GENERALIZED $t$-STRUCTURES: $t$-STRUCTURES FOR SHEAVES OF $dg$-MODULES OVER A SHEAF OF $dg$-ALGEBRAS AND DIAGONAL $t$-STRUCTURES

ROY JOSHUA

Abstract. $t$-structures, in the abstract, apply to any triangulated category. However, for the most part, they have been studied so far only in the context of sheaves of modules over sites provided with sheaves of rings. In this paper we define and study $t$-structures for categories of modules over sites provided with sheaves of $dg$-algebras and $E_{\infty}$-$dg$-algebras. A close variant, as we show, are the diagonal $t$-structures that come up in the context of filtered derived categories and also in the context of crystalline cohomology (as in the work of Ekedahl). All of this is carried out in the unified frame-work of aisles. We consider several examples: equivariant derived categories and derived categories associated to algebraic stacks, motivic derived categories, the derived categories of presheaves of spectra on Grothendieck sites as well as crystalline derived categories.

Table of Contents

1. Introduction
2. The basic contexts.
3. Construction of (standard) $t$-structures
4. Cell and CW-cell modules
5. $t$-structures for filtered derived categories
6. The crystalline case
7. Further examples
8. References
1. Introduction.

$t$-structures were originally introduced in [BBD] to define and study perverse sheaves and soon afterwards in [Ek] to study crystalline cohomology problems. Since then, they have appeared in other contexts (see [De], [KM], [J1] for example). They appear prominently in certain conjectures (see [Jan]) on algebraic cycles where a formalism similar to $l$-adic derived categories for algebraic cycles is formulated. Questions on the existence of generalized $t$-structures have been around for over 25 years: see, for example, [Bo] where questions on the existence of generalized intersection cohomology theories were raised.

In this paper we utilize the techniques of pre-aisles and aisles introduced in [KV] to provide a painless way to define and study (generalized) $t$-structures for many of the above contexts. As applications of our work, we also show how to apply our results to several of the above situations. Our interest in the material discussed here was awakened by some questions posed by N. Ramachandran to the author in 2003 while both of us were members of the MPI, Bonn. We thank Ramachandran for raising these questions and to the MPI for its hospitality.

The following theorem is typical of our results in the paper. (The notion of pre-aisles and aisles are discussed below.) For the purposes of this introduction we may assume the site $S$ is the small Zariski, Nisnevich or étale sites associated to a given Noetherian scheme $S$.

**Theorem 1.1.** Let $(S, \mathcal{R})$ denote a ringed site as in [2.3] so that the underlying site $S$ is essentially small and has enough points. Let $C(S, \mathcal{R})$ denote the category of all unbounded complexes of sheaves of $\mathcal{R}$-modules on $S$. Moreover we assume the following:

(i) for each object $U$ in the site $S$, there exists a large enough integer $N$ (depending on $U$) so that $H^i(U, F_U) = 0$ for all $i > N$ and all sheaves of $\mathcal{R}$-modules $F$ on the site $S$ and

(ii) for all filtered direct systems $\{F\alpha|\alpha\}$ of sheaves of $\mathcal{R}$-modules and every object $U$ in the site $S$, $\colim_{\alpha} H^*(U, F\alpha) \cong H^*(U, \colim F\alpha)$.

Let $\mathcal{A}$ be a sheaf of $E_\infty$-dgas or dgas on the ringed site $(S, \mathcal{R})$. Let $D\text{Mod}(S, \mathcal{A})^{\leq 0}$ denote the pre-aisle in $D\text{Mod}(S, \mathcal{A})$ generated by $j(U)^*\mathcal{A}[[n]]$, $n \geq 0$, $U$ in the site $S$.

Then (i) $D\text{Mod}(S, \mathcal{A})^{\leq 0}$ is an aisle in $D\text{Mod}(S, \mathcal{A})$, i.e. defines a $t$-structure on $D\text{Mod}(S, \mathcal{A})$.

(ii) Assume next the hypotheses of [2.3] hold. i.e. We will assume that $\mathcal{A}$ is provided with an augmentation $\mathcal{A} \to \mathcal{R}$ which is assumed to be a map of sheaves of $E_\infty$-dgas and that $\mathcal{A}$ is connected, i.e. $\mathcal{A}^i = 0$ for $i < 0$ and $\mathcal{A}^0 = \mathcal{R}$.

Then (a) $j_U!(\mathcal{A}[[U]])$ $D\text{Mod}(S, \mathcal{A})^{\leq 0}$ $D\text{Mod}(S, \mathcal{A})^{\geq 0} = \text{the heart of the above } t\text{-structure, where } j_U : U \to S$ is the structure map of the object. (b) Moreover, every object $M$ in $D\text{Mod}(S, \mathcal{A})^{\leq 0}$ satisfies the property that the natural map $\tau^{\leq 0}(R\overset{L}{\otimes}_{\mathcal{A}} M) \to R\overset{L}{\otimes}_{\mathcal{A}} M$ is a quasi-isomorphism in $D\text{Mod}(S, \mathcal{R})$. In other words, the functor $R\overset{L}{\otimes}_{\mathcal{A}}(\quad)$ $D\text{Mod}(S, \mathcal{A}) \to D\text{Mod}(S, \mathcal{R})^{\leq 0}$ to $D\text{Mod}(S, \mathcal{R})^{\geq 0}$.

We briefly mention two examples here, which are discussed in more detail in the last section of the paper.

**Example 1.2.** Utilizing the $E_\infty$-structure on the motivic complexes (see [J1]), it is possible to define motivic derived categories as derived categories of dg-modules over the motivic $E_\infty$-dga.

**Example 1.3.** The equivariant derived categories of $l$-adic sheaves on a scheme provided with an action by a smooth group-scheme is shown to be equivalent to the derived category of sheaves over a sheaf of dgas in certain cases (see [7] and [Mil]) and such an equivalence is conjectured to hold under fairly general hypotheses: see [So]. We obtain the following result in this context (which is essentially an extension to positive characteristics the result in [7]; such an extension needs the isovariant étale site considered in [11] and [J2]. (These are briefly recalled in the last section where the following theorem is discussed in more detail.)

**Theorem 1.4.** Let $X$ denote a projective toric variety for the action of a torus $T$ over an algebraically closed field of characteristic $p \geq 0$. Let $l$ denote a fixed prime different from $p$. Let $p : [X/T]_{\text{iso.et}} \to [X/T]_{\text{iso.et}}$ denote the map of sites associated to the quotient stack $[X/T]$, and let $\mathcal{A} = Rp_*\mathcal{Q}_l$ denote the sheaf of dgas on $[X/T]_{\text{iso.et}}$. Then the following hold:

(i) The points of the site $[X/T]_{\text{iso.et}}$ correspond to $T$-orbits and the stalk of $\mathcal{A}$ at such a point $\bar{p} = Tp$, $p \in X$, is given by $\mathcal{A}_{\bar{p}} = H^*(BT_p, \mathcal{Q}_l)$ where $T_p$ denotes the stabilizer at $p$. 
(ii) \( A \) is formal in the sense that \( A \simeq \mathcal{H}^*(A) \) and

(iii) one obtains an equivalence of derived categories (bounded below complexes): \( D^b_+ (X, \mathbb{Q}) \simeq D_+ ([X/T]_{iso}, \mathcal{A}) \).

The \( t \)-structure on the right-hand-side is defined as in the last theorem and corresponds to the usual \( t \)-structure on the left-hand-side.

We conclude this introduction by recalling the notion of pre-aisles and aisles (see [KV1], [KV2] and [TLSS]).

1.0.1. Let \( \mathcal{T} \) be a triangulated category whose translation functor is denoted by \((-)[1]\) and its iterates by \((-)[n]\), with \( n \in \mathbb{Z} \). A \( t \)-structure on \( \mathcal{T} \) in the sense of ([BBD, Définition 1.3.1]) is a pair of full subcategories \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) such that, denoting \( \mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[n] \) and \( \mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[n] \), the following conditions hold:

1. For \( \mathcal{X} \in \mathcal{T}^{\leq 0} \) and \( \mathcal{Y} \in \mathcal{T}^{\geq 1} \), \( \operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0 \).
2. \( \mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1} \) and \( \mathcal{T}^{\geq 0} \supset \mathcal{T}^{\geq 1} \).
3. For each \( \mathcal{X} \in \mathcal{T} \) there is a distinguished triangle \( \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to \mathcal{X} \) if \( \mathcal{X} \) and \( \mathcal{Z} \) belong to one of these categories, so does \( \mathcal{Y} \).

The subcategories \( \mathcal{T}^{\leq n} \) and \( \mathcal{T}^{\geq n} \), in general, are not triangulated subcategories but they come close. In fact, each subcategory \( \mathcal{T}^{\leq n} \) has the structure of a suspended category in the sense of Keller and Vossieck [7]. Let us recall this definition.

An additive category \( \mathcal{U} \) is suspended if and only if it is graded by an additive translation functor \( T \) (sometimes called shifting) and there is class of diagrams of the form \( \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to \mathcal{X} \) (often denoted simply \( \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \)) called distinguished triangles such that the following axioms, analogous to those for triangulated categories in Verdier's [V, p. 266] hold:

1. (SP1) Every triangle isomorphic to a distinguished one is distinguished. For \( \mathcal{X} \in \mathcal{U} \), \( 0 \to \mathcal{X} \xrightarrow{id} \mathcal{X} \to 0 \) is a distinguished triangle. Every morphism \( u : \mathcal{X} \to \mathcal{Y} \) can be completed to a distinguished triangle \( \mathcal{X} \xrightarrow{u} \mathcal{Y} \to \mathcal{Z} \to \mathcal{X} \).
2. (SP2) \( \mathcal{X} \xrightarrow{u} \mathcal{Y} \to \mathcal{Z} \) is a distinguished triangle in \( \mathcal{U} \) then so is \( \mathcal{Y} \to \mathcal{Z} \to \mathcal{T} \mathcal{X} \xrightarrow{Tu} \mathcal{T} \mathcal{Y} \).
3. (SP3) = (TR3) in Verdier's loc. cit.
4. (SP4) = (TR4) in Verdier's loc. cit.

The main difference with triangulated categories is that the translation functor in a suspended category may not have an inverse and therefore some objects cannot be shifted back. The formulation of axioms (SP1) and (SP2) reflect this fact. If \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) is a \( t \)-structure on a triangulated category \( \mathcal{T} \), the aisle \( \mathcal{T}^{\leq 0} \) is a suspended subcategory of \( \mathcal{T} \) whose distinguished triangles are diagrams in \( \mathcal{T}^{\leq 0} \) that are distinguished triangles in \( \mathcal{T} \) (Proposition 1.5). Moreover, the aisle \( \mathcal{T}^{\geq 0} \) determines the \( t \)-structure because the co-aisle \( \mathcal{T}^{\geq 0} \) is recovered as \((\mathcal{T}^{\leq 0})^\perp[1]\). The terminology “aisle” and “co-aisle” comes from [7].

We will call a suspended subcategory \( \mathcal{U} \) of a triangulated category \( \mathcal{T} \) where the triangulation in \( \mathcal{U} \) is given by the triangles which are distinguished in \( \mathcal{T} \) and the shift functor is induced by the one in \( \mathcal{T} \), a \( pre-aisle \). We see easily that to check that a full subcategory \( \mathcal{U} \) of \( \mathcal{T} \) is a pre-aisle, it is enough to verify that

- For any \( \mathcal{X} \in \mathcal{U} \), \( \mathcal{X}[1] \) is also in \( \mathcal{U} \).
- Given a distinguished triangle \( \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \), if \( \mathcal{X} \) and \( \mathcal{Z} \) belong to \( \mathcal{U} \), then so does \( \mathcal{Y} \).

Once these two facts hold for \( \mathcal{U} \), the verification of axioms (SP1) through (SP4) is immediate.

The following are the key techniques we use to construct \( t \)-structures in this paper.
Theorem 1.6. ([KV2] Section 1) A suspended subcategory $U$ of a triangulated category $T$ is an aisle (i.e. $(U, U^{+}[1])$ is a t-structure on $T$) if and only if the canonical inclusion functor $U 	o T$ has a right adjoint.

Definition 1.7. Let $T$ be a triangulated category. An object $E$ of $T$ is called compact if the functor $\text{Hom}_T(E, -)$ commutes with arbitrary (small) co-products. Another way of expressing the condition is that a map from $E$ to a co-product factors through a finite subcoproduct.

