ANNALS OF K-THEORY

vol. 9 no. 3 2024

λ-ring structures on the *K*-theory of algebraic stacks

Roy Joshua and Pablo Pelaez

msp

A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 9 (2024), No. 3, pp. 519–581
DOI: 10.2140/akt.2024.9.519

λ-ring structures on the *K*-theory of algebraic stacks

Roy Joshua and Pablo Pelaez

We consider the K-theory of smooth algebraic stacks, establish λ and γ operations, and show that the higher K-theory of such stacks is always a pre-λ-ring, and is a λ-ring if every coherent sheaf is the quotient of a vector bundle. As a consequence, we are able to define Adams operations and absolute cohomology for smooth algebraic stacks satisfying this hypothesis. We also obtain a comparison of the absolute cohomology with the equivariant higher Chow groups in certain special cases.

1. Introduction 519
2. K-theory and G-theory of quotient stacks and algebraic stacks 527
3. The G-construction and relative K-theory 530
4. Another model for the homotopy fiber of the G-construction 535
5. Lambda operations on the higher K-theory of algebraic stacks 539
6. The lambda-ring structure on higher K-theory 555
7. γ-operations and absolute cohomology 567
8. Examples 570
Appendix A. Key theorems of Waldhausen K-theory 573
Appendix B. Simplicial and cosimplicial objects, chain complexes 576
Acknowledgments 578
References 579

1. Introduction

In the case of smooth schemes of finite type over a field, the existence of λ-operations on algebraic K-theory enables one to define absolute cohomology as the eigenspace for the Adams operations on rational algebraic K-theory. In this paper we investigate the corresponding situation for algebraic stacks, beginning with λ-operations.

To begin with, it ought to be pointed out that it has been an open question whether there exist λ and Adams operations on the higher K-theory of algebraic stacks. The first result in this paper is an affirmative answer to this question, at least for many smooth quotient stacks; in fact, we show that the higher K-groups

Joshua thanks the Simons Foundation for support.
MSC2020: 14C35, 14D23, 19E08.
Keywords: algebraic stacks, K-theory, lambda operations.

of smooth algebraic stacks are pre-λ-rings, and that for algebraic stacks where every coherent sheaf is the quotient of a vector bundle, they are in fact λ-rings. Though the last *resolution property* is closely related to being a quotient stack, our proof is stack-theoretic in that it does not require an explicit presentation of the stack as a quotient stack. In fact, finding a presentation for a stack with the above resolution property as a quotient stack may be quite involved depending on the situation. Moreover, invoking a minimal number of background results proved on the K-theory and G-theory of algebraic stacks as in [Joshua 2012], the proofs here are quite straightforward and only a little background on stacks is required.

Before we proceed any further, it seems important to point out why it is essential to work with the K-theory of perfect complexes, using the machinery of categories with cofibrations and weak equivalences in the sense of [Waldhausen 1985]. The main difficulty is that the Quillen K-theory of the category of vector bundles even on a general smooth scheme, let alone on an algebraic space or algebraic stack, does not have good properties like Poincaré duality or Mayer–Vietoris property. Here Poincaré duality refers to the identification between the K-theory of the exact category of vector bundles with the K-theory of the exact category of coherent sheaves when the stack is regular or smooth. We begin with a few basic definitions so as to be able to state this in a precise manner.

Definition 1.0.1 [Kratzer 1980a, page 240; Weibel 2013, page 98][1]. A *pre*-λ-ring R is a commutative ring with unit and provided with maps $\lambda^i : \mathrm{R} \to \mathrm{R}$, $i \geq 0$, (which are in general not homomorphisms) so that (i) $\lambda^0(r) = 1$, for all $r \in \mathrm{R}$, (ii) $\lambda^1 = \mathrm{id}$, and (iii) $\lambda^n(r+s) = \sum_{i=0}^{n} \lambda^i(r) \cdot \lambda^{n-i}(s)$ for all $r, s \in \mathrm{R}$. A *pre-λ-ring without unit* is a commutative ring R without a unit element and provided with maps $\lambda^i : \mathrm{R} \to \mathrm{R}$, $i > 0$ satisfying the conditions (ii) and (iii). (For (iii) to make sense, we use the convention that $\lambda^0(r) \cdot \lambda^n(s) = \lambda^n(s)$ and $\lambda^n(r) \cdot \lambda^0(s) = \lambda^n(r)$, for all $r \in \mathrm{R}$, $s \in \mathrm{R}$.) Given a pre-λ ring R, a *pre-λ-algebra* over R is a commutative ring S (not necessarily with a unit) provided with the structure of a module over R, and so that $\mathrm{R} \oplus \mathrm{S}$ gets the structure of a pre-λ-ring with the following operations:

- The sum on $\mathrm{R} \oplus \mathrm{S}$ is the obvious sum induced by the sum on R and S.
- The product on $\mathrm{R} \oplus \mathrm{S}$ is defined by $(r, s) \cdot (r', s') = (r \cdot r', r \cdot s' + r' \cdot s + s \cdot s')$, where the products $r \cdot s'$ and $r' \cdot s$ are formed using the module structure of S over R, and the product $s \cdot s'$ is formed using the product on S.
- One is given maps $\lambda^i : \mathrm{R} \oplus \mathrm{S} \to \mathrm{R} \oplus \mathrm{S}$, $i \geq 0$ so that the following hold:

(i) For all $i \geq 0$, λ^i restricted to R identifies with the given operation λ^i on R.

(ii) For all $i > 0$, λ^i restricted to S maps to S, and

[1] Please note that what we call a pre-λ-ring is called a λ-ring and what we call a λ-ring is called a special λ-ring in [Weibel 2013].

(iii) $\lambda^n(r,s) = \big(\lambda^n(r), \sum_{i=0}^{n-1} \lambda^i(r)\cdot\lambda^{n-i}(s)\big)$ for all $r \in \mathrm{R}$, $s \in \mathrm{S}$.

The product $\lambda^i(r).\lambda^{n-i}(s)$ uses the R-module structure of S.

A pre-λ-ring R is a *λ-ring*, if $\lambda^n(1) = 0$ for $n > 1$, and for certain universal polynomials $\mathrm{P}_{k,l}$ and P_k with integral coefficients defined as in [Atiyah and Tall 1969, page 258] the following equations hold:

$$\begin{aligned} \lambda^k(r\cdot s) &= \mathrm{P}_k(\lambda^1(r),\dots,\lambda^k(r);\lambda^1(s),\dots,\lambda^k(s)) \quad \text{and} \\ \lambda^k(\lambda^l(r)) &= \mathrm{P}_{k,l}(\lambda^1(r),\dots,\lambda^{k\cdot l}(r)), \qquad r,s\in\mathrm{R}. \end{aligned} \tag{1.1}$$

One defines a *λ-ring without unit* to be a pre-λ-ring without unit satisfying the relations in (1.1). If R is a λ-ring and S is a pre-λ-algebra over R, we say S is a *λ-algebra* over R if the relations above also hold for $\lambda^k(\lambda^l(r+s))$, and for $\lambda^k((r+s)\cdot(r'+s'))$ if $r, r' \in \mathrm{R}$ and $s, s' \in S$, that is, $\mathrm{R}\oplus\mathrm{S}$ is a λ-ring with the operations defined above.

Given an algebraic stack $\mathcal{S}$, $\mathrm{K}(\mathcal{S})$ will denote the space obtained by applying the constructions of [Waldhausen 1985] to the category of perfect complexes on the stack $\mathcal{S}$; see Definition 2.0.2. For a closed algebraic substack $\mathcal{S}'$ of $\mathcal{S}$, $\mathrm{K}_{\mathcal{S}'}(\mathcal{S})$ will denote the *space* defining the higher algebraic K-theory of $\mathcal{S}$ with supports in $\mathcal{S}'$ as in Definition 2.0.2. Then we obtain the theorem stated below, which is one of the main results of this paper.

Theorem 1.1.1. (i) *Let $\mathcal{S}$ denote a smooth algebraic stack of finite type over a regular Noetherian base scheme S. Then $\pi_0(\mathrm{K}(\mathcal{S}))$ is a pre-λ-ring.*

(ii) *For $\mathcal{S}'$ denoting a closed algebraic substack, $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$, for each fixed $n \geq 0$, is a pre-λ-algebra over the pre-λ-ring $\pi_0(\mathrm{K}(\mathcal{S}))$.*

The above pre-λ-ring structure is compatible with pull-backs: that is, if $f : \tilde{\mathcal{S}} \to \mathcal{S}$ is a map of smooth algebraic stacks and $\tilde{\mathcal{S}}' = \tilde{\mathcal{S}}\times_{\mathcal{S}}\mathcal{S}'$, then the induced map $f^* : \pi_0(\mathrm{K}(\mathcal{S})) \to \pi_0(\mathrm{K}(\tilde{\mathcal{S}}))$ is a map of pre-λ-rings, and the induced map $f^* : \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})) \to \pi_n(\mathrm{K}_{\tilde{\mathcal{S}}'}(\tilde{\mathcal{S}}))$, for each fixed $n \geq 0$, is a map of pre-λ-algebras over $\pi_0(\mathrm{K}(\mathcal{S}))$. The λ-operations are homomorphisms on $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ for all $n > 0$.

(iii) *In case every coherent sheaf on the smooth stack $\mathcal{S}$ is the quotient of a vector bundle, each $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$, for $n \geq 0$, is a λ-algebra over $\pi_0(\mathrm{K}(\mathcal{S}))$ in the above sense.*

Remark 1.1.2. The proof of Theorem 1.1.1 is split into two parts: the first part that discusses the proofs of the two statements (i) and (ii) appear at the end of Section 5. All of Section 6 is devoted to a proof of statement (iii).

The following is a quick summary of the techniques adopted in this paper to prove the above theorem and Theorems 1.1.3 and 1.1.4 discussed below. First, we invoke the technique in [Bloch and Lichtenbaum 1997], whereby higher K-groups can be reduced to certain relative Grothendieck groups. This needs to make use of the homotopy property for the K-theory of algebraic stacks (see [Joshua 2012, Theorem 5.17]), which makes it necessary to restrict to smooth stacks. But then we need to interpret the relative Grothendieck groups in terms of a relative form of the Gillet–Grayson undelooping adapted to the Waldhausen setting. Considerable effort (in fact, all of Section 3) is needed to carry this out. These, together with some well-known arguments due to Grayson [1989, Section 7], suffice to put a pre-λ-ring structure on the higher K-theory of all *smooth* algebraic stacks of finite type over any regular Noetherian base scheme.

In order to verify that the higher K-theory of algebraic stacks form λ-rings, we are forced to restrict to smooth stacks that have the *resolution property*, namely where every coherent sheaf is the quotient of a vector bundle. In fact, the Quillen K-theory of vector bundles and the K-theory of perfect complexes are known to be isomorphic *only when they have the resolution property*: *namely*, *the property that every coherent sheaf is the quotient of a vector bundle*; see [Thomason and Trobaugh 1990, 8.6. Exercise] for the example of a scheme X, which is the union of two copies of the affine n-space $\mathbb{A}^n$, for $n \geq 2$, glued along $\mathbb{A}^n - \{0\}$. This scheme does not have the resolution property, as the resolution property would imply that the diagonal morphism is affine, and it is not in this case. More details on the resolution property may be found in [Totaro 2004]; see also [Joshua 2012, Proposition 2.8]. One may also see [SGA 6 1971, Exposé II] and [Brenner and Schröer 2003] for related results. Finally we adapt certain arguments of Gillet and Soulé [1987] to prove that the corresponding relative Grothendieck groups are λ-rings.

