GENERALIZED t-STRUCTURES: t-STRUCTURES FOR SHEAVES OF dg-MODULES OVER A SHEAF OF dg-ALGEBRAS AND DIAGONAL t-STRUCTURES

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Abstract. t-structures, in the abstract, apply to any triangulated category. However, for the most part, they have been studied so far only in the context of sheaves of modules over sites provided with sheaves of rings. In this paper we define and study t-structures for categories of modules over sites provided with sheaves of dgas and E_∞-dgas. A close variant, as we show, are the diagonal t-structures that come up in the context of filtered derived categories and also in the context of crystalline cohomology (as in the work of Ekedahl). All of this is carried out in the unified frame-work of *aisles*. We consider several examples: equivariant derived categories and derived categories associated to algebraic stacks, motivic derived categories, the derived categories of presheaves of spectra on Grothendieck sites as well as crystalline derived categories.

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1. **Introduction.**

t-structures were originally introduced in [BBD] to define and study perverse sheaves and soon afterwords in [Ek] to study crystalline cohomology problems. Since then, they have appeared in other contexts (see [De], [KM], [J-1] for example). They appear prominently in certain conjectures (see [Jan]) on algebraic cycles where a formalism similar to l-adic derived categories for algebraic cycles is formulated. Questions on the existence of generalized t-structures have been around for over 25 years: see, for example, [Bo] where questions on the existence of generalized intersection cohomology theories were raised.

In this paper we utilize the techniques of *pre-aisles and aisles* introduced in [KV1] to provide a painless way to define and study (generalized) t-structures for many of the above contexts. As applications of our work, we also show how to apply our results to several of the above situations. Our interest in the material discussed here was awakened by some questions posed by N. Ramachandran to the author in 2003 while both of us were members of the MPI, Bonn. We thank Ramachandran for raising these questions and to the MPI for its hospitality.

The following theorem is typical of our results in the paper. (The notion of *pre-aisles* and *aisles* are discussed below.) For the purposes of this introduction we may assume the site $\mathcal{S}$ is the small Zariski, Nisnevich or étale sites associated to a given Noetherian scheme $\mathcal{S}$.

Theorem 1.1. *Let $(\mathcal{S}, \mathcal{R})$ denote a ringed site as in 2.1 so that the underlying site $\mathcal{S}$ is essentially small and has enough points. Let $C(\mathcal{S}, \mathcal{R})$ denote the category of all unbounded complexes of sheaves of $\mathcal{R}$-modules on $\mathcal{S}$. Moreover we assume the following:*

(i) for each object U in the site $\mathcal{S}$, there exists a large enough integer N (depending on U) so that $H^i(U, F_{|U}) = 0$ for all $i > N$ and all sheaves of $\mathcal{R}$-modules F on the site $\mathcal{S}$ and

(ii) for all filtered direct systems $\{F_\alpha|\alpha\}$ of sheaves of $\mathcal{R}$-modules and every object U in the site $\mathcal{S}$, $\underset{\alpha}{colim} H^*(U, F_\alpha) \cong H^*(U, \underset{\alpha}{colim} F_\alpha)$.

Let $\mathcal{A}$ be a sheaf of E_∞-dgas or dgas on the ringed site $(\mathcal{S}, \mathcal{R})$. Let $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ denote the pre-aisle in $DMod(\mathcal{S}, \mathcal{A})$ generated by $j_{U!} jU^*(\mathcal{A}[n])$, $n \geq 0$, U in the site $\mathcal{S}$.

Then (i) $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ is an aisle in $DMod(\mathcal{S}, \mathcal{A})$, i.e. defines a t-structure on $DMod(\mathcal{S}, \mathcal{A})$.

(ii) Assume next the hypotheses of 2.2 hold. i.e. We will assume that $\mathcal{A}$ is provided with an augmentation $\mathcal{A} \rightarrow \mathcal{R}$ which is assumed to be a map of sheaves of E_∞-dgas and that $\mathcal{A}$ is connected, i.e. $\mathcal{A}^i = 0$ for $i < 0$ and $\mathcal{A}^0 = \mathcal{R}$.

Then (a) $j_{U!}(\mathcal{A}_{|U}) \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\leq 0} \cap DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$ = the heart of the above t-structure, where $j_U : U \rightarrow \mathcal{S}$ is the structure map of the object . (b) Moreover, every object M in $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ satisfies the property that the natural map $\tau_{\leq 0}(\mathcal{R} \overset{L}{\underset{\mathcal{A}}{\otimes}} M) \rightarrow \mathcal{R} \overset{L}{\underset{\mathcal{A}}{\otimes}} M$ is a quasi-isomorphism in $DMod(\mathcal{S}, \mathcal{R})$. In other words, the functor $\mathcal{R} \overset{L}{\underset{\mathcal{A}}{\otimes}} (\quad) : DMod(\mathcal{S}, \mathcal{A}) \rightarrow DMod(\mathcal{S}, \mathcal{R})$ sends $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ to $DMod(\mathcal{S}, \mathcal{R})^{\leq 0}$.

We briefly mention two examples here, which are discussed in more detail in the last section of the paper.

Example 1.2. Utilizing the E_∞-structure on the motivic complexes (see [J-1]), it is possible to define *motivic derived categories* as derived categories of dg-modules over the motivic E_∞-dga.

Example 1.3. The equivariant derived categories of l-adic sheaves on a scheme provided with an action by a smooth group-scheme is shown to be equivalent to the derived category of sheaves over a sheaf of dgas in certain cases (see [**?**] and [Guil]) and such an equivalence is conjectured to hold under fairly general hypotheses: see [So]. We obtain the following result in this context (which is essentially an extension to positive characteristics the result in [**?**]: such an extension needs the isovariant étale site considered in [T] and [J-2]. (These are briefly recalled in the last section where the following theorem is discussed in more detail.)

Theorem 1.4. *Let X denote a projective toric variety for the action of a torus T over an algebraically closed field of characteristic $p \geq 0$. Let l denote a fixed prime different from p. Let $p : [X/T]_{lis.et} \rightarrow [X/T]_{iso.et}$ denote the map of sites associated to the quotient stack $[X/T]$. and let $\mathcal{A} = Rp_*(\underline{\mathbb{Q}}_l)$ denote the sheaf of dgas on $[X/T]_{iso.et}$. Then the following hold:*

(i) The points of the site $[X/T]_{iso.et}$ correspond to T-orbits and the stalk of $\mathcal{A}$ at such a point $\bar{p} = Tp$, $p \varepsilon X$, is given by $\mathcal{A}_{\bar{p}} = \mathbb{H}^*(BT_p, \mathbb{Q}_l)$ where T_p denotes the stabilizer at p.

(ii) $\mathcal{A}$ is formal in the sense that $\mathcal{A} \simeq \mathcal{H}^*(\mathcal{A})$ and

(iii) one obtains an equivalence of derived categories (bounded below complexes): $D_+^T(X, \mathbb{Q}_l) \simeq D_+([X/T]_{iso.et}, \mathcal{A})$. The t-structure on the right-hand-side is defined as in the last theorem and corresponds to the usual t-structure on the left-hand-side.

We conclude this introduction by recalling the notion of pre-aisles and aisles (see [KV1], [KV2] and [TLSS]).

1.0.1. Let $\mathcal{T}$ be a triangulated category whose translation functor is denoted by $(-)[1]$ and its iterates by $(-)[n]$, with $n \in \mathbb{Z}$. A *t-structure* on $\mathcal{T}$ in the sense of ([BBD, Définition 1.3.1]) is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ such that, denoting $\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$, the following conditions hold:

(t1) For $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$, $\mathcal{H}om_{\mathcal{T}}(X, Y) = 0$.
(t2) $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 0} \supset \mathcal{T}^{\geq 1}$.
(t3) For each $X \in \mathcal{T}$ there is a distinguished triangle $A \to X \to B \xrightarrow{+} A[1]$ with$A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 1}$.

The subcategory $\mathcal{T}^{\leq 0}$ is called the *aisle* of the t-structure, and $\mathcal{T}^{\geq 0}$ is called the *co-aisle*. As usual for a subcategory $\mathcal{C} \subset \mathcal{T}$ we denote the associated orthogonal subcategories as $\mathcal{C}^{\perp} = \{Y \in \mathcal{T} / \mathcal{H}om_{\mathcal{T}}(Z, Y) = 0, \ \forall Z \in \mathcal{C}\}$ and ${}^{\perp}\mathcal{C} = \{Z \in \mathcal{T} / \mathcal{H}om_{\mathcal{T}}(Z, Y) = 0, \ \forall Y \in \mathcal{C}\}$. The following are immediate formal consequences of the definition.

Proposition 1.5. *Let $\mathcal{T}$ be a triangulated category, $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ a t-structure in $\mathcal{T}$, and $n \in \mathbb{Z}$, then*

1. *$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ is a pair of* orthogonal *subcategories of $\mathcal{T}$, i.e. $\mathcal{T}^{\geq 1} = \mathcal{T}^{\leq 0 \perp}$ and $\mathcal{T}^{\leq 0} = {}^{\perp}\mathcal{T}^{\geq 1}$.*
2. *The subcategories $\mathcal{T}^{\leq n}$ are stable for positive translations and the subcategories $\mathcal{T}^{\geq n}$ are stable for negative translations.*
3. *The canonical inclusion $\mathcal{T}^{\leq n} \to \mathcal{T}$ has a right adjoint denoted $\tau^{\leq n}$, and $\mathcal{T}^{\geq n} \to \mathcal{T}$ a left adjoint denoted $\tau^{\geq n}$. Moreover, $X \in \mathcal{T}^{\leq n}$ if, and only if, $\tau^{\geq n+1}(X) = 0$. Similarly results hold for $\mathcal{T}^{\geq n}$.*
4. *For X in $\mathcal{T}$ there is a distinguished triangle $\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \xrightarrow{+} \tau^{\leq 0}X[1]$*
5. *The subcategories $\mathcal{T}^{\leq n}$ and $\mathcal{T}^{\geq n}$ are stable under extensions, i.e. given a distinguished triangle $X \to Y \to Z \xrightarrow{+}$, if X and Z belong to one of these categories, so does Y.*

The subcategories $\mathcal{T}^{\leq n}$ and $\mathcal{T}^{\geq n}$, in general, are not triangulated subcategories but they come close. In fact, each subcategory $\mathcal{T}^{\leq n}$ has the structure of a *suspended category* in the sense of Keller and Vossieck [**?**]. Let us recall this definition.

An additive category $\mathcal{U}$ is suspended if and only if is graded by an additive translation functor T (sometimes called *shifting*) and there is class of diagrams of the form $X \to Y \to Z \to TX$ (often denoted simply $X \to Y \to Z \xrightarrow{+}$) called *distinguished triangles* such that the following axioms, analogous to those for triangulated categories in Verdier's [V, p. 266] hold:

(SP1) Every triangle isomorphic to a distinguished one is distinguished. For $X \in \mathcal{U}$, $0 \to X \xrightarrow{id} X \to 0$ is a distinguished triangle. Every morphism $u : X \to Y$ can be completed to a distinguished triangle $X \xrightarrow{u} Y \to Z \to TX$
(SP2) If $X \xrightarrow{u} Y \to Z \to TX$ is a distinguished triangle in $\mathcal{U}$ then so is $Y \to Z \to TX \xrightarrow{Tu} TY$.
(SP3) = (TR3) in Verdier's *loc. cit.*
(SP4) = (TR4) in Verdier's *loc. cit.*

The main difference with triangulated categories is that the translation functor in a suspended category may not have an inverse and therefore some objects can not be shifted back. The formulation of axioms (SP1) and (SP2) reflect this fact. If $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure on a triangulated category T, the aisle $\mathcal{T}^{\leq 0}$ is a suspended subcategory of $\mathcal{T}$ whose distinguished triangles are diagrams in $\mathcal{T}^{\leq 0}$ that are distinguished triangles in $\mathcal{T}$ (Proposition 1.5). Moreover, the aisle $\mathcal{T}^{\leq 0}$ determines the t-structure because the co-aisle $\mathcal{T}^{\geq 0}$ is recovered as $(\mathcal{T}^{\leq 0})^{\perp}[1]$. The terminology "aisle" and "co-aisle" comes from [**?**].

We will call a suspended subcategory $\mathcal{U}$ of a triangulated category $\mathcal{T}$ where the triangulation in $\mathcal{U}$ is given by the triangles which are distinguished in $\mathcal{T}$ and the shift functor is induced by the one in $\mathcal{T}$, a *pre-aisle*. We see easily that to check that a full subcategory $\mathcal{U}$ of $\mathcal{T}$ is a pre-aisle, it is enough to verify that

- For any X in $\mathcal{U}$, $X[1]$ is also in $\mathcal{U}$.
- Given a distinguished triangle $X \to Y \to Z \to X[1]$, if X and Z belong to $\mathcal{U}$, then so does Y.

Once these two facts hold for $\mathcal{U}$, the verification of axioms (SP1) through (SP4) is immediate.

The following are the key techniques we use to construct t-structures in this paper.

Theorem 1.6. *([KV2, Section 1]) A suspended subcategory $\mathcal{U}$ of a triangulated category $\mathcal{T}$ is an aisle (i.e. $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ is a t-structure on $\mathcal{T}$) if and only if the canonical inclusion functor $\mathcal{U} \to \mathcal{T}$ has a right adjoint.*

Definition 1.7. Let $\mathcal{T}$ be a triangulated category. An object E of $\mathcal{T}$ is called *compact* if the functor $\mathcal{H}om_{\mathcal{T}}(E, -)$ commutes with arbitrary (small) co-products. Another way of expressing the condition is that a map from E to a co-product factors through a finite subcoproduct.

