overset{o}{U}{}^2 \to \overset{o}{U}{}^2$ is the map interchanging the two factors which are indexed by 1 and 3. Let $\tau'_{1,3} : \overset{o}{U}{}^3 \to \overset{o}{U}{}^3$ denote the corresponding map that interchanges the factors indexed by 1 and 3. (Recall that the above factors are indexed by 1, 2 and 3.) The $\times$ on the right-hand-side denotes the cross-product - see [Iver] chapter II, section 10. Now the last term may be identified with

$$(6.3.1)\ \hat{p}_{1,3*}{\tau'_{1,3}}^*(\Delta^*(\tilde{p}_{1,2}^*(\alpha) \times \tilde{p}_{2,3}^*(\beta))) = \hat{p}_{1,3*}(\Delta^*({\tau'_{1,3}}^*\tilde{p}_{1,2}^*(\alpha) \times {\tau'_{1,3}}^*\tilde{p}_{2,3}^*(\beta)))$$
$$= \hat{p}_{1,3*}(\Delta^*(\tilde{p}_{2,3}^*(\tau^*(\alpha)) \times \tilde{p}_{1,2}^*(\tau^*(\beta))))$$

The last equality follows from the commutative squares:

$$\begin{array}{ccc} \overset{o}{U}{}^3 & \xrightarrow{\tau'_{1,3}} & \overset{o}{U}{}^3 \\ {\scriptstyle p_{2,3}}\downarrow & & \downarrow{\scriptstyle p_{1,2}} \\ \overset{o}{U}{}^2 & \xrightarrow{\tau} & \overset{o}{U}{}^2 \end{array} \qquad \begin{array}{ccc} \overset{o}{U}{}^3 & \xrightarrow{\tau'_{1,3}} & \overset{o}{U}{}^3 \\ {\scriptstyle p_{1,2}}\downarrow & & \downarrow{\scriptstyle p_{2,3}} \\ \overset{o}{U}{}^2 & \xrightarrow{\tau} & \overset{o}{U}{}^2 \end{array}$$

Now the last term in (6.3.1) may be identified with

$$(6.3.2)\ \hat{p}_{1,3*}(\Delta^*T^*(\tilde{p}_{1,2}^*(\tau^*(\beta)) \times \tilde{p}_{2,3}^*(\tau^*(\alpha))))$$

Here $T : \overset{o}{U}{}^3 \times \overset{o}{U}{}^3 \to \overset{o}{U}{}^3 \times \overset{o}{U}{}^3$ is the map that interchanges the two factors. Observe that any $K \epsilon D_b^{c,G}(\overset{oo}{U};\ \mathbb{Q})$ is provided with pairings $\underline{\mathbb{Q}} \otimes i_*(K) \to i*(K)$ and $i_*(K) \otimes \underline{\mathbb{Q}} \to i_*(K)$ so that the triangle:

$$(6.3.3)\quad \begin{array}{ccc} \underline{\mathbb{Q}} \otimes i_*(K) & \xrightarrow[T^*]{\simeq} & i_*(K) \otimes \underline{\mathbb{Q}} \\ & \searrow \quad \swarrow & \\ & K & \end{array}$$

commutes. We therefore obtain the commutative triangle:

$$(6.3.4)\quad \begin{array}{ccc} \mathbb{H}^*_{G,p_{2,3}^{-1}(\overset{oo}{U})}(\overset{o}{U}{}^3;\mathbb{Q}) \otimes \mathbb{H}^*_{G,p_{1,2}^{-1}(\overset{oo}{U})}(\overset{o}{U}{}^3;p_{1,2}^*i_*(K)) & \xrightarrow{T^*} & \mathbb{H}^*_{G,p_{1,2}^{-1}(\overset{oo}{U})}(\overset{o}{U}{}^3;p_{1,2}^*i_*(K)) \otimes \mathbb{H}^*_{G,p_{2,3}^{-1}(\overset{oo}{U})}(\overset{o}{U}{}^3;\mathbb{Q}) \\ & \searrow & \downarrow \\ & & \mathbb{H}^*_{G,\overset{ooo}{U}}(\overset{o}{U}{}^3;i_{1,2,3*}i_{1,2,3}^*p_{1,2}^*(i_*(K))) \end{array}$$

where the two unmarked maps are Δ^* and are induced by the inclined maps in (6.3.3). (On the left (right) we have used the pairing $\underline{\mathbb{Q}} \otimes p_{1,2}^* i_*(K) \rightarrow i_{1,2,3*} i_{1,2,3}^* p_{1,2}^*(i_*(K))$ $(p_{1,2}^* i_*(K) \otimes \underline{\mathbb{Q}} \rightarrow i_{1,2,3*} i_{1,2,3}^* p_{1,2}^*(i_*(K))$, respectively).) Recall (see [Iver] p. 128) that the cup-product defined as the composition of the pull-back by Δ and the cross-product is graded commutative. Therefore, (using the pairing on the right in (6.3.4)), one may now identify the term in (6.3.2) with