Here is a comparison of our results with other related results in the literature. There is a great deal of literature on the λ-ring structure and related operations on the higher K-theory of schemes, including schemes that are possibly singular; see [Kratzer 1980b; 1980a; Levine 1997; Gillet and Soulé 1999; Lecomte 1998]. We will not discuss such results any further, except to point out that when restricted to the category of (suitably nice schemes) our results in the present paper reduce to these. While it has been known for sometime, especially after [Grayson 1989], how to define λ-operations for the Quillen K-theory of exact categories or the Waldhausen analogue of it, it was not clear that the required relations are satisfied. For example, in [Grayson 1989], λ-operations are defined for the K-theory of the exact category of vector bundles, but it was left open whether they satisfied the required properties to define a λ-ring in general. Even for quotient stacks, or equivalently for schemes provided with an action by a smooth group-scheme, it has not been known till very recently if there exists a λ-ring structure on their higher K-theory. In fact, this was posed as a conjecture in the literature. In [Köck 1998, (2.5) Proposition], it was shown that the Higher K-theory of quotient stacks is a pre-λ-ring and satisfies the first relation in (1.1). It was conjectured there (see [Köck 1998, (2.7) Conjecture] that the second relation in (1.1) is also satisfied by the higher K-theory of quotient stacks and it remained open till the very recent preprint [Köck and Zanchetta 2021].

Most of the recent progress in this area in the literature follows the relatively recent results of Grayson defining higher K-theory using binary complexes as in [Grayson 2012]. Making use of this approach, the authors of [Harris et al. 2017] prove the existence of a λ-ring structure for the higher K-theory of schemes, including ones that are possibly singular, but still left open the corresponding question for equivariant algebraic K-theory or for the algebraic K-theory of (quotient) stacks. (There is also the work of Riou [2010] and Zanchetta [2021], which provide a λ-ring structure on the higher K-theory of certain classes of schemes, including ones that are possibly singular: while these also involve a reduction to Grothendieck groups, this approach does not seem to extend to any larger category than schemes.) In the very recent preprint, [Köck and Zanchetta 2021, Theorem 5.1], the authors provide a proof that the second relation in (1.1) is also satisfied by higher equivariant K-theory, making essential use of binary complexes.

Therefore, the corresponding question for the higher algebraic K-theory of algebraic stacks in general has not been even looked at in the literature so far.[2] [3] Here are some of the main features of our work:

- Our first result in Theorem 1.1.1, proving the existence of a pre-λ-ring structure on the higher Algebraic K-theory of all smooth Algebraic stacks satisfying certain mild finiteness conditions therefore is the first positive result for smooth algebraic stacks in general.

Secondly, the second statement in Theorem 1.1.1 has the following features:

- We prove the existence of a λ-ring structure on the higher K-theory of smooth algebraic stacks satisfying the *resolution property*. While this property is closely related to the stack being a quotient stack, it is not always equivalent to being a quotient stack (see [Totaro 2004, Theorems 1.1 and 1.2] as well as the discussion following [Totaro 2004, Proposition 1.3] for a precise comparison), and our proof does *not* require the stack to be a quotient stack.

- Even when the resolution property holds and one knows the given stack is a quotient stack, finding an explicit presentation for the stack as a quotient stack

[2] λ-operations can be defined at the level of Grothendieck groups for quotient stacks quite easily; see [Edidin et al. 2017] where they define such operations on the Grothendieck groups of certain inertia stacks associated to smooth quotient stacks that are Deligne–Mumford. In case the stack is a quotient stack of the form $[X/T]$ for a split torus T, the inertia stack is a disjoint union of quotient stacks of the form $[X^t/T]$, $t \in T$; see [Stacks 2005–, 95.17: Examples of inertia stacks, see Tag 0373]. In this case, Theorem 1.1.1 would extend the λ-operations of [Edidin et al. 2017] to the higher K-theory of these inertia stacks, even when the stack $[X/T]$ is not Deligne–Mumford.

[3] Quotient stacks of the form $[X/G]$, for a scheme X and an affine group scheme G are quite special, and there are as many algebraic stacks which are not such global quotient stacks.

for the action of an affine group scheme on a scheme is rather involved, and our techniques do not require knowing explicitly any such presentation.

• Moreover, as shown in Theorem 1.1.1(iii), our results on λ-algebra structures hold for *the relative case* also, that is, also for the higher K-theory with supports in a closed substack. (The results of [Köck and Zanchetta 2021, Theorem 5.1], as stated are only for the absolute case.) This is important, as we are then able to obtain certain long-exact sequences in the associated absolute cohomology as in Theorem 1.1.4.

• Finally, our techniques do not make use of the definition of higher algebraic K-theory using binary complexes, but instead use more traditional (topological) methods, such as a relative form of the Gillet–Grayson $\mathrm{G}_\bullet$-construction. The price we pay for this may be that we have to restrict to smooth stacks, so that the higher K-theory of these stacks have the homotopy property, and we are able to invoke the methods of [Bloch and Lichtenbaum 1997] to reduce higher K-theory to certain relative Grothendieck groups.

Making use of Theorem 1.1.1, we define γ *and Adams operations* on the higher K-groups of algebraic stacks that satisfy the property that every coherent sheaf is the quotient of a vector bundle; further, making use of these operations we are also able to define the *absolute cohomology with* $\mathbb{Q}$*-coefficients* for such algebraic stacks. These results may be summarized in the following theorems.

Theorem 1.1.3. *Let $\mathcal{S}$ denote a smooth algebraic stack as in Theorem 1.1.1 having the resolution property, and let $\mathcal{S}'$ denote a closed algebraic substack*:

(i) *Then there are γ and Adams operations on each $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ that satisfy the* (*usual*) *relations*

$$\gamma^1 = \mathrm{id}, \quad \gamma^k(K) = \sum_{k'+k''=k} \gamma^{k'}(K').\gamma^{k''}(K''),$$

if $K = K' + K''$ in the λ-ring $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$, and if $\mathcal{S}' = \mathcal{S}$, then $\gamma^0(K) = [\mathcal{O}_{\mathcal{S}}] =$ the class of the structure sheaf $\mathcal{O}_{\mathcal{S}}$ for any $K \in w\,\mathrm{Perf}_{fl}(\mathcal{S})$. Moreover, for certain universal polynomials $\mathrm{Q}_{k,l}$ and Q_k with integral coefficients (*see* [Atiyah and Tall 1969, page 262]) *the following relations hold*:

$$\gamma^k(\gamma^l(\alpha)) = \mathrm{Q}_{k,l}(\gamma^1(\alpha), \dots, \gamma^{k.l}(\alpha))$$

and

$$\gamma^k(\alpha.\beta) = \mathrm{Q}_k(\gamma^1(\alpha), \dots, \gamma^k(\alpha); \gamma^1(\beta), \dots, \gamma^k(\beta)),$$

for $\alpha, \beta \in \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$.

The Adams operations ψ^k preserve the additive and multiplicative structures on $\pi_*(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$. The Adams operations and the γ-filtration are natural with respect to pull-back. The graded piece $gr_n(\pi_*(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))\otimes\mathbb{Q})$ is the eigenspace for the induced action of ψ^k with eigenvalue k^n.

(ii) *In case there exists a coarse moduli space* $\mathfrak{M}$ ($\mathfrak{M}'$) *for the stack* $\mathcal{S}$(*the substack* $\mathcal{S}'$, *respectively*) *as an algebraic space, the* γ*-filtration on* $\pi_*(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})) \otimes \mathbb{Q}$ *above is compatible with the* γ*-filtration on* $\pi_*(\mathrm{K}_{\mathfrak{M}'}(\mathfrak{M})) \otimes \mathbb{Q}$.

One important distinction from the corresponding situation for schemes is that the γ-filtration is almost never *nilpotent* for algebraic stacks, as may be seen in Remarks 7.2.1. Though the hypothesis that *the resolution property* holds seems strong, it is clearly satisfied by many quotient stacks thanks to the work of Thomason [1986; 1987a; 1987b] and the work of Totaro [2004].

We will define *absolute cohomology* by

$$\mathrm{H}^i_{\mathcal{S}',\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(j)) = \mathrm{gr}^j(\pi_{2j-i}(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})) \otimes \mathbb{Q});$$

see Definition 7.3.1 for more details.

Theorem 1.1.4 (localization theorem for absolute cohomology). *Let* $\mathcal{S}$ *denote a smooth algebraic stack as in Theorem 1.1.1, and let* $\mathcal{S}'_0 \subseteq \mathcal{S}'_1$ *denote two closed algebraic substacks. We will further assume that every coherent sheaf on the stack* $\mathcal{S}$ *is the quotient of a vector bundle. Then one obtains the long exact sequence of absolute cohomology groups*:

$$\cdots \to \mathrm{H}^n_{\mathcal{S}'_0,\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(i)) \to \mathrm{H}^n_{\mathcal{S}'_1,\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(i)) \to \mathrm{H}^n_{\mathcal{S}'_1-\mathcal{S}'_0,\mathrm{abs}}(\mathcal{S}-\mathcal{S}'_0, \mathbb{Q}(i)) \\ \to \mathrm{H}^{n+1}_{\mathcal{S}'_0,\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(i)) \to \cdots .$$

Here is the layout of the paper. Section 2 is a quick review of the basic properties of the K-theory and G-theory of algebraic stacks proved in [Joshua 2007; 2012]. We make a special effort here in order to make the paper accessible to readers who are primarily interested in the case of quotient stacks.

Section 3 introduces a *key technique*: we obtain an explicit description of relative K-theory in terms of a *relative version* of the Gillet–Grayson G-construction (see [Gillet and Grayson 1987] and [Grayson 1989]), adapted to the setting of categories with cofibrations and weak equivalences by the methods of [Gunnarsson et al. 1992] and [Gunnarsson and Schwänzl 2002]. (Though there is another description of relative K-theory due to Waldhausen [1985, Definition 1.5.4], that description does not use the G-construction and therefore we cannot use it in our context.)

A nontrivial issue that shows up in this section is the difficulty of finding a categorical model for path spaces, that is, a categorical construction whose nerve gives the usual path space. This is possible with the Waldhausen $\mathrm{S}_\bullet$-construction, as is shown in [Gunnarsson et al. 1992, Section 2], but does not extend to the $\mathrm{G}_\bullet$-construction. We circumvent this issue by defining the path space only after topological realization. On the other hand, the mapping cone construction (which is in a sense dual to the path space construction) readily extends to functors between simplicial categories. We invoke this construction in Section 4 to establish an additivity theorem for relative K-theory defined using a relative form of the $\mathrm{G}_\bullet$-construction.

Section 5 is a key section, where we start by defining λ-operations on the Waldhausen K-theory of perfect complexes in the framework of the $G_\bullet$-construction. Since λ-operations are nonadditive, we make use of simplicial methods to do this, as in [Dold and Puppe 1961] and [Gillet and Soulé 1987]. Making use of techniques developed in the earlier sections, this is then extended to relative K-theory defined using the relative $G_\bullet$-construction. The above techniques, along with the technique of reducing higher K-theory to relative Grothendieck groups (as in [Bloch and Lichtenbaum 1997]) enable us to prove the first part of the main theorem. This puts a pre-λ-ring structure on the higher K-theory of all smooth algebraic stacks that are of finite type over any regular Noetherian base scheme, thereby completing the proofs of the first two statements in Theorem 1.1.1.

In order to verify that the higher K-groups of algebraic stacks form λ-rings, we are forced to restrict to smooth algebraic stacks that have the *resolution property*, namely where every coherent sheaf is the quotient of a vector bundle. Finally we adapt certain arguments of [Gillet and Soulé 1987] to prove that the corresponding relative Grothendieck groups are λ-rings. These occupy all of Section 6 and complete the proof of the last statement of Theorem 1.1.1.

In Section 7 we define and study γ-operations and absolute cohomology for algebraic stacks. This section also contains the proofs of Theorems 1.1.3 and 1.1.4. We conclude with several explicit examples in Section 8. This section already has a brief comparison of absolute cohomology with the equivariant higher Chow groups in a few special cases. As pointed out here, the relationship with the equivariant higher Chow groups needs the machinery of derived completion (as in [Carlsson and Joshua 2023]) in general. Therefore, we have decided to explore this in a sequel, where we also plan to discuss Riemann–Roch theorems. A couple of short appendices are added to make the paper self-contained. Appendix A summarizes the main results of Waldhausen K-theory. Appendix B summarized some well-known relations between simplicial objects, cosimplicial objects and chain complexes in abelian categories.