Theorem 1.8. *([TLSS]) Let $\mathcal{S} = \{E_\alpha / \alpha \in A\}$ a set of compact objects in a triangulated category $\mathcal{T}$. Let $\mathcal{U}$ the smallest co-complete (i.e. closed under all small sums) pre-aisle of $\mathcal{T}$ which contains the family $\mathcal{S}$. Then $\mathcal{U}$ is an aisle in $\mathcal{T}$.*

2. **The Basic Contexts**

In this section we discuss three of the different contexts we consider in this paper. A fourth one, namely that of crystalline derived categories is discussed separately in section 5. We begin by discussing a frame-work that is common to two of the basic contexts.

2.1. **The common frame-work.** Let $\mathcal{S}$ denote a *site* with the following properties: (i) it is essentially small (ii) has enough points. We will denote the points of the site $\mathcal{S}$ by $\bar{\mathcal{S}}$. In addition we will assume that every object U in the site $\mathcal{S}$ is quasi-compact and that the site $\mathcal{S}$ is locally coherent in the sense of [SGA]4 Exposé VI (2.3). (Recall the latter notion is defined as follows: an object U in $\mathcal{S}$ is *quasi-separated* if for any two maps $V \to U$ and $W \to U$, the fibered product $V \underset{U}{\times} W$ is quasi-compact. An object U is *coherent* if it is both quasi-compact and quasi-separated. A site with a terminal object X is *coherent* if every object quasi-separated in $\mathcal{S}$ is quasi-separated over the terminal object X and the terminal object X is coherent. Given the site $\mathcal{S}$ and an object X in $\mathcal{S}$, the site $\mathcal{S}/X$ will denote the site whose objects are morphisms $u : U \to X$ and morphisms are morphisms in $\mathcal{S}$ over X. We say a site $\mathcal{S}$ is *locally coherent* if it has a covering $\{U_i | i\}$ so that each of the sites $\mathcal{S}/U_i$ is coherent.

The main observation we make now is the following:

(2.1.1)
$$\underset{\alpha}{\text{colim}} H^n(U, F_\alpha) \cong H^n(U, \underset{\alpha}{\text{colim}} F_\alpha)$$

for each U in the site $\mathcal{S}$ and for each filtered direct system $\{F_\alpha | \alpha\}$ of abelian sheaves on $\mathcal{S}$ and for each n.

2.1.2. We will let $\mathcal{R}$ denote either one of the following: (i) a sheaf of commutative Noetherian rings (or graded commutative Noetherian rings) with unit on the site $\mathcal{S}$ or (ii) the constant sphere spectrum Σ^0. We will let $C(\mathcal{S}, \mathcal{R})$ denote the category of all unbounded complexes of $\mathcal{R}$-modules (with differentials of degree +1) in the first case and the category of all sheaves of spectra on the site $\mathcal{S}$ in the second case. (In case $\mathcal{R}$ is graded, we will assume that $\mathcal{R} = \underset{i \epsilon \mathbb{Z}}{\oplus} R_i$ and that $C(\mathcal{S}, \mathcal{R})$ will denote the category of complexes of sheaves of graded modules over $\mathcal{R}$: a sheaf of graded modules $M = \underset{i \epsilon \mathbb{Z}}{\oplus} M_i$. For a sheaf of graded $\mathcal{R}$-modules M, $M(t)$ will denote the object with a shift of grading given by: $M(t)_i = M_{t+i}$.)

We will let $\mathcal{R} \otimes \Delta[1]$ denote the following object in $C(\mathcal{S}, \mathcal{R})$: if $\mathcal{R}$ is a sheaf of rings, then this is the obvious chain complex associated to the simplicial object defined by $n \mapsto \underset{\alpha \epsilon \Delta[n]}{\oplus} \mathcal{R}$ (and with the obvious structure maps). If $\mathcal{R}$ denotes Σ^0, then this is the suspension spectrum associated to the pointed simplicial set $\Delta[1]_+$. Observe that we have canonical morphisms $d_i : \mathcal{R} \cong \mathcal{R} \otimes \Delta[0] \to \mathcal{R} \otimes \Delta[1]$, $i = 0$, $i = 1$. If F is a complex of abelian sheaves on $\mathcal{S}$, $\mathcal{H}^n(F)$ will denote the cohomology sheaf in degree n of the complex F; in case F is a sheaf of spectra, this will denote $\pi_{-n}(F)$ = the sheaf of $-n$-th homotopy groups of F.

Proposition 2.1. *Assume the following hypothesis:*

for each U in the site $\mathcal{S}$, there exists an integer $N > 0$ so that $H^n(U, F) = 0$ for all $n > N$ and all abelian sheaves F

Let $\{F_\alpha | \alpha\}$ denotes a filtered direct system of complexes in $C(\mathcal{S}, \mathcal{R})$ where $C(\mathcal{S}, \mathcal{R})$ denotes the category as above.

Then one obtains the quasi-isomorphism:

(2.1.3)
$$colim \mathbb{H}^n(U, F_\alpha) \cong \mathbb{H}^n(U, colim F_\alpha)$$

for each n and each U in the site $\mathcal{S}$.

Proof. This follows by comparing the spectral sequences

$$E_2^{s,t} = \underset{\alpha}{\mathrm{colim}} H^s(U, \mathcal{H}^t(F_\alpha)) \Rightarrow \underset{\alpha}{\mathrm{colim}} \mathbb{H}^{s+t}(U, F_\alpha) \text{ and}$$

$$E_2^{s,t} = H^s(U, \mathcal{H}^t(\underset{\alpha}{\mathrm{colim}} F_\alpha)) \Rightarrow \mathbb{H}^{s+t}(U, \underset{\alpha}{\mathrm{colim}} F_\alpha)$$

Since both spectral sequences converge strongly under the above hypotheses, and one obtains an isomorphism at the E_2-terms by (2.1.3), the required isomorphism of the abutments follows. □

Definitions 2.2. (i) Let $\mathcal{R}$ denote a sheaf of commutative Noetherian rings with 1 on the site $\mathcal{S}$. A sheaf of dgas will mean a sheaf $\mathcal{A}$ of differential graded algebras on the site $\mathcal{S}$, with values in $C(\mathcal{S}, \mathcal{R})$. A sheaf of E_∞ dgas will similarly mean an unbounded complex $\mathcal{A}$ in $C(\mathcal{S}, \mathcal{R})$ which is a sheaf of algebras over an E_∞-operad.

(ii) In addition to these situations, we will also consider cases where $\mathcal{A}$ is a sheaf of E_∞-ring spectra on the site $\mathcal{S}$. (An E_∞-ring spectrum will mean an object in the category of spectra that is also an algebra over an E_∞-operad.) Denoting the constant sheaf of sphere spectra by $\mathcal{R}$, such sheaves of $E-\infty$-ring spectra may be viewed as E_∞-ring objects of $C(\mathcal{S}, \mathcal{R})$.)

2.1.4. *Basic conventions.* Henceforth $\mathcal{A}$ will denote a sheaf of E_∞-algebras or a sheaf of E_∞-ring spectra. $Mod(\mathcal{S}, \mathcal{A})$ will denote the category of sheaves of E_∞-modules over $\mathcal{A}$. The obvious pairing $Mod(\mathcal{S}, \mathcal{A}) \times Mod(\mathcal{S}, \mathcal{A}) \rightarrow C(\mathcal{S}, \mathcal{R})$ will be denoted $\otimes$. (Here $\mathcal{R}$ will denote a sheaf of commutative Noetherian rings with 1 in case $\mathcal{A}$ is a sheaf of E_∞-algebras and will denote the constant sheaf of sphere spectra Σ^0 in case $\mathcal{A}$ is a sheaf of E_∞-ring spectra.) We will refer to E_∞-ring spectra (sheaves of E_∞-ring spectra) as E_∞-dgas (sheaves of E_∞-dgas, respectively).

Next we consider the homotopy category associated to $Mod(\mathcal{S}, \mathcal{A})$: this will have the same objects as $Mod(\mathcal{S}, \mathcal{A})$ but morphisms will be homotopy classes of morphisms where a homotopy H between two morphisms $f, g : K \rightarrow L$ is a morphism $K \otimes \Delta[1] \rightarrow L$ so that $f = H \circ d_0$ and $g = H \circ d_1$. We define a morphism $f : K \rightarrow L$ to be a *quasi-isomorphism* if $f : K \rightarrow L$ induces an isomorphism on the cohomology sheaves. A diagram $K' \xrightarrow{f} K \xrightarrow{g} K''$ is a *distinguished triangle* if there is a map from the mapping cone, $Cone(f)$ to K'' that is a quasi-isomorphism. (Observe that $Cone(f) \varepsilon Mod(\mathcal{S}, \mathcal{A})$. The *derived category* $DMod(\mathcal{S}, \mathcal{A})$ is the category obtained by inverting these quasi-isomorphisms.

Proposition 2.3. *$DMod(\mathcal{S}, \mathcal{A})$ is a triangulated category with the above structure*

Proof. is skipped and left as an exercise. □

For each U in the site $\mathcal{S}$, let $j_U^* : C(\mathcal{S}, \mathcal{R}) \rightarrow C(\mathcal{S}/U, \mathcal{R})$ denote the obvious restriction functor; let $j_{U!}$ (Rj_{U*}) denote the left-adjoint (right-adjoint) to j_U^*.

Proposition 2.4. *Assume the hypotheses as in Proposition 2.1 with $C(\mathcal{S}, \mathcal{R})$ denoting any one of the categories considered there.*

Let $\mathcal{A}$ denotes a sheaf of E_∞-dgas in $C(\mathcal{S}, \mathcal{R})$. Then, for each U in the site $\mathcal{S}$, $j_{U!} j_U^*(\mathcal{A})[n]$ is a compact object in $D(Mod(\mathcal{S}, \mathcal{A}))$.

Proof. Let $M \varepsilon DMod(\mathcal{S}, \mathcal{A})$. Let $RHom_{\mathcal{A}}$ ($\mathcal{RH}om_{\mathcal{A}}$) denote the external (internal) Hom in the derived category $DMod(\mathcal{S}, \mathcal{A})$. Then

$$RHom_{\mathcal{A}}(j_{U!} j_U^*(\mathcal{A})[n], M) = R\Gamma(\mathcal{S}, \mathcal{RH}om_{\mathcal{A}}(j_{U!} j_U^*(\mathcal{A})[n], M) = R\Gamma(\mathcal{S}_{|U}, \mathcal{RH}om_{j_U^*(\mathcal{A})}(j_U^*(\mathcal{A}), M[-n]))$$

$$= R\Gamma(\mathcal{S}_{|U}, \mathcal{RH}om_{\mathcal{R}_{|U}}(\mathcal{R}_{|U}, M[-n])) = R\Gamma(\mathcal{S}_{|U}, M[-n]).$$

Now let $\{M_\alpha | \alpha\}$ denote a direct system of objects in $DMod(\mathcal{S}, \mathcal{A})$. In view of the identifications in the last paragraph, one observes, using Proposition 2.1 that $\underset{\alpha}{\mathrm{colim}} RHom(j_{U!} j_U^*(\mathcal{A})[n], M_\alpha) \cong RHom(j_{U!} j_U^*(\mathcal{A})[n], \underset{\alpha}{\mathrm{colim}} M_\alpha)$. This proves the proposition. □

Examples 2.5. One may consider the following as typical examples where the last proposition applies:

1. $R =$ the constant sheaf $\mathbb{Q}$ and $\mathcal{A}$ is any sheaf of dgas in $C(\mathcal{S}, \mathbb{Q})$ with $\mathcal{S}$ the big Zariski, étale, Nisnevich or qfh sites associated to schemes of finite types over a Noetherian base scheme of finite Krull dimension.
2. $R =$ the constant sheaf $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{Z}/p$ for some prime p and $\mathcal{A}$ is any sheaf of dgas in $C(\mathcal{S}, \mathcal{R})$ with $\mathcal{S}$ the big Zariski Nisnevich or qfh sites associated to schemes of finite type over a Noetherian base scheme of finite Krull dimension.

(3) In addition to the above one may let $\mathcal{S}$ denote any site that has finite cohomological dimension with respect to all abelian sheaves, for example, the transcendental site associated to any complex algebraic variety. Now $\mathcal{R}$ may denote any sheaf of of commutative rings or the constant sheaf of sphere spectra and $\mathcal{A}$ any E_∞-ring object in $C(\mathcal{S}, \mathcal{R})$.

2.2. **Derived categories of modules over sites provided with sheaves of dgas and E_∞-dgas.** We will make the following basic hypotheses throughout this sub-section:

$\mathcal{A}$ is a sheaf of E_∞-dgas or dgas on the ringed site $(\mathcal{S}, \mathcal{R})$. We will assume that $\mathcal{A}$ is provided with an augmentation $\mathcal{A} \to \mathcal{R}$ which is assumed to be a map of sheaves of E_∞-dgas and that $\mathcal{A}$ is *connected*, i.e. $\mathcal{A}^i = 0$ for $i < 0$ and $\mathcal{A}^0 = \mathcal{R}$.