Theorem 1.8. ([LLS]) Let $S = \{E_\alpha / \alpha \in A\}$ a set of compact objects in a triangulated category $T$. Let $\mathcal{U}$ the smallest co-complete (i.e. closed under all small sums) pre-aisle of $T$ which contains the family $S$. Then $\mathcal{U}$ is an aisle in $T$.

2. The Basic Contexts

In this section we discuss three of the different contexts we consider in this paper. A fourth one, namely that of crystalline derived categories is discussed separately in section 5. We begin by discussing a frame-work that is common to two of the basic contexts.

2.1. The common frame-work. Let $S$ denote a site with the following properties: (i) it is essentially small (ii) has enough points. We will denote the points of the site $S$ by $\tilde{S}$. In addition we will assume that every object $U$ in the site $S$ is quasi-compact and that the site $S$ is locally coherent in the sense of [SGA]4 Exposé VI (2.3). (Recall the latter notion is defined as follows: an object $U$ in $S$ is quasi-separated if for any two maps $V \to U$ and $W \to U$, the fibered product $V \times_W U$ is quasi-compact. An object $U$ is coherent if it is both quasi-compact and quasi-separated. A site with a terminal object $X$ is coherent if every object quasi-separated in $S$ is quasi-separated over the terminal object $X$ and the terminal object $X$ is coherent. Given the site $S$ and an object $X$ in $S$, the site $S/X$ will denote the site whose objects are morphisms $u: U \to X$ and morphisms are morphisms in $S$ over $X$. We say a site $S$ is locally coherent if it has a covering $\{U_i\}$ so that each of the sites $S/U_i$ is coherent.

The main observation we make now is the following:

\begin{equation}
\text{colim}_\alpha H^n(U, F_\alpha) \cong H^n(U, \text{colim}_\alpha F_\alpha)
\end{equation}

for each $U$ in the site $S$ and for each filtered direct system $\{F_\alpha\}$ of abelian sheaves on $S$ and for each $n$.

2.1.2. We will let $\mathcal{R}$ denote either one of the following: (i) a sheaf of commutative Noetherian rings (or graded commutative Noetherian rings) with unit on the site $S$ or (ii) the constant sphere spectrum $\Sigma^0$. We will let $C(S, \mathcal{R})$ denote the category of all unbounded complexes of $\mathcal{R}$-modules (with differentials of degree +1) in the first case and the category of all sheaves of spectra on the site $S$ in the second case. (In case $\mathcal{R}$ is graded, we will assume that $\mathcal{R} = \bigoplus_i R_i$ and that $C(S, \mathcal{R})$ will denote the category of complexes of sheaves of graded modules over $\mathcal{R}$: a sheaf of graded modules $M = \bigoplus_{i \in \mathbb{Z}} M_i$. For a sheaf of graded $\mathcal{R}$-modules $M$, $M(t)$ will denote the object with a shift of grading given by: $M(t)_i = M_{i+t}$.)

We will let $\mathcal{R} \otimes \Delta[1]$ denote the following object in $C(S, \mathcal{R})$: if $\mathcal{R}$ is a sheaf of rings, then this is the obvious chain complex associated to the simplicial object defined by $n \mapsto \bigoplus_{\alpha \in \Delta[n]} \mathcal{R}$ (and with the obvious structure maps). If $\mathcal{R}$ denotes $\Sigma^0$, then this is the suspension spectrum associated to the pointed simplicial set $\Delta[1]_+$. Observe that we have canonical morphisms $d_i: \mathcal{R} \cong \mathcal{R} \otimes \Delta[0] \to \mathcal{R} \otimes \Delta[1]$, $i = 0$, $i = 1$. If $F$ is a complex of abelian sheaves on $S$, $H^n(F)$ will denote the cohomology sheaf in degree $n$ of the complex $F$; in case $F$ is a sheaf of spectra, this will denote $\pi_{-n}(F) = \text{the sheaf of } \sim n\text{-th homotopy groups of } F$.

Proposition 2.1. Assume the following hypothesis:

for each $U$ in the site $S$, there exists an integer $N > 0$ so that $H^n(U, F) = 0$ for all $n > N$ and all abelian sheaves $F$

Let $\{F_\alpha\}$ denotes a filtered direct system of complexes in $C(S, \mathcal{R})$ where $C(S, \mathcal{R})$ denotes the category as above.

Then one obtains the quasi-isomorphism:

\begin{equation}
\text{colim}_\alpha \mathbb{H}^n(U, F_\alpha) \cong \mathbb{H}^n(U, \text{colim}_\alpha F_\alpha)
\end{equation}

for each $n$ and each $U$ in the site $S$. 
Proof. This follows by comparing the spectral sequences

\[ E_2^{s,t} = \colim_{\alpha} H^s(U, H^t(F_{\alpha})) \Rightarrow \colim_{\alpha} \mathbb{H}^{s+t}(U, F_{\alpha}) \] and

\[ E_2^{s,t} = H^s(U, H^t(\colim F_{\alpha})) = \mathbb{H}^{s+t}(U, \colim F_{\alpha}) \]

Since both spectral sequences converge strongly under the above hypotheses, and one obtains an isomorphism at the $E_2$-terms by (2.1.3), the required isomorphism of the abutments follows. \qed

**Definitions 2.2.** (i) Let $\mathcal{R}$ denote a sheaf of commutative Noetherian rings with 1 on the site $\mathcal{S}$. A sheaf of dgas will mean a sheaf $\mathcal{A}$ of differential graded algebras on the site $\mathcal{S}$, with values in $C(\mathcal{S}, \mathcal{R})$. A sheaf of $E_\infty$ dgas will similarly mean an unbounded complex $\mathcal{A}$ in $C(\mathcal{S}, \mathcal{R})$ which is a sheaf of algebras over an $E_\infty$-operad.

(ii) In addition to these situations, we will also consider cases where $\mathcal{A}$ is a sheaf of $E_\infty$-ring spectra on the site $\mathcal{S}$. (An $E_\infty$-ring spectrum will mean an object in the category of spectra that is also an algebra over an $E_\infty$-operad.) Denoting the constant sheaf of sphere spectra by $\mathcal{R}$, such sheaves of $E-\infty$-ring spectra may be viewed as $E_\infty$-ring objects of $C(\mathcal{S}, \mathcal{R})$.)

2.1.4. **Basic conventions.** Henceforth $\mathcal{A}$ will denote a sheaf of $E_\infty$-algebras or a sheaf of $E_\infty$-ring spectra. $\text{Mod}(\mathcal{S}, \mathcal{A})$ will denote the category of sheaves of $E_\infty$-modules over $\mathcal{A}$. The obvious pairing $\text{Mod}(\mathcal{S}, \mathcal{A}) \times \text{Mod}(\mathcal{S}, \mathcal{A}) \rightarrow C(\mathcal{S}, \mathcal{R})$ will be denoted $\otimes$. (Here $\mathcal{R}$ will denote a sheaf of commutative Noetherian rings with 1 in case $\mathcal{A}$ is a sheaf of $E_\infty$-algebras and will denote the constant sheaf of sphere spectra $\Sigma^0$ in case $\mathcal{A}$ is a sheaf of $E_\infty$-ring spectra.) We will refer to $E_\infty$-ring spectra (sheaves of $E_\infty$-ring spectra) as $E_\infty$-dgas (sheaves of $E_\infty$-dgas, respectively).

Next we consider the homotopy category associated to $\text{Mod}(\mathcal{S}, \mathcal{A})$: this will have the same objects as $\text{Mod}(\mathcal{S}, \mathcal{A})$ but morphisms will be homotopy classes of morphisms where a homotopy $H$ between two morphisms $f, g : K \rightarrow L$ is a morphism $K \otimes \Delta[1] \rightarrow L$ so that $f = H \circ d_0$ and $g = H \circ d_1$. We define a morphism $f : K \rightarrow L$ to be a quasi-isomorphism if $f : K \rightarrow L$ induces an isomorphism on the cohomology sheaves. A diagram $K' \rightarrow K \rightarrow K''$ is a distinguished triangle if there is a map from the mapping cone, $\text{ Cone}(f)$ to $K''$ that is a quasi-isomorphism. (Observe that $\text{ Cone}(f) \in \text{ Mod}(\mathcal{S}, \mathcal{A})$. The derived category $\text{ DMod}(\mathcal{S}, \mathcal{A})$ is the category obtained by inverting these quasi-isomorphisms.

**Proposition 2.3.** $\text{ DMod}(\mathcal{S}, \mathcal{A})$ is a triangulated category with the above structure.

Proof. is skipped and left as an exercise. \qed

For each $U$ in the site $\mathcal{S}$, let $j_U^! : C(\mathcal{S}, \mathcal{R}) \rightarrow C(\mathcal{S}/U, \mathcal{R})$ denote the obvious restriction functor; let $j_U$ (or $j_U^*$) denote the left-adjoint (right-adjoint) to $j_U^!$.

**Proposition 2.4.** Assume the hypotheses as in Proposition 2.1 with $C(\mathcal{S}, \mathcal{R})$ denoting any one of the categories considered there.

Let $\mathcal{A}$ denotes a sheaf of $E_\infty$-dgas in $C(\mathcal{S}, \mathcal{R})$. Then, for each $U$ in the site $\mathcal{S}$, $j_U^! j_U^! (\mathcal{A})[n]$ is a compact object in $\text{ DMod}(\mathcal{S}, \mathcal{A})$.

Proof. Let $M \in \text{ DMod}(\mathcal{S}, \mathcal{A})$. Let $\text{ RHom}_\mathcal{A}$ (or $\text{ RHom}_\mathcal{A}$) denote the external (internal) Hom in the derived category $\text{ DMod}(\mathcal{S}, \mathcal{A})$. Then

\[ \text{ RHom}_\mathcal{A}(j_U^! j_U^! (\mathcal{A})[n], M) = \text{ RGamma}(\mathcal{S}/U, \text{ RHom}_\mathcal{A}(j_U^! j_U^! (\mathcal{A})[n], M) = \text{ RGamma}(\mathcal{S}/U, \text{ RHom}_\mathcal{A}(j_U^! (\mathcal{A}), M[-n])) \]

\[ = \text{ RGamma}(\mathcal{S}/U, \text{ RHom}_\mathcal{A}(\mathcal{R}/U, M[-n])) = \text{ RGamma}(\mathcal{S}/U, M[-n]). \]

Now let $\{ M_\alpha \}_\alpha$ denote a direct system of objects in $\text{ DMod}(\mathcal{S}, \mathcal{A})$. In view of the identifications in the last paragraph, one observes, using Proposition 2.1 that $\colim_{\alpha} \text{ RHom}(j_U^! j_U^! (\mathcal{A})[n], M_\alpha) \cong \text{ RHom}(j_U^! j_U^! (\mathcal{A})[n], \colim_{\alpha} M_\alpha)$. This proves the proposition. \qed

**Examples 2.5.** One may consider the following as typical examples where the last proposition applies:

(1) $R = \text{ the constant sheaf } \mathbb{Q}$ and $\mathcal{A}$ is any sheaf of dgas in $C(\mathcal{S}, \mathbb{Q})$ with $\mathcal{S}$ the big Zariski, étale, Nisnevich or qfh sites associated to schemes of finite types over a Noetherian base scheme of finite Krull dimension.

(2) $R = \text{ the constant sheaf } \mathbb{Q}, \mathbb{Z}$ or $\mathbb{Z}/p$ for some prime $p$ and $\mathcal{A}$ is any sheaf of dgas in $C(\mathcal{S}, \mathcal{R})$ with $\mathcal{S}$ the big Zariski Nisnevich or qfh sites associated to schemes of finite type over a Noetherian base scheme of finite Krull dimension.
2.2. Derived categories of modules over sites provided with sheaves of dgas and $E_\infty$-dgas. We will make the following basic hypotheses throughout this sub-section:

$\mathcal{A}$ is a sheaf of $E_\infty$-dgas or dgas on the ringed site $(\mathcal{S}, \mathcal{R})$. We will assume that $\mathcal{A}$ is provided with an augmentation $\mathcal{A} \to \mathcal{R}$ which is assumed to be a map of sheaves of $E_\infty$-dgas and that $\mathcal{A}$ is connected, i.e. $\mathcal{A}^i = 0$ for $i < 0$ and $\mathcal{A}^0 = \mathcal{R}$.