Quick summary of the notational terminology. The basic terminology on algebraic stacks as well as algebraic K-theory is discussed in the beginning of Section 2: therefore, we do not repeat them here. The Gillet–Grayson G-construction will be denoted $G_\bullet$, while the Waldhausen S-construction will be denoted $S_\bullet$: these are discussed in Section 3. Nerve denotes the functor sending a small category to the simplicial set which is its nerve: this appears in Sections 3 and 4. Given an exact category $\boldsymbol{A}$, $\mathrm{Cos.mixt}(\boldsymbol{A})$ will denote the category of cosimplicial-simplicial objects in $\boldsymbol{A}$: this set-up is used in the definition of the derived functors of the exterior power operation in Section 5. Given a simplicial set $X_\bullet$, $\mathrm{sub}_k\, X_\bullet$ produces a multisimplicial set of order k: this is discussed in [Grayson 1989, Section 4] and recalled in Section 5 along with certain other functors such as Ξ.

2. K-theory and G-theory of quotient stacks and algebraic stacks: a quick review

This section is a quick summary of the basic results on the K-theory and G-theory of algebraic stacks proved in [Joshua 2007; 2012]. Assuming these results, there is very little new stack-theoretic material needed in the later sections so that several of the basic results of this paper are no harder to state and prove for general Artin stacks than for the special case of quotient stacks.

We will fix a regular Noetherian base scheme S throughout the paper and will consider only objects defined and finitely presented over S.

Definition 2.0.1. (i) An *algebraic stack* $\mathcal{S}$ will mean an algebraic stack (of Artin type) which is finitely presented over a regular Noetherian base scheme S. An *action* of a group scheme G on a stack $\mathcal{S}$ will mean morphisms $\mu, \mathrm{pr}_2 : \mathrm{G} \times \mathcal{S} \to \mathcal{S}$ and $e : \mathcal{S} \to \mathrm{G} \times \mathcal{S}$ satisfying the usual relations.

(ii) A quotient stack [X/G] will denote the Artin stack associated to the action of a smooth affine group-scheme G on an algebraic space X, both defined over S.[4]

It is shown in [Joshua 2003, Appendix] that if G is a smooth group scheme acting on an algebraic stack $\mathcal{S}$, a quotient stack $[\mathcal{S}/G]$ exists as an algebraic stack. In this case, there is an equivalence between the category of G-equivariant $\mathcal{O}_{\mathcal{S}}$-modules on $\mathcal{S}$ and the category of $\mathcal{O}_{[\mathcal{S}/G]}$-modules; see [Joshua 2003, Appendix]. Therefore, one may incorporate the equivariant situation into the following discussion by considering quotient stacks of the form $[\mathcal{S}/G]$.

We have chosen to work mostly with the lisse-étale site (see [Laumon and Moret-Bailly 2000, Chapter 12; Olsson 2007]), though it seems possible to work instead with the smooth site. Observe that if $\mathcal{S}$ is an algebraic stack, the underlying category of $\mathcal{S}_{\mathrm{lis-et}}$ is the same as the underlying category of the smooth site $\mathcal{S}_{smt}$, whose objects are smooth maps $u : \mathrm{U} \to \mathcal{S}$, with U an algebraic space. The coverings of an object $u : \mathrm{U} \to \mathcal{S}$ in the site $\mathcal{S}_{\mathrm{lis-et}}$ are étale surjective maps $\{u_i : \mathrm{U}_i \to \mathrm{U} \mid i\}$. We will provide $\mathcal{S}_{\mathrm{lis-et}}$ with the structure sheaf $\mathcal{O}_{\mathcal{S}}$. One defines a sheaf of $\mathcal{O}_{\mathcal{S}}$-modules M on $\mathcal{S}_{\mathrm{lis-et}}$ to be *cartesian* as in [Laumon and Moret-Bailly 2000, Definition 12.3], that is, if for each map $\phi : \mathrm{U} \to \mathrm{V}$ in $\mathcal{S}_{\mathrm{lis-et}}$, the induced map $\phi^{-1}(M_{|\mathrm{V}_{\mathrm{et}}}) \to M_{|\mathrm{U}_{\mathrm{et}}}$ is an isomorphism. In fact, it suffices to have this property for all smooth maps ϕ. *In this paper, we will restrict to complexes of $\mathcal{O}_{\mathcal{S}}$-modules M whose cohomology sheaves are all cartesian.*

Definition 2.0.2. (i) *Throughout the paper, unless explicitly mentioned to the contrary, a complex will mean a cochain complex, that is, where the differentials*

[4] In fact, the reader may observe as in the discussion in Definition 2.0.4 that working with quotient stacks corresponds to working in the equivariant framework, and does not require any special knowledge of stack-theoretic machinery.

are of degree $+1$. A bounded complex of $\mathcal{O}_{\mathcal{S}}$-modules M is *strictly perfect*, if its cohomology sheaves are all cartesian and locally on the site $\mathcal{S}_{\text{lis-et}}$, M is a bounded complex of locally free coherent $\mathcal{O}_{\mathcal{S}}$-modules. The complex M is *perfect* if the cohomology sheaves are all cartesian, and locally on the site $\mathcal{S}_{\text{lis-et}}$, M is quasiisomorphic to a strictly perfect complex of $\mathcal{O}_{\mathcal{S}}$-modules.

(ii) M is *pseudocoherent*, if it is locally quasiisomorphic to a bounded above complex of $\mathcal{O}_{\mathcal{S}}$-modules with bounded coherent cohomology sheaves, which are cartesian. (One may readily prove that if M is perfect, it is pseudocoherent. Observe that the usual definition of pseudocoherence as in [SGA 6 1971] does not require the cohomology sheaves to be bounded; we have included this hypothesis in the definition of pseudocoherence mainly for convenience.)

(iii) Let $\mathcal{S}'$ denote a closed algebraic substack of $\mathcal{S}$. Then the category of all perfect (pseudocoherent, strictly perfect) complexes with supports contained in $\mathcal{S}'$, along with quasiisomorphisms forms a category with cofibrations and weak equivalences (see Definition 3.0.1): the *cofibrations* are those maps that are *degree-wise split monomorphisms*. It will be denoted by $\mathrm{Perf}_{\mathcal{S}'}(\mathcal{S})$ ($\mathrm{Pseudo}_{\mathcal{S}'}(\mathcal{S})$, $\mathrm{StPerf}_{\mathcal{S}'}(\mathcal{S})$, respectively); the K-theory *space* (G-theory *space*) of $\mathcal{S}$ with supports in $\mathcal{S}'$ will be defined to be the K-theory space of the category with cofibrations and weak equivalences $\mathrm{Perf}_{\mathcal{S}'}(\mathcal{S})$ ($\mathrm{Pseudo}_{\mathcal{S}'}(\mathcal{S})$, respectively) and denoted $\mathrm{K}_{\mathcal{S}'}(\mathcal{S})$ ($G_{\mathcal{S}'}(\mathcal{S})$, respectively): the weak equivalences in these categories with cofibrations and weak equivalences are quasiisomorphisms. We distinguish these from the corresponding K-theory spectra which will be denoted $\boldsymbol{K}_{\mathcal{S}'}(\mathcal{S})$ ($\boldsymbol{G}_{\mathcal{S}'}(\mathcal{S})$, respectively).

We also let $\mathrm{Perf}_{fl,\mathcal{S}'}(\mathcal{S})$ ($\mathrm{Pseudo}_{fl,\mathcal{S}'}(\mathcal{S})$) denote the full subcategory of $\mathrm{Perf}_{\mathcal{S}'}(\mathcal{S})$ ($\mathrm{Pseudo}_{\mathcal{S}'}(\mathcal{S})$) consisting of complexes of flat $\mathcal{O}_{\mathcal{S}}$-modules in each degree. Observe from [Illusie 1971, Chapitre I, Théorème 4.2.1.1] that flat $\mathcal{O}_{\mathcal{S}}$-modules have the *additional property that they are direct limits of finitely generated flat submodules at each stalk*. (Observe also that the existence of flat resolutions and the Waldhausen approximation theorem (see Theorem A.0.4) imply that one obtains a weak equivalence: $\mathrm{K}(\mathrm{Perf}_{\mathcal{S}'}(\mathcal{S})) \simeq \mathrm{K}(\mathrm{Perf}_{fl,\mathcal{S}'}(\mathcal{S}))$.)

Definition 2.0.3. We define a sheaf of $\mathcal{O}_{\mathcal{S}}$-modules on $\mathcal{S}_{\text{lis-et}}$ to be *quasicoherent* with respect to a given atlas, if its restriction to the étale site of *the given* atlas for $\mathcal{S}$ is quasicoherent. Coherent sheaves and locally free coherent sheaves are defined similarly. (Observe that this is slightly different from the usage in [Laumon and Moret-Bailly 2000], where a quasicoherent sheaf is also assumed to be cartesian as in [loc. cit., Definition 12.3]. However, such a definition would then make it difficult to define a quasicoherator that converts a complex of $\mathcal{O}_{\mathcal{S}}$-modules to a complex of quasicoherent $\mathcal{O}_{\mathcal{S}}$-modules. This justifies our choice. Since we always restrict to complexes of $\mathcal{O}_{\mathcal{S}}$-modules whose cohomology sheaves are cartesian, the present definition works out in practice to be more or less equivalent to the one in [loc. cit.].) An $\mathcal{O}_{\mathcal{S}}$-module will always mean a sheaf of $\mathcal{O}_{\mathcal{S}}$-modules on $\mathcal{S}_{\text{lis-et}}$. The category of $\mathcal{O}_{\mathcal{S}}$-modules will be denoted $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ (or $\mathrm{Mod}(\mathcal{S}_{\text{lis-et}}, \mathcal{O}_{\mathcal{S}})$ to be more precise).

Let $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ ($\mathrm{QCoh}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, $\mathrm{Coh}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$) denote the category of all $\mathcal{O}_{\mathcal{S}}$-modules (all quasicoherent $\mathcal{O}_{\mathcal{S}}$-modules, all coherent $\mathcal{O}_{\mathcal{S}}$-modules, respectively).

Let X denote a scheme of finite type over a regular Noetherian base scheme S and let G denote a smooth affine group scheme of finite type over S acting on X. Then we point out that, if one restricts to quotient stacks of the form $[\mathrm{X}/G]$, then we may choose to work with the following (somewhat more familiar) choices.

Definition 2.0.4 (the case of quotient stacks). Assuming the above framework, let $\mathrm{Pseudo}([\mathrm{X}/\mathrm{G}]) = \mathrm{Pseudo}^{\mathrm{G}}(\mathrm{X})$ where the right-hand side denotes the category of bounded above complexes of G-equivariant $\mathcal{O}_{\mathrm{X}}$-modules (on the Zariski site of X), with bounded coherent cohomology sheaves. Similarly, one may let $\mathrm{Perf}([\mathrm{X}/\mathrm{G}]) = \mathrm{Perf}^{\mathrm{G}}(\mathrm{X})$ denote the category of complexes of G-equivariant $\mathcal{O}_{\mathrm{X}}$-modules that are locally quasiisomorphic on the Zariski site of X to bounded complexes of locally free $\mathcal{O}_{\mathrm{X}}$-modules with bounded coherent cohomology sheaves.

In this case we may replace $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ ($\mathrm{QCoh}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, $\mathrm{Coh}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$) by the category $\mathrm{Mod}^{\mathrm{G}}(\mathrm{X})$ ($\mathrm{QCoh}^{\mathrm{G}}(\mathrm{X})$, $\mathrm{Coh}^{\mathrm{G}}(\mathrm{X})$), which will denote the category of all G-equivariant $\mathcal{O}_{\mathrm{X}}$-modules (G-equivariant quasicoherent $\mathcal{O}_{\mathrm{X}}$-modules, G-equivariant coherent $\mathcal{O}_{\mathrm{X}}$-modules, respectively). Moreover, in this context, *cartesian sheaves* of $\mathcal{O}_{\mathcal{S}}$-modules correspond to sheaves of $\mathcal{O}_X$-modules that are G-equivariant.