A particularly interesting case is when $\mathcal{A}$ denotes the rational motivic dga (constructed in [J-1]): in this case the above condition is equivalent to the Beilinson-Soulé vanishing conjecture. More generally one may assume $\mathcal{A}$ is a sheaf of bi-graded dgas, $\mathcal{A} = \oplus_{r,s}\mathcal{A}(s)^r$ with r denoting the degree of the chain complex and s another index we call the weight, so that $\mathcal{A}^\bullet = \oplus \mathcal{A}^\bullet(s)$, i.e. each $\mathcal{A}^\bullet(s)$ is a sub-complex of $\mathcal{A}^\bullet$. Then, if $\mathcal{A}^\bullet(0) = \mathcal{R}[0]$ one clearly obtains an augmentation in the above sense. Observe that these conditions are met by the (integral) motivic E_∞-dga. The connectedness is then equivalent to an integral form of the Beilinson-Soulé vanishing conjecture.

We will let $D_+(Mod(\mathcal{S}, \mathcal{A}))$ $(D_-(Mod(\mathcal{S}, \mathcal{A})))$ denote the full sub-category of $DMod(\mathcal{S}, \mathcal{A})$ consisting of complexes that are bounded below (above, respectively).

This frame-work will provide one class of examples we consider.

2.3. **Presheaves of E_∞-module spectra and CW-cell modules.** This frame-work will provide the second class of examples we consider. Throughout the following discussion we will assume the basic situation of 2.2 where $\mathcal{R} = \underset{i\varepsilon\mathbb{Z}}{\oplus} R(i)$ is a sheaf of graded commutative rings with 1 *or* where $\mathcal{R}$ denotes the constant sheaf of sphere spectra. In the first case, $\mathcal{R}[s](t)$ will denote the complex concentrated in degree s and given by the sheaf $\mathcal{R}(t)$ there; in the second case when $\mathcal{R}$ denotes the constant sheaf of sphere spectra Σ^0, it will denote the s-fold suspension Σ^s. Assume that we are in the first situation and that $\mathcal{A}\varepsilon C(\mathcal{S}, \mathcal{R})$ is an E_∞-ring object. Then $A[s](t) = \mathcal{A}\underset{\mathcal{R}}{\otimes}\mathcal{R}[s](t)$.

Definitions 2.6. (i) We will assume henceforth, *but only in this section*, that the sheaf of E_∞-dgas $\mathcal{A}$ is -1-*connected*, i.e. $\mathcal{H}^s(\mathcal{A}) = 0$ for all $s \geq 1$. (This terminology is derived from the case where $\mathcal{A}^i = B_{-i}$ for a chain-complex $\mathcal{B}$ (i.e. one whose differentials are of degree -1.)) In this case, the theory developed below is entirely similar to the homotopy theory of CW-complexes. We say a sheaf of $\mathcal{A}$-modules M is $-n$-*connected* if $\mathcal{H}^i(M) = 0$ for all $i \geq n$. Since $\mathcal{H}om_{\mathcal{A}}(\mathcal{A}, M) \cong \mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M) \cong M$, this is equivalent to $\mathcal{H}^i(\mathcal{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{A}, M)) \cong \mathcal{H}^i(\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M)) = 0$ for all $i \geq n$. We say M is *connected if it is $-n$-connected for some $n > 0$.*

(ii) A map $f : M' \to M$ in $DMod(\mathcal{S}, \mathcal{A})$ is a k-*equivalence* if the induced map $\mathcal{H}^i(f) : \mathcal{H}^i(M') \to \mathcal{H}^i(M)$ is an isomorphism for all $i > k$ and an epimorphism for $i = k$.

Definition 2.7. (i) A *free* $\mathcal{R}$-module is an object $M\varepsilon C(\mathcal{S}, \mathcal{R})$ so that M is given by a sum $\underset{s_U,t_U\varepsilon\mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{R})[s_U](t_U)$, where U ranges over the objects of the site $\mathcal{S}$. A *free* $\mathcal{A}$-module is an object $M\varepsilon DMod(\mathcal{S}, \mathcal{A})$ so that M is given by a sum $\underset{s_U,t_U\varepsilon\mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{R})[s_U](t_U)$, where U ranges over the objects of the site $\mathcal{S}$. We call s_U (t_U) the *dimension (weight)* of the free module $j_{U!}j_U^*(\mathcal{A})[s_U](t_U)$.

(ii)An $\mathcal{R}$-module M is a *cone* $\mathcal{R}$-module if $M = Cone(id : \underset{s_U,t_U\varepsilon\mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{R})[s_U](t_U) \to \underset{s_U,t_U\varepsilon\mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{R})[s_U](t_U))$ for some free $\mathcal{R}$-module $\underset{s_U,t_U\varepsilon\mathbb{Z}}{\oplus} j_{U!}j_U^*(\mathcal{R})[s_U](t_U)$. A *cone* $\mathcal{A}$-module is defined similarly. A *cell-module* $M\varepsilon C(\mathcal{S}, \mathcal{R})$ is an object $M\varepsilon C(\mathcal{S}, \mathcal{R})$ provided with a decreasing filtration $\{F_iM | i \leq 0\}$ by sub-objects in $C(\mathcal{S}, \mathcal{R})$ so that $F_0(M)$ is a free $\mathcal{R}$-module for some fixed integer N and each successive quotient $F_iM/F_{i+1}M$ is also a free $\mathcal{R}$-module, for all $i \leq 0$. Moreover F_iM is the mapping cone of a map $f_i : \mathcal{F}_i \to F_{i+1}M$ of a map in $C(\mathcal{S}, \mathcal{R}))$ with $\mathcal{F}_i$ a free $\mathcal{R}$-module. (Observe that this mapping cone may be realized as a quotient of $F_{i+1}M \oplus Cone(\mathcal{F}_i)$.) In this case we say that F_iM is obtained from $F_{i+1}M$ by *attaching free $\mathcal{R}$-cell modules* for each summand in $\mathcal{F}_i$. One defines *cell $\mathcal{A}$-modules* $M\varepsilon D(Mod(\mathcal{S}, \mathcal{A}))$ similarly.

(iii) A $CW\,\mathcal{R}$-module is a cell $\mathcal{R}$-module $M\varepsilon C(\mathcal{S}, \mathcal{R})$ so that the dimension of each of the summands $j_{U!}j_U^*(\mathcal{R})[s_U](t_U)$ in $F_iM/F_{i+1}M$ are *strictly greater* than the dimension of each of the summands $j_{U!}j_U^*(\mathcal{R})[s'_U](t'_U)$ in F_{i-1}/F_iM. One defines CW $\mathcal{A}$-modules similarly.

2.4. **Filtered derived categories.** $C(\mathcal{S},\mathcal{R})$ will denote the category considered in 2.1.2 and $\mathcal{A}$ will denote an E_∞-dga in $C(\mathcal{S},\mathcal{R})$. Next we will consider objects $M\varepsilon Mod(\mathcal{S},\mathcal{A})$ provided with *a decreasing filtration* F (i.e. $F_i(M) \subseteq F_{i-1}(M)$) indexed by the non-negative integers. Morphisms between two such objects will be morphisms in $Mod(\mathcal{S},\mathcal{A})$ that preserve the filtrations. $FMod(\mathcal{S},\mathcal{A})$ will denote this category of filtered objects and filtration-preserving maps. We will invert maps $f : M' \to M$ in $FMod(\mathcal{S},\mathcal{A})$ that induce quasi-isomorphisms on each F_i to define the corresponding derived category: this will be denoted $DFMod(\mathcal{S},\mathcal{A})$.

Observe that when $\mathcal{A} = \mathcal{R}$, we obtain the usual filtered derived category of complexes of $\mathcal{R}$-modules provided with decreasing filtrations.

This will be another of the basic situations we consider.

3. Construction of (standard) t-structures on sites provided with E_∞ sheaves of dgas

In this section we will provide each of the basic situations considered in the last section with standard t-structures.

Proposition 3.1. *Assume the hypotheses as in Proposition (2.4) and that* $\mathcal{A}\varepsilon C(\mathcal{S},\mathcal{R})$ *is a sheaf of* E_∞ *dgas. Let* $\mathbf{C} = \{j_{U!}(\mathcal{A}_{|U})[n] | U\varepsilon\mathcal{S}, n \geq 0\}$.

(i) Then $\mathbf{C}$ *is a set of compact objects in the triangulated category* $DMod(\mathcal{S},\mathcal{A})$.

Let $Coh(\mathbf{C})^{\leq 0}$ *denote the smallest sub-category of* $Mod(\mathcal{S},\mathcal{A})$ *containing all of* $\mathbf{C}$ *and closed under the following operations: (i) finite sums, (ii) mapping cones, (iii) translations* [1] *and (iv) extensions (i.e. if* $F' \xrightarrow{f} F \to Cone(f) \to F'[1]$ *is a distinguished triangle with* F', $Cone(f)$ *and* $F'[1]\varepsilon Coh(\mathcal{S})^{\leq 0}$, *then* $F\varepsilon Coh(\mathbf{C})^{\leq 0}$.

(ii) Let $QCoh(\mathbf{C})^{\leq 0}$ *denote the full sub-category of* $Mod(\mathcal{S},\mathcal{A})$ *consisting of filtered colimits* $\underset{\alpha}{\mathrm{colim}} F_\alpha$, *each* $F_\alpha\varepsilon Coh(\mathbf{C})^{\leq 0}$ *and with the indexing set for the filtered colimit being small. Then the smallest co-complete pre-aisle containing all of* $\mathbf{C}$ *identifies with* $QCoh(\mathbf{C})^{\leq 0}$.

(iii) Let $\mathrm{Comp}(\mathbf{C})$ *denote the full sub-category of* $QCoh(\mathbf{C})^{\leq 0}$ *containing* $Coh(\mathbf{C})^{\leq 0}$ *and closed under summands. Now* $\mathrm{Comp}(\mathbf{C})$ *identifies with the full sub-category of compact objects in* $QCoh(\mathbf{C})^{\leq 0}$.

Proof. The first assertion follows from Proposition 2.4. Clearly $QCoh(\mathbf{C})^{\leq 0}$ is closed under all small sums: any such sum may be written as a filtered colimit of finite sums. Next observe that the sub-category $Coh(\mathbf{C})^{\leq 0}$ consists of compact objects. Now it suffices to show that $QCoh(\mathbf{C})^{\leq 0}$ is closed under mapping cones, translations [1] and extensions. Let $F' = \underset{i}{\mathrm{colim}}\{F'_i | i\varepsilon I\} \to F = \underset{j}{\mathrm{colim}}\{F_i | i\varepsilon J\}$ denote a map of objects in $QCoh(\mathbf{C})^{\leq 0}$ with each F'_i, $F_i\varepsilon Coh(\mathbf{C})^{\leq 0}$. Since each F'_i is a compact object, one observes that for each $i\varepsilon I$, there exists an index $j_i\varepsilon J$ so that the map $F'_i \to F' \to \underset{j}{\mathrm{colim}}\{F_j | j\}$ factors through F_{j_i}. Therefore, after re-indexing $F = \{F_j | j\varepsilon J\}$ one may assume that both F' and F are indexed by the same indexing set I and the map f is given by a map $\{f_i : F'_i \to F_i | i\varepsilon I\}$. Clearly the mapping cone $Cone(f_i)\varepsilon Coh(\mathbf{C})^{\leq 0}$ and therefore $Cone(f) \cong \underset{\to\, i}{\lim}\, Cone(f_i)\varepsilon QCoh(\mathbf{C})^{\leq 0}$. Similarly one may show that $QCoh(\mathbf{C})^{\leq 0}$ is closed under the translations [1]. Next consider an extension: $F' \xrightarrow{f} F \xrightarrow{g} F'' \xrightarrow{h} F'[1]$ with F', $F''\varepsilon QCoh(\mathbf{C})^{\leq 0}$. Now F identifies with $Cone(h)[-1]$. Clearly the argument above shows that one may write the map h as $\underset{i}{\mathrm{colim}} h_i : F''_i \to F'_i[1]$; therefore one has a diagram of extensions $F'_i \to Cone(h_i)[-1] \to G_i \to F'_i[1]$. Therefore $Cone(h_i)[-1]\varepsilon Coh(\mathbf{C})^{\leq 0}$; since $F \cong \underset{i}{\mathrm{colim}} Cone(h_i)[-1]$, it follows that $F\varepsilon QCoh(\mathbf{C})^{\leq 0}$. Therefore $QCoh(\mathbf{C})^{\leq 0}$ is a co-complete pre-aisle containing all of $\mathbf{C}$. This proves (ii).

(iii). Let $F = \underset{\to\, i}{\lim} F_i$ denote an object in $QCoh(\mathbf{C})^{\leq 0}$ which is a compact object. Then the identity map of F must factor through some finite colimit $F' = \underset{\to\, j}{\lim} F_j$, so that F is a split summand of F' which clearly belongs to $Coh(\mathbf{C})^{\leq 0}$. This proves the last assertion. ☐

One of the main results we prove in this paper is the following:

Theorem 3.2. *Assume the hypotheses as in Proposition (2.4) and that* $\mathcal{A}\varepsilon C(\mathcal{S},\mathcal{R})$ *is a sheaf of* E_∞ *dgas. Let* $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ *denote the pre-aisle in* $DMod(\mathcal{S},\mathcal{A})$ *generated by* $j_{U!}jU^*(\mathcal{A}[n])$, $n \geq 0$, U *in the site* $\mathcal{S}$.

Then (i) $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ *is an aisle in* $DMod(\mathcal{S},\mathcal{A})$ *, i.e. defines a* t*-structure on* $DMod(\mathcal{S},\mathcal{A})$.