A particularly interesting case is when $\mathcal{A}$ denotes the rational motivic dga (constructed in [J-1]): in this case the above condition is equivalent to the Beilinson-Soulé vanishing conjecture. More generally one may assume $\mathcal{A}$ is a sheaf of bi-graded dgas, $\mathcal{A} = \oplus_{r,s} \mathcal{A}(s)^r$ with $r$ denoting the degree of the chain complex and $s$ another index we call the weight, so that $\mathcal{A}^s = \oplus \mathcal{A}^s(s)$, i.e. each $\mathcal{A}^s(s)$ is a sub-complex of $\mathcal{A}^s$. Then, if $\mathcal{A}^s(0) = \mathcal{R}(0)$ one clearly obtains an augmentation in the above sense. Observe that these conditions are met by the (integral) motivic $E_\infty$-dga. The connectedness is then equivalent to an integral form of the Beilinson-Soulé vanishing conjecture.

We will let $D_+(\text{Mod}(\mathcal{S}, \mathcal{A}))$ ($D_-(\text{Mod}(\mathcal{S}, \mathcal{A}))$) denote the full sub-category of $D\text{Mod}(\mathcal{S}, \mathcal{A})$ consisting of complexes that are bounded below (above, respectively).

This framework will provide one class of examples we consider.

2.3. Presheaves of $E_\infty$-module spectra and CW-cell modules. This framework will provide the second class of examples we consider. Throughout the following discussion we will assume the basic situation of 2.2 where $\mathcal{R} = \oplus_{i \in \mathbb{Z}} \mathcal{R}(i)$ is a sheaf of graded commutative rings with $1$ or where $\mathcal{R}$ denotes the constant sheaf of sphere spectra.

In the first case, $\mathcal{R}[s](t)$ will denote the complex concentrated in degree $s$ and given by the sheaf $\mathcal{R}(t)$ there; in the second case when $\mathcal{R}$ denotes the constant sheaf of sphere spectra $\mathcal{R}^0$, it will denote the $s$-fold suspension $\Sigma^s$. Assume that we are in the first situation and that $\mathcal{A} \in \mathcal{C}(\mathcal{S}, \mathcal{R})$ is an $E_\infty$-ring object. Then $\mathcal{A}[s](t) = \mathcal{A} \otimes \mathcal{R}[s](t)$.

Definitions 2.6. (i) We will assume henceforth, but only in this section, that the sheaf of $E_\infty$-dgas $\mathcal{A}$ is $-1$-connected, i.e. $\mathcal{H}^i(\mathcal{A}) = 0$ for all $s \geq 1$. (This terminology is derived from the case where $\mathcal{A}^i = B_{-1}$ for a chain-complex $B$ (i.e. one whose differentials are of degree $-1$).) In this case, the theory developed below is entirely similar to the homotopy theory of CW-complexes. We say a sheaf of $\mathcal{A}$-modules $M$ is $-n$-connected if $\mathcal{H}^i(M) = 0$ for all $i \geq n$. Since $\text{Hom}_\mathcal{A}(\mathcal{A}, M) \cong \text{Hom}_\mathcal{R}(\mathcal{R}, M) \cong M$, this is equivalent to $\mathcal{H}^i(\text{Hom}_\mathcal{A}(\mathcal{A}, M)) \cong \mathcal{H}^i(\text{Hom}_\mathcal{R}(\mathcal{R}, M)) = 0$ for all $i \geq n$. We say $M$ is connected if it is $-n$-connected for some $n > 0$.

(ii) A map $f : M' \to M$ in $D\text{Mod}(\mathcal{S}, \mathcal{A})$ is a $k$-equivalence if the induced map $\mathcal{H}^i(f) : \mathcal{H}^i(M') \to \mathcal{H}^i(M)$ is an isomorphism for all $i > k$ and an epimorphism for $i = k$.

Definition 2.7. (i) A free $\mathcal{R}$-module is an object $\mathcal{M} \in \mathcal{C}(\mathcal{S}, \mathcal{R})$ so that $M$ is given by a sum $\oplus_{s_U, t_U \in \mathbb{Z}} j_{U!} j_U^* (\mathcal{R})(s_U)(t_U)$, where $U$ ranges over the objects of the site $\mathcal{S}$. A free $\mathcal{A}$-module is an object $\mathcal{M} \in D\text{Mod}(\mathcal{S}, \mathcal{A})$ so that $M$ is given by a sum $\oplus_{s_U, t_U \in \mathbb{Z}} j_{U!} j_U^* (\mathcal{R})(s_U)(t_U)$, where $U$ ranges over the objects of the site $\mathcal{S}$. We call $s_U$ ($t_U$) the dimension (weight) of the free module $j_{U!} j_U^* (\mathcal{A})(s_U)(t_U)$.

(ii) An $\mathcal{R}$-module $M$ is a cone $\mathcal{R}$-module if $M = \text{Cone(id : } \oplus_{s_U, t_U \in \mathbb{Z}} j_{U!} j_U^* (\mathcal{R})(s_U)(t_U) \to \oplus_{s_U, t_U \in \mathbb{Z}} j_{U!} j_U^* (\mathcal{R})(s_U)(t_U) )$ for some free $\mathcal{R}$-module $\oplus_{s_U, t_U \in \mathbb{Z}} j_{U!} j_U^* (\mathcal{R})(s_U)(t_U)$. A cone $\mathcal{A}$-module is defined similarly. A cell-module $\mathcal{M} \in \mathcal{C}(\mathcal{S}, \mathcal{R})$ is an object $\mathcal{M} \in \mathcal{C}(\mathcal{S}, \mathcal{R})$ provided with a decreasing filtration $\{F_i M | i \leq 0\}$ by sub-objects in $\mathcal{C}(\mathcal{S}, \mathcal{R})$ so that $F_0(M)$ is a free $\mathcal{R}$-module for some fixed integer $N$ and each successive quotient $F_i M / F_{i+1}M$ is also a free $\mathcal{R}$-module, for all $i \leq 0$. Moreover $F_i M$ is the mapping cone of a map $f_i : F_i \to F_{i+1}M$ of a map in $\mathcal{C}(\mathcal{S}, \mathcal{R})$ with $F_i$ a free $\mathcal{R}$-module. (Observe that this mapping cone may be realized as a quotient of $F_{i+1}M \oplus \text{Cone}(F_i)$.) In this case we say that $F_i M$ is obtained from $F_{i+1} M$ by attaching free $\mathcal{R}$-cell modules for each summand in $F_i$. One defines cell $\mathcal{A}$-modules $\mathcal{M} \in D(\text{Mod}(\mathcal{S}, \mathcal{A}))$ similarly.

(iii) A $CW$ $\mathcal{R}$-module is a cell $\mathcal{R}$-module $\mathcal{M} \in \mathcal{C}(\mathcal{S}, \mathcal{R})$ so that the dimension of each of the summands $j_{U!} j_U^* (\mathcal{R})(s_U)(t_U)$ in $F_i M / F_{i+1} M$ are strictly greater than the dimension of each of the summands $j_{U!} j_U^* (\mathcal{R})(s_U')(t_U')$ in $F_{i-1} M / F_i M$. One defines $CW$ $\mathcal{A}$-modules similarly.
2.4. Filtered derived categories. $\mathcal{C}(\mathcal{S}, \mathcal{R})$ will denote the category considered in 2.1.2 and $\mathcal{A}$ will denote an $E_\infty$-dgas in $\mathcal{C}(\mathcal{S}, \mathcal{R})$. Next we will consider objects $\mathcal{M} \in \text{Mod}(\mathcal{S}, \mathcal{A})$ provided with a decreasing filtration $F$ (i.e. $F_i(M) \subseteq F_{i-1}(M)$) indexed by the non-negative integers. Morphisms between two such objects will be morphisms in $\text{Mod}(\mathcal{S}, \mathcal{A})$ that preserve the filtrations. $F\text{Mod}(\mathcal{S}, \mathcal{A})$ will denote this category of filtered objects and filtration-preserving maps. We will invert maps $f : M' \to M$ in $F\text{Mod}(\mathcal{S}, \mathcal{A})$ that induce quasi-isomorphisms on each $F_i$ to define the corresponding derived category: this will be denoted $DF\text{Mod}(\mathcal{S}, \mathcal{A})$.

Observe that when $\mathcal{A} = \mathcal{R}$, we obtain the usual filtered derived category of complexes of $\mathcal{R}$-modules provided with decreasing filtrations.

This will be another of the basic situations we consider.

3. Construction of (standard) $t$-structures on sites provided with $E_\infty$ sheaves of dgas

In this section we will provide each of the basic situations considered in the last section with standard $t$-structures.

Proposition 3.1. Assume the hypotheses as in Proposition (2.4) and that $\mathcal{A} \in \mathcal{C}(\mathcal{S}, \mathcal{R})$ is a sheaf of $E_\infty$ dgas. Let $\mathcal{C} = \{J_{U}(\mathcal{A}[U])[n]|U \in \mathcal{S}, n \geq 0\}$.

(i) Then $\mathcal{C}$ is a set of compact objects in the triangulated category $D\text{Mod}(\mathcal{S}, \mathcal{A})$.

Let $\text{Coh}(\mathcal{C}) \leq 0$ denote the smallest sub-category of $\text{Mod}(\mathcal{S}, \mathcal{A})$ containing all of $\mathcal{C}$ and closed under the following operations: (i) finite sums, (ii) mapping cones, (iii) translations [1] and (iv) extensions (i.e. if $F' \to F \to \text{Cone}(f) \to F'[1]$ is a distinguished triangle with $F'$, $\text{Cone}(f)$ and $F'[1] \in \text{Coh}(\mathcal{C}) \leq 0$, then $\text{F Coh}(\mathcal{C}) \leq 0$.

(ii) Let $\text{Q Coh}(\mathcal{C}) \leq 0$ denote the full sub-category of $\text{Mod}(\mathcal{S}, \mathcal{A})$ consisting of filtered colimits $\text{colim}_t F_t$, each $F_t \in \text{Coh}(\mathcal{C}) \leq 0$ and with the indexing set for the filtered colimit being small. Then the smallest co-complete pre-aisle containing all of $\mathcal{C}$ identifies with $\text{Q Coh}(\mathcal{C}) \leq 0$.

(iii) Let $\text{Comp}(\mathcal{C})$ denote the full sub-category of $\text{Q Coh}(\mathcal{C}) \leq 0$ containing $\text{Coh}(\mathcal{C}) \leq 0$ and closed under summands. Now $\text{Comp}(\mathcal{C})$ identifies with the full sub-category of compact objects in $\text{Q Coh}(\mathcal{C}) \leq 0$.

Proof. The first assertion follows from Proposition (2.4). Clearly $\text{Q Coh}(\mathcal{C}) \leq 0$ is closed under all small sums: any such sum may be written as a compact filtered sum of finite sums. Next observe that the sub-category $\text{Coh}(\mathcal{C}) \leq 0$ consists of compact objects. Now it suffices to show that $\text{Q Coh}(\mathcal{C}) \leq 0$ is closed under mapping cones, translations [1] and extensions. Let $F' = \text{colim}_i \{F_i|\epsilon I\} \to F = \text{colim}_j \{F_j|\epsilon I\}$ denote a map of objects in $\text{Q Coh}(\mathcal{C}) \leq 0$ with each $F_i$, $F_j \in \text{Coh}(\mathcal{C}) \leq 0$. Since each $F_i$ is a compact object, one observes that for each $\epsilon I$, there exists an index $j_\epsilon \epsilon J$ so that the map $F'_i \to F_i \to \text{colim}_j \{F_j|\epsilon I\}$ factors through $F_j$. Therefore, after re-indexing $F = \{F_j|\epsilon I\}$ one may assume that both $F'$ and $F$ are indexed by the same indexing set $I$ and the map $f$ is given by a map $\{f_i : F'_i \to F_i|\epsilon I\}$. Clearly the mapping cone $\text{Cone}(f_i) \epsilon \text{Coh}(\mathcal{C}) \leq 0$ and therefore $\text{Cone}(f) \cong \text{lim}_i \text{Cone}(f_i) \epsilon \text{Q Coh}(\mathcal{C}) \leq 0$. Similarly one may show that $\text{Q Coh}(\mathcal{C}) \leq 0$ is closed under the translations [1]. Next consider an extension: $F' \cap F' \cap F' \cap F' \cap F' [1]$ with $F'$, $F'' \epsilon \text{Q Coh}(\mathcal{C}) \leq 0$. Now $F$ identifies with $\text{Cone}(\epsilon)[1]$. Clearly the argument above shows that one may write the map $h$ as $\text{colim}_i : F''_i \to F'_i[1]$; therefore one has a diagram of extensions $F' \to \text{Cone}(\epsilon)[1] \to G_i \to F'_i[1]$. Therefore $\text{Cone}(\epsilon)[1] \epsilon \text{Coh}(\mathcal{C}) \leq 0$; since $F \cong \text{colim}_i \text{Cone}(\epsilon)[1]$, it follows that $\epsilon \text{Q Coh}(\mathcal{C}) \leq 0$. Therefore $\text{Q Coh}(\mathcal{C}) \leq 0$ is a co-complete pre-aisle containing all of $\mathcal{C}$. This proves (ii).