Let $\boldsymbol{A}$ denote any of the abelian categories considered in Definitions 2.0.3 or 2.0.4. Let $\mathrm{C}^b_{\mathrm{cc}}(\boldsymbol{A})$ ($\mathrm{C}^b_{\mathrm{cart}}(\boldsymbol{A})$) denote the category of all bounded complexes of objects in $\boldsymbol{A}$ with cohomology sheaves that are cartesian and coherent (cartesian, respectively). Similarly, we will let $\mathrm{C}_{\mathrm{bcc}}(\boldsymbol{A})$ denote the full subcategory of complexes in $\boldsymbol{A}$ with cohomology sheaves that are cartesian, coherent and vanish in all but finitely many degrees. These are all bi-Waldhausen categories (see Definition 3.0.1(iv)) with the same structure as above, that is, with cofibrations (fibrations) being maps of complexes that are degree-wise split monomorphisms (degree-wise split epimorphisms, respectively), and weak equivalences being maps that are quasiisomorphisms.

We summarize in the next theorem several basic properties of the K-theory and G-theory of such stacks proven elsewhere; see [Joshua 2007, Section 2; 2012, Sections 2, 3 and 5].

Theorem 2.0.5. (i) $\mathcal{S} \mapsto \pi_*(\mathrm{K}(\mathcal{S}))$ *is a contravariant functor from the category of algebraic stacks and morphisms of finite type to the category of graded rings.*

(ii) *Let $\mathcal{S}$ denote a smooth algebraic stack. Then the natural map* $\mathrm{K}(\mathcal{S}) \to \mathrm{G}(\mathcal{S})$ *is a weak equivalence. In case $\mathcal{S}'$ is a closed algebraic substack of $\mathcal{S}$, the natural map* $\mathrm{K}_{\mathcal{S}'}(\mathcal{S}) \to \mathrm{G}(\mathcal{S}')$ *is a weak equivalence.*

(iii) *The obvious inclusion functors*

$$\mathrm{C}^b_{\mathrm{cart}}(\mathrm{Coh}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})) \to \mathrm{C}^b_{\mathrm{cc}}(\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})) \to \mathrm{C}_{\mathrm{bcc}}(\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})) \to \mathrm{Pseudo}(\mathcal{S})$$

induce weak equivalences on taking the corresponding K-theory spaces.

(iv) *Assume that every coherent sheaf on the algebraic stack $\mathcal{S}$ is the quotient of a vector bundle. Then the obvious map* $\mathrm{K}_{\mathrm{naive}}(\mathcal{S}) \to \mathrm{K}(\mathcal{S})$ *is a weak equivalence*, *where* $\mathrm{K}_{\mathrm{naive}}(\mathcal{S}) = \mathrm{K}(\mathrm{StPerf}(\mathcal{S}))$.

Examples. • Assume the base scheme is a field k and G is a linear algebraic group. On a quotient stack [X/G], where the scheme X is assumed to be G-quasiprojective (that is, admits a G-equivariant locally closed immersion into a projective space $\mathbb{P}^n$ on which G acts linearly), every coherent sheaf is the quotient of a vector bundle. This follows from the work of Thomason [1987b] (see and also Theorem 8.0.1).

• Converse (see [Totaro 2004, Theorems 1.1 and 1.2] and [Edidin et al. 2001, Theorem 2.18]): Any smooth Deligne–Mumford stack $\mathcal{S}$ over a Noetherian base scheme, with generically trivial stabilizer is a quotient stack, [X/G] for an algebraic space X. If in addition, the stack $\mathcal{S}$ is defined over a field, and the coarse moduli space is a scheme with affine diagonal, then the stack has the resolution property. If $\mathcal{S}$ is a normal Noetherian algebraic stack over Spec $\mathbb{Z}$ whose stabilizer groups at closed points are affine, and every coherent sheaf on $\mathcal{S}$ is a quotient of a vector bundle, then $\mathcal{S}$ is a quotient stack.

Theorem 2.0.6 [Joshua 2012, Section 5]. (i) (closed immersion) *Let* $i : \mathcal{S}' \to \mathcal{S}$ *denote the closed immersion of an algebraic substack. Then the obvious map* $\mathrm{G}(\mathcal{S}') = \mathrm{K}(\mathrm{C}^b_{\mathrm{cart}}(\mathrm{Coh}(\mathcal{S}'))) \to \mathrm{K}(\mathrm{C}^b_{\mathrm{cart},\mathcal{S}'}(\mathrm{Coh}(\mathcal{S})))$ *is a weak equivalence*, *where* $\mathrm{C}^b_{\mathrm{cart},\mathcal{S}'}(\mathrm{Coh}(\mathcal{S}))$ *denotes the full subcategory of* $\mathrm{C}^b(\mathrm{Coh}(\mathcal{S}))$ *of complexes whose cohomology sheaves are cartesian and have supports in* $\mathcal{S}'$.

(ii) (localization) *Let* $i : \mathcal{S}' \to \mathcal{S}$ *denote a closed immersion of algebraic stacks with open complement* $j : \mathcal{S}'' \to \mathcal{S}$. *Then one obtains the fibration sequence* $\mathrm{G}(\mathcal{S}') \to \mathrm{G}(\mathcal{S}) \to \mathrm{G}(\mathcal{S}'') \to \sum \mathrm{G}(\mathcal{S}')$.

(iii) (homotopy property) *Let $\mathcal{S}$ denote an algebraic stack and let* $\pi : \mathcal{S} \times \mathbb{A}^1 \to \mathcal{S}$ *denote the obvious projection. Then* $\pi^* : \mathrm{G}(\mathcal{S}) \to \mathrm{G}(\mathcal{S} \times \mathbb{A}^1)$ *is a weak equivalence.*

3. The *G*-construction and relative *K*-theory

The G-construction is an undelooping of K-theory. Recall that the Waldhausen K-theory (defined as in [Waldhausen 1985]) involves a delooping of K-theory given by the $S_\bullet$-construction. However, this means to define the K-groups, one needs to perform an undelooping. One way to do this is to simply take the loop-space on the space produced by the $\mathrm{S}_\bullet$-construction. The G-construction is a way to perform this instead at the categorical level. Such a construction is in fact needed to obtain a presentation of (higher) K-theory groups, as well as in being able to define λ-operations on higher K-theory.

Though the basics of such a construction are outlined in [Gunnarsson et al. 1992] for categories with cofibrations and weak equivalences (and in [Gillet and Grayson 1987] for the Q-construction on exact categories), their construction cannot be used in this paper, because of the following issue. In this paper, a key technique we use is to reduce higher K-theory to the Grothendieck group of a relative K-theory space. We would need a suitable G-construction that applies to this relative K-theory space. The construction appearing in [Gunnarsson et al. 1992, Definition 2.2] (and which is related to the one in [Gillet and Grayson 1987]), only applies to the absolute case. Therefore, we provide a somewhat different, but related G-construction that applies to relative K-theory and is a suitable relative variant of the one considered in [Gunnarsson et al. 1992, Definition 2.2]. The main difference stems from the fact that for the $S_\bullet$-construction applied to a category with cofibrations and weak equivalences $\boldsymbol{B}$, a path space may be defined readily by shifting the constituent categories in the simplicial category $S_\bullet(\boldsymbol{B})$ by 1 and throwing away the face map d_0. This works fine since $S_0(\boldsymbol{B})$ is a point, so that the resulting path space is simplicially contractible. But with $G_\bullet(\boldsymbol{B})$, $G_0(\boldsymbol{B}) = \boldsymbol{B} \times \boldsymbol{B}$, so that the above shifting technique does not define a path space for $wG_\bullet(\boldsymbol{B})$.

Instead, we directly define the homotopy fiber of a map on the $G_\bullet$-construction as in (3.2) and make use of that to define the relative K-theory space using the $G_\bullet$-construction.

We will presently recall the *G-construction for categories with cofibrations and weak equivalences* from [Gunnarsson et al. 1992, Section 2].

Definition 3.0.1. First, a category is *pointed*, if it is equipped with a distinguished *zero object*: this zero object will often be denoted $*$. Then, *a category with cofibrations and weak equivalences* will mean the following throughout the paper; see [Waldhausen 1985, 1.1]:

(i) A *pointed category* $\boldsymbol{A}$, provided with a subcategory $co\boldsymbol{A}$ of cofibrations satisfying the axioms [loc. cit., Cof.1 through Cof.3 in 1.1] and also provided with a subcategory $w\boldsymbol{A}$ of weak equivalences satisfying the axioms [loc. cit., Weq.1 and Weq.2 in 1.2], as well as the *Saturation and Extension axioms* in [Waldhausen 1985, 1.2]. We will often refer to this as *a Waldhausen category.*[5]

(ii) A subcategory of *fibrations* will denote a subcategory of the pointed category $\boldsymbol{A}$ satisfying the dual of the axioms [loc. cit., Cof.1 through Cof.3 in 1.1]. A *bi-Waldhausen category* will denote a pointed category $\boldsymbol{A}$ provided with a

[5]This terminology is consistent with [loc. cit.], and is convenient, though we are told Waldhausen personally does not prefer to use it.

subcategory of cofibrations, a subcategory of fibrations and a subcategory of weak equivalences, satisfying the above axioms, as well as the dual of [loc. cit., Weq.2 in 1.2].

(iii) A functor $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$ between categories with cofibrations (fibrations) and weak equivalences is an *exact functor* if it preserves the subcategories of cofibrations (fibrations) and weak equivalences.

Given a category $\boldsymbol{A}$ with cofibrations and weak equivalences, $w\mathrm{S}_\bullet(\boldsymbol{A})$ will denote the simplicial category (that is, a simplicial object in the category of all small categories), so that the objects of $w\mathrm{S}_n(\boldsymbol{A})$ are sequences $A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$ of cofibrations in $\boldsymbol{A}$ (with a choice of subquotients A_i/A_{i-1}). A morphism in this category between

$$A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n \quad \text{and} \quad B_1 \rightarrowtail B_2 \rightarrowtail \cdots \rightarrowtail B_n$$

is given by a sequence of weak equivalences $A_i \to B_i$ and $A_i/A_{i-1} \to B_i/B_{i-1}$, compatible with the given cofibrations. The *path-object* $\mathrm{P}w\mathrm{S}_\bullet(\boldsymbol{A})$ associated to $w\mathrm{S}_\bullet(\boldsymbol{A})$ is the simplicial category given by $\mathrm{P}w\mathrm{S}_\bullet(\boldsymbol{A})_n = w\mathrm{S}_{n+1}(\boldsymbol{A})$, together with the functor $d_0 : w\mathrm{S}_{n+1}(\boldsymbol{A}) \to w\mathrm{S}_n(\boldsymbol{A})$. The face map d_i (the degeneracy s_i) of the simplicial category $\mathrm{P}w\mathrm{S}_\bullet(\boldsymbol{A})$ is the face map d_{i+1} (the degeneracy s_{i+1}, respectively) of the simplicial category $w\mathrm{S}_\bullet(\boldsymbol{A})$; see [Waldhausen 1985, page 341]. Then the *G-construction on* $\boldsymbol{A}$ is the simplicial category defined by the fibered product $w\mathrm{G}_\bullet(\boldsymbol{A}) = \mathrm{P}w\mathrm{S}_\bullet(\boldsymbol{A}) \times_{d_0, w\mathrm{S}_\bullet(\boldsymbol{A}), d_0} \mathrm{P}w\mathrm{S}_\bullet(\boldsymbol{A})$, with the cofibrations and weak equivalences defined in the obvious manner using these structures on $\mathrm{P}w\mathrm{S}_\bullet(\boldsymbol{A})$ and $w\mathrm{S}_\bullet(\boldsymbol{A})$.