(ii) Assume next the hypotheses of 2.2 hold. Then (a) $j_{U!}(\mathcal{A}_{|U})\varepsilon DMod(\mathcal{S},\mathcal{A})^{\leq 0} \cap DMod(\mathcal{S},\mathcal{A})^{\geq 0}$ *= the heart of the above t-structure. (b) Moreover, every object M in* $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ *satisfies the property that the natural map* $\tau_{\leq 0}(\mathcal{R}\overset{L}{\underset{\mathcal{A}}{\otimes}}M) \rightarrow \mathcal{R}\overset{L}{\underset{\mathcal{A}}{\otimes}}M$ *is a quasi-isomorphism in* $DMod(\mathcal{S},\mathcal{R})$*. In other words, the functor* $\mathcal{R}\overset{L}{\underset{\mathcal{A}}{\otimes}}(\quad) : DMod(\mathcal{S},\mathcal{A}) \rightarrow DMod(\mathcal{S},\mathcal{R})$ *sends* $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ *to* $DMod(\mathcal{S},\mathcal{R})^{\leq 0}$*.*

Proof. The first statement is clear from the last proposition in view of Theorem 1.8. We will now prove the remaining statements. To prove (a), observe that $RHom_{\mathcal{A}}(j_{U!}(\mathcal{A}_{|U})[n], j_{V!}(\mathcal{A}_{|V})) \simeq R\Gamma(U\underset{X}{\times}V, \mathcal{A}_{U\underset{X}{\times}V}[-n])$ so that

$$Hom_{DMod(\mathcal{S},\mathcal{A})}(j_{U!}(\mathcal{A}_{|U})[n], j_{V!}(\mathcal{A}_{|V})) = H^0(RHom_{\mathcal{A}}(j_{U!}(\mathcal{A}_{|U})[n], j_{V!}(\mathcal{A}_{|V})) = H^0(R\Gamma(U\underset{X}{\times}V, \mathcal{A}_{U\underset{X}{\times}V}[-n])) = 0$$

for any $n \geq 1$. The last equality follows from the hypothesis that $\mathcal{A}^i = 0$ for all $i < 0$. This proves $j_{V!}(\mathcal{A}_V)\varepsilon(DMod(\mathcal{S},\mathcal{A})^{\leq -1})^{\perp} = DMod(\mathcal{S},\mathcal{A})^{\geq 0}$.

Since $j_{V!}(\mathcal{A}_V)\varepsilon DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ by definition, the assertion (a) in (ii) is proved.

By the definition of $D(Mod(\mathcal{S},\mathcal{A})^{\leq 0}$ above (and Proposition 3.1 above), there exist a sequence $\{M_i|i\varepsilon I\}$ in $Coh(\mathbf{C})^{\leq 0}$ so that $M \cong \underset{i}{\mathrm{colim}}\{M_i|i\}$.

One of our key observations now is that each $M_i\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R}\varepsilon Coh(\mathcal{S},\mathcal{R})^{\leq 0}$. This is clear if $M_i = j_{U!}(\mathcal{A}_{|U}[n])$ for some $U\varepsilon\mathcal{S}$ and $n \geq 0$: in this case $M\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R} \cong j_{U!}(\mathcal{R}_{|U}[n])$. In general, recall that M is obtained by finitely many operations from the set $\{j_{U!}(\mathcal{A}_{|U})[n]|U\varepsilon\mathcal{S}, n \geq 0\}$ where the allowed operations are finite sums, mapping cones, translations [1] and extensions. Since $Coh(\mathcal{S},\mathcal{R})^{\leq 0}$ is closed under these operations, one may show readily that each $M_i\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R}\varepsilon Coh(\mathcal{S},\mathcal{R})^{\leq 0}$. Next recall that $QCoh(\mathcal{S},\mathcal{R})^{\leq 0}$ is the set of objects obtained as filtered colimits of objects in $Coh(\mathcal{S},\mathcal{R})^{\leq 0}$. Therefore $M\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R} = (\underset{i}{\mathrm{colim}}M_i)\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R} \cong \underset{i}{\mathrm{colim}}(M_i\overset{L}{\underset{\mathcal{A}}{\otimes}}\mathcal{R})\varepsilon QCoh(\mathcal{S},\mathcal{R})f^{\leq 0}$. This completes the proof of the second statement and hence that of the theorem. □

Remark 3.3. Next recall from Definitions 2.2(iii) that in case $\mathcal{R}$ denotes the constant sheaf of sphere spectra, and $\mathcal{A}\varepsilon C(\mathcal{S},\mathcal{R})$ is any E_∞-ring object, there is an augmentation $\mathbb{Z}(\mathcal{A}) \rightarrow \mathbb{Z}$ of sheaves of E_∞-algebras. Moreover the functor $\mathbb{Z} : D(Mod(\mathcal{S},\mathcal{A})) \rightarrow D(Mod(\mathcal{S},\mathbb{Z}(\mathcal{A}))$ is a functor of triangulated categories and $\mathbb{Z}(\mathcal{A})\varepsilon C(\mathcal{S},\mathbb{Z})$ is an E_∞ ring-object.

In this case, every object M in $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ satisfies the property that the natural map $\tau_{\leq 0}(\underline{\mathbb{Z}}\overset{L}{\underset{\mathbb{Z}(\mathcal{A})}{\otimes}}\mathbb{Z}(M)) \rightarrow \underline{\mathbb{Z}}\overset{L}{\underset{\mathbb{Z}(\mathcal{A})}{\otimes}}\mathbb{Z}(M)$ is a quasi-isomorphism in $DMod(\mathcal{S},\underline{\mathbb{Z}})$. In other words, the functor $\underline{\mathbb{Z}}\overset{L}{\underset{\mathbb{Z}(\mathcal{A})}{\otimes}}(\quad) : DMod(\mathcal{S},\mathcal{A}) \rightarrow DMod(\mathcal{S},\underline{\mathbb{Z}})$ sends $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ to $DMod(\mathcal{S},\underline{\mathbb{Z}})^{\leq 0}$.

Examples 3.4. Observe that the theorem applies to all the examples considered in 2.5.

Remark 3.5. It does not seem possible to say, in general, that $DMod(\mathcal{S},\mathcal{A})^{\leq 0}$ is the full sub-category of $DMod(\mathcal{S},\mathcal{A})$ that the functor $\mathcal{R}\overset{L}{\underset{\mathcal{A}}{\otimes}}(\quad)$ sends to $DMod(\mathcal{S},\mathcal{R})^{\leq 0}$. Nevertheless, the above theorem shows that one has meaningful, non-trivial t-structures defined on the category $DMod(\mathcal{S},\mathcal{A})$.

In the next section we proceed to define and study the notion of *constructibility* in the category $D(Mod(\mathcal{S},\mathcal{A}))$, in particular how the t-structures defined above relate to the full sub-category of constructible objects.

Definition 3.6. Assume the above situation. We will define $\sigma_{\leq n} : DMod(\mathcal{S},\mathcal{A}) \rightarrow DMod(\mathcal{S},\mathcal{A})^{\leq n}$ as right-adjoint to the obvious inclusion $DMod(\mathcal{S},\mathcal{A})^{\leq n} \subseteq DMod(\mathcal{S},\mathcal{A})$. Let $K\varepsilon DMod(\mathcal{S},\mathcal{A})$. We will define $\sigma_{\geq n+1}(K)$ by requiring $\sigma_{\leq n}(K) \rightarrow K \rightarrow \sigma_{\geq n+1}(K) \rightarrow \sigma_{\leq n}K[1]$ to be a distinguished triangle. Then $\sigma_{\geq n+1}$ will be right-adjoint to the obvious inclusion $DMod(\mathcal{S},\mathcal{A})^{\geq n+1} \rightarrow DMod(\mathcal{S},\mathcal{A})$.

3.1. **Constructibility and t-structures.**

Definition 3.7. In general, we will say that a sheaf of $\mathcal{A}$-modules M is *of finite type* if *is a compact object* in $DMod(\mathcal{S},\mathcal{A})$. In case $\mathcal{R}$ is the constant sheaf of rings associated to a commutative ring R, we will say that a sheaf of $\mathcal{A}$-modules M is *a constructible $\mathcal{A}$-module*, if it is of finite type. The full sub-category of compact objects in $DMod(\mathcal{S}, \mathcal{A})$ will be denoted $D_c(Mod(\mathcal{S}, \mathcal{A}))$.

Proposition 3.8. *The truncation functors $\sigma_{\leq n}$ and $\sigma_{\geq n+1}$ as in Definition 3.6 preserve compactness and hence the property of being of finite type as well as being constructible.*

Proof. In view of the fact that $\sigma_{\leq n}(K) \rightarrow K \rightarrow \sigma_{\geq n+1}(K) \rightarrow \sigma_{\leq n}(K)[1]$ is a distinguished triangle for any $K \varepsilon DMod(\mathcal{S}, \mathcal{A})$, it suffices to prove this for the functor $\sigma_{\geq 0}$.

First observe that the filtered colimit of a collection $\{K_i | i \varepsilon I\}$ of objects $K_i \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$ also belongs to $DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$. To see this, consider

$$Hom_{DMod(\mathcal{S},\mathcal{A})}(j_{U!}(\mathcal{A}_{|U})[n], \lim_{\rightarrow i} K_i) = H^0(RHom_{\mathcal{A}}(j_{U!}(\mathcal{A}_{|U})[n], \lim_{\rightarrow i} K_i))$$

$$\cong H^0(R\Gamma(U, \lim_{\rightarrow i} j_U^*(K_i)[-n])) \cong \lim_{\rightarrow i} H^0(R\Gamma(U, j_U^*(K_i)[-n]) = \lim_{\rightarrow i} H^0(RHom_{\mathcal{A}}(j_{U!}\mathcal{A}_{|U})[n], K_i)) = 0$$

for all $n \geq 1$. The last equality is from the hypothesis that each $K_i \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$ and the isomorphism prior to that follows from our hypotheses on the site as in 2.1.

Let $\phi : DMod(\mathcal{S}, \mathcal{A})^{\geq 0} \rightarrow DMod(\mathcal{S}, \mathcal{A})$ denote the obvious inclusion functor. Clearly ϕ commutes with filtered colimits since the former category is a full sub-category of the latter and the former category is itself closed under the formation of filtered colimits as we just showed.

Next let $M \varepsilon DMod(\mathcal{S}, \mathcal{A})$ denote a compact object and let $\{K_i | i \varepsilon I\}$ be a collection of objects $K_i \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$. Then:

$$RHom_{\mathcal{A}}(\sigma_{\geq 0}(M), \lim_{\rightarrow i} K_i) \cong RHom_{\mathcal{A}}(M, \phi(\lim_{\rightarrow i} K_i))$$

$$\cong RHom_{\mathcal{A}}(M, \lim_{\rightarrow i} \phi(K_i)) \cong \lim_{\rightarrow i} RHom_{\mathcal{A}}(M, \phi(K_i))$$

$$\cong \lim_{\rightarrow i} RHom_{\mathcal{A}}(\sigma_{\geq 0}(M), K_i).$$

These prove that $\sigma_{\geq 0}$ preserves compactness. $\square$

Corollary 3.9. *In case $\mathcal{A}$ is provided with an augmentation $\mathcal{A} \rightarrow \mathcal{R}$, then the functor $M \mapsto M \overset{L}{\underset{\mathcal{A}}{\otimes}} \mathcal{R}$, $DMod(\mathcal{S}, \mathcal{A}) \rightarrow DMod(\mathcal{S}, \mathcal{R})$ induces a functor $D_c(Mod(\mathcal{S}, \mathcal{A}))^{\leq 0} \rightarrow D_c(Mod(\mathcal{S}, \mathcal{R}))^{\leq 0}$.*

Proof. In view of Theorem 3.2 (ii)(b), it suffices to show that the above functor preserves compactness for objects in $QCoh(\mathbf{C})^{\leq 0}$ as in Proposition 3.1. As shown there, the compact objects in $QCoh(\mathcal{S}, \mathcal{A})^{\leq 0}$ identify with split summands of objects in $Coh(\mathcal{S}, \mathcal{A})^{\leq 0}$. Clearly the functor $M \mapsto M \overset{L}{\underset{\mathcal{A}}{\otimes}} \mathcal{R}$ sends split summands of objects in $Coh(\mathcal{S}, \mathcal{A})^{\leq 0}$ to split summands of objects in $Coh(\mathcal{S}, \mathcal{R})^{\leq 0}$. $\square$

Let $\mathcal{C} = DMod(\mathcal{S}, \mathcal{A})^{\leq 0} \cap DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$ denote the heart of the above t-structure on $DMod(\mathcal{S}, \mathcal{A})$. Observe that this is an abelian category. Let $\mathcal{C}_c = \mathcal{C} \cap D_c(Mod(\mathcal{S}, \mathcal{A}))$, i.e. the full sub-category of all objects in the heart that are also compact.

Theorem 3.10. *$\mathcal{C}_c$ is an additive sub-category of $\mathcal{C}$ closed under extensions.*

Proof. This is clear since any short-exact sequence $M' \rightarrow M \rightarrow M''$ in $\mathcal{C}$ corresponds to a distinguished triangle in $DMod(\mathcal{S}, \mathcal{A})$ with each M', M and M'' in $\mathcal{C}$. Moreover, in such a short-exact sequence $M' \rightarrow M \rightarrow M''$, M is compact if both M' and M'' are. $\square$

3.2. **Example: Non-standard t-structures, generalized perverse sheaves and perverse extensions.** In this section we will show briefly how to define generalized perverse sheaves and perverse extensions of generalized perverse sheaves.