$$\hat{p}_{1,3*}((-1)^{n.m} \Delta^*(\tilde{p}_{1,2}^*(\tau^*(\beta)) \times \tilde{p}_{2,3}^*(\tau^*(\alpha)))) = (-1)^{n.m} \hat{p}_{1,3*}(\Delta^*(\tilde{p}_{1,2}^*(\tau^*(\beta)) \times \tilde{p}_{2,3}^*(\tau^*(\alpha))))$$

$$= (-1)^{n.m}(\tau^*(\beta) * \tau^*(\alpha)). \quad \square$$

(6.3.4)*Remark.* This result clearly requires that K have a left and right module structure over $\mathbb{Q}$, which is clear. This clearly fails if $\mathbb{Q}$ is replaced by a locally constant sheaf of rings that is in general non-commutative; therefore (6.3) does not generalize to the case of unequal parameters.

(6.4) Assume throughout the rest of the paper that $\mathbb{H}_n^G(\overset{oo}{U};\, \mathbb{Q}) = 0$ for all *odd* n. Let M denote a *left-module* over the convolution algebra $H_*^G(\overset{oo}{U};\, \mathbb{Q})$. Using the anti-involution τ^* we may convert the above left-module-structure into a right-module-structure as follows:

(6.4.1) $m \bullet h = \tau^*(h) \circ m$,

Here $m \in M$, $h \in H_*^G(\overset{oo}{U};\, \mathbb{Q})$; the operation $\bullet$ is the new one while $\circ$ is the given operation of $H_*^G(\overset{oo}{U};\, \mathbb{Q})$ on M from the left. We will let $M(\tau)$ denote M viewed as a *right-module* over $H_*^G(\overset{oo}{U};\, \mathbb{Q})$ using the above structure.

(6.5.1) Let $L \in D_b^{c,G}(\overset{o}{U};\, \mathbb{Q})$. Now the group $\mathbb{Z}/2.\mathbb{Z}$ acts trivially on $\overset{o}{U}$ and L. If $\Delta : \overset{o}{U} \rightarrow \overset{oo}{U} = \overset{o}{U} \underset{U}{\times} \overset{o}{U}$ denotes the obvious map, it is clear $\Delta_*(L) \in D_b^{G,\mathbb{Z}/2.\mathbb{Z}}(\overset{oo}{U};\, \mathbb{Q})$ and that the action of $\mathbb{Z}/2.\mathbb{Z}$ on the latter is trivial.

(6.5.2) Now the pairing $\underline{\mathbb{Q}} \otimes L \rightarrow L$ $(L \otimes \underline{\mathbb{Q}} \rightarrow L)$ induces the structure of a left-module (right-module, respectively) over the convolution algebra $H_*^G(\overset{oo}{U};\, \mathbb{Q})$ on $\mathbb{H}_G^*(\overset{oo}{U};\, \Delta_*(L))$. The left-module-structure (the right-module-structure) on $\mathbb{H}_G^*(\overset{oo}{U};\, \Delta_*(L))$ will be denoted by $\mathbb{H}_G^*(\overset{oo}{U};\, \Delta_*(L))_L$ $(\mathbb{H}_G^*(\overset{oo}{U};\, \Delta_*(L))_R$, respectively).

(6.5.3) **Theorem**. Assume the above situation. Now we obtain the isomorphism of *right-modules* over $H_*^G(\overset{oo}{U};\, \mathbb{Q})$:

$$\mathbb{H}_G^*(\overset{oo}{U};\, \Delta_*(L))_R \cong \mathbb{H}_G^*(\overset{oo}{U};\, \Delta_*(L))_L(\tau)$$

Proof. This follows immediately from (6.3) and (6.4.1) since $\mathbb{Z}/2.\mathbb{Z}$ acts trivially on $\mathbb{H}_G^*(\overset{oo}{U};\, \Delta_*(L))$. In more detail, the right-module-structure on $\mathbb{H}_G^*(\overset{oo}{U};\, \Delta_*(L))_R$ is given as follows. Let $h \in H_*^G(\overset{oo}{U};\, \mathbb{Q})$ and $m \in \mathbb{H}_G^*(\overset{oo}{U};\, \Delta_*(L))$. Now $m * h = \tau^*(h) * m$. The last equality follows from (6.3). (Recall $\tau^*(m) = m$ and that $H_i^G(\overset{oo}{U};\, \mathbb{Q}) = 0$ for all *odd* i. Therefore $deg(h)$ and $deg(m).deg(h)$ are even.) By (6.4.1) $\tau^*(h) * m = m \bullet h$. $\square$

Remark. Observe that this theorem applies to all the modules constructed by the functors J^* and $J^!$ in (5.3.2). We will apply this in the forthoming paper [J-6] to provide a general construction of *self-dual* modules starting with equivariant perverse sheaves on the variety U.