(iii). Let $F = \text{lim}_i F_i$ denote an object in $\text{Q Coh}(\mathcal{C}) \leq 0$ which is a compact object. Then the identity map of $F$ must factor through some finite colimit $F' = \text{lim}_i F_i$, so that $F$ is a split summand of $F'$ which clearly belongs to $\text{Coh}(\mathcal{C}) \leq 0$. This proves the last assertion.

One of the main results we prove in this paper is the following:

Theorem 3.2. Assume the hypotheses as in Proposition (2.4) and that $\mathcal{A} \in \mathcal{C}(\mathcal{S}, \mathcal{R})$ is a sheaf of $E_\infty$ dgas. Let $D\text{Mod}(\mathcal{S}, \mathcal{A}) \leq 0$ denote the pre-aisle in $D\text{Mod}(\mathcal{S}, \mathcal{A})$ generated by $j_{U} \cap j_{U}(\mathcal{A}[n])$, $n \geq 0$, $U$ in the site $\mathcal{S}$.

Then (i) $D\text{Mod}(\mathcal{S}, \mathcal{A}) \leq 0$ is an aisle in $D\text{Mod}(\mathcal{S}, \mathcal{A})$, i.e. defines a $t$-structure on $D\text{Mod}(\mathcal{S}, \mathcal{A})$. 

Definition 3.7. In general, we will say that a sheaf of constructibility.

(ii) Assume next the hypotheses of Definition 3.7 hold. Then (a) \( j_U!(\mathcal{A}_U) \varepsilon \text{DMog}(S, A)^{\leq 0} \cap \text{DMog}(S, A)^{=0} = \text{the heart of the above } t\text{-structure.} \) (b) Moreover, every object \( M \) in \( \text{DMog}(S, A)^{\leq 0} \) satisfies the property that the natural map \( \tau_{\leq 0}(\mathcal{R} \otimes M) \to \mathcal{R} \otimes M \) is a quasi-isomorphism in \( \text{DMog}(S, \mathcal{R}) \). In other words, the functor \( \mathcal{R} \otimes (\quad ) : \text{DMog}(S, A) \to \text{DMog}(S, \mathcal{R}) \) sends \( \text{DMog}(S, A)^{\leq 0} \) to \( \text{DMog}(S, \mathcal{R})^{\leq 0} \).

Proof. The first statement is clear from the last proposition in view of Theorem [LS]. We will now prove the remaining statements. To prove (a), observe that \( R\text{Hom}_A(j_U!(\mathcal{A}_U)[n], j_U!(\mathcal{A}_V)) \simeq R\Gamma(U \times V, \mathcal{A}_U \times V[-n]) \) so that

\[
\text{Hom}_{\text{DMog}(S, A)}(j_U!(\mathcal{A}_U)[n], j_U!(\mathcal{A}_V)) = H^0(R\text{Hom}_A(j_U!(\mathcal{A}_U)[n], j_U!(\mathcal{A}_V))) = H^0(R\Gamma(U \times V, \mathcal{A}_U \times V[-n])) = 0
\]

for any \( n \geq 1 \). The last equality follows from the hypothesis that \( A^i = 0 \) for all \( i < 0 \). This proves \( j_U!(\mathcal{A}_V) \varepsilon (\text{DMog}(S, A)^{\leq 1}) = \text{DMog}(S, A)^{\geq 0} \).

Since \( j_U!(\mathcal{A}_V) \varepsilon \text{DMog}(S, A)^{\leq 0} \) by definition, the assertion (a) in (ii) is proved.

By the definition of \( \text{DMog}(S, A)^{\leq 0} \) above and Proposition 3.1 above, there exist a sequence \( \{M_i | \varepsilon I \} \) in \( \text{Coh}(C)^{\leq 0} \) so that \( M \cong \colim_i M_i \).

Remark 3.3. Next recall from Definitions [2, 3] (iii) that in case \( \mathcal{R} \) denotes the constant sheaf of sphere spectra, and \( A \varepsilon C(S, \mathcal{R}) \) is any \( E_\infty \)-ring object, there is an augmentation \( Z(A) \to \mathbb{Z} \) of sheaves of \( E_\infty \)-algebras. Moreover the functor \( Z : \text{D}(\text{Mod}(S, A)) \to \text{D}(\text{Mod}(S, Z(A))) \) is a functor of triangulated categories and \( Z(A) \varepsilon C(S, \mathcal{Z}) \) is an \( E_\infty \)-ring-object.

In this case, every object \( M \) in \( \text{DMog}(S, A)^{\leq 0} \) satisfies the property that the natural map \( \tau_{\leq 0}(\mathbb{Z} \otimes_Z \mathbb{Z}(M)) \to \mathbb{Z} \otimes Z(M) \) is a quasi-isomorphism in \( \text{DMog}(S, \mathbb{Z}) \). In other words, the functor \( \mathbb{Z} \otimes (\quad ) : \text{DMog}(S, A) \to \text{DMog}(S, \mathbb{Z}) \) sends \( \text{DMog}(S, A)^{\leq 0} \) to \( \text{DMog}(S, \mathbb{Z})^{\leq 0} \).

Examples 3.4. Observe that the theorem applies to all the examples considered in 2.5.

Remark 3.5. It does not seem possible to say, in general, that \( \text{DMog}(S, A)^{\leq 0} \) is the full sub-category of \( \text{DMog}(S, A) \) that the functor \( \mathbb{Z} \otimes (\quad ) \) sends to \( \text{DMog}(S, \mathcal{R})^{\leq 0} \). Nevertheless, the above theorem shows that one has meaningful, non-trivial \( t\)-structures defined on the category \( \text{DMog}(S, A) \).

In the next section we proceed to define and study the notion of constructibility in the category \( \text{D}(\text{Mod}(S, A)) \), in particular how the \( t\)-structures defined above relate to the full sub-category of constructible objects.

Definition 3.6. Assume the above situation. We will define \( \sigma_{\leq n} : \text{D}(\text{Mod}(S, A)) \to \text{D}(\text{Mod}(S, A))^{\leq n} \) as right-adjoint to the obvious inclusion \( \text{D}(\text{Mod}(S, A))^{\leq n} \subseteq \text{D}(\text{Mod}(S, A)) \). Let \( K \varepsilon \text{D}(\text{Mod}(S, A)) \). We will define \( \sigma_{\geq n+1}(K) \) by requiring \( \sigma_{\leq n}(K) \to K \to \sigma_{\geq n+1}(K) \to \sigma_{\leq n}K[1] \) to be a distinguished triangle. Then \( \sigma_{\geq n+1} \) will be right-adjoint to the obvious inclusion \( \text{D}(\text{Mod}(S, A))^{\geq n+1} \to \text{D}(\text{Mod}(S, A)) \).

3.1. Constructibility and \( t\)-structures.

Definition 3.7. In general, we will say that a sheaf of \( \mathcal{A} \)-modules \( M \) is of finite type if is a compact object in \( \text{DMog}(S, A) \). In case \( \mathcal{R} \) is the constant sheaf of rings associated to a commutative ring \( R \), we will say that a sheaf
of $A$-modules $M$ is a constructible $A$-module, if it is of finite type. The full sub-category of compact objects in $DMod(S,A)$ will be denoted $D_c(Mod(S,A))$.

**Proposition 3.8.** The truncation functors $\sigma_{\leq n}$ and $\sigma_{\geq n+1}$ as in Definition 3.6 preserve compactness and hence the property of being of finite type as well as being constructible.

**Proof.** In view of the fact that $\sigma_{\leq n}(K) \rightarrow K \rightarrow \sigma_{\geq n+1}(K) \rightarrow \sigma_{\leq n}(K)$ is a distinguished triangle for any $K \in DMod(S,A)$, it suffices to prove this for the functor $\sigma_{\geq 0}$.

First observe that the filtered colimit of a collection $\{K_i|i \in I\}$ of objects $K_i \in DMod(S,A)_{\geq 0}$ also belongs to $DMod(S,A)_{\geq 0}$. To see this, consider

$$\text{Hom}_{DMod(S,A)}(j_!(A[U])[n], \lim_i K_i) = H^0(\text{RHom}_A(j_!(A[U])[n], \lim_i K_i))$$

$$\cong H^0(\text{R}(U, \lim j^*_i(K_i)[-n])) \cong \lim_i H^0(\text{R}(U, j^*_i(K_i)[-n])) = \lim_i H^0(\text{RHom}_A(j_!(A[U])[n], K_i)) = 0$$

for all $n \geq 1$. The last equality is from the hypothesis that each $K_i \in DMod(S,A)_{\geq 0}$ and the isomorphism prior to that follows from our hypotheses on the site as in 2.1.

Let $\phi : DMod(S,A)_{\geq 0} \rightarrow DMod(S,A)$ denote the obvious inclusion functor. Clearly $\phi$ commutes with filtered colimits since the former category is a full sub-category of the latter and the former category is itself closed under the formation of filtered colimits as we just showed.

Next let $M \in DMod(S,A)$ denote a compact object and let $\{K_i|i \in I\}$ be a collection of objects $K_i \in DMod(S,A)_{\geq 0}$. Then:

$$\text{RHom}_A(\sigma_{\geq 0}(M), \lim_i K_i) \cong \text{RHom}_A(M, \phi(\lim_i K_i))$$

$$\cong \text{RHom}_A(M, \lim_i \phi(K_i)) \cong \lim_i \text{RHom}_A(M, \phi(K_i))$$

$$\cong \lim_i \text{RHom}_A(\sigma_{\geq 0}(M), K_i).$$

These prove that $\sigma_{\geq 0}$ preserves compactness.

**Corollary 3.9.** In case $A$ is provided with an augmentation $A \rightarrow R$, then the functor $M \mapsto M \otimes_A R$, $DMod(S,A) \rightarrow DMod(S,R)$ induces a functor $D_c(Mod(S,A))_{\leq 0} \rightarrow D_c(Mod(S,R))_{\leq 0}$.

**Proof.** In view of Theorem 3.2 (ii)(b), it suffices to show that the above functor preserves compactness for objects in $QCoh(C)_{\leq 0}$ as in Proposition 3.1. As shown there, the compact objects in $QCoh(S,A)_{\leq 0}$ identify with split summands of objects in $Coh(S,A)_{\leq 0}$. Clearly the functor $M \mapsto M \otimes_A R$ sends split summands of objects in $Coh(S,A)_{\leq 0}$ to split summands of objects in $Coh(S,R)_{\leq 0}$. 

Let $C = DMod(S,A)_{\leq 0} \cap DMod(S,A)_{\geq 0}$ denote the heart of the above $t$-structure on $DMod(S,A)$. Observe that this is an abelian category. Let $C_c = C \cap D_c(Mod(S,A))$, i.e. the full sub-category of all objects in the heart that are also compact.

**Theorem 3.10.** $C_c$ is an additive sub-category of $C$ closed under extensions.

**Proof.** This is clear since any short-exact sequence $M' \rightarrow M \rightarrow M''$ in $C$ corresponds to a distinguished triangle in $DMod(S,A)$ with each $M'$, $M$ and $M''$ in $C$. Moreover, in such a short-exact sequence $M' \rightarrow M \rightarrow M''$, $M$ is compact if both $M'$ and $M''$ are.

3.2. Example: Non-standard $t$-structures, generalized perverse sheaves and perverse extensions. In this section we will show briefly how to define generalized perverse sheaves and perverse extensions of generalized perverse sheaves.