Remark 3.0.2. Recall that the map $d^0 : [n] \to [n+1]$ in Δ omits 0 in its image. Therefore, one may readily see that the description of $w\mathrm{G}_n\boldsymbol{A}$ as in [Gunnarsson and Schwänzl 2002, Section 1, (2)] holds, which is also the same as that in [Grayson 1989, Section 3]. An n simplex of this simplicial category is given by two $n+1$ simplices of $w\mathrm{S}_\bullet\boldsymbol{A}$ whose successive quotients are provided with compatible isomorphisms. In particular, a vertex of this simplicial category is given by a pair of objects in $w\boldsymbol{A}$.

Let $w\mathrm{Cof}$ denote the category whose objects are categories with cofibrations and weak equivalences in the above sense. Since the construction $\boldsymbol{A} \mapsto w\mathrm{G}_\bullet(\boldsymbol{A})$ is covariantly functorial, we will view it as a functor

$$w\mathrm{G}_\bullet : w\mathrm{Cof} \to \Delta^{\mathrm{op}} - w\mathrm{Cof}, \tag{3.1}$$

where $\Delta^{\mathrm{op}} - w\mathrm{Cof}$ denotes the category of all simplicial objects in $w\mathrm{Cof}$.

We will apply these constructions to various categories with cofibrations and weak equivalences we encounter, for example, the following ones. Let $\mathcal{S}$ denote an algebraic stack and let $\mathrm{Perf}(\mathcal{S})$ denote the category of perfect complexes on $\mathcal{S}$. For what follows, one may let $\mathcal{S}'$ denote a closed algebraic substack of the given stack

$\mathcal{S}$ and consider $\mathrm{Perf}_{\mathcal{S}'}(\mathcal{S})$ also in the place of $\mathrm{Perf}(\mathcal{S})$; however, for the most part we will explicitly discuss only the case where $\mathcal{S}' = \mathcal{S}$.

One may readily verify that the category $\mathrm{Perf}(\mathcal{S})$ is pseudoadditive (see [Gunnarsson et al. 1992, Definition 2.3]: observe that it suffices to show that if $A \rightarrowtail C$ is a degree-wise split monomorphism of complexes, then the natural maps $C \oplus_A C \to C \times C/A \leftarrow C \oplus C/A$ are quasiisomorphisms. In fact, the second map is clearly an isomorphism and one may show that the first map is an isomorphism in each degree as follows. The assumption that $A \rightarrowtail C$ is a degree-wise split monomorphism shows that one has an isomorphism in each degree n: $C^n \cong A^n \oplus C^n/A^n$. This then implies that in each degree n, one obtains the isomorphism: $C^n \oplus_{A^n} C^n \cong C^n \oplus (C^n/A^n)$. It is shown in [Gunnarsson et al. 1992, Theorem 2.6] that $|w\mathrm{G}_\bullet(\boldsymbol{A})| \simeq \Omega(|wS_\bullet(\boldsymbol{A})|)$, provided $\boldsymbol{A}$ is *pseudoadditive*.

Next let $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$ denote an *exact functor* of categories with cofibrations and weak equivalences. We will assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ are both pseudoadditive categories. First we let $|w\mathrm{G}_\bullet(\boldsymbol{A})|$ ($|w\mathrm{G}_\bullet(\boldsymbol{B})|$) denote the topological realization of the diagonal of the bisimplicial set obtained by taking the nerve of the simplicial category $w\mathrm{G}_\bullet(\boldsymbol{A})$ ($w\mathrm{G}_\bullet(\boldsymbol{B})$, respectively). Since $w\mathrm{G}_\bullet(\boldsymbol{B})$ is a simplicial category, the nerve functor Nerve applied to it produces a bisimplicial set. $\Delta\,\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B}))$ will denote its diagonal. Now one may observe that each 0-simplex of the simplicial space $\Delta\,\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B}))$, which is a pair of objects $(P, Q) \in \boldsymbol{B}$, defines a connected component of the space $|w\mathrm{G}_\bullet(\boldsymbol{B})|$. We will choose for each connected component of $|w\mathrm{G}_\bullet(\boldsymbol{B})|$ a 0-simplex that will remain fixed throughout the following discussion, and will serve as the base point for that component.

We next consider the *path space* $\mathrm{P}(|w\mathrm{G}_\bullet(\boldsymbol{B})|)$ of pointed paths: it will consist of paths $p : \mathrm{I} = |\Delta[1]| \to |w\mathrm{G}_\bullet(\boldsymbol{B})|$, so that $p(1)$ is at the chosen base point for some connected component of $|w\mathrm{G}_\bullet(\boldsymbol{B})|$. Clearly the map sending a path p to $p(0)$ defines a map $\pi : \mathrm{P}(|w\mathrm{G}_\bullet(\boldsymbol{B})|) \to |w\mathrm{G}_\bullet(\boldsymbol{B})|$. We define $w\mathrm{G}(\boldsymbol{f})$ by the pullback square:

$$\begin{tikzcd}
w\mathrm{G}(\boldsymbol{f}) \arrow[r] \arrow[d] & \mathrm{P}(|w\mathrm{G}_\bullet(\boldsymbol{B})_*|) \arrow[d, "\pi"] \\
|w\mathrm{G}_\bullet(\boldsymbol{A})| \arrow[r, "|w\mathrm{G}_\bullet(\boldsymbol{f})|"] & |w\mathrm{G}_\bullet(\boldsymbol{B})|,
\end{tikzcd} \tag{3.2}$$

where $\mathrm{P}(|w\mathrm{G}_\bullet(\boldsymbol{B})_*|)$ denotes the path component of $\mathrm{P}(|w\mathrm{G}_\bullet(\boldsymbol{B})|)$ that is sent by π to the path component of $|w\mathrm{G}_\bullet(\boldsymbol{B})|$ containing the base point $(*, *)$, with $*$ denoting the base point of the category $\boldsymbol{B}$.

3.3. ***The connected components of $w\mathrm{G}(\boldsymbol{f})$.*** Observe that each triple $((P, Q), (\bar{P}, \bar{Q}), p)$ defines a connected component of $w\mathrm{G}(\boldsymbol{f})$, where (P, Q) is a 0-simplex of $w\mathrm{G}_\bullet(\boldsymbol{A})$, $(\bar{P}, \bar{Q})$ is a 0-simplex of $w\mathrm{G}_\bullet(\boldsymbol{B})$ in the same connected component of the base point $(*, *)$ of $|w\mathrm{G}_\bullet(\boldsymbol{B})|$ so that

$$\boldsymbol{f}(P) = \bar{P}, \quad \boldsymbol{f}(Q) = \bar{Q}, \quad \text{and} \tag{3.4}$$

p is a path in $|w\mathrm{G}_\bullet(\boldsymbol{B})|$ joining the vertex $(\bar{P}, \bar{Q})$ to the base point$(*, *)$.

One may observe that the 1-simplices of the simplicial set $\Delta(\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B})))$, will be given by *pairs of commutative squares*:

$$\begin{tikzcd} \bar{P}^1_0 \arrow[r, "\simeq"] \arrow[d, tail] & \bar{P}^2_0 \arrow[d, tail] \\ \bar{P}^1_1 \arrow[r, "\simeq"] & \bar{P}^2_1 \end{tikzcd} \qquad \begin{tikzcd} \bar{Q}^1_0 \arrow[r, "\simeq"] \arrow[d, tail] & \bar{Q}^2_0 \arrow[d, tail] \\ \bar{Q}^1_1 \arrow[r, "\simeq"] & \bar{Q}^2_1 \end{tikzcd} \tag{3.5}$$

together with an isomorphism $\bar{P}^j_1/\bar{P}^j_0 \cong \bar{Q}^j_1/\bar{Q}^j_0$, for $j = 1, 2$, where $(\bar{P}^j_i, \bar{Q}^j_i)$ are 0-simplices in $w\mathrm{G}_\bullet(\boldsymbol{B})$, the vertical maps are cofibrations, while the horizontal maps are weak equivalences.

The path components of the topological space $|w\mathrm{G}_\bullet(\boldsymbol{B})|$ correspond to the path components of the simplicial set $\Delta(\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B})))$. Therefore, we obtain the following equivalent description of the connected components of $w\mathrm{G}(\boldsymbol{f})$ by viewing the path p in (3.4) as a *zig-zag*-path (see, for example: [Gabriel and Zisman 1967, Chapter II, 7.3]) in $\Delta(\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B})))$

$$(\bar{P}, \bar{Q}) = (\bar{P}_0, \bar{Q}_0) \rightarrowtail (\bar{P}_1, \bar{Q}_1) \leftarrowtail (\bar{P}_2, \bar{Q}_2) \rightarrowtail \\ \cdots \leftarrowtail (\bar{P}_{m-1}, \bar{Q}_{m-1}) \rightarrowtail (\bar{P}_m, \bar{Q}_m) = (*, *), \tag{3.6}$$

where each arrow $(\bar{P}_i, \bar{Q}_i) \rightarrowtail (\bar{P}_{i+1}, \bar{Q}_{i+1})$ $((\bar{P}_i, \bar{Q}_i) \leftarrowtail (\bar{P}_{i+1}, \bar{Q}_{i+1}))$ is a 1-simplex of $\Delta(\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B})))$ in the above sense. (To see this, observe that such a zig-zag path in $\Delta(\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B})))$ corresponds to a simplicial map $q : \mathrm{I}_n \to \Delta(\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B})))$, and therefore to a map on the realizations: $p : \mathrm{I} = |\mathrm{I}_n| \to |\Delta(\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B})))|$. Here I_n is the simplicial set considered in [Gabriel and Zisman 1967, 2.5.1].)

Example 3.6.1. The basic application of the construction in (3.2) is to the following situation. Let $\mathcal{S}_1$ denote a smooth algebraic substack of the given stack $\mathcal{S}$ and let $\Delta[n] = \mathrm{Spec}\big(\mathcal{O}_S[x_0, \dots, x_n]/\sum x_i - 1\big)$, where S is the base scheme, which is assumed to be a regular Noetherian scheme. We may let $\mathcal{S}_0 = \mathcal{S}_1 \times_{\mathrm{S}} \Delta[n]$. Since $\Delta[n] \cong \mathbb{A}^n$, $\mathcal{S}_0$ is also smooth. Then the following are closed substacks of $\mathcal{S}_0$:

(i) $\mathcal{S}_1 \times_{\mathrm{S}} \delta\Delta[n]$, where $\delta\Delta[n] = \bigcup_{i=0,\dots,n} \delta^i\Delta[n]$ with $\delta^i\Delta[n]$ denoting the i-th face of $\Delta[n]$.

(ii) $\mathcal{S}_1 \times_{\mathrm{S}} \Sigma$, where $\Sigma = \bigcup_{i=0,\dots,n-1} \delta^i\Delta[n]$.

(iii) $\mathcal{S}_2 \times_{\mathrm{S}} \Delta[n]$, where $\mathcal{S}_2$ is any closed algebraic substack of $\mathcal{S}_1$.

In each of the above cases, one may let f denote the corresponding closed immersion, and $\boldsymbol{f}$ denote the corresponding functor of categories with cofibrations and weak equivalences.

4. Another model for the homotopy fiber of the G-construction

It is clearly preferable to obtain a categorical model for the homotopy fiber, whose realization identifies with the homotopy fiber of the realizations constructed in the last section. Here the difficulty is with obtaining a suitable model for the path space, which seems to be possible only in special cases, like in the case of the $\mathrm{S}_\bullet$-construction. On the other hand, it is relatively straightforward to obtain a model for the homotopy cofiber, which we proceed to discuss next.

For this we begin with a rather general construction. First, a simplicial category will denote a simplicial object in the category of all (small) categories, rather than a category that is simplicially enriched. Then a functor $f_\bullet : \mathrm{S}'_\bullet \to \mathrm{S}_\bullet$ between simplicial categories will denote a collection of functors $\{f_n : \mathrm{S}'_n \to \mathrm{S}_n \mid n\}$ so that they commute with the face maps and degeneracies. Let $*$ denote a chosen category with just one object denoted $*$, and only one morphism, namely the identity morphism of $*$.