Appendix.

Let G denote a complex linear algebraic group acting on a variety X and let H denote a closed subgroup. We assume that G and H are in general *not necessarily connected.* Let H act on $G \times X$ by $h.(g, x) = (g.h^{-1}, hx)$, $h \epsilon H$, $g \epsilon G$ and $x \epsilon X$. Then a geometric quotient $G \underset{H}{\times} X$ exists for this action and the map $s : G \times X \to G \underset{H}{\times} X$ is smooth with fibers isomorphic to H. Now G acts on $G \times X$ by translation on the first factor; this induces a G-action on $G \underset{H}{\times} X$ as well. One verifies that the map s is equivariant for these actions of G. Let $p : G \underset{H}{\times} X \to X$ denote the map induced by the map $G \times X \to X$ which is defined by $(g, x) \to (g.x)$. One verifies that p is G-equivariant for the G-action on $G \underset{H}{\times} X$ as in (6.2.3) and the G-action on X. It follows that p defines a map $\bar{p} : EG \underset{G}{\times} (G \underset{H}{\times} X) \to EG \underset{G}{\times} X$. Let $r : G \times X \to G \underset{G}{\times} X = X$ denote the projection to the second factor.

Next let $G \times H$ act on $G \times X$ by $(g_1, h_1).(g, x) = (g_1 g h_1^{-1}, h_1 x)$, g_1, $g \epsilon G$, $h_1 \epsilon H$ and $x \epsilon X$. We observe that the maps r and s are such that we obtain the commutative squares:

$$\begin{array}{ccc} (G \times H) \times (G \times X) & \longrightarrow & G \times X \\ {\scriptstyle pr_1 \times s}\downarrow & & {\scriptstyle s}\downarrow \\ G \times (G \underset{H}{\times} X) & \longrightarrow & G \underset{H}{\times} X \end{array} \qquad \begin{array}{ccc} (G \times H) \times (G \times X) & \longrightarrow & G \times X \\ {\scriptstyle pr_2 \times r}\downarrow & & {\scriptstyle r}\downarrow \\ H \times X & \longrightarrow & X \end{array}$$

It follows that r and s induce maps $\bar{r} : E(G \times H) \underset{G \times H}{\times} (G \times X) \to EH \underset{H}{\times} X$ and $\bar{s} : E(G \times H) \underset{G \times H}{\times} (G \times X) \to EG \underset{G}{\times} (G \underset{H}{\times} X)$.

Let $\Delta : H \to G \times H$ denote the diagonal and let $j : X \to G \times X$ denote the map $x \to (e, x)$ where e is the identity element of G. We now observe that the square

$$\begin{array}{ccc} H \times X & \longrightarrow & X \\ {\scriptstyle \Delta \times j}\downarrow & & {\scriptstyle j}\downarrow \\ (G \times H) \times (G \times X) & \longrightarrow & G \times X \end{array}$$

commutes. It follows that j and Δ induce a map $\bar{j} : EH \underset{H}{\times} X \to E(G \times H) \underset{G}{\times} (G \times X)$; one checks readily that $\bar{r} \circ \bar{j} =$ the identity. We denote $\bar{s} \circ \bar{j}$ by $\bar{i}$.

(A.1)**Theorem**. Under the above assumptions the functors :

$$D_b^{c,H}(X; \mathbb{Q}) \xrightarrow{\bar{r}^*} D_b^{c,G \times H}(G \times X; \mathbb{Q}) \text{ and } D_b^{c,G}(G \underset{H}{\times} X; \mathbb{Q}) \xrightarrow{\bar{s}^*} D_b^{c,G \times H}(G \times X; \mathbb{Q})$$

are equivalences of categories. Hence so are the functors $\bar{j}^*$ and $\bar{i}^*$.

Proof. This theorem is proved in [J-5] Theorem (6.4) under the hypotheses that G and H are connected. The same proof applies here verbatim; we will however outline a proof that the functors are fully-faithful. (See [J-7] (6.4) for the remaining details.) Observe first that the fibers of each r_n (s_n) is isomorphic to $G^{\times n}$ ($H^{\times n}$, respectively). Therefore observe that the fibers of the simplicial map $\bar{r}$ ($\bar{s}$) are isomorphic to the simplicial space EG (EH, respectively); hence these have trivial cohomology with respect to any locally constant abelian sheaf. (Observe that, since EG and EH are contractible, any locally constant sheaf on them is actually constant.)