Assume one is given a stratified site, $S$, i.e. one is provided with a collection of locally closed smooth subobjects $X_i$ of the terminal object $X$ of $X$. Let $S_i = S \times X_i$. By taking the unions of the strata one defines a finite increasing filtration $S_0 \rightarrow S_1 \rightarrow \cdots \rightarrow S_n = S$, where each $j_i$ is an open immersion. Let a (perversity)
function \( p : \{ S_{i+1} - S_i \} \to \) (integers) be given. Let \( \mathcal{A} \) denote sheaf of \( E_\infty \) dgas on \( S \). Now \( DMod(S, \mathcal{A}) \) and \( DMod(S_i - S_{i-1}, A_{S_i - S_{i-1}}) \) will denote the obvious derived categories of \( \mathcal{A} \)-modules. Assume that every object in each of these categories has finite cohomological dimension for each object in the corresponding site. Now one may glue together standard \( t \)-structures on each stratum, shifted suitably by the perversity \( p(S_i - S_{i-1}) \) to define a non-standard \( t \)-structure on \( DMod(S, \mathcal{A}) \) exactly as in \[ \text{[BBD] Chapter 1}. \] Accordingly we will define

\[
\begin{align*}
DMod(S, \mathcal{A})^{\leq 0} &= \{ K \in DMod(S, \mathcal{A}) | j_i^*(K) \in DMod(S_i - S_{i-1}, A_{S_i - S_{i-1}}) \leq p(S_i - S_{i-1}) \} \\
DMod(S, \mathcal{A})^{\geq 0} &= \{ K \in DMod(S, \mathcal{A}) | j_i^!(K) \in DMod(S_i - S_{i-1}, A_{S_i - S_{i-1}}) \geq p(S_i - S_{i-1}) \} 
\end{align*}
\]

**Theorem 3.11.** (i) The above structures define a \( t \)-structure on \( DMod(S, \mathcal{A}) \) with the aisle \( DMod(S, \mathcal{A})^{\leq 0} \) and co-aisle \( DMod(S, \mathcal{A})^{\geq 0} \).

(ii) Given a generalized perverse sheaf \( P \in DMod(S_0, A_{S_0})^{\leq 0} \land DMod(S_0, A_{S_0})^{\geq 0} \), there exists extensions \( P \in DMod(S, \mathcal{A})^{\leq 0} \land DMod(S, \mathcal{A})^{\geq 0} \) of \( P_0 \), i.e. \( j_i^*(P) \approx P \).

(iii) Given a generalized perverse sheaf \( P \in DMod(S_i, A_{S_i})^{\leq 0} \land DMod(S_i, A_{S_i})^{\geq 0} \), the extension \( P \in DMod(S, \mathcal{A})^{\leq 0} \land DMod(S, \mathcal{A})^{\geq 0} \) of \( P_0 \) is unique if \( j_i^*(P) \in DMod(S_i - S_{i-1}, A_{S_i - S_{i-1}}) \leq p(S_i - S_{i-1}) - 1 \) and \( j_i^!(P) \in DMod(S_i - S_{i-1}, A_{S_i - S_{i-1}}) \leq p(S_i - S_{i-1}) + 1 \).

**Proof.** This is essentially the argument in \[ \text{[BBD] Theorem 1.4.10}. \] we provide some details mainly for the sake of completeness. We will restrict to the case where there are only two strata, i.e. \( S_1 = S \). Let \( K \in DMod(S, \mathcal{A})^{\leq 0} \) and \( L \in DMod(S, \mathcal{A})^{\geq 1} \). To show \( Hom(K, L) = 0 \), one may argue as follows. First one observes the existence of the distinguished triangle:

\[
j_i^!(K) \to K \to i_*i^*(K) \to j_i^*(K)[1]
\]

where \( j : S_0 \to S \) and \( i : S - S_0 \to S \) are the obvious maps. This provides us with the long-exact-sequence:

\[
\cdots \to H^0(\mathcal{R}Hom_A(i_*i^*(K), L)) \to H^0(\mathcal{R}Hom_A(K, L)) \to H^0(\mathcal{R}Hom_A(j_i^!(K), L)) \to H^1(\mathcal{R}Hom_A(i_*i^*(K), L)) \to \cdots
\]

Now \( H^0(\mathcal{R}Hom_A(i_*i^*(K), L)) \cong H^0(\mathcal{R}Hom_A(i^*(K), i^!(L)) \cong 0 \) and \( H^0(\mathcal{R}Hom_A(j_i^!(K), L)) \cong H^0(\mathcal{R}Hom_A(j_i^!(K), j_i^!(L))) \cong 0 \) by our hypotheses. Therefore \( H^0(\mathcal{R}Hom_A(K, L)) = 0 \) as well. It is clear from the definitions that \( DMod(S, \mathcal{A})^{\leq n} \subseteq DMod(S, \mathcal{A})^{\leq n+1} \) and similarly \( DMod(S, \mathcal{A})^{\geq n+1} \subseteq DMod(S, \mathcal{A})^{\geq n} \).

We define the functor \( \sigma_{\leq 0} \) as follows. Let \( K \in DMod(S, \mathcal{A}) \). We let \( Y = \) the canonical homotopy fiber of the map \( K \to Rj_\ast j_i^!(K) \). Then we define \( \sigma_{\leq 0}(K) \) to be the canonical homotopy fiber of the map \( Y \to \sigma_{\geq 1}j_i^!(Y) \). \( \sigma_{\geq 1} \) is defined to be the mapping cone of the obvious map \( \sigma_{\geq 0}(K) \to K \). These prove (i).

Give \( P_0 \) as in (ii), we let \( P = \sigma_{\leq 0}(Rj_\ast(P_0)) \). To see this is an extension of \( P_0 \) proceed as follows. Let \( \phi_i : DMod(S_i, A_{S_i})^{\leq 0} \to DMod(S_i, A_{S_i})^{\geq 0} \) denote the obvious (inclusion). We will denote \( \phi_1 \) by \( \phi \). The definition of the above aisles shows that \( \phi \circ j_i = j_i \circ \phi_0 \). Since \( j_i \) is right-adjoint to \( j_i \), respectively, it follows that \( j_i^! \) commutes with \( \sigma_{\leq 0} \). Since \( j_i \circ Rj_\ast = \) the identity, we observe that \( j_i^!(P) \approx \sigma_{\leq 0}(P_0) \approx P_0 \) where the last identification is by the assumptions on \( P_0 \).

To show \( P \) is in fact a generalized perverse sheaf, one first observes the distinguished triangle: \( i_*Ri^!(\sigma_{\leq 0}(Rj_\ast(P_0))) \to \sigma_{\leq 0}(Rj_\ast(P_0)) \to Rj_\ast(P_0) \to i_*Ri^!(\sigma_{\leq 0}(Rj_\ast(P_0))[1]) \). (The identification \( Rj_\ast(j_i^!(\sigma_{\leq 0}(Rj_\ast(P_0)))) \approx Rj_\ast(P_0) \) follows from the observation above that \( \sigma_{\leq 0}(Rj_\ast(P_0)) \) is an extension of \( P_0 \).) One also has a distinguished triangle \( \sigma_{\leq 0}(Rj_\ast(P_0)) \to Rj_\ast(P_0) \to \sigma_{\leq 1}Rj_\ast(P_0) \to \sigma_{\leq 0}(Rj_\ast(P_0)) \). Therefore, one obtains the quasi-isomorphism \( \sigma_{\geq 1}(Rj_\ast(P_0)) \approx i_*Ri^!(\sigma_{\leq 0}(Rj_\ast(P_0))[1]) \) and hence \( Ri^!(\sigma_{\leq 0}(Rj_\ast(P_0))) \in DMod(S - S_0, A_{S - S_0})^{\geq 0} \). This proves the required assertion and completes the proof of (ii).

(iii) Let \( P \) denote a generalized perverse sheaf extending the perverse sheaf \( P_0 \in DMod(S_0, A_{S_0})^{\leq 0} \land DMod(S_0, A_{S_0})^{\geq 0} \) satisfying the hypotheses in (iii). The key diagram is:
triangles. Now the hypotheses imply that $i^!(P) \in \text{DMod}(S - S_0, A_S - S_0)$ and that $i^!(P) \in \text{DMod}(S - S_0, S - S_0)_{\leq p(S - S_0) + 1}$. Therefore, the following lemma shows that $i_!i^!(P)[1] \simeq i_!(i_g \circ j)^!(P)).$ This implies that $P$ identifies with the canonical homotopy fiber of the map $R_{j_!}(P_0) \to i_*\sigma_{\geq p(S - S_0)}(R_{j_!}(P_0)).$ Therefore it is unique. This completes the proof of the theorem. □

Lemma 3.12. Let $A \to B \to C \to A[1]$ denote a distinguished triangle in $\text{DMod}(S, A)$ and let $n$ be an integer so that the natural map $C \to \sigma_{\geq n}(C)$ is a quasi-isomorphism. Then the natural map $\sigma_{\leq n-1}(A) \to \sigma_{\leq n-1}(B)$ is a quasi-isomorphism.

Proof. Let $K \in \text{DMod}(S, A)_{\leq n}$. Since $C \to \sigma_{\geq n}C$ is a quasi-isomorphism, it follows that $\text{Hom}_{\text{DMod}(S, A)}(K, C) = 0 = \text{Hom}_{\text{DMod}(S, A)}(K, C)[-1]$. Therefore the map $A \to B$ induces an isomorphism

$$\text{Hom}_{\text{DMod}(S, A)}(K, A) \cong \text{Hom}_{\text{DMod}(S, A)}(K, B).$$

Now the definition of the functor $\sigma_{\leq n-1}$ as right-adjoint to the inclusion $\text{DMod}(S, A)_{\leq n-1} \to \text{DMod}(S, A)$ shows that the induced map $\sigma_{\leq n-1}(A) \to \sigma_{\leq n-1}(B)$ is a quasi-isomorphism. □

4. Cell and CW-cell modules

Throughout this section we will assume the basic hypotheses as in [23], i.e. $A$ is $-1$-connected or equivalently $\mathcal{H}^i(A) = 0$ for all $s > 0$. We will assume that $\mathcal{R}$ is a constant sheaf. In this section we develop the basic theory of cell and CW-cell-modules over a sheaf of $\mathbb{E}_N$-dgas.

Now the following basic results show that the theory of $\text{CW} - A$-modules is indeed similar to the homotopy theory of $\text{CW}$-complexes: see, for example, [Gray, Chapter 16].

4.0. Convention. : Henceforth, we will denote $j_U j_U^*(\mathcal{R}) (j_U j_U^*(A))$ by $\mathcal{R}_U (A_U$, respectively).

Proposition 4.1. Throughout let $P$, $Q$ be $A$-modules. (i) Let $g : A_U[n - 1] \to P$, $f : P \to R$ be $A$-maps and assume $\mathcal{H}^i(Q) = 0$ for $i = -n + 1$. Then there exists a covering $\{V_\alpha \to U[\alpha]\}$ of $U$ so that each $f|_{V_\alpha}$ extends to a map $\text{Cone}(g|_{V_\alpha}) \to R|_{V_\alpha}$ where $\text{Cone}(g|_{V_\alpha})$ denotes the mapping cone of $g|_{V_\alpha}$. (In this case we say that $f$ extends locally to a map from $\text{Cone}(g)$ to $R$.)

(ii) Let $S$ denote a set of integers and $(P, Q)$ a relative $\text{CW} - A$-module, i.e. $Q$ is obtained from $P$ by attaching free $A$-modules $A_U[na,U]$ with $na \in S$. Suppose $R$ is an $A$-module so that $\mathcal{H}^{-i}(R) = 0$ for all $i \in S$. Then any map $f : P \to R$ of $A$-modules admits a local extension $\tilde{f} : Q \to R$, i.e. there exists a covering $\{V_\alpha \to S[\alpha]\}$ so that each restriction $f|_{V_\alpha} : P|_{V_\alpha} \to R|_{V_\alpha}$ extends to a map $\tilde{f}|_{V_\alpha} : Q|_{V_\alpha} \to R|_{V_\alpha}$.

(iii) Suppose that there exists a covering $\{V_\alpha \to S\}$ so that $(P|_{V_\alpha}, Q|_{V_\alpha})$ is a relative $\text{CW} - A$-module in the above sense so that $Q|_{V_\alpha}$ is obtained from $P|_{V_\alpha}$ by attaching free $A$-cells in dimensions $\leq -n$. Then $\mathcal{H}^i(Q/P) = 0$ for all $i > -n$.

(iv) If $Q$ is a $\text{CW} - A$-module so that $F_{i+1}(Q) = F_i(Q)$ for all $i \geq -n + 1$, then $\mathcal{H}^i(Q) = 0$ for all $i > -n$.

Proof. For each point $p$ of the site $S$, the only obstruction to extending $f_p$ to $\text{Cone}(g_p)$ is that the composition $f_p \circ g_p$ be null-homotopic: this is clear since $H^0(R\text{Hom}_{A_p}(A_p[n - 1], Q_p)) = H^{-n+1}(Q_p) = 0$ by the hypothesis. This proves (i). To prove (ii) one uses (i) as a starting point to handle the case when $Q_p$ is obtained from $P_p$ by attaching a single $A_p$-cell. In general one uses Zorn’s lemma as in [Gray Corollary 16.3].