Assume one is given a stratified site, $\mathcal{S}$, i.e. one is provided with a collection of locally closed smooth sub-objects X_i of the terminal object X of X. Let $\mathcal{S}'_i = \mathcal{S} \underset{X}{\times} X_i$. By taking the unions of the strata one defines a finite increasing filtration $\mathcal{S}_0 \overset{j_1}{\rightarrow} \mathcal{S}_1 \overset{j_2}{\rightarrow} \cdots \overset{j_n}{\rightarrow} \mathcal{S}_n = \mathcal{S}$, where each j_i is an open immersion. Let a (perversity) function $p : \{S_{i+1} - S_i | i\} \rightarrow (integers)$ be given. Let $\mathcal{A}$ denote sheaf of E_∞ dgas on $\mathcal{S}$. Now $DMod(\mathcal{S}, \mathcal{A})$ and $DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{\mathcal{S}_i - \mathcal{S}_{i-1}})$ will denote the obvious derived categories of $\mathcal{A}$-modules. Assume that every object in each of these categories has finite cohomological dimension for each object in the corresponding site. Now one may glue together standard t-structures on each stratum, shifted suitably by the perversity $p(\mathcal{S}_i - \mathcal{S}_{i-1})$ to define a non-standard t-structure on $DMod(\mathcal{S}, \mathcal{A})$ exactly as in [BBD, Chapter 1]. Accordingly we will define

$$DMod(\mathcal{S}, \mathcal{A})^{\leq 0} = \{K \varepsilon DMod(\mathcal{S}, \mathcal{A}) | j_i^*(K) \varepsilon DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{|\mathcal{S}_i - \mathcal{S}_{i-1}})^{\leq p(\mathcal{S}_i - \mathcal{S}_{i-1})}\}$$
$$DMod(\mathcal{S}, \mathcal{A})^{\geq 0} = \{K \varepsilon DMod(\mathcal{S}, \mathcal{A}) | j_i^!(K) \varepsilon DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{\mathcal{S}_i - \mathcal{S}_{i-1}})^{\geq p(\mathcal{S}_i - \mathcal{S}_{i-1})}\} \tag{3.2.1}$$

Theorem 3.11. *(i) The above structures define a t-structure on $DMod(\mathcal{S}, \mathcal{A})$ with the aisle $DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ and co-aisle $DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$.*

(ii) Given a generalized perverse sheaf $P_0 \varepsilon DMod(\mathcal{S}_0, \mathcal{A}_{|\mathcal{S}_0})^{\leq 0} \cap DMod(\mathcal{S}_0, \mathcal{A}_{|\mathcal{S}_0})^{\geq 0}$, there exists extensions $P \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\leq 0} \cap DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$ of P_0, i.e. $j_0^*(P) \simeq P$.

(iii) Given a generalized perverse sheaf $P_0 \varepsilon DMod(\mathcal{S}_0, \mathcal{A}_{|\mathcal{S}_0})^{\leq 0} \cap DMod(\mathcal{S}_0, \mathcal{A}_{|\mathcal{S}_0})^{\geq 0}$, the extension $P \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\leq 0} \cap DMod(\mathcal{S}, \mathcal{A})^{\geq 0}$ of P_0 is unique if $j_i^*(P) \varepsilon DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{|\mathcal{S}_i - \mathcal{S}_{i-1}})^{\leq p(\mathcal{S}_i - \mathcal{S}_{i-1}) - 1}$ and $j_i^!(P) \varepsilon DMod(\mathcal{S}_i - \mathcal{S}_{i-1}, \mathcal{A}_{|\mathcal{S}_i - \mathcal{S}_{i-1}})^{\leq p(\mathcal{S}_i - \mathcal{S}_{i-1}) + 1}$.

Proof. This is essentially the argument in [BBD, Theorem 1.4.10]: we provide some details mainly for the sake of completeness. We will restrict to the case where there are only two strata, i.e. $\mathcal{S}_1 = \mathcal{S}$. Let $K \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\leq 0}$ and $L \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\geq 1}$. To show $Hom(K, L) = 0$, one may argue as follows. First one observes the existence of the distinguished triangle:

$$j_!(j^*(K)) \rightarrow K \rightarrow i_* i^*(K) \rightarrow j_! j^*(K)[1]$$

where $j : \mathcal{S}_0 \rightarrow \mathcal{S}$ and $i : \mathcal{S} - \mathcal{S}_0 \rightarrow \mathcal{S}$ are the obvious maps. This provides us with the long-exact-sequence:

$$\cdots \rightarrow H^0(RHom_{\mathcal{A}}(i_* i^*(K), L)) \rightarrow H^0(RHom_{\mathcal{A}}(K, L)) \rightarrow H^0(RHom_{\mathcal{A}}(j_! j^*(K), L)) \rightarrow H^1(RHom_{\mathcal{A}}(i_* i^*(K), L)) \cdots$$

Now $H^0(RHom_{\mathcal{A}}(i_* i^*(K), L)) \cong H^0(RHom_{\mathcal{A}}(i^*(K), i^!(L)) \cong 0$ and $H^0(RHom_{\mathcal{A}}(j_! j^*(K), L))$ $\cong H^0(RHom_{\mathcal{A}}(j^*(K), j^*(L))) \cong 0$ by our hypotheses. Therefore $H^0(RHom_{\mathcal{A}}(K, L)) = 0$ as well. It is clear from the definitions that $DMod(\mathcal{S}, \mathcal{A})^{\leq n} \subseteq DMod(\mathcal{S}, \mathcal{A})^{\leq n+1}$ and similarly $DMod(\mathcal{S}, \mathcal{A})^{\geq n+1} \subseteq DMod(\mathcal{S}, \mathcal{A})^{\geq n}$.

We define the functor $\sigma_{\leq 0}$ as follows. Let $K \varepsilon DMod(\mathcal{S}, \mathcal{A})$ be given. We let $Y =$ the canonical homotopy fiber of the map $K \rightarrow Rj_*(\sigma_{\geq 1} j^*(K))$. Then we define $\sigma_{\leq 0}(K) =$ the canonical homotopy fiber of the map $Y \rightarrow i_*(\sigma_{\geq 1} i^*(Y))$. $\sigma_{\geq 1}(K)$ is defined to be the mapping cone of the obvious map $\sigma_{\leq 0}(K) \rightarrow K$. These prove (i).

Give P_0 as in (ii), we let $P = \sigma_{\leq 0}(Rj_*(P_0))$. To see this is an extension of P_0 we proceed as follows. Let $\phi_i : DMod(\mathcal{S}_i, \mathcal{A}_{|\mathcal{S}_i})^{\leq 0} \rightarrow DMod(\mathcal{S}_i, \mathcal{A}_{|\mathcal{S}_i})$ denote the obvious (inclusion). We will denote ϕ_1 by ϕ. The definition of the above aisles shows that $\phi \circ j_! = j_! \circ \phi_0$. Since j^* ($\sigma_{\leq 0}$) is right-adjoint to $j_!$ (ϕ, respectively), it follows that j^* commutes with $\sigma_{\leq 0}$. Since $j^* \circ Rj_* =$ the identity, we observe that $j^*(P) \simeq \sigma_{\leq 0}(P_0) \simeq P_0$ where the last identification is by the assumptions on P_0.

To show P is in fact a generalized perverse sheaf, one first observes the distinguished triangle: $i_* Ri^!(\sigma_{\leq 0} Rj_*(P_0)) \rightarrow \sigma_{\leq 0} Rj_*(P_0) \rightarrow Rj_*(P_0) \rightarrow i_* Ri^!(\sigma_{\leq 0} Rj_*(P_0))[1]$. (The identification $Rj_*(j^*(\sigma_{\leq 0}(Rj_*(P_0)))) \simeq Rj_*(P_0)$ follows from the observation above that $\sigma_{\leq 0} Rj_*(P_0)$ is an extension of P_0.) One also has a distinguished triangle $\sigma_{\leq 0} Rj_*(P_0) \rightarrow Rj_*(P_0) \rightarrow \sigma_{\geq 1} Rj_*(P_0) \rightarrow \sigma_{\leq 0} Rj_*(P_0)[1]$. Therefore, one obtains the quasi-isomorphism $\sigma_{\geq 1}(Rj_*(P_0) \simeq i_* Ri!(\sigma_{\leq 0} Rj_*(P_0))[1]$ and hence $Ri^!(\sigma_{\leq 0} Rj_*(P_0)) \varepsilon DMod(\mathcal{S} - \mathcal{S}_0, \mathcal{A}_{|\mathcal{S} - \mathcal{S}_0})^{\geq 0}$. This proves the required assertion and completes the proof of (ii).

(iii) Let P denote a generalized perverse sheaf extending the perverse sheaf $P_0 \varepsilon DMod(\mathcal{S}_0, \mathcal{A}_{|\mathcal{S}_0})^{\leq 0} \cap DMod(\mathcal{S}_0, \mathcal{A}_{|\mathcal{S}_0})^{\geq 0}$ satisfying the hypotheses in (iii). The key diagram is:

$$\begin{tikzcd} & & i_* i^*(P) \arrow[dr] & & \\ & P \arrow[ur] \arrow[dr] & & i_*(Rj_*/j_!)j^*(P) \arrow[dr] & \\ j_!(j^*(P)) \arrow[ur] & & Rj_*j^*(P) \arrow[ur] & & i_*Ri^!(P)[1] \end{tikzcd}$$

where the left-side, right-side and the diagram $P \to Rj_*j^*(P) \to i_*(Rj_*/j_!)j^*(P) \to P[1]$ are distinguished triangles. Now the hypotheses imply that $i^*(P)\varepsilon DMod(\mathcal{S}-\mathcal{S}_0, \mathcal{A}_{\mathcal{S}-\mathcal{S}_0})^{\leq p(\mathcal{S}-\mathcal{S}_0)-1}$ and that $Ri^!(P)\varepsilon DMod(\mathcal{S}-\mathcal{S}_0, \mathcal{A}_{\mathcal{S}-\mathcal{S}_0})^{\geq p(\mathcal{S}-\mathcal{S}_0)+1}$. Therefore, the following lemma shows that $i_*Ri^!(P)[1] \simeq i_*(\sigma_{\geq p(\mathcal{S}-\mathcal{S}_0)}(Rj_*/j_!)j^*(P)) \simeq i_*(\sigma_{\geq p(\mathcal{S}-\mathcal{S}_0)}Rj_*j^*(P))$. This implies that P identifies with the canonical homotopy fiber of the map $Rj_*(P_0) \to i_*\sigma_{\geq p(\mathcal{S}-\mathcal{S}_0)}i^*(Rj_*(P_0))$. Therefore it is unique. This completes the proof of the theorem. $\square$

Lemma 3.12. *Let $A \to B \to C \to A[1]$ denote a distinguished triangle in $DMod(\mathcal{S}, \mathcal{A})$ and let n be an integer so that the natural map $C \to \sigma_{\geq n}(C)$ is a quasi-isomorphism. Then the natural map $\sigma_{\leq n-1}(A) \to \sigma_{\leq n-1}(B)$ is a quasi-isomorphism.*

Proof. Let $K\varepsilon DMod(\mathcal{S}, \mathcal{A})^{\leq n-1}$. Since $C \to \sigma_{\geq n}C$ is a quasi-isomorphism, it follows that $Hom_{DMod(\mathcal{S},\mathcal{A})}(K, C) = 0 = Hom_{DMod(\mathcal{S},\mathcal{A})}(K, C[-1])$. Therefore the map $A \to B$ induces an isomorphism
$Hom_{DMod(\mathcal{S},\mathcal{A})}(K, A) \overset{\cong}{\to} Hom_{DMod(\mathcal{S},\mathcal{A})}(K, B)$.

Now the definition of the functor $\sigma_{\leq n-1}$ as right-adjoint to the inclusion $DMod(\mathcal{S}, \mathcal{A})^{\leq n-1} \to DMod(\mathcal{S}, \mathcal{A})$ shows that the induced map $\sigma_{\leq n-1}(A) \to \sigma_{\leq n-1}(B)$ is a quasi-isomorphism. $\square$

4. **Cell and CW-cell modules**

Throughout this section we will assume the basic hypotheses as in 2.3, i.e. $\mathcal{A}$ is -1-connected or equivalently $\mathcal{H}^s(\mathcal{A}) = 0$ for all $s > 0$. We will assume that $\mathcal{R}$ is a constant sheaf. In this section we develop the basic theory of cell and CW cell-modules over a sheaf of E_∞-dgas.

Now the following basic results show that the theory of $CW-\mathcal{A}$-modules is indeed similar to the homotopy theory of CW-complexes: see, for example, [Gray, Chapter 16].

4.0.2. *Convention.* : Henceforth, we will denote $j_{U!}j_U^*(\mathcal{R})$ $(j_{U!}j_U^*(\mathcal{A}))$ by $\mathcal{R}_U$ $(\mathcal{A}_U$, respectively).