Since $(EG)_0 = G$ any constructible G-equivariant abelian sheaf on EG is locally constant. (Recall that a sheaf $F = \{F_n|n\}$ on a simplicial space $X.$ is locally constant if (i) F_0 is locally constant on X_0 and (ii) F has descent as in (1.2.1).) If K is a constructible G-equivariant abelian sheaf on $EG\underset{G}{\times}(G\underset{H}{\times}X)$, $\bar{s}^*(K)$ is a $G \times H$-equivariant constructible sheaf on $E(G\times H)\underset{G\times H}{\times}(G\times X)$. It follows that that the cohomology sheaves of $\bar{s}^*(K)$ are locally constant on the fibers of $\bar{s}$; recall these fibers were observed to be $\cong EH$ *which is contractible irrespective of whether H is connected or not.* Therefore, the fibers of $\bar{s}$ are acyclic with respect to $\bar{s}^*(K)$. Now it suffices to observe that each of the maps $\bar{s}_n$ is cohomologically proper (i.e. base-change in cohomology holds with respect to any map $\alpha : Z \to (EG\underset{G}{\times}(G\underset{H}{\times}X))_n$ and for the pair $(\bar{s}_n,\ \bar{s}_n^*(K_n)))$ to be able to apply Corollary (A.9) of [J-5] that shows the natural map $K = \{K_n|n\} \to \{Rs_{n*}s_n^*K_n|n\}$ induces an isomorphism

(A.1.1) $H^t(EG\underset{G}{\times}(G\underset{H}{\times}X);\ K) \simeq H^t(E(G\times H)\underset{G\times H}{\times}(G\times X);\ \bar{s}^*K))$

As these isomorphisms are natural in K they induce a map of the hypercohomology spectral sequences proving thereby that such an isomorphism holds for any $K^\cdot \epsilon\ D^G(G\underset{H}{\times}X)$. (Recall that the above derived categories consist of bounded complexes.) If $P,\ Q\ \epsilon\ D_b^{c,G}(G\underset{H}{\times}X;\ \mathbb{Q})$ one lets $K = \mathcal{RH}om(P,Q)$; this shows that $\bar{s}^*$ induces an isomorphism:

$$Hom_{D_b^{c,G}(G\underset{H}{\times}X;\mathbb{Q})}(P,\ Q) \overset{\cong}{\to} Hom_{D_b^{c,G\times H}(G\times X;\mathbb{Q})}(\bar{s}^*(P),\ \bar{s}^*(Q))$$

This proves $\bar{s}^*$ is fully-faithful even if G and H need not be connected. The proof that $\bar{r}^*$ is fully-faithful is similar. Since $\bar{r}\circ\bar{j}$ = the identity, it follows that $\bar{j}^*$ is also fully-faithful. Since $\bar{s}\circ\bar{j} = \bar{i}$, it also follows that the map $\bar{i}^*$ is fully-faithful. □

Next we consider the map $i : EH\underset{H}{\times}X \to EG\underset{G}{\times}X$. Let $i^* : D_b^{c,G}(X;\ \mathbb{Q}) \to D_b^{c,H}(X;\ \mathbb{Q})$ be the obvious restriction functor. Let $\mathbb{D}_{EG\underset{G}{\times}X}$ ($\mathbb{D}_{EH\underset{H}{\times}X}$) denote the dualizing complex (as defined in (1.4.2)) of the category $D_b^{c,G}(X;\ \mathbb{Q})$ ($D_b^{c,H}(X;\ \mathbb{Q})$, respectively); let $\mathbb{D}_{EG\underset{G}{\times}X} = \mathcal{RH}om(\quad, \mathbb{D}_{EG\underset{G}{\times}X})$ ($\mathbb{D}_{EH\underset{H}{\times}X} = \mathcal{RH}om(\quad, \mathbb{D}_{EH\underset{H}{\times}X})$, respectively). If $Ri^! = \mathbb{D}_{EH\underset{H}{\times}X}\circ i^*\circ\mathbb{D}_{EG\underset{G}{\times}X}$, then one obtains the natural identification:

(A.2) $Ri^! \simeq i^* : D_b^{c,G}(X;\ \mathbb{Q}) \to D_b^{c,H}(X;\ \mathbb{Q})$

Proof is clear from the observation that $\mathbb{D}_{(EG\underset{G}{\times}X)_n} = \underline{\mathbb{Q}}^{\otimes(n)}\otimes\mathbb{D}_X$ and $\mathbb{D}_{(EH\underset{H}{\times}X)_n} = \underline{\mathbb{Q}}^{\otimes(n)}\otimes\mathbb{D}_X$, where $\mathbb{D}_X$ denotes the dualizing complex for the category $D_b^c(X;\ \mathbb{Q})$.

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Department of Mathematics,
The Ohio State University,
Columbus, Ohio-43210, USA.