(iii) It is enough to assume that $Q$ is obtained by attaching finitely many free $A$-cells to $P$. In this case one uses an ascending induction on the number of these cells and the exact sequence $\mathcal{H}^i(Q'/P) \to \mathcal{H}^i(Q/P) \to \mathcal{H}^i(Q'/Q')$ where $Q'$ is obtained from $P$ by attaching one less free $A$-cell. Observe that key use is made of the hypothesis that $\mathcal{H}^i(A) = 0$ for all $i > 0$: in fact, the last assertion is false if this is not the case. Clearly (iii) implies (iv). □
Theorem 4.2. Let $\mathcal{M} \in \text{DMod}(\mathcal{S}, \mathcal{A})$ so that $\mathcal{M}$ is connected in the above sense. Then there exists a CW-cell $\mathcal{A}$-module $P(M) \in \text{DMod}(\mathcal{S}, \mathcal{A})$ with a map $P(M) \to M$ which is a quasi-isomorphism. (We say $P(M) \to M$ is a CW-$\mathcal{A}$-resolution.) Moreover, if $M' \to M$ is a map between two such objects in $\text{DMod}(\mathcal{S}, \mathcal{A})$, there exist CW-$\mathcal{A}$-resolutions $P(M') \to M'$, $P(M) \to M$ and a map $P(M') \to P(M)$ preserving the given filtrations so that one obtains a commutative square

$$
P(M') \xrightarrow{P(M)} P(M) \\
M' \xrightarrow{M} M$$

Proof. Assume that $\mathcal{H}^i(M) = 0$ and $\mathcal{H}^i(M') = 0$ for all $i > N$. For each class $[\alpha_N] \in \mathcal{H}^N(M)$, let $\alpha_N : \mathcal{A}_U[-N] \to j_U j_U^* (M) \to M$ denote a map representing $[\alpha_N]$. Now let $P_N(M) = \bigoplus_{[\alpha_N] \in \mathcal{H}^N(M)} \mathcal{A}_U[-N]$; we will map this to $M$ by mapping the summand indexed by $[\alpha_N]$ by the corresponding map $\alpha_N$ to $M$. We will denote this map $P_N(M) \to M$ by $p_N(M)$.

Consider the cone $\mathcal{A}$-module $\text{Cone}(P_N(M))$ and also the mapping cone $\text{Cone}(p_N(M))$. Observe that one has the distinguished triangle: $P_N(M) \xrightarrow{p_N(M)} M \to \text{Cone}(p_N(M)) \to P_N(M)[1]$ which results in the long-exact sequence:

$$\cdots \to \mathcal{H}^i(\text{Cone}(P_N(M))) \to \mathcal{H}^i(M) \to \mathcal{H}^i(\text{Cone}(p_N(M))) \to \mathcal{H}^{i+1}(P_N(M)) \to \cdots$$

Since $\mathcal{H}^{i+k}(P_N(M)) = 0$ for all $k > 0$ and $\mathcal{H}^N(P_N(M)) \to \mathcal{H}^N(M)$ is a surjection by our choice of $P_N(M)$, it follows that

(4.0.3) \quad $\mathcal{H}^i(\text{Cone}(p_N(M))) = 0, i \geq N$

denoting $p_N(M) : P_N(M) \to M$ is an $N$-equivalence.

4.0.4. Now replace $M$ by $\text{Cone}(p_N(M))$ and for each class $[\alpha_{-N+1}] \in \mathcal{H}^{N-1}(\text{Cone}(p_N(M)))$, let $\alpha_{-N+1} : \mathcal{A}_{U_{-N+1}}[-N+1] \to \text{Cone}(p_N(M)) = \text{Cyl}(p_N(M))/P_N(M)$ denote a representative.

The map $\alpha_{-N+1}$ may be viewed as a map of pairs $(\text{Cone}(\mathcal{A}_{U_{-N+1}}[-N]), \mathcal{A}_{U_{-N+1}}[-N]) \to (\text{Cyl}(p_N(M)), P_N(M))$. Observe that $\mathcal{A}_{U_{-N+1}}[-N]$ maps naturally to $\text{Cone}(\mathcal{A}_{U_{-N+1}}[-N])$ with the cokernel $\simeq \mathcal{A}_{U_{-N+1}}[-N+1]$. Let

$$\alpha_{-N+1} : \mathcal{E}_{-N+1} \to (\text{Cyl}(p_N(M)), P_N(M))$$

denote the obvious map.

We let $P_{N-1}(M) = (P_N(M) \oplus \oplus_{\alpha_{-N+1}, \text{Cone}(\mathcal{A}_{U_{-N+1}}[-N])}/\sim$: here $\sim$ is the relation where we identify the summand $\mathcal{A}_{U_{-N+1}}[-N]$ of the corresponding $\text{Cone}(\mathcal{A}_{U_{-N+1}}[-N])$ with its image in $P_N(M)$. We map the pair $(P_{N-1}(M), P_N(M))$ to $(\text{Cyl}(p_N(M)), M)$ by mapping the summand $P_N(M)$ by the obvious inclusion into $\text{Cyl}(p_N(M))$ and the summand $\text{Cone}(\mathcal{A}_{U_{-N+1}}[-N])$ by the map $\alpha_{-N+1}$. We will denote this map by $p_{N-1}(M)$. Let $\pi_N : \text{Cyl}(p_N(M)) \to M$ denote the obvious map and let $p_{N-1}(M) = \pi_N \circ p_{N-1}(M)'$. Then $p_{N-1}(M)/p_N(M) = p_N(M)$.

Let $\text{Cyl}(p_{N-1}(M))'$ denote the mapping cylinder of $p_{N-1}(M)'$. Now one has the long-exact-sequence:

$$\cdots \to \mathcal{H}^{i-1}(\text{Cyl}(p_{N-1}(M))'/P_{N-1}(M)) \to \mathcal{H}^i(P_{N-1}(M)/P_N(M))/\mathcal{H}^{i-1}(\text{Cyl}(p_{N-1}(M))'/P_{N-1}(M)) \cong \mathcal{H}^i(\text{Cyl}(p_{N-1}(M))'/P_{N-1}(M)) \to \cdots$$

By construction $\mathcal{H}^{N-1}(p_{N-1}(M))'$ is surjective, $\mathcal{H}^i(\text{Cyl}(p_{N-1}(M))'/P_{N-1}(M))) = \mathcal{H}^i(\text{Cone}(p_N(M))) = 0$ for all $i > N$ and $\mathcal{H}^i(P_{N-1}(M)/P_N(M)) = 0$ for all $i > N-1$. Therefore, it follows that $\mathcal{H}^i(\text{Cyl}(p_{N-1}(M))'/P_{N-1}(M))) = 0$ for all $i > N-1$. i.e. the map $p_{N-1}(M)$ is an $N-1$-equivalence.

We may therefore, continue the inductive construction and define $p_k(M)$ as a CW-cell $\mathcal{A}$-module provided with a map $p_k(M) : p_k(M) \to M$, $k \leq N$ which is a $k$-equivalence. Finally one lets $P(M) = \colim k \to \infty p_k(M)$ along with
the map \( p(M) : P(M) \to M \) defined as \( \lim_{k \to -\infty} p_k(M) \). One verifies immediately that \( p(M) \) is a quasi-isomorphism: clearly \( P(M) \) is a CW-cell \( \mathcal{A} \)-module. This proves the first statement in the theorem.

To make the construction of CW-\( \mathcal{A} \)-resolutions functorial, we will need to make the following modifications to the arguments above. Instead of choosing representative cohomology classes as in (4.0.3), we choose all possible maps \( j_U j_U^! (A)[-N] \to M \), for all \( U \) in the site \( \mathcal{S} \). (Since our site \( \mathcal{S} \) is assumed to be essentially small this causes no difficulties.) We will then need to repeat the same construction in 4.0.4 and at every stage of the inductive process. Next observe that since both \( M' \) and \( M \) are assumed to be connected, we may choose a large enough \( N \) so that \( H^i(M) = H^i(M') = 0 \) for all \( i > N \). Now the construction of the resolution \( P(M) \) is made functorial in \( M \). Moreover the construction of the CW-\( \mathcal{A} \) resolution by descending induction shows that the induced map \( P(f) : P(M') \to P(M) \) preserves the CW-filtration. This proves the second statement in the theorem. \( \square \)

**Definition 4.3.** Let \( D\text{Mod}^{\text{cw}}(\mathcal{S}, \mathcal{A}) \) denote the category whose objects are all CW-\( \mathcal{A} \)-modules and morphisms are morphisms that preserve the given cell-filtrations. A morphism between two CW-\( \mathcal{A} \)-modules will be called a quasi-isomorphism if it is a quasi-isomorphism in the underlying category \( D\text{Mod}(\mathcal{S}, \mathcal{A}) \). The corresponding derived category obtained by inverting these quasi-isomorphisms will be denoted \( D\text{Mod}^{\text{cw}}(\mathcal{S}, \mathcal{A}) \).

**Corollary 4.4.** The obvious functor \( D\text{Mod}^{\text{cw}}(\mathcal{S}, \mathcal{A}) \to D\text{Mod}(\mathcal{S}, \mathcal{A}) \) is an equivalence of categories.

**Proof.** This follows from the last Theorem. \( \square \)

The following is a key result of this section.

**Theorem 4.5.** Let \( M \in D(\text{Mod}(\mathcal{S}, \mathcal{A})) \) denote a constructible \( \mathcal{A} \)-module. Then \( H^i(\sigma_{\leq n}(M)) = 0 \) for all \( i > n \) and \( \cong H^i(M) \) if \( i \leq n \), i.e. the functor \( \sigma_{\leq n} \) in Definition 3.6 identifies with the functor that kills the cohomology in degrees above \( n \).

**Proof.** This is a direct consequence of Proposition 4.1(iv): this shows that one may kill all the cohomology of \( M \) in degrees lower than a fixed integer \( -n \) by attaching Cone-\( \mathcal{A} \)-modules (as in 2.7(ii)) with the summands \( j_U j_U^! (A)[s_U](t_U) \), \( s_U \geq n + 1 \). Therefore the truncation functor \( \sigma_{\geq -n} \) identifies with the functor that kills the cohomology in degrees lower than \( -n \). \( \square \)

**Remark 4.6.** Therefore, it follows \( \mathcal{A}_U \) cannot belong to the heart of this \( t \)-structure unless \( \mathcal{A} \) is concentrated in degree 0.

5. \( t \)-structures for filtered derived categories adapted to filtered modules over sheaves of filtered dgas

In this section we will adapt and extend the discussions in [BBD] Chapter 3 and [Beil] Appendix to define a \( t \)-structure for the filtered derived category of filtered sheaves of dg-modules over a fixed given sheaf of dgas \( \mathcal{A} \): this will be such that each \( \mathcal{A}_U \) (with the obvious trivial filtration) will belong to the heart of the above \( t \)-structure.

We will henceforth assume the basic situation of section 2 and consider the filtered derived category \( D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A}) \) of objects \( M \) in \( (\mathcal{S}, \mathcal{A}) \) provided with descending (i.e. non-increasing) filtrations \( \{ F_M | M \in \mathcal{Z} \} \) with each \( F_M \in \text{Mod}(\mathcal{S}, \mathcal{A}) \) and \( F_i(M) = * \) for \( i > 0 \) (depending on \( M \)). Observe that to any such filtration \( F \), one may associate an ascending (i.e. non-decreasing) filtration \( F' \) given by \( F'_i M = F_{i-M} \). Now \( F \) and \( F' \) clearly determine each other.

**Theorem 5.1.** (See [BBD] Chapter 3 and also [Beil] Appendix.) Let 

\[
D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A})^{\leq 0} = \{ (M, F) \in D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A}) | \text{grd}_F^P(M) \in D\text{Mod}(\mathcal{S}, \mathcal{A})^{\leq 1} \} \text{ and }
\]

\[
D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A})^{\geq 0} = \{ (M, F) \in D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A}) | \text{grd}_F^P(M) \in D\text{Mod}(\mathcal{S}, \mathcal{A})^{\geq 1} \}
\]

where the \( t \)-structure on \( D\text{Mod}(\mathcal{S}, \mathcal{A}) \) is defined as in Theorem 3.2.