Proposition 4.1. *Throughout let P, Q be $\mathcal{A}$-modules. (i) Let $g : \mathcal{A}_U[n-1] \to P$, $f : P \to R$ be $\mathcal{A}$-maps and assume $\mathcal{H}^i(Q) = 0$ for $i = -n+1$. Then there exists a covering $\{V_\alpha \to U|\alpha\}$ of U so that each $f_{|V_\alpha}$ extends to a map $Cone(g_{|V_\alpha}) \to R_{|V_\alpha}$ where $Cone(g_{|V_\alpha}))$ denotes the mapping cone of $g_{|V_\alpha}$. (In this case we say that f extends locally to a map from $Cone(g)$ to R.)*

(ii) Let S denote a set of integers and (P, Q) a relative $CW-\mathcal{A}$-module, i.e. Q is obtained from P by attaching free $\mathcal{A}$-modules $\mathcal{A}_U[n_{\alpha,U}]$ with $n_{\alpha,U}\varepsilon S$. Suppose R is an $\mathcal{A}$-module so that $\mathcal{H}^{-i}(R) = 0$ for all $i\varepsilon S$. Then any map $f : P \to R$ of $\mathcal{A}$-modules admits a local extension $\tilde{f} : Q \to R$, i.e. there exists a covering $\{V_\alpha \to \mathcal{S}|\alpha\}$ so that each restriction $f_{|V_\alpha} : P_{|V_\alpha} \to R_{|V_\alpha}$ extends to a map $\tilde{f}_{|V_\alpha} : Q_{|V_\alpha} \to R_{|V_\alpha}$.

(iii) Suppose that there exists a covering $\{V_\alpha \to \mathcal{S}\}$ so that $(P_{|V_\alpha}, Q_{|V_\alpha})$ is a relative $CW-\mathcal{A}$-module in the above sense so that $Q_{|V_\alpha}$ is obtained from $P_{|V_\alpha}$ by attaching free-$\mathcal{A}$-cells in dimensions $\leq -n$. Then $\mathcal{H}^i(Q/P) = 0$ for all $i > -n$.

(iv) If Q is a CW-$\mathcal{A}$-module so that $F_{i+1}(Q) = F_i(Q)$ for all $i \geq -n+1$, then $\mathcal{H}^i(Q) = 0$ for all $i > -n$.

Proof. For each point p of the site $\mathcal{S}$, the only obstruction to extending f_p to $Cone(g_p)$ is that the composition $f_p \circ g_p$ be null-homotopic: this is clear since $H^0(RHom_{\mathcal{A}_p}(\mathcal{A}_p[n-1], Q_p)) = H^{-n+1}(Q_p) = 0$ by the hypothesis. This proves (i). To prove (ii) one uses (i) as a starting point to handle the case when Q_p is obtained from P_p by attaching a single $\mathcal{A}_p$-cell. In general one uses Zorn's lemma as in [Gray, Corollary 16.3].

(iii) It is enough to assume that Q is obtained by attaching finitely many free $\mathcal{A}$-cells to P. In this case one uses an ascending induction on the number of these cells and the exact sequence $\mathcal{H}^i(Q'/P) \to \mathcal{H}^i(Q/P) \to \mathcal{H}^i(Q/Q')$ where Q' is obtained from P by attaching one *less* free $\mathcal{A}$-cell. Observe that key use is made of the hypothesis that $\mathcal{H}^i(\mathcal{A}) = 0$ for all $i > 0$: in fact, the last assertion is false if this is not the case. Clearly (iii) implies (iv). $\square$

Theorem 4.2. *Let $\mathcal{M} \varepsilon DMod(\mathcal{S}, \mathcal{A})$ so that $\mathcal{M}$ is connected in the above sense. Then there exists a CW-cell $\mathcal{A}$-module $P(M) \varepsilon DMod(\mathcal{S}, \mathcal{A})$ with a map $P(M) \to M$ which is a quasi-isomorphism. (We say $P(M) \to M$ is a CW-$\mathcal{A}$-resolution.) Moreover, if $M' \to M$ is a map between two such objects in $DMod(\mathcal{S}, \mathcal{A})$, there exist CW-$\mathcal{A}$-resolutions $P(M') \to M'$, $P(M) \to M$ and a map $P(M') \to P(M)$ preserving the given filtrations so that one obtains a commutative square*

$$\begin{array}{ccc} P(M') & \longrightarrow & P(M) \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M \end{array}$$

Proof. Assume that $\mathcal{H}^i(M) = 0$ and $\mathcal{H}^i(M') = 0$ for all $i > N$. For each class $[\alpha_N] \varepsilon \mathcal{H}^N(M)$, let $\alpha_N : \mathcal{A}_U[-N] \to j_{U!}j_U^*(M) \to M$ denote a map representing $[\alpha_N]$. Now let $P_N(M) = \underset{[\alpha_N]\varepsilon \mathcal{H}^N(M)}{\oplus} \mathcal{A}_U[-N]$: we will map this to M by mapping the summand indexed by $[\alpha_N]$ by the corresponding map α_N to M. We will denote this map $P_N(M) \to M$ by $p_N(M)$.

Consider the cone $\mathcal{A}$-module $Cone(P_N(M))$ and also the mapping cone $Cone(p_N(M))$. Observe that one has the distinguished triangle: $P_N(M) \overset{p_N(M)}{\to} M \to Cone(p_N(M)) \to P_N(M)[1]$ which results in the long-exact sequence:

$$\cdots \to \mathcal{H}^i(P_N(M)) \to \mathcal{H}^i(M) \to \mathcal{H}^i(Cone(p_N(M))) \to \mathcal{H}^{i+1}(P_N(M)) \to \cdots$$

Since $\mathcal{H}^{N+k}(P_N(M)) = 0$ for all $k > 0$ and $\mathcal{H}^N(P_N(M)) \to \mathcal{H}^N(M)$ is a surjection by our choice of $P_N(M)$, it follows that

$$\mathcal{H}^i(Cone(p_N(M)) = 0, i \geq N \tag{4.0.3}$$

i.e. the map $p_N : P_N(M) \to M$ is an N-equivalence.

4.0.4. Now replace M by $Cone(p_N(M))$ and for each class $[\alpha_{-N+1}] \varepsilon \mathcal{H}^{N-1}(Cone(p_N(M))$, let $\alpha_{-N+1} : \mathcal{A}_{U_{\alpha_{-N+1}}}[-N+1] \to Cone(p_N(M)) = Cyl(p_N(M))/P_N(M)$ denote a representative.

The map α_{-N+1} may be viewed as a map of pairs $(Cone(\mathcal{A}_{U_{\alpha_{-N+1}}}[-N]), \mathcal{A}_{U_{\alpha_{-N+1}}}[-N]) \to (Cyl(p_N(M)), P_N(M))$. Observe that $\mathcal{A}_{U_{\alpha_{-N+1}}}[-N]$ maps naturally to $Cone(\mathcal{A}_{U_{\alpha_{-N+1}}}[-N]$ with the cokernel $\simeq \mathcal{A}_{U_{\alpha_{-N+1}}}[-N+1]$. Let

$$\underset{[\alpha_{-N+1}]\varepsilon \mathcal{H}^{N-1}(Cone(p_N(M)))}{\oplus} \alpha_{-N+1} : \underset{[\alpha_{-N+1}]\varepsilon \mathcal{H}^{N-1}(Cone(p_N(M)))}{\oplus} (Cone(A_{U_{\alpha_{-N+1}}}[-N], \mathcal{A}_{U_{\alpha_{-N+1}}}[-N])$$
$$\to (Cyl(p_N(M)), P_N(M))$$

denote the obvious map.

We let $P_{N-1}(M) = (P_N(M) \oplus \oplus_{\alpha_{-N+1}} Cone(\mathcal{A}_{U_{\alpha_{-N+1}}}[-N]))/ \sim$: here $\sim$ is the relation where we identify the summand $\mathcal{A}_{U_{\alpha_{-N+1}}}[-N]$ of the corresponding $Cone(U_{\alpha_{-N+1}}[-N])$ with its image in $P_N(M)$. We map the pair $(P_{N-1}(M), P_N(M))$ to $(Cyl(p_N(M)), M)$ by mapping the summand $P_N(M)$ by the obvious inclusion into $Cyl(p_N(M))$ and the summand $Cone(\mathcal{A}_{U_{\alpha_{-N+1}}}[-N])$ by the map α_{-N+1}. We will denote this map by $p_{N-1}(M)'$. Let $\pi_N : Cyl(p_N(M)) \to M$ denote the obvious map and let $p_{N-1}(M) = \pi_N \circ p_{N-1}(M)'$. Then $p_{N-1}(M)_{|P_N(M)} = p_N(M)$.

Let $Cyl(p_{N-1}(M)')$ denote the mapping cylinder of $p_{N-1}(M)'$. Now one has the long-exact-sequence:

$$\cdots \to \mathcal{H}^{i-1}(Cyl(p_{N-1}(M)')/P_{N-1}(M)) \to \mathcal{H}^i(P_{N-1}(M)/P_N(M)) \overset{\mathcal{H}^i(p_{N-1}(M)')}{\to} \mathcal{H}^i(Cyl(p_{N-1}(M)')/P_N(M)) \cong \mathcal{H}^i(Cyl(p_N(M)/P_N(M))) \to \cdots$$

By construction $\mathcal{H}^{N-1}(p_{N-1}(M)')$ is surjective, $\mathcal{H}^i(Cyl(p_N(M)/P_N(M))) = \mathcal{H}^i(Cone(p_N(M))) = 0$ for all $i \geq N$ and $\mathcal{H}^i(P_{N-1}(M)/P_N(M)) = 0$ for all $i > N-1$. Therefore, it follows that $\mathcal{H}^i(Cyl(p_{N-1}(M)')/P_{N-1}(M)) = 0$ for all $i \geq N-1$. i.e. the map $p_{N-1}(M)$ is an $N-1$-equivalence.

We may therefore, continue the inductive construction and define $P_k(M)$ as a CW-cell $\mathcal{A}$-module provided with a map $p_k(M) : P_k(M) \to M$, $k \leq N$ which is a k-equivalence. Finally one lets $P(M) = \underset{k \to -\infty}{\operatorname{colim}} P_k(M)$ along with the map $p(M) : P(M) \rightarrow M$ defined as $\underset{k \rightarrow -\infty}{\operatorname{colim}} p_k(M)$. One verifies immediately that $p(M)$ is a quasi-isomorphism: clearly $P(M)$ is a CW-cell $\mathcal{A}$-module. This proves the first statement in the theorem.

To make the construction of CW-$\mathcal{A}$-resolutions functorial, we will need to make the following modifications to the arguments above. Instead of choosing representative cohomology classes as in (4.0.3), we choose all possible maps $j_{U!}j_U^*(\mathcal{A})[-N] \rightarrow M$, for all U in the site $\mathcal{S}$. (Since our site $\mathcal{S}$ is assumed to be essentially small this causes no difficulties.) We will then need to repeat the same construction in 4.0.4 and at every stage of the inductive process. Next observe that since both M' and M are assumed to be connected, we may choose a large enough N so that $\mathcal{H}^i(M) = \mathcal{H}^i(M') = 0$ for all $i > N$. Now the construction of the resolution $P(M)$ is made functorial in M. Moreover the construction of the CW-$\mathcal{A}$ resolution by descending induction shows that the induced map $P(f) : P(M') \rightarrow P(M)$ preserves the CW-filtration. This proves the second statement in the theorem. □

Definition 4.3. Let $Mod^{cw}(\mathcal{S}, \mathcal{A})$ denote the category whose objects are all $CW - \mathcal{A}$-modules and morphisms are morphisms that preserve the given cell-filtrations. A morphism between two $CW - \mathcal{A}$-modules will be called a quasi-isomorphism if it is a quasi-isomorphism in the underlying category $DMod(\mathcal{S}, \mathcal{A})$. The corresponding derived category obtained by inverting these quasi-isomorphisms will be denoted $DMod^{cw}(\mathcal{S}, \mathcal{A})$.

Corollary 4.4. *The obvious functor $DMod^{cw}(\mathcal{S}, \mathcal{A}) \rightarrow DMod(\mathcal{S}, \mathcal{A})$ is an equivalence of categories.*

Proof. This follows from the last Theorem. □

The following is a key result of this section.

Theorem 4.5. *Let $M \varepsilon D(Mod(\mathcal{S}, \mathcal{A}))$ denote a constructible $\mathcal{A}$-module. Then $\mathcal{H}^i(\sigma_{\leq n}(M)) = 0$ for all $i > n$ and $\cong \mathcal{H}^i(M)$ if $i \leq n$, i.e. the functor $\sigma_{\leq n}$ in Definition 3.6 identifies with the functor that kills the cohomology in degrees above n.*

Proof. This is a direct consequence of Proposition 4.1(iv): this shows that one may kill all the cohomology of M in degrees lower than a fixed integer $-n$ by attaching Cone-$\mathcal{A}$-modules (as in 2.7(ii)) with the summands $j_{U!}j_U^*(\mathcal{A})[s_U](t_U)$, $s_U \geq n+1$. Therefore the truncation functor $\sigma_{\geq -n}$ identifies with the functor that kills the cohomology in degrees lower than $-n$. □

Remark 4.6. Therefore, it follows $\mathcal{A}_U$ *cannot* belong to the heart of this t-structure unless $\mathcal{A}$ is concentrated in degree 0.

5. t-structures for filtered derived categories adapted to filtered modules over sheaves of filtered dgas

In this section we will adapt and extend the discussions in [BBD, Chapter 3] and [Beil, Appendix] to define a t-structure for the filtered derived category of filtered shaves of dg-modules over a fixed given sheaf of dgas $\mathcal{A}$: this will be such that each $\mathcal{A}_U$ (with the obvious trivial filtration) will belong to the heart of the above t-structure.