Definition 4.0.1. Let $f : \mathrm{S}'_\bullet \to \mathrm{S}_\bullet$ denote a functor between two simplicial categories. Then we let $\mathrm{Cone}(f)_\bullet$ denote the simplicial category that is given in degree n by the category:

$$\mathrm{Cone}(f_\bullet)_n = * \sqcup \left(\bigsqcup_{\alpha \in \Delta[1]_n - \{(0,\dots,0),(1,\dots,1)\}} \mathrm{S}'_n \right) \sqcup \mathrm{S}_n, \tag{4.1}$$

where we regard S_n as indexed by $(1, \dots, 1) \in \Delta[1]_n$ and $*$ as indexed by $(0, \dots, 0) \in \Delta[1]_n$, with the face maps and degeneracies induced from those of $\mathrm{S}'_\bullet$, $\mathrm{S}_\bullet$ and those of $\Delta[1]$. More precisely, we define the face map $d_i : \mathrm{Cone}(f)_n \to \mathrm{Cone}(f)_{n-1}$ by:

(i) The summand $*$ is sent to the summand * by the identity.

(ii) The summand S_n indexed by $(1, \dots, 1) \in \Delta[1]_n$ is sent to the summand S_{n-1} indexed by $(1, \cdots, 1) \in \Delta[1]_{n-1}$ by the face map $d_i^{S_\bullet}$.

(iii) If $d_i(\alpha) = \alpha'$, with $\alpha \in \Delta[1]_n - \{(0, \dots, 0), (1, \dots, 1)\}$ and $\alpha' \in \Delta[1]_{n-1} - \{(0, \dots, 0), (1, \dots, 1)\}$, then d_i sends the summand S'_n indexed by α to the summand S'_{n-1} indexed by α' by the face map $d_i^{\mathrm{S}'_\bullet}$.

(iv) If $d_i(\alpha) = (0, \dots, 0)$, $\alpha \in \Delta[1]_n - \{(0, \dots, 0), (1, \dots, 1)\}$, then d_i sends the S'_n indexed by α to $*$.

(v) If $d_i(\alpha) = (1, \dots, 1)$, $\alpha \in \Delta[1]_n - \{(0, \dots, 0), (1, \dots, 1)\}$, then d_i sends the summand S'_n indexed by α to S_{n-1} by $f_{n-1} \circ d_i^{\mathrm{S}'_\bullet} = d_i^{\mathrm{S}_\bullet} \circ f_n : \mathrm{S}'_n \to \mathrm{S}_{n-1}$.

We define the degeneracy $s_i : \mathrm{Cone}(f)_{n-1} \to \mathrm{Cone}(f)_n$ by:

(i) The summand $*$ is sent to the summand * by the identity.

(ii) The summand S_{n-1} indexed by $(1, \dots, 1) \in \Delta[1]_{n-1}$ is sent to the summand S_n indexed by $(1, \dots, 1) \in \Delta[1]_n$ by the degeneracy $s_i^{S_\bullet}$.

(iii) The summand S'_{n-1} indexed by $\alpha \in \Delta[1]_{n-1} - \{(0, \dots, 0), (1, \dots, 1)\}$ is sent to the summand S'_n indexed by $s_i(\alpha) \in \Delta[1]_n - \{(0, \dots, 0), (1, \dots, 1)\}$ by $s_i : \mathrm{S}'_{n-1} \to \mathrm{S}'_n$.

We skip the verification that, so defined, $\mathrm{Cone}(f)_\bullet$ is a simplicial category, together with a natural functor $\mathrm{S}_\bullet \to \mathrm{Cone}(f)_\bullet$, sending S_n to the summand in $\mathrm{Cone}(f)_n$ indexed by $(1, \dots, 1) \in \Delta[1]_n$. In fact, one may also define a bisimplicial category

$$\mathrm{Cone}(f)_{\bullet,\bullet} \tag{4.2}$$

so that in bidegree (n, m) one has S'_m (S_m) replacing S'_n (S_n, respectively) in (4.1), and with the face maps and degeneracies defined suitably. Then the simplicial category in (4.1) will be the diagonal of this bisimplicial category.

Proposition 4.2.1. *Let $f : \mathrm{S}'_\bullet \to \mathrm{S}_\bullet$ denote a functor of simplicial categories. Let* $\mathrm{Cone}(\Delta\,\mathrm{Nerve}(f))$ *denote the mapping cone of the map of simplicial sets*

$$\Delta\,\mathrm{Nerve}(f) : \Delta\,\mathrm{Nerve}(\mathrm{S}'_\bullet) \to \Delta\mathrm{Nerve}(\mathrm{S}_\bullet),$$

which is defined as in Definition 4.0.1, with the simplicial set $\mathrm{Nerve}(\mathrm{S}'_n)$ ($\mathrm{Nerve}(\mathrm{S}_n)$) *replacing* S'_n (S_n, *respectively*)*. Then* $\Delta\,\mathrm{Nerve}(\mathrm{Cone}(f)_\bullet)$ *can be identified with* $\mathrm{Cone}(\Delta\,\mathrm{Nerve}(f))$.

Proof. One may readily observe from its definition that the nerve functor Nerve commutes with finite coproducts. Therefore, it follows that

$$(\Delta\,\mathrm{Nerve}(\mathrm{Cone}(f)))_n = \mathrm{Nerve}_n(*) \sqcup \Bigg(\bigsqcup_{\alpha \in \Delta[1]_n - \{(0,\dots,0),(1,\dots,1)\}} \mathrm{Nerve}_n(\mathrm{S}'_n) \Bigg) \sqcup \mathrm{Nerve}_n(\mathrm{S}_n),$$

while $\mathrm{Cone}(\Delta\,\mathrm{Nerve}(f))_n$ is also given by the same set. We skip the verification that the structure maps for both simplicial sets $\Delta\,\mathrm{Nerve}(\mathrm{Cone}(f))$ and $\mathrm{Cone}(\Delta\,\mathrm{Nerve}(f))$ are the same. $\square$

Next let $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$ denote an *exact* functor between categories with cofibrations and weak equivalences.

Theorem 4.2.2. *Let $w\mathrm{G}_\bullet(f) : w\mathrm{G}_\bullet(\boldsymbol{A}) \to w\mathrm{G}_\bullet(\boldsymbol{B})$ denote the induced functor of the G-constructions. Then one obtains the natural identification*:

$$\begin{aligned}&\Delta\,\mathrm{Nerve}(\mathrm{Cone}(w\mathrm{G}_\bullet(f)))\\ &\qquad\qquad = \mathrm{Cone}(\Delta\,\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{A})) \xrightarrow{\Delta\,\mathrm{Nerve}(w\mathrm{G}_\bullet(f))} \Delta\,\mathrm{Nerve}(w\mathrm{G}_\bullet(\boldsymbol{B}))).\end{aligned}$$

Proof. This is clear from Proposition 4.2.1. □

Corollary 4.2.3. *Assume the same hypotheses as in Theorem 4.2.2. Then we obtain the natural weak equivalence*:

$$w\mathrm{G}(f) \simeq \Omega(|\mathrm{Nerve}(\mathrm{Cone}(w\mathrm{G}_\bullet(f)))|),$$

where $w\mathrm{G}(f)$ is the space defined in (3.2).

Proof. This is clear in view of Theorem 4.2.2 and the observation that $w\mathrm{G}(f)$ is in fact an infinite loop space. Since $w\mathrm{G}(f)$ is the homotopy fiber of a map induced by $w\mathrm{G}_\bullet(f) : w\mathrm{G}_\bullet(\boldsymbol{A}) \to w\mathrm{G}_\bullet(\boldsymbol{B})$, it suffices to observe that $|w\mathrm{G}_\bullet(\boldsymbol{A})|$ and $|w\mathrm{G}_\bullet(\boldsymbol{B})|$ are both infinite loop spaces.

Here are some additional details. First, observe that for any category $\boldsymbol{C}$ with cofibrations and weak equivalences, there is a natural map

$$|w\mathrm{G}_\bullet(\boldsymbol{C})| \to \Omega|w\mathrm{S}_\bullet(\boldsymbol{C})|.$$

Applying this to the functor $f : \boldsymbol{A} \to \boldsymbol{B}$ of categories with cofibrations and weak equivalences, we obtain the homotopy commutative diagram:

$$\begin{tikzcd}
{|w\mathrm{G}_\bullet(\boldsymbol{A})|} \arrow[r] \arrow[d] & {|w\mathrm{G}_\bullet(\boldsymbol{B})|} \arrow[r] \arrow[d] & {|\mathrm{Nerve}(\mathrm{Cone}(w\mathrm{G}_\bullet(f)))|} \arrow[d] \\
{\Omega|w\mathrm{S}_\bullet(\boldsymbol{A})|} \arrow[r] & {\Omega|w\mathrm{S}_\bullet(\boldsymbol{B})|} \arrow[r] & {\Omega|\mathrm{Nerve}(\mathrm{Cone}(w\mathrm{S}_\bullet(f)))|,}
\end{tikzcd}$$

where the vertical maps are weak equivalences. The bottom row is clearly a diagram of infinite loop spaces. Therefore, the homotopy fiber of the first map in the second row is $\Omega^2|\mathrm{Nerve}(\mathrm{Cone}(w\mathrm{S}_\bullet(f)))|$. Since $|w\mathrm{G}_\bullet(f)|$ is the homotopy fiber of the map $|w\mathrm{G}_\bullet(\boldsymbol{A})| \to |w\mathrm{G}_\bullet(\boldsymbol{B})|$, one sees that it is weakly equivalent to $\Omega^2|\mathrm{Nerve}(\mathrm{Cone}(w\mathrm{S}_\bullet(f)))|$, which in turn is weakly equivalent to $\Omega|\mathrm{Nerve}(\mathrm{Cone}(w\mathrm{G}_\bullet(f)))|$; see [Gunnarsson et al. 1992] or [Waldhausen 1985] for further details. □

Next we consider additivity on exact sequences for $\mathrm{Cone}(w\mathrm{G}_\bullet(f))$. Let $f : \boldsymbol{A} \to \boldsymbol{B}$ denote an exact functor of categories with cofibrations and weak equivalences, and let

$$g = w\mathrm{G}_\bullet(f) : w\mathrm{G}_\bullet(\boldsymbol{A}) \to w\mathrm{G}_\bullet(\boldsymbol{B}) \tag{4.3}$$

denote the induced functor of simplicial categories. Let $\mathrm{E}(\boldsymbol{A})$ ($\mathrm{E}(\boldsymbol{B})$) denote the Waldhausen category of cofibration sequences, that is, the category whose objects are short exact sequences of the form $F' \rightarrowtail F \twoheadrightarrow F''$ in $\boldsymbol{A}$ ($\boldsymbol{B}$, respectively). These induce the simplicial categories $\mathrm{E}(w\mathrm{G}_\bullet(\boldsymbol{A})) = w\mathrm{G}_\bullet(\mathrm{E}(\boldsymbol{A}))$ and $\mathrm{E}(w\mathrm{G}_\bullet(\boldsymbol{B})) = w\mathrm{G}_\bullet(\mathrm{E}(\boldsymbol{B}))$. Let $\mathrm{E}(\boldsymbol{g}) = \mathrm{E}w\mathrm{G}_\bullet(\boldsymbol{f}) : \mathrm{E}w\mathrm{G}_\bullet(\boldsymbol{A}) \to \mathrm{E}w\mathrm{G}_\bullet(\boldsymbol{B})$. This is a functor between the simplicial categories $\mathrm{E}w\mathrm{G}_\bullet(\boldsymbol{A})$ and $\mathrm{E}w\mathrm{G}_\bullet(\boldsymbol{B})$. We consider its mapping cone, $\mathrm{Cone}(\mathrm{E}(\boldsymbol{g})) = \mathrm{E}(\mathrm{Cone}(\boldsymbol{g}))$ as in Definition 4.0.1. Let

$$\mathrm{Cone}(w\mathrm{G}_\bullet(\boldsymbol{f})) \boxtimes \mathrm{Cone}(w\mathrm{G}_\bullet(\boldsymbol{f})) \tag{4.4}$$

denote the simplicial category given in degree n by

$$* \sqcup \left(\bigsqcup_{\alpha \in \Delta[1]_n - \{(0,\dots,0),(1,\cdots,1)\}} (w\mathrm{G}_n(\boldsymbol{A}) \times w\mathrm{G}_n(\boldsymbol{A})) \right) \sqcup (w\mathrm{G}_n(\boldsymbol{B}) \times w\mathrm{G}_n(\boldsymbol{B})), \tag{4.5}$$

with the face maps and degeneracies defined as in Definition 4.0.1. We define a functor of simplicial categories

$$\Phi : \mathrm{Cone}(\mathrm{E}(\boldsymbol{g})) = \mathrm{E}(\mathrm{Cone}(\boldsymbol{g})) \to \mathrm{Cone}(w\mathrm{G}_\bullet(\boldsymbol{f})) \boxtimes \mathrm{Cone}(w\mathrm{G}_\bullet(\boldsymbol{f})), \tag{4.6}$$

which will be induced by the projections to either factor, that is, sending a cofiber sequence $(X \rightarrowtail Z \twoheadrightarrow Y) \mapsto (X, Y)$.

Theorem 4.6.1 (additivity theorem, I). *Assuming the above situation, the functor Φ induces a weak equivalence on taking the Nerve.*

Proof. For the proof it is convenient to view the Cone construction for a map of simplicial categories as first defining a bisimplicial category (as in (4.2)), and then taking its diagonal. Therefore, we reduce to considering a corresponding functor $\Phi_{\bullet,\bullet}$ of bisimplicial categories: then the proof reduces to showing that the corresponding functors $w\mathrm{G}_\bullet\mathrm{E}(\boldsymbol{A}) \to w\mathrm{G}_\bullet(\boldsymbol{A}) \times w\mathrm{G}_\bullet(\boldsymbol{A})$ and $w\mathrm{G}_\bullet\mathrm{E}(\boldsymbol{B}) \to w\mathrm{G}_\bullet(\boldsymbol{B}) \times w\mathrm{G}_\bullet(\boldsymbol{B})$ are weak equivalences. But this is proven in [Gunnarsson et al. 1992, Theorem 2.10]. □

Definition 4.6.2. Let $\boldsymbol{A}$, $\boldsymbol{B}$ denote categories with cofibrations and weak equivalences. Let $F', F, F'' : \boldsymbol{A} \to \boldsymbol{B}$ denote exact functors. Then $F' \rightarrowtail F \twoheadrightarrow F''$ is a cofibration sequence, if for each object $A \in \boldsymbol{A}$, $F'(A) \rightarrowtail F(A) \twoheadrightarrow F''(A)$ is a cofibration sequence in $\boldsymbol{B}$.

Corollary 4.6.3 (additivity theorem for the mapping cone). *Let $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$ denote an exact functor of categories with cofibrations and weak equivalences and let $\boldsymbol{g} = w\mathrm{G}_\bullet(\boldsymbol{f})$:*

(i) *Assume $\{X'_m \rightarrowtail Z'_m \twoheadrightarrow Y'_m \mid m\}$ is a cofibration sequence in $w\mathrm{G}_\bullet(\boldsymbol{A})$, (that is, an object of $\mathrm{E}(w\mathrm{G}_\bullet(\boldsymbol{A}))$) and $\{X_m \rightarrowtail Z_m \twoheadrightarrow Y_m \mid m\}$ is a cofibration sequence in $w\mathrm{G}_\bullet(\boldsymbol{B})$, (that is, an object of $\mathrm{E}(w\mathrm{G}_\bullet(\boldsymbol{B}))$) so that they are compatible under the functor $\boldsymbol{g}$, that is, $\boldsymbol{g}(X'_m) \cong X_m$, $\boldsymbol{g}(Y'_m) \cong Y_m$ and $\boldsymbol{g}(Z'_m) \cong Z_m$, for all m, where $\cong$ denotes isomorphisms. Then the above data provides a cofibration sequence in $\mathrm{Cone}(w\mathrm{G}_\bullet(\boldsymbol{f}))$.*

(ii) *Let $F' \rightarrowtail F \twoheadrightarrow F''$ denote a cofibration sequence of exact functors from the Waldhausen category $\boldsymbol{C}$ to $\boldsymbol{A}$, and let $\bar{F}' \rightarrowtail \bar{F} \twoheadrightarrow \bar{F}''$ denote a cofibration sequence of exact functors from the Waldhausen category $\boldsymbol{C}$ to $\boldsymbol{B}$, so that the diagram of functors*

$$\begin{tikzcd} \mathrm{f}\circ F' \arrow[r, tail] \arrow[d, "\cong"] & \mathrm{f}\circ F \arrow[r, two heads] \arrow[d, "\cong"] & \mathrm{f}\circ F'' \arrow[d, "\cong"] \\ \bar{F}' \arrow[r, tail] & \bar{F} \arrow[r, two heads] & \bar{F}'' \end{tikzcd} \tag{4.7}$$

commutes. Denoting by $\mathfrak{F}$ ($\mathfrak{F}'$, $\mathfrak{F}''$) the induced functor defined on $\boldsymbol{C}$ and taking values in $\mathrm{Cone}(w\mathrm{G}_\bullet(f))$ *induced by the pair* $((F, \bar{F}), (F', \bar{F}'), (F'', \bar{F}''),$ *we obtain a weak equivalence*:

$$\mathfrak{F} \simeq \mathfrak{F}' \vee \mathfrak{F}'',$$

where $\mathfrak{F}' \vee \mathfrak{F}''$ is the functor defined by $(\mathfrak{F}' \vee \mathfrak{F}'')(C) = \mathfrak{F}'(C) \vee \mathfrak{F}''(C)$ for all objects $C \in \boldsymbol{C}$. In other words, we obtain a weak equivalence $\mathfrak{F}(C) \simeq \mathfrak{F}'(C) \vee \mathfrak{F}''(C)$ for all objects $C \in \boldsymbol{C}$.

Proof. (i) follows readily from the definition of $\mathrm{Cone}(w\mathrm{G}_\bullet(\boldsymbol{f}))$ as in Definition 4.0.1. (ii) follows from Theorem 4.6.1, along the same lines as the proof of the corresponding result in [Waldhausen 1985, Proposition 1.3.2]. □

Remark 4.7.1. Corollary 4.2.3 translates the additivity theorem in Corollary 4.6.3 to an additivity theorem for the homotopy fiber of the K-theory spaces associated to an exact functor of categories with cofibrations and weak equivalences. This is then invoked in a key step showing that the relative Higher K-groups of smooth algebraic stacks have a pre-λ-ring structure: see (5.20).

5. Lambda operations on the higher *K*-theory of algebraic stacks: the prelambda ring structure

We first recall briefly the construction of lambda and Adams operations for Waldhausen style K-theory in [Gunnarsson and Schwänzl 2002]. This involves first finding an undelooping of algebraic K-theory at the categorical level considered first in [Gunnarsson et al. 1992] and then defining operations corresponding to the exterior powers at the level of categories with cofibrations and weak equivalences. The definitions of these operations have been given in [Gunnarsson and Schwänzl 2002] for categories with cofibrations and weak equivalences and based on the approach in [Grayson 1989] (which is worked out in the framework of the Quillen K-theory of exact categories), but under the assumption that the power operations preserve weak equivalences. To make sure that this hypothesis is satisfied in our context, we consider the left derived functors of these operations in the sense of [Dold and Puppe 1961] and [Illusie 1971, Chapitre I]; see also [Soulé 1992, pages 22–27] for a readable account. The techniques of the last 2 sections, and the technique of [Bloch and Lichtenbaum 1997] whereby higher K-groups can be reduced to a relative form of Grothendieck groups then enable us to show that the higher K-groups of all smooth algebraic stacks are pre-λ-rings.

We next recall the definition of $\bigwedge^k$ (as in [Grayson 1989]). Let $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ denote the category of all $\mathcal{O}_{\mathcal{S}}$-modules. Let $\mathrm{Filt}_k(\mathcal{S})$ denote the category whose objects are sequences of split monomorphisms in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$:

$$M = M_{0,1} \rightarrowtail M_{0,2} \rightarrowtail \cdots \rightarrowtail M_{0,k}$$

together with subquotients $M_{i,j} = M_{0,j}/M_{0,i} \in \mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, for $i < j$. We let

$$\bigwedge^k(M) = M_{0,1} \wedge M_{0,2} \wedge \cdots \wedge M_{0,k}, \tag{5.1}$$

which is defined locally on $\mathcal{S}$ as the quotient of $(M_{0,1} \otimes M_{0,2} \otimes \cdots \otimes M_{0,k})$ by the submodule generated by terms of the form $m_1 \otimes m_2 \otimes \cdots \otimes m_i \otimes m_{i+1} \otimes \cdots \otimes m_n$, with $m_i = m_{i+1}$ for some i. One may verify readily that the functor $\bigwedge^k$ applied to an object M in $\mathrm{Filt}_k(\mathcal{S})$, where each $M_{0,i}$ is *flat* and is the direct limit of its finitely generated flat submodules, will produce $\bigwedge^k(M)$ which is also *flat*. (To see this, one may localize on $\mathcal{S}$ to reduce to the case of a local ring, in which case flat and finitely generated implies free.)

Henceforth, we will denote $\bigwedge^k(M)$ by $\bigwedge^k(M_{0,1}, \dots, M_{0,k})$. Now one may observe that there is a natural map

$$M_{0,1} \otimes \cdots \otimes M_{0,k} \to \bigwedge^k(M_{0,1}, \dots, M_{0,k}).\text{[6]} \tag{5.2}$$

Next we proceed to consider the derived functor of the exterior power; we follow [Dold and Puppe 1961] or [Illusie 1971, Chapitre I, Section 4] in this. Observe that the exterior power $\bigwedge^k$ is a nonadditive functor, and therefore one needs to use simplicial techniques in defining its derived functors. Let $\mathrm{P}_{fl}(\mathcal{S})$ denote the full subcategory of flat $\mathcal{O}_{\mathcal{S}}$-modules which are the direct limits of their finitely generated flat submodules. Let $\mathrm{Perf}_{fl}(\mathcal{S})$ denote the full subcategory of perfect complexes that consist of flat $\mathcal{O}_{\mathcal{S}}$-modules in each degree, which are also the direct limits of their finitely generated flat submodules. If $\mathcal{S}_0$ is a closed algebraic substack of $\mathcal{S}$, $\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ will denote the full subcategory of $\mathrm{Perf}_{fl}(\mathcal{S})$ consisting of complexes of flat $\mathcal{O}_{\mathcal{S}}$-modules with supports in $\mathcal{S}_0$.

Lemma 5.5.1 below shows the existence of functorial flat resolutions, so that one may restrict to flat $\mathcal{O}_{\mathcal{S}}$-modules without loss of generality. Then one obtains an

[6] Definitions (5.1) and (5.2) are defined on $\mathrm{Filt}_k(\mathcal{S})$ for convenience. We will be in fact applying these often to objects belonging to $w\mathrm{S}_k\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ which are filtered objects satisfying more restrictive conditions.

imbedding

$$i : \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \to \mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S})), \tag{5.3}$$

where $\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S}))$ denotes the category of all cosimplicial-simplicial objects of $\mathrm{P}_{fl}(\mathcal{S})$ as follows; see [Illusie 1971, Chapitre I, 4.1]. Let $\cdots \to K^{-n} \to K^{-n+1} \to \cdots \to K^0 \to K^1 \to \cdots \to K^m \to \cdots$ denote an object in $\mathrm{Perf}_{fl}(\mathcal{S})$. One first sends it to the double complex in the second quadrant with $\cdots \to K^{-n} \to \cdots K^0$ along the negative x-axis, with K^0 in position $(0, 0)$, and the complex $K^0 \to K^1 \to \cdots \to K^m \to \cdots$ along the positive y-axis. Next one applies de-normalization functors (see Appendix B) that produce the cosimplicial-simplicial object, $i(K)$ in $\mathrm{P}_{fl}(\mathcal{S})$ from this, that is, an object in $\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S}))$.

Let $\mathrm{N} = \mathrm{N}^v \circ \mathrm{N}_h$ denote the normalization functor as in (B.2) and let Tot denote the functor defined in (B.4). Recall $\mathrm{N} = \mathrm{N}^v \circ \mathrm{N}_h$ sends a cosimplicial-simplicial object to a double complex, and Tot denotes taking the total complex of the corresponding double complex. We define a morphism $K' \to K$ in $\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S}))$ to be a quasiisomorphism, if the induced map on applying the functor $\mathrm{Tot} \circ \mathrm{N}$ is a quasiisomorphism. One may now verify that the composition $\mathrm{Tot} \circ \mathrm{N} \circ i = \mathrm{id}$ so that the functor i in fact induces a faithful functor of the associated derived categories obtained by inverting quasiisomorphisms. (Observe from Appendix B that the normalization functor N and the functor i preserve flatness.)

Let $w\mathrm{S}_k\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ denote the category whose objects are sequences of cofibrations

$$\mathrm{K}_{0,1} \rightarrowtail \mathrm{K}_{0,2} \cdots \rightarrowtail \mathrm{K}_{0,k}$$

together with choices of subquotients $\mathrm{K}_{i,j} = \mathrm{K}_{0,j}/\mathrm{K}_{0,i} \in w\mathrm{Perf}_{fl}(\mathcal{S})$, for $i < j$. One defines $w\mathrm{S}_k(\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S}))$ similarly. We define

$$\begin{aligned} {\textstyle\bigwedge}^k_{cs} : w\mathrm{S}_k(\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S}))) &\to w\,\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S})), \\ {\textstyle\bigwedge}^k : w\mathrm{S}_k\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) &\to w\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \end{aligned} \tag{5.4}$$

as functors of categories with cofibrations and weak equivalences in the following manner.

Let $K = \mathrm{K}_{0,1} \rightarrowtail \mathrm{K}_{0,2} \rightarrowtail \cdots \rightarrowtail \mathrm{K}_{0,k} \in w\mathrm{S}_k(\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S})))$ ($\in w\mathrm{S}_k\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$, respectively). We define ${\textstyle\bigwedge}^k_{cs}$ to be the functor induced by applying ${\textstyle\bigwedge}^k : \mathrm{Filt}_k(\mathrm{P}_{fl}(\mathcal{S})) \to \mathrm{P}_{fl}(\mathcal{S})$ in each cosimplicial-simplicial degree. The functor ${\textstyle\bigwedge}^k : w\mathrm{S}_k\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \to \mathrm{wPerf}_{fl,\mathcal{S}_0}(\mathcal{S})$ is defined by

$${\textstyle\bigwedge}^k(K) = \mathrm{Tot} \circ \mathrm{N}\big({\textstyle\bigwedge}^k_{cs}(i(K))\big), \quad K \in w\mathrm{S}_k\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}), \tag{5.5}$$

where $\mathrm{N} = \mathrm{N}^v \circ \mathrm{N}_h$ once again.

Lemma 5.5.1 (functorial flat resolutions). *Let $\mathcal{S}_0$ denote a closed algebraic substack of the given stack $\mathcal{S}$. Then there exists a functor $\mathfrak{F} : w\mathrm{S}_\bullet \mathrm{Perf}_{\mathcal{S}_0}(\mathcal{S}) \to w\mathrm{S}_\bullet \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ having the following properties. Let $\mathrm{U} : w\mathrm{S}_\bullet \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \to w\mathrm{S}_\bullet \mathrm{Perf}_{\mathcal{S}_0}(\mathcal{S})$ denote the obvious forgetful functor. Then, there exists a natural transformation $U \circ \mathfrak{F} \to \mathrm{id}$ so that for each $M \in w\mathrm{S}_\bullet \mathrm{Perf}_{\mathcal{S}_0}(\mathcal{S})$, the corresponding map $(U \circ \mathfrak{F})(M) \to M$ is a quasiisomorphism.*

Proof. Recall that the stack $\mathcal{S}$ is of finite type over the regular Noetherian base scheme S. Therefore, the lisse-étale site $\mathcal{S}_{\mathrm{lis-et}}$ is essentially small. Given an $M \in \mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, one may define

$$\mathfrak{F}(M) = \bigoplus_{U \in \mathcal{S}_{\mathrm{lis-et}}} \ \bigoplus_{\phi \in \mathrm{Hom}(j_{U!} j_U^*(\mathcal{O}_{\mathcal{S}}), M)} j_{U!,\phi} j_U^*(\mathcal{O}_{\mathcal{S}}).$$

(Here $j_{U!,\phi} j_U^*(\mathcal{O}_{\mathcal{S}}) = j_{U!} j_U^*(\mathcal{O}_{\mathcal{S}})$.) We define a surjection $\epsilon_{-1} : \mathfrak{F}(M) \to M$ by mapping the summand indexed by $\phi \in \mathrm{Hom}(j_{U!} j_U^*(\mathcal{O}_{\mathcal{S}}), M)$ to M by the map ϕ. Given a map $f : M' \to M$ in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, one defines the induced map $\mathfrak{F}(f) : \mathfrak{F}(M') \to \mathfrak{F}(M)$ by sending the summand $j_{U!,\phi} j_U^*(\mathcal{O}_{\mathcal{S}})$ to $j_{U!,f\circ\phi} j_U^*(\mathcal{O}_{\mathcal{S}})$ by the identity map. Now one may readily see that the assignment $M \mapsto \mathfrak{F}(M)$ is functorial in M. Moreover, if $M' \to M$ is a split monomorphism in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$, the induced map $\mathfrak{F}(M') \to \mathfrak{F}(M)$ is also a split monomorphism. One may repeatedly apply the functor $\mathfrak{F}$ to the kernel of ϵ_{-1} to obtain a resolution $\mathfrak{F}_\bullet(M) \to M$. It follows, therefore, that the functor $\mathfrak{F}$ induces a functor $\mathrm{S}_\bullet \mathrm{Perf}_{\mathcal{S}_0}(\mathcal{S}) \to \mathrm{S}_\bullet \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ that preserves cofibrations (that is, degree-wise split monomorphisms) and weak equivalences. ☐

Recall that we only consider complexes of sheaves of $\mathcal{O}$-modules whose cohomology sheaves are cartesian. The following proposition shows that the exterior power operations preserve cofibrations, weak equivalences and the property that the cohomology sheaves are cartesian.

Proposition 5.5.2. *The functor $\bigwedge^k : w\mathrm{S}_k \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \to w\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ is a functor of categories with cofibrations and weak equivalences.*

Proof. First observe that exterior powers preserve degree-wise split monomorphisms, so that these functors in fact preserve cofibrations. That they preserve weak equivalences follows essentially from the observation that the exterior power $\bigwedge^k : w\mathrm{S}_k \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \to w\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ is in fact a derived functor; see [Illusie 1971, Chapter I, Proposition 4.2.1.3]. The same proposition [loc. cit., Proposition 4.2.1.3] applies to the case where all the monomorphisms $\mathrm{K}_{0,i} \to \mathrm{K}_{0,i+1}$ are the identity maps. This Proposition may be extended to apply to the situation in hand as follows. Recall that a map $f : K = \mathrm{K}_{0,1} \rightarrowtail \mathrm{K}_{0,2} \rightarrowtail \cdots \rightarrowtail \mathrm{K}_{0,k} \to L = L_{0,1} \rightarrowtail L_{0,2} \rightarrowtail \cdots \rightarrowtail L_{0,k}$ in $w\mathrm{S}_k \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ is a weak equivalence, if the induced maps $f_{0,i} : \mathrm{K}_{0,i} \to L_{0,i}$ are all quasiisomorphisms. One may replace the mapping cone $\mathrm{Cone}(f)$ by K, and assume each $\mathrm{K}_{0,i}$ is acyclic. Then the same argument as in the proof of [loc. cit., Chapitre I, Proposition 4.2.1.3] applies to show that K is the locally filtered colimit (locally filtered in the sense of [loc. cit., 2.2.5, Chapitre I]) of complexes $\mathrm{K}_\alpha = \mathrm{K}_{0,1}(\alpha) \rightarrowtail \mathrm{K}_{0,2}(\alpha) \rightarrowtail \cdots \rightarrowtail \mathrm{K}_{0,k}(\alpha)$, with the property that each $\mathrm{K}_{0,i}(\alpha)$ is a bounded complex of finitely generated free $\mathcal{O}^{\mathrm{str}.h}_{\mathcal{S},p}$-modules at each geometric point p, and that there is a chain null-homotopy of each K_α at each stalk. (Here $\mathcal{O}^{\mathrm{str}.h}_{\mathcal{S},p}$ denotes the strict henselization of $\mathcal{O}_{\mathcal{S}}$ at the geometric point p.)

In fact, one may apply [loc. cit., Chapitre I, Proposition 4.2.1.3] to see that each $\mathrm{K}_{0,i}$ is such a locally filtered colimit. By reindexing, we may assume that we obtain a locally filtered direct system of complexes $\{\mathrm{K}_{0,1}(\alpha) \to \mathrm{K}_{0,1}(\alpha) \to \cdots \to \mathrm{K}_{0,k}(\alpha) \mid \alpha\}$ so that each $\mathrm{K}_{0,i}$ is such a filtered colimit. Then one may replace $\mathrm{K}_{0,i}(\alpha)$ for $i \geq 2$ by the mapping cylinder (see [Thomason and Trobaugh 1990, 1.1.2]) of the given map $\mathrm{K}_{0,i-1}(\alpha) \to \mathrm{K}_{0,i}(\alpha)$, so that one may assume the maps $\mathrm{K}_{0,i-1}(\alpha) \to \mathrm{K}_{0,i}(\alpha)$, for all $i \geq 2$ are cofibrations. This observation will then provide the required extension of [Illusie 1971, Chapitre I, Proposition 4.2.1.3] to complexes provided with a finite increasing filtration by subcomplexes. Since $\bigwedge^k$ commutes with taking stalks, filtered colimits, and preserves chain homotopies, it follows that $\bigwedge^k : w\mathrm{S}_k \operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \to w\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ preserves weak equivalences.

Now it suffices to show that $\bigwedge^k(K)$ has cohomology sheaves which are cartesian. In view of [Olsson 2007, Lemma 3.6], it suffices to show the following: if $f : U \to V$ denotes a *smooth* map between schemes in $\mathcal{S}_{\mathrm{lis-et}}$, then $f^*\mathcal{H}^i\big(\bigwedge^k(\mathrm{K}_{|V_{\mathrm{et}}})\big) \simeq \mathcal{H}^i\big(\bigwedge^k(\mathrm{K}_{|U_{\mathrm{et}}})\big)$ for all i. The definition of $\bigwedge^k$ above shows that

$$f^*\big(\textstyle\bigwedge^k(\mathrm{K}_{|V_{\mathrm{et}}})\big) \cong \bigwedge^k(f^*(\mathrm{K}_{|V_{\mathrm{et}}})) \cong \bigwedge^k(\mathrm{K}_{|U_{\mathrm{et}}}).$$

The last quasiisomorphism follows from the observation that $\bigwedge^k$ preserves quasiisomorphisms. Next observe that f^* is an exact functor since f is smooth. Therefore, taking cohomology sheaves commutes with f^*, proving that $\mathcal{H}^i\big(\bigwedge^k(\mathrm{K}_{|U_{\mathrm{et}}})\big) \cong \mathcal{H}^i\big(f^*\big(\bigwedge^k(\mathrm{K}_{|V_{\mathrm{et}}})\big)\big) \cong f^*\big(\mathcal