Then there exists a unique \( t \)-structure on \( D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A}) \) with \( D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A})^{\leq 0} \) (\( D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A})^{\geq 0} \)) the corresponding aisle (co-aisle, respectively). The truncation functors on \( D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A}) \) corresponding to the above \( t \)-structure will be denoted \( \sigma_{\leq n} \) and \( \sigma_{\geq n} \), respectively.

**Proof.** Let \( (M, F) \in D\text{F}\_\text{Mod}(\mathcal{S}, \mathcal{A}) \) be a fixed object where \( F \) denotes a descending filtration on \( M \). We define \( \sigma_{\leq 0}(M, F) \) using descending induction on \( F_i(M) \). Let \( N >> 0 \) be such that \( F_{N+1}(M) = * \) while \( F_N(M) \neq * \). We let \( \sigma_{\leq N}^F(M, F) \) be defined by \( \sigma_{\leq N}^F(F_N(M)) = \sigma_{\leq N}(F_N(M)) \) where the last functor \( \sigma_{\leq N} \) is the one defined
in Theorem 3.2. Assume we have defined \( \sigma^F_{\leq n}(F_k(M)) \) and \( \sigma^F_{\leq n-1}(F_k(M)[1]) \) for all \( k \geq n \), where \( n \leq N \) is a fixed integer so that \( \sigma^F_{\leq n-1}(F_k(M)[1]) \equiv (\sigma^F_{\leq n}(F_k(M)))[1] \). Let

\[
\sigma^F_{\leq n}(F_n-1(M)/F_n(M)) = \sigma^F_{\leq n}(F_n-1(M)/F_n(M)) [1]
\]

with the functor \( \sigma_{\leq n} \) defined as in Theorem 3.2. Now observe that one has a distinguished triangle: \( F_n(M) \to F_n-1(M)/F_n(M) \to F_n(M)[1] \). Therefore we obtain a natural map \( \sigma^F_{\leq n-1}(F_n-1(M)/F_n(M)) \to \sigma^F_{\leq n-1}(F_n(M)[1]) \equiv (\sigma^F_{\leq n}(F_n(M)))[1] \). We let \( \sigma^F_{\leq n}(F_n(M)) \) be the canonical homotopy fiber of the map \( \sigma^F_{\leq n}(F_n(M)) \to (\sigma^F_{\leq n}(F_n(M)))[1] \). Therefore we obtain a distinguished triangle

\[
\sigma^F_{\leq n}(F_n(M)) \to \sigma^F_{\leq n-1}(F_n(M)) \to \sigma^F_{\leq n-1}(F_n(M)/F_n(M)) \to (\sigma^F_{\leq n}(F_n(M)))[1]
\]

We may now continue with the induction and define \( \sigma_{\leq k}(F_k(M)) \) for all \( k \leq N \). We let \( \sigma_{\leq 0}(M,F) \) be defined by \( F_k(\sigma_{\leq 0}(M,F)) = \sigma^F_{\leq k}(F_k(M)) \): after replacing \( \sigma^F_{\leq n-1}(F_n-1(M)) \) by the mapping cylinder of the above map \( \sigma^F_{\leq n}(F_n(M)) \to \sigma^F_{\leq n-1}(F_n-1(M)) \), one may observe that \( \sigma_{\leq k+1}(F_{k+1}(M)) \) is a sub-object of \( \sigma_{\leq k}(F_k(M)) \). By descending induction, one may also show that there is a natural map \( \sigma_{\leq 0}(M,F) \to (M,F) \) of objects in \( D_F Mod(S,A) \). The distinguished triangle (5.0.5) shows that \( \text{grd}^F_{k-1}(\sigma_{\leq 0}(M,F)) \equiv \sigma^F_{\leq n-1}(F_n-1(M)/F_n(M)) = \sigma_{\leq n}(F_n-1(M)/F_n(M)) - \sigma_{\leq 0}(M,F) \in D_F Mod(S,A)^{\leq 0} \) as defined above. We define \( \sigma_{\geq 1}(M,F) \) so that one has a distinguished triangle: \( \sigma_{\leq 0}(M,F) \to (M,F) \to \sigma_{\geq 1}(M,F) \to \sigma_{\leq 0}(M,F)[1] \) in \( D_F Mod(S,A) \).

These verify all the axioms of a t-structure as in [1.0.1] except the first. To see this one uses the spectral sequence in [BBD] (3.1.3.5). Then \( \text{Hom}_{D_F Mod(S,A)}((M,F),(M',F')) \) will identify with the \( E^0_{1,0} \)-term of this spectral sequence. The latter will be trivial if \( (M,F) \in D_F Mod(S,A)^{\leq 0} \) and \( (M',F') \in D_F Mod(S,A)^{\geq 1} \), proving that under the same hypothesis, \( \text{Hom}_{D_F Mod(S,A)}((M,F),(M',F')) = 0 \). This proves the theorem.

**Remarks 5.2.**

1. The case when \( A = R \) and when the filtrations are finite is considered in [Beil].

2. Any object in \( D Mod(S,R) \) may be provided with the \( \text{b\'e\text{t}e} \)-filtration as in [BBD] Chapter 3: this is the decreasing filtration defined as follows on any \( Kc(S,R) \): \( F_i(K) = \text{the sub-complex} \ 0 \to K^{i+1} \to K^{i+2} \to \cdots \). Now \( \text{grd}^F_i(K) = K^i \) if \( j = i \) and \( 0 = 0 \). Therefore it follows readily that any \( K \in D Mod(S,R) \) which is bounded above and provided with the \( \text{b\'e\text{t}e} \)-filtration \( F \) belongs to the heart of the t-structure on \( D_F Mod(S,R) \) considered above. In particular, any sheaf of dgas \( A \) which is bounded above and provided with the \( \text{b\'e\text{t}e} \)-filtration belongs to this heart. Therefore one may extend the formalism of perverse sheaves, extending perverse sheaves etc. as in [3.2] above to this setting. We skip the details.

6. **The crystalline case**

We will begin by recalling the basic framework from [Ek] and [Ill]. The Raynaud ring is the graded \( W \)-ring, \( W = W(k) = \text{the ring of Witt vectors of a perfect field} k \) of characteristic \( p \) and generated by \( F, V \) in degree 0 and \( d \) in degree 1 with the following relations:

\[
(6.0.7) \quad FV = VF = p, Fa = a^p F, V = Va^p, \quad da = ad, FDV = d, d^2 = 0, aeW
\]

Here \((-)^p\) is the Frobenius automorphism of \( W \). Given a scheme \( X \) defined over \( k \), one may adapt the definition above to define a sheaf of rings \( R \) on the Zariski site of \( X \), called the sheaf of Raynaud rings on \( X \). One may then extend the discussion below to the Zariski site of the given scheme \( X \). However, for the sake of simplicity, we will keep \( X = \text{Spec} k \) throughout the following discussion.

We let \( R_i, i = 0, 1 \), denote the piece of \( R \) in degree \( i \). In view of the above relations, one observes that a left \( R \)-module \( M \) is the same as a complex of \( R_0 \)-modules and where the differential \( d : M^n \to M^{n+1} \) satisfies \( FDV = d \). Moreover any left \( R_0 \)-module can be viewed as a left \( R \)-module concentrated in degree 0. Henceforth an \( R \)-module will mean a left \( R \)-module. Given an \( R \)-module \( M, M(n) \) will denote the \( R \)-module defined by \( M(n)^i = M^{n+i} \) and the differential \( d \) given by \((-)^p d \).

A complex of \( R \)-modules can be viewed as a double complex \( M^{**} \) where the first degree (called the horizontal direction) corresponds to the \( R \)-grading. The second degree will be called the vertical direction. Thus \( M^{**n} \) denotes the \( n \)-th row of \( M^{**} \) and this is an \( R \)-module. Observe that one may take the cohomology of the double complex
with respect to the vertical differential: these cohomology objects will be all $R$-modules. Thus $H^n_D(M^{••})$ denotes the $n$-th (vertical) cohomology. We define the derived category of $R$-modules, $D(R)$ to be the category of all complexes of $R$-modules where we invert vertical quasi-isomorphisms. We let the derived category of all $R$-modules be denoted $D(R)$.

Next recall the diagonal $t$-structures of [Ek]. First, for each $R$-module $M$ and an integer $n$, one defines the $R$-module $\tilde{\tau}_{\leq n} M$ by

\[(6.0.7) \tilde{\tau}_{\leq n} M = (\cdots \to M^{n-1} \overset{d}{\to} M^n \to F^\infty B^{n+1} \to 0)\]

where $B^{n+1} = \text{Im}(d^n : M^n \to M^{n+1})$ and $F^\infty B^{n+1} = \bigcup_{i \geq 0} F^i B^{n+1}$. Clearly this is a sub-$R$-module of $M$. Next $\tilde{\tau}_{\geq n+1} M$ is defined to be the quotient $M/\tilde{\tau}_{\leq n} M$. A complex of $R$-modules $M \in D(R)^{\leq 0}$ if for each $n$, the natural map $\tilde{\tau}_{\leq n} H^v_n(M^{••}) \to H^v_n(M^{••})$ is an isomorphism of $R$-modules. We say a complex of $R$-modules $M \in D(R)^{\geq 0}$ if the natural maps $H^v_n(M^{••}) \to \tilde{\tau}_{\geq 0}(H^v_n(M^{••}))$ is an isomorphism.

It was shown in [Ek] that this defines a $t$-structure on $D(R)$, the diagonal $t$-structure with the heart of the $t$-structure given by $D(R)^{\leq 0} \cap D(R)^{\geq 0}$. We proceed to show that the derived category $D(R)$ is compactly generated and that the above $t$-structure is defined by a family of compact objects as in Theorem 1.8.

For each pair of integers $i$ and $j$ we define the complex $R(-j)[i]$ of $R$-modules which is the following (double) complex: we put the $R$-module $R(-j)$ (viewed as a complex) as the $-i$-th row and put zeros elsewhere. Observe that $H^v_n(R(-j)[i]) = R(-j)$ if $n = -i$ and 0 otherwise. Therefore, if $j \leq i$, $R(-j)[i] \in D(R)^{\leq 0}$. We also consider the complex of $R$-modules, $R_0(-i)[i]$ which is the following (double) complex: we put the $R$-module $R_0(-i)$ as the $-i$-th row and put zeros elsewhere.

The following is the main result of this section.

**Theorem 6.1.** $D(R)^{\leq 0}$ is the smallest pre-aisle generated by the $R$-modules $R(-j)[i]$, $j \leq i$ and $R_0(-i)[i]$. These $R$-modules are compact objects in $D(R)$ and hence the above pre-aisle is an aisle which defines the diagonal $t$-structure.

**Proof.** To see that each $R(-j)[i]$ is compact, observe that $\text{RHom}_R(R(-j)[i], M) = \text{RHom}_R(R, M[j][i]) = M(j)[-i]$. Therefore, the above $\text{RHom}$ commutes with filtered colimits in the argument $M$. To see that each $R_0(-i)[i]$ is compact, observe that $\text{RHom}_R(R_0(i)[-i], M) = \text{Ker}(d : M^i \to M^{i+1})$ viewed as an $R$-module in the obvious manner. Since finite inverse limits commute with filtered colimits, one can see that $\text{RHom}_R(R_0(i)[-i], M)$ commutes with colimits in the second argument. These prove that the objects $R(-j)[i]$, $j \leq i$ and $R_0(-i)[i]$ are compact objects in the derived category $D\text{Mod}(R)$.

The main part of the rest of the proof consists in showing that given any complex of $R$-modules $M \in D(R)^{\leq 0}$, it can be constructed from the $R$-modules $R(-j)[i]$, $j \leq i$ and $R_0(-i)[i]$.

**Step 1.** Here we will assume the given complex of $R$-modules is concentrated in one row, say the $-i$-th. i.e. We may assume the given complex of $R$-modules is of the form $M[i]$, for some $R$-module $M$. We will now show that one may find a resolution $P_0$ of $M$ by $R$-modules, so that each $P_n$ is a sum of $R$-modules of the form $R(-j)$, $j \leq i$ and $R_0(-i)$. Since $M[i] \in D(R)^{\leq 0}$, the natural map $\tilde{\tau}_{\leq i}(M) \to M$ is an isomorphism. Therefore, $M^j = 0$ for $j > i + 1 = F^\infty B^i$ for $j = i + 1$. For each $j \leq i$, and each element $m_j \in M^j$, one may define a map $R(-j) \to M$ by sending $1 \in R_0$ to $m_j$ and then by extending the map in the obvious way to all of $R(-j)$. For each element $m_i$ that maps to zero by the differential, one may define a map $R_0(-i) \to M$ by sending $1 \in R_0$ to $m_i$. Let $P_0$ denote the resulting sum of the $R$-modules $R(-j)$, $j \leq i$ and $R_0(-i)$. Clearly the maps defined above provide a surjection $d_{-1} : P_0 \to M$ of $R$-modules. Now $P_0[i] \in D(R)^{\leq 0}$ by construction.