We will henceforth assume the basic situation of section 2 and consider the *filtered derived category* $DF\_Mod(\mathcal{S}, \mathcal{A})$ of objects M in $C(\mathcal{S}, \mathcal{A})$ provided with *descending* (i.e. non-increasing) filtrations $\{F_iM | i \varepsilon \mathbb{Z}\}$ with each $F_iM \varepsilon Mod(\mathcal{S}, \mathcal{A})$ and $F_i(M) = *$ for $i >> 0$ (depending on M). Observe that to any such filtration F, one may associate an ascending (i.e. non-decreasing) filtration F' given by $F'_iM = F_{-i}M$. Now F and F' clearly determine each other.

Theorem 5.1. *(See* [BBD, Chapter 3] *and also* [Beil, Appendix]*.) Let*

$$DF\_Mod(\mathcal{S}, \mathcal{A})^{\leq 0} = \{(M, F) \varepsilon DF\_Mod(\mathcal{S}, \mathcal{A}) | grd_F^i(M) \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\leq i}\} \text{ and}$$

$$DF\_Mod(\mathcal{S}, \mathcal{A})^{\geq 0} = \{(M, F) \varepsilon DF\_Mod(\mathcal{S}, \mathcal{A}) | grd_F^i(M) \varepsilon DMod(\mathcal{S}, \mathcal{A})^{\geq i}\}$$

where the t-structure on $DMod(\mathcal{S}, \mathcal{A})$ is defined as in Theorem 3.2.

Then there exists a unique t-structure on $DF\_Mod(\mathcal{S}, \mathcal{A})$ with $DF\_Mod(\mathcal{S}, \mathcal{A})^{\leq 0}$ ($DF\_Mod(\mathcal{S}, \mathcal{A})^{\geq 0}$) the corresponding aisle (co-aisle, respectively). The truncation functors on $DF\_Mod(\mathcal{S}, \mathcal{A})$ corresponding to the above t-structure will be denoted $\sigma_{\leq n}$ and $\sigma_{>n}$, respectively.

Proof. Let $(M, F) \varepsilon DF{-}Mod(\mathcal{S}, \mathcal{A})$ be a fixed object where F denotes a descending filtration on M. We define $\sigma_{\leq 0}(M, F)$ using descending induction on $F_i(M)$. Let $N >> 0$ be such that $F_{N+1}(M) = *$ while $F_N(M) \neq *$. We let $\sigma^F_{\leq N}(M)$ on $F_N(M)$ be defined by $\sigma^F_{\leq N}(F_N(M)) = \sigma_{\leq N}(F_N(M))$ where the last functor $\sigma_{\leq N}$ is the one defined in Theorem 3.2. Assume we have defined $\sigma^F_{\leq k} F_k(M)$ and $\sigma^F_{\leq k-1}(F_k(M)[1])$ for all $k \geq n$, where $n \leq N$ is a fixed integer so that $\sigma^F_{\leq k-1}(F_k(M)[1]) \cong (\sigma^F_{\leq k}(F_k(M)))[1]$. Let

$$\sigma^F_{\leq n}(F_{n-1}(M)/F_n(M)) = \sigma_{\leq n}(F_{n-1}(M)/F_n(M)) \tag{5.0.5}$$

with the functor $\sigma_{\leq n}$ defined as in Theorem 3.2. Now observe that one has a distinguished triangle: $F_n(M) \rightarrow F_{n-1}(M) \rightarrow F_{n-1}(M)/F_n(M) \rightarrow F_n(M)[1]$. Therefore we obtain a natural map $\sigma^F_{\leq n-1}(F_{n-1}(M)/F_n(M)) \rightarrow \sigma^F_{\leq n-1}(F_n(M)[1]) \cong (\sigma^F_{\leq n}(F_n(M)))[1]$. We let $\sigma^F_{\leq n-1}(F_{n-1}(M))$ be the canonical homotopy fiber of the map ($\sigma^F_{\leq n-1}(F_{n-1}(M)/F_n(M)) \rightarrow (\sigma^F_{\leq n}(F_n(M)))[1]$). Therefore, we obtain a distinguished triangle

$$\sigma^F_{\leq n}(F_n(M)) \rightarrow \sigma^F_{\leq n-1}(F_{n-1}(M)) \rightarrow \sigma^F_{\leq n-1}(F_{n-1}(M)/F_n(M)) \rightarrow (\sigma^F_{\leq n}(F_n(M)))[1] \tag{5.0.6}$$

We may now continue with the induction and define $\sigma_{\leq k}(F_k M)$ for all $k \leq N$. We let $\sigma_{\leq 0}(M, F)$ be defined by $F_k(\sigma_{\leq 0}(M, F)) = \sigma^F_{\leq k}(F_k(M))$: after replacing $\sigma^F_{\leq n-1}(F_{n-1}(M))$ by the mapping cylinder of the above map $\sigma^F_{\leq n}(F_n(M)) \rightarrow \sigma^F_{\leq n-1}(F_{n-1}(M))$, one may observe that $\sigma_{\leq k+1}(F_{k+1}(M))$ is a sub-object of $\sigma_{\leq k}(F_k(M))$. By descending induction, one may also show that there is a natural map $\sigma_{\leq 0}(M, F) \rightarrow (M, F)$ of objects in $DF\_Mod(\mathcal{S}, \mathcal{A})$. The distinguished triangle (5.0.6) shows that $grd^F_{n-1}(\sigma_{\leq 0}(M, F)) \simeq \sigma^F_{\leq n-1}(F_{n-1}(M)/F_n(M)) = \sigma_{\leq n-1}(F_{n-1}(M)/F_n(M))$ so that $\sigma_{\leq 0}(M, F) \varepsilon DF\_Mod(\mathcal{S}, \mathcal{A})^{\leq 0}$ as defined above. We define $\sigma_{\geq 1}(M, F)$ so that one has a distinguished triangle: $\sigma_{\leq 0}(M, F) \rightarrow (M, F) \rightarrow \sigma_{\geq 1}(M, F) \rightarrow \sigma_{\leq 0}(M, F)[1]$ in $DF\_Mod(\mathcal{S}, \mathcal{A})$.

These verify all the axioms of a t-structure as in 1.0.1 except the first. To see this one uses the spectral sequence in [BBD, (3.1.3.5)]. Then $Hom_{DF\_Mod(\mathcal{S},\mathcal{A})}((M, F), (M', F'))$ will identify with the $E_1^{0,0}$-term of this spectral sequence. The latter will be trivial if $(M, F) \varepsilon DF\_Mod(\mathcal{S}, \mathcal{A})^{\leq 0}$ and $(M', F') \varepsilon DF\_Mod(\mathcal{S}, \mathcal{A})^{\geq 1}$, proving that under the same hypothesis, $Hom_{DF\_Mod(\mathcal{S},\mathcal{A})}((M, F), (M', F')) = 0$. This proves the theorem. ☐

Remarks 5.2. 1. The case when $\mathcal{A} = \mathcal{R}$ and when the filtrations are finite is considered in [Beil].

2. Any object in $DMod(\mathcal{S}, \mathcal{R})$ may be provided with the *béte*-filtration as in [BBD, Chapter 3]: this is the decreasing filtration defined as follows on any $K \varepsilon C(\mathcal{S}, \mathcal{R})$: $F_i(K) =$ the sub-complex $0 \rightarrow K^{i+1} \rightarrow K^{i+2} \rightarrow \cdots$. Now $(grd^F_i(K))^j = K^i$ if $j = i$ and $= 0$ otherwise. Therefore it follows readily that any $K \varepsilon DMod(\mathcal{S}, \mathcal{R})$ which is bounded above and provided with the *béte*-filtration F belongs to the heart of the t-structure on $DF\_Mod(\mathcal{S}, \mathcal{R})$ considered above. In particular, any sheaf of dgas $\mathcal{A}$ which is bounded above and provided with the *béte*-filtration belongs to this heart. Therefore one may extend the formalism of perverse sheaves, extending perverse sheaves etc. as in 3.2 above to this setting. We skip the details.

6. **The crystalline case**

We will begin by recalling the basic framework from [Ek] and [Ill]. The Raynaud ring is the graded W-ring, $W = W(k) =$ the ring of Witt vectors of a perfect field k of characteristic p and generated by F, V in degree 0 and d in degree 1 with the following relations:

$$\begin{aligned} FV = VF = p, Fa = a^{\sigma} F, V = V a^{\sigma}, \\ da = ad, FdV = d, d^2 = 0, a \varepsilon W \end{aligned} \tag{6.0.7}$$

Here $(-)^{\sigma}$ is the Frobenius automorphism of W. Given a scheme X defined over k, one may adapt the definition above to define a sheaf of rings $\mathcal{R}$ on the Zariski site of X, called the sheaf of Raynaud rings on X. One may then extend the discussion below to the Zariski site of the given scheme X. However, for the sake of simplicity, we will keep $X = Spec\, k$ throughout the following discussion.

We let R_i, $i = 0, 1$, denote the piece of R in degree i. In view of the above relations, one observes that a left R-module M is the same as a complex of R_0-modules and where the differential $d : M^n \rightarrow M^{n+1}$ satisfies $FdV = d$. Moreover any left R_0-module can be viewed as a left R-module concentrated in degree 0. Henceforth an R-module will mean a left R-module. Given an R-module M, $M(n)$ will denote the R-module defined by $M(n)^i = M^{n+i}$ and the differential d given by $(-1)^n d$.

A complex of R-modules can be viewed as a double complex $M^{\bullet\bullet}$ where the first degree (called the horizontal direction) corresponds to the R-grading. The second degree will be called the vertical direction. Thus $M^{\bullet,n}$ denotes the n-th row of $M^{\bullet\bullet}$ and this is an R-module. Observe that one may take the cohomology of the double complex with respect to the vertical differential : these cohomology objects will be all R-modules. Thus $H_v^n(M^{\bullet\bullet})$ denotes the n-th (vertical) cohomology. We define the derived category of R-modules, $D(R)$ to be the category of all complexes of R-modules where we invert vertical quasi-isomorphisms. We let the derived category of all R-modules be denoted $D(R)$.

Next recall the *diagonal t-structures* of [Ek]. First, for each R-module M and an integer n, one defines the R-module $\tilde{\tau}_{\le n}M$ by

$$\tilde{\tau}_{\le n}M = (\cdots \to M^{n-1} \overset{d}{\to} M^n \to F^\infty B^{n+1} \to 0) \tag{6.0.7}$$

where $B^{n+1} = Im(d^n : M^n \to M^{n+1})$ and $F^\infty B^{n+1} = \underset{i\ge 0}{\cup} F^i B^{n+1}$. Clearly this is a sub-$R$-module of M. Next $\tilde{\tau}_{\ge n+1}M$ is defined to be the quotient $M/\tilde{\tau}_{\le n}M$. A complex of R-modules $M \varepsilon D(R)^{\le 0}$ if for each n, the natural map $\tilde{\tau}_{\le n}H_v^{-n}(M^{\bullet\bullet}) \to H_v^{-n}(M^{\bullet\bullet})$ is an isomorphism of R-modules. We say a complex of R-modules $M \varepsilon D(R)^{\ge 0}$ if the natural maps $H_v^{-n}(M^{\bullet\bullet}) \to \tilde{\tau}_{\ge 0}(H_v^{-n}(M^{\bullet\bullet}))$ is an isomorphism.

It was shown in [Ek] that this defines a t-structure on $D(R)$, *the diagonal t-structure* with the *heart of the t-structure* given by $D(R)^{\le 0} \cap D(R)^{\ge 0}$. We proceed to show that the derived category $D(R)$ is compactly generated and that the above t-structure is defined by a family of compact objects as in Theorem 1.8.

For each pair of integers i and j we define the complex $R(-j)[i]$ of R-modules which is the following (double) complex: we put the R-module $R(-j)$ (viewed as a complex) as the $-i$-th row and put zeros elsewhere. Observe that $H_v^n(R(-j)[i]) = R(-j)$ if $n = -i$ and 0 otherwise. Therefore, if $j \le i$, $R(-j)[i] \varepsilon D(R)^{\le 0}$. We also consider the complex of R-modules, $R_0(-i)[i]$ which is the following (double) complex: we put the R-module $R_0(-i)$ as the $-i$-th row and put zeros elsewhere.

The following is the main result of this section.

Theorem 6.1. *$D(R)^{\le 0}$ is the smallest pre-aisle generated by the R-modules $R(-j)[i]$, $j \le i$ and $R_0(-i)[i]$. These R-modules are compact objects in $D(R)$ and hence the above pre-aisle is an aisle which defines the diagonal t-structure.*

Proof. To see that each $R(-j)[i]$ is compact, observe that $RHom_{\mathcal{R}}(R(-j)[i], M) = RHom_{\mathcal{R}}(R, M(j)[-i]) = M(j)[-i]$. Therefore, the above $RHom$ commutes with filtered colimits in the argument M. To see that each $R_0(-i)[i]$ is compact, observe that $RHom_{\mathcal{R}}(R_0(i)[-i], M) = Ker(d : M^i \to M^{i+1})$ viewed as an R-module in the obvious manner. Since finite inverse limits commute with filtered colimits, one can see that $RHom_R(R_0(i)[-i], \quad)$ commutes with colimits in the second argument. These prove that the objects $R(-j)[i]$, $j \le i$ and $R_0(-i)[i]$ are compact objects in the derived category $DMod(R)$.

The main part of the rest of the proof consists in showing that given any complex of R-modules $M \varepsilon D(R)^{\le 0}$, it can be constructed from the R-modules $R(-j)[i]$, $j \le i$ and $R_0(-i)[i]$.

Step 1. Here we will assume the given complex of R-modules is concentrated in one row, say the $-i$-th. i.e. We may assume the given complex of R-modules is of the form $M[i]$, for some R-module M. We will now show that one may find a resolution P_0 of M by R-modules, so that each P_n is a sum of R-modules of the form $R(-j)$, $j \le i$ and $R_0(-i)$. Since $M[i] \varepsilon D(R)^{\le 0}$, the natural map $\tilde{\tau}_{\le i}(M) \to M$ is an isomorphism. Therefore, $M^j = 0$ for $j > i+1$, $= F^\infty B^i$ for $j = i+1$. For each $j \le i$, and each element $m_j \varepsilon M^j$, one may define a map $R(-j) \to M$ by sending $1 \varepsilon R_0$ to m_j and then by extending the map in the obvious way to all of $R(-j)$. For each element m_i that maps to zero by the differential, one may define a map $R_0(-i)$ to M by sending $1 \varepsilon R_0$ to m_i. Let P_0 denote the resulting sum of the R-modules $R(-j)$, $j \le i$ and $R_0(-i)$. Clearly the maps defined above provide a surjection $d_{-1} : P_0 \to M$ of R-modules. Now $P_0[i] \varepsilon D(R)^{\le 0}$ by construction.

Moreover, if $K_0 = kernel(d_{-1})$, then $K_0[i] \varepsilon D(R)^{\le 0}$ as well. For this, it suffices to show that the natural map $\tilde{\tau}_{\le i}(K_0) \to K_0$ is an isomorphism. For this it suffices to check that $K_0^{i+1} = Im(d : K_0^i \to K_0^{i+1})$: this is an immediate consequence of our definition of P_0. Therefore, one may repeat the above construction with K_0 replacing M to define P_1 and K_1. Inductively we may define the sequence P_n in the same manner. Observe that the map $d_n : P_n \to P_{n-1}$ factors through $K_{n-1} = kernel(d : P_{n-1} \to P_{n-2})$ so that the sequence $\{P_n | n\}$ provides a resolution of M. Then $P[i]$ provides the required resolution of $M[i]$.

Step 2. Next we will assume that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules so that all three $M'[i]$, $M[i]$ and $M''[i]$ belong to $D(R)^{\le 0}$. We may observe readily that we can find resolutions $P'_\bullet$ of M', $P_\bullet$ of M and $P''_\bullet$ of M'' together with maps $P'_\bullet \to P_\bullet$ and $P_\bullet \to P''_\bullet$ (over the given map $M' \to M$ and $M \to M''$, respectively) so that for each n, the sequence $0 \to P'_n \to P_n \to P''_n \to 0$ is exact and where each P'_n, P_n and P''_n is a sum of R-modules of the form $R(-j)$, $j \le i$ and $R_0(-i)$. (Since this is only a minor modification of a standard argument, we skip the remaining details.)

Step 3. Here we are given a bounded complex M of R-modules, i.e. there exist integers n and N, $n \le N$ so that $M^{\bullet j} = 0$ for $j < n$ or for $j > N$. By using induction on $N - n$, it suffices to assume $N = n + 2$. In other words, we are given a diagram $M' \overset{\phi}{\to} M \overset{\psi}{\to} M''$ of R-modules with $\psi \circ \phi = 0$. We will show that in this case, we may find resolutions $P'_\bullet$ of M', $P_\bullet$ of M and $P''_\bullet$ of M'' with maps $\phi_\bullet : P'_\bullet \to P_\bullet$ and $\psi_\bullet : P_\bullet \to P''_\bullet$ so that the compositions $\psi_n \circ \phi_n = 0$ for all n. Moreover each P'_n, P_n and P''_n are a sum of terms of the form $R(-j)[i]$, $j \le i$ and $R_0(-i)[i]$.

We will make use of Step 2 to accomplish this. For this, observe that one has the following short exact sequences: $0 \to ker(\phi) \to M' \to Im(\phi) \to 0$, $0 \to Im(\phi) \to ker(\psi) \to Ker(\psi)/Im(\phi) \to 0$, $0 \to ker(\psi) \to M \to Im(\psi) \to 0$ and $0 \to Im(\psi) \to M'' \to M''/Im(\psi) \to 0$. By applying Step 2 to these short sequence in successive order, one may construct the required resolutions.

Final Step. Finally we consider the case where we are given any complex M of R modules, not necessarily bounded. We may view this as a complex in the vertical direction. For each integer $N \ge 0$, let $M[-N, N]$ denote the naive truncation of the complex M to degrees in the range $-N$ to N, i.e. $M[-N, N]^{\bullet j} = M^{\bullet j}$, $-N \le j \le N$. The last step shows how to find resolutions $P_\bullet[-N, N]$ of $M[-N, N]$ that are compatible as N varies over all non-negative integers. Now we let $P_\bullet = \lim_{N \to \infty} P_\bullet[-N, N]$. $\square$

7. Further Examples

7.1. **Example 1: Motivic Derived Categories.** We first show how motivic derived categories may be defined in this framework. We will fix a ground field k, of arbitrary characteristic p throughout the paper and will only consider smooth schemes of finite type over k. This category will be denoted $(smt.schms)$. When provided with the big Zariski (Nisnevich, étale) topologies, we obtain the big-sites $(smt.schms)_{Zar}$, $((smt.schms)_{Nis}$, $(smt.schms)_{et}$, respectively). $\mathbb{Z} = \underset{r}{\oplus} \mathbb{Z}(r)$ will denote the integral *motivic complex* on the sites $(smt.schms)_{Zar}$ and $(smt.schms)_{Nis}$. l will denote a number prime to p and $\mathbb{Z}/l = \underset{r}{\oplus} \mathbb{Z}/l(r)$ denote the corresponding $mod - l$ motivic complex with $\mathbb{Z}/l_{et}$ the corresponding complex on the big-étale site $(smt.schms)_{et}$. $\mathbb{Q} = \underset{r}{\oplus} \mathbb{Q}(r) = \underset{r}{\oplus} \mathbb{Z} \otimes \mathbb{Q}$. In general, we will fix a commutative Noetherian ring R and consider $\mathbb{Z} \otimes \mathcal{R}$: this is a sheaf of E_∞-dgas over the ring R and we will denote this by $\mathcal{A}$ throughout. Observe that now one has augmentations $\mathcal{R} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{R}$ the composition of which is the identity. (Here $\mathcal{R}$ denotes the obvious constant sheaf associated to R.)

7.2. $D((smt.schms)_{Zar}, \mathcal{A})$**,** $D((smt.schms)_{Nis}, \mathcal{A})$ **and** $D((smt.schms)_{et}, \mathbb{Z}/l)$**.** We will consider explicitly only the derived category $D((smt.schms)_{Nis}, \mathcal{A})$, since the other two may be defined similarly with appropriate modifications. We will let $Sh((smt.schms)_{Nis}, \mathcal{R})$ denote the category of all sheaves of $\mathcal{R}$-modules on the site $(smt.schms)_{Nis}$. Similarly $Sh(X_{Nis}, \mathcal{R})$ will denote the category of all sheaves of $\mathcal{R}$-modules on the site X_{Nis} for a given scheme X. $C(Sh((smt.schms)_{Nis}, \mathcal{R}))$ ($C(Sh(X_{Nis}, \mathcal{R}))$) will denote the category of all (unbounded) complexes of objects in $Sh((smt.schms)_{Nis}, \mathcal{R})$ ($Sh(X_{Nis}, \mathcal{R})$, respectively). We first define $Mod((smt.schms)_{Nis}, \mathcal{A})$ to consist of all complexes of sheaves K on $(smt.schms)_{Nis}$ with the following properties:

(i) $K = \underset{r}{\oplus} K(r)$ has homotopy invariant cohomology sheaves and
(ii) K has the structure of a complex of sheaves of E_∞-modules over the sheaf of E_∞-dgas $\mathcal{A}$.

A morphism $f : K' \to K$ between two such objects will be a map that preserves the last structure. The objects of the derived category $D((smt.schms)_{Nis}, \mathcal{A})$ are the same as those of $Mod((smt.schms)_{Nis}, \mathcal{A})$.

Given a scheme $X \varepsilon (smt.schms)$, we let $(smt.schms/X)$ denote the sub-category of $(smt.schms)$ that are of finite type over X with morphisms $Y' \to Y$ being morphisms of smooth schemes compatible with the given maps to X. The site $(smt.schms/X)_{Zar}$ $((smt.schms/X)_{Nis}$, $(smt.schms/X)_{et})$ is the corresponding *big* site and will be often denoted X_{Zar} (X_{Nis}, X_{Et}, respectively). $Mod(X_{Nis}, \mathcal{A})$ will denote the corresponding derived category of complexes K defined on the site X_{Nis} that in addition have the structure of complexes of sheaves of E_∞-modules over $\mathcal{A}_{|X}$ = the restriction of $\mathcal{A}$ to $((smt.schms)/X)_{Nis}$. $D(X_{Nis}, \mathcal{A})$ will denote the corresponding derived category.

7.3. $D^{gm}(X_{Zar}, \mathcal{A})$, $D^{gm}(X_{Nis}, \mathcal{A})$ **and** $D^{gm}(X_{Et}, \mathbb{Z}/l)$. Again we will explicitly consider only the derived category $D^{gm}(X_{Nis}, \mathcal{A})$. For each smooth scheme Y quasi-projective over X and with structure map $f : Y \to X$, we consider $Rf_*(\mathcal{A}_{|Y})$. For a locally closed sub-scheme Y_0 of Y with immersion $j : Y_0 \to Y$, we also consider $Rf_*(j_!(\mathcal{A}_{|Y}))$. We let $D^{gm}(X_{Nis}, \mathcal{A})$ be the full subcategory of $D(X_{Nis}, \mathcal{A})$ generated by such objects.

Definition 7.1. (a) Let $D^{gm}_c(X_{Nis}, \mathcal{A})$ denote the smallest sub-category of $D^{gm}(X_{Nis}, \mathcal{A})$ containing all of $\mathbf{C} = \{Rf_*(j_!(\mathcal{A}_{|Y}))|j : Y_0 \to Y, f\}$ where j and f are as before and closed under the following operations: (i) finite sums, (ii) mapping cones, (iii) translations [1] and (iv) extensions (i.e. if $F' \xrightarrow{f} F \to Cone(f) \to F'[1]$ is a distinguished triangle with F', $Cone(f)$ and $F'[1] \varepsilon D^{gm}_c(X_{Nis}, \mathcal{A})$, then $F \varepsilon D^{gm}_c(X_{Nis}, \mathcal{A})$.)

(b) Let $D^{gm}_{s.c}(X_{Nis}, \mathcal{A})$ denote the smallest sub-category of $D^{gm}(X_{Nis}, \mathcal{A})$ containing all of $\{(j_!(\mathcal{A}_{|Y}))|j : X_0 \to X\}$ where j is a locally closed immersion as before and closed under the following operations: (i) finite sums, (ii) mapping cones, (iii) translations [1] and (iv) extensions (as before).

These derived categories are studied in detail in a forthcoming paper, [J-3], and we relate them to Voevodsky's derived category of geometric motives.

7.4. **Example 2: Equivariant Derived Categories.** We will fix an algebraically closed field k of characteristic $p \geq 0$. Let G denote the action of a smooth group scheme on a scheme X of finite type over k. Now $[X/G]$ will denote the associated *quotient stack*: see [**?**, LMB]or example. One associates several sites to the stack $[X/G]$: $[X/G]_{lis.et}$ denotes the site whose objects are smooth maps $s : S \to [X/G]$ with S an algebraic stack of finite type over k and where the coverings of a given object $s : S \to [X/G]$ are étale coverings. One defines the *iso-variant étale* site $[X/G]_{iso.et}$ as follows: the objects are G-isovariant etale maps $Y \to X$ (these correspond to iso-variant étale maps $[Y/G] \to [X/G]$ of the associated stacks. Sending an iso-variant étale map to the same map viewed simply as an étale map defines a map of sites: $p_* : [X/G]_{lis.et} \to [X/G]_{iso.et}$. It is shown in [J-2] that the latter site has enough points and the points correspond to G-orbits of geometric points of X.

Next assume that the group G is a torus T and X is a toric variety associated to T. Now T acts with finitely many orbits on X so that there are only finitely many points on the site $[X/T]_{iso.et}$, each corresponding to the T-orbits of geometric points of X. In characteristic zero it is possible to define a topological space corresponding to the site $[X/T]_{iso.et}$ and this is the approach adopted in [Lun]. However, this approach clearly fails in positive characteristic and necessitates the use of the site $[X/T]_{iso.et}$.

Next given a geometric point $\bar{x}$ of X, it has a T-stable neighborhood of the form $T\bar{x} \times V_{\bar{x}}$ where $V_{\bar{x}}$ denotes an affine toric variety for $T_{\bar{x}}$ which contracts $T_{\bar{x}}$-equivariantly to $\bar{x}$. Therefore one may readily compute $Rp_*(\underline{\mathbb{Q}}_l)$ and show that one has the isomorphism:

$$Rp_*(\mathbb{Q}_l)_{T\bar{x}} \simeq H^*(BT_{\bar{x}}; \mathbb{Q}_l)$$

Moreover one knows that for toric varieties, the stabilizers $T_{\bar{x}}$ are all connected. Now it follows readily as in [Lun] that one has the equivalence of categories:

$$D_+([X/T]_{iso.et}, Rp_*(\mathbb{Q}_l)) \simeq D_+([X/T]_{lis.et}, \mathbb{Q}_l).$$

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