Moreover, if $K_0 = \text{kernel}(d_{-1})$, then $K_0[i] \in D(R)^{\leq 0}$ as well. For this, it suffices to show that the natural map $\tilde{\tau}_{\leq i}(K_0) \to K_0$ is an isomorphism. For this it suffices to check that $K_0^{i+1} = \text{Im}(d : K_0^i \to K_0^{i+1})$: this is an immediate consequence of our definition of $P_0$. Therefore, one may repeat the above construction with $K_0$ replacing $M$ to define $P_1$ and $K_1$. Inductively we may define the sequence $P_n$ in the same manner. Observe that the map $d_n : P_n \to P_{n-1}$ factors through $K_{n-1} = \text{kernel}(d : P_{n-1} \to P_{n-2})$ so that the sequence $\{P_n[n]\}$ provides a resolution of $M$. Then $P[i]$ provides the required resolution of $M[i]$.

**Step 2.** Next we will assume that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $R$-modules so that all three $M'[i], M[i]$ and $M''[i]$ belong to $D(R)^{\leq 0}$. We may observe readily that we can find resolutions $P'_n$ of $M'$, $P_n$ of $M$ and $P''_n$ of $M''$ together with maps $P'_n \to P_n$ and $P_n \to P''_n$ (over the given map $M' \to M$ and $M \to M''$,
respectively) so that for each $n$, the sequence $0 \to P'_n \to P_n \to P''_n \to 0$ is exact and where each $P'_n$, $P_n$ and $P''_n$ is a sum of $R$-modules of the form $R(-j)$, $j \leq i$ and $R_0(-i)$. (Since this is only a minor modification of a standard argument, we skip the remaining details.)

**Step 3.** Here we are given a bounded complex $M$ of $R$-modules, i.e. there exist integers $n$ and $N$, $n \leq N$ so that $M^{(n)} = 0$ for $j < n$ or for $j > N$. By using induction on $N - n$, it suffices to assume $N = n + 2$. In other words, we are given a diagram $M' \xrightarrow{\phi} M \xrightarrow{\psi} M''$ of $R$-modules with $\psi \circ \phi = 0$. We will show that in this case, we may find resolutions $P'_n$ of $M'$, $P_n$ of $M$ and $P''_n$ of $M''$ with maps $\phi'_n : P'_n \to P_n$ and $\psi'_n : P'_n \to P''_n$ so that the compositions $\psi'_n \circ \phi'_n = 0$ for all $n$. Moreover each $P'_n$, $P_n$ and $P''_n$ are a sum of terms of the form $R(-j)[i]$, $j \leq i$ and $R_0(-i)[i]$.

We will make use of Step 2 to accomplish this. For this, observe that one has the following short exact sequences:

$$0 \to \ker(\phi) \to M' \to \im(\phi) \to 0, \quad 0 \to \im(\phi) \to \ker(\psi) \to \ker(\psi) / \im(\phi) \to 0, \quad 0 \to \ker(\psi) \to M \to \im(\psi) \to 0$$

and

$$0 \to \im(\psi) \to M'' \to M'' / \im(\psi) \to 0.$$  

By applying Step 2 to these short sequence in successive order, one may construct the required resolutions.

**Final Step.** Finally we consider the case where we are given any complex $M$ of $R$ modules, not necessarily bounded. We may view this as a complex in the vertical direction. For each integer $N \geq 0$, let $M[-N,N]$ denote the naive truncation of the complex $M$ to degrees in the range $-N$ to $N$, i.e. $M[-N,N]^{(n)} = M^{(n)}$, $-N \leq j \leq N$. The last step shows how to find resolutions $P_n[-N,N]$ of $M[-N,N]$ that are compatible as $N$ varies over all non-negative integers. Now we let $P_n = \lim_{N \to -\infty} P_n[-N,N]$.  

\[\square\]

7. Further Examples

7.1. **Example 1: Motivic Derived Categories.** We first show how motivic derived categories may be defined in this framework. We will fix a ground field $k$, of arbitrary characteristic $p$ throughout the paper and will only consider smooth schemes of finite type over $k$. This category will be denoted $(smt.schms)$. When provided with the big Zariski (Nisnevich, étale) topologies, we obtain the big-sites $(smt.schms)_{zar}$, $(smt.schms)_{Nis}$, $(smt.schms)_{et}$, respectively. $\mathbb{Z} = \oplus \mathbb{Z}(r)$ will denote the integral motivic complex on the sites $(smt.schms)_{zar}$ and $(smt.schms)_{Nis}$. $l$ will denote a number prime to $p$ and $\mathbb{Z}/l = \oplus \mathbb{Z}/l(r)$ denote the corresponding mod $l$ motivic complex with $\mathbb{Z}/l_{et}$ the corresponding complex on the big-étale site $(smt.schms)_{et}$. $\mathbb{Q} = \oplus \mathbb{Q}(r) = \oplus \mathbb{Z} \otimes \mathbb{Q}$. In general, we will fix a commutative Noetherian ring $R$ and consider $\mathbb{Z} \otimes R$: this is a sheaf of $E_{\infty}$-dgs over the ring $R$ and we will denote this by $A$ throughout. Observe that now one has augmentations $R \to A$ and $A \to R$ the composition of which is the identity. (Here $R$ denotes the obvious constant sheaf associated to $R$.)

7.2. $D((smt.schms)_{zar}, A)$, $D((smt.schms)_{Nis}, A)$ and $D((smt.schms)_{et}, \mathbb{Z}/l)$. We will consider explicitly only the derived category $D((smt.schms)_{Nis}, A)$, since the other two may be defined similarly with appropriate modifications. We will let $\text{Sh}((smt.schms)_{Nis}, R)$ denote the category of all sheaves of $\mathcal{R}$-modules on the site $(smt.schms)_{Nis}$. Similarly $\text{Sh}(X_{Nis}, \mathcal{R})$ will denote the category of all sheaves of $\mathcal{R}$-modules on the site $X_{Nis}$ for any smooth scheme $X$. $C(\text{Sh}(X_{Nis}, \mathcal{R}))$ will denote the category of all (unbounded) complexes of objects in $\text{Sh}(X_{Nis}, \mathcal{R})$ (respectively). We first define $\text{Mod}(smt.schms)_{Nis}, A)$ to consist of all complexes of sheaves $K$ on $(smt.schms)_{Nis}$ with the following properties:

(i) $K = \oplus K(r)$ has homotopy invariant cohomology sheaves and

(ii) $K$ has the structure of a complex of sheaves of $E_{\infty}$-modules over the sheaf of $E_{\infty}$-dgs $A$.

A morphism $f : K' \to K$ between two such objects will be a map that preserves the last structure. The objects of the derived category $D((smt.schms)_{Nis}, A)$ are the same as those of $\text{Mod}(smt.schms)_{Nis}, A)$.

Given a scheme $X \in (smt.schms)$, we let $(smt.schms/X)$ denote the sub-category of $(smt.schms)$ that are of finite type over $X$ with morphisms $Y' \to Y$ being morphisms of smooth schemes compatible with the given maps to $X$. The site $(smt.schms/X)_{zar}$ $(smt.schms/X)_{Nis}$ $(smt.schms/X)_{et}$) is the corresponding big site and will be often denoted $X_{zar}$ $X_{Nis}$, $X_{et}$, respectively. $\text{Mod}(X_{Nis}, A)$ will denote the corresponding derived category of complexes $K$ defined on the site $X_{Nis}$ that in addition have the structure of complexes of sheaves of $E_{\infty}$-modules over $A_{X} = \text{the restriction of } A$ to $(smt.schms/X)_{Nis}$. $D(X_{Nis}, A)$ will denote the corresponding derived category.
7.3. $D^{gm}(X_{zar}, A)$, $D^{gm}(X_{nis}, A)$ and $D^{gm}(X_{et}, \mathbb{Z}/l)$. Again we will explicitly consider only the derived category $D^{gm}(X_{nis}, A)$. For each smooth scheme $Y$ quasi-projective over $X$ and with structure map $f : Y \to X$, we consider $Rf_*(A_Y)$. For a locally closed sub-scheme $Y_0$ of $Y$ with immersion $j : Y_0 \to Y$, we also consider $Rf_*(j_!(A_Y))$. We let $D^{gm}(X_{nis}, A)$ be the full subcategory of $D(X_{nis}, A)$ generated by such objects.

Definition 7.1. (ii) Let $D^{gm}(X_{nis}, A)$ denote the smallest sub-category of $D^{gm}(X_{nis}, A)$ containing all of $C = \{ Rf_*(j_!(A_Y)) | j : Y_0 \to Y, f \}$ where $j$ and $f$ are as before and closed under the following operations: (i) finite sums, (ii) mapping cones, (iii) translations [1] and (iv) extensions (i.e. if $F \to Cone(f) \to F'[1]$ is a distinguished triangle with $F'$, $Cone(f)$ and $F'[1]$, then $F \in D^{gm}(X_{nis}, A)$.)

(b) Let $D^{gm}_*(X_{nis}, A)$ denote the smallest sub-category of $D^{gm}(X_{nis}, A)$ containing all of $\{ j_!(A_Y) | j : X_0 \to X \}$ where $j$ is a locally closed immersion as before and closed under the following operations: (i) finite sums, (ii) mapping cones, (iii) translations [1] and (iv) extensions (as before).

These derived categories are studied in detail in a forthcoming paper, [J-3], and we relate them to Voevodsky’s derived category of geometric motives.

7.4. Example 2: Equivariant Derived Categories. We will fix an algebraically closed field $k$ of characteristic $p \geq 0$. Let $G$ denote the action of a smooth group scheme on a scheme $X$ of finite type over $k$. Now $[X/G]$ will denote the associated quotient stack: see [?, LMB] for example. One associates several sites to the stack $[X/G]$: $[X/G]_{nis, et}$ denotes the site whose objects are smooth maps $s : S \to [X/G]$ with $S$ an algebraic stack of finite type over $k$ and where the coverings of a given object $s : S \to [X/G]$ are étale coverings. One defines the iso-variant étale site $[X/G]_{iso, et}$ as follows: the objects are $G$-isovariant étale maps $Y \to X$ (these correspond to iso-variant étale maps $[Y/G] \to [X/G]$ of the associated stacks. Sending an iso-variant étale map to the same map viewed simply as an étale map defines a map of sites: $p_* : [X/G]_{iso, et} \to [X/G]_{isol, et}$. It is shown in [J-2] that the latter site has enough points and the points correspond to $G$-orbits of geometric points of $X$.

Next assume that the group $G$ is a torus $T$ and $X$ is a toric variety associated to $T$. Now $T$ acts with finitely many orbits on $X$ so that there are only finitely many points on the site $[X/T]_{iso, et}$, each corresponding to the $T$-orbits of geometric points of $X$. In characteristic zero it is possible to define a topological space corresponding to the site $[X/T]_{iso, et}$ and this is the approach adopted in [Lan]. However, this approach clearly fails in positive characteristic and necessitates the use of the site $[X/T]_{iso, et}$.

Next given a geometric point $\bar{x}$ of $X$, it has a $T$-stable neighborhood of the form $T\bar{x} \times V_2$ where $V_2$ denotes an affine toric variety for $T_2$ which contracts $T_\bar{x}$-equivariantly to $\bar{x}$. Therefore one may readily compute $R_{p_*}(\mathcal{Q})$ and show that one has the isomorphism:

$$R_{p_*}(\mathcal{Q})|_{T\bar{x}} \simeq H^*(BT_2; \mathcal{Q})$$

Moreover one knows that for toric varieties, the stabilizers $T_\bar{x}$ are all connected. Now it follows readily as in [Lan] that one has the equivalence of categories:

$$D_+([X/T]_{iso, et}, R_{p_*}(\mathcal{Q})) \simeq D_+([X/T]_{iso, et}, \mathcal{Q}).$$

References


**Department of Mathematics, Ohio State University, Columbus, Ohio, 43210, USA.**

_E-mail address: joshua@math.ohio-state.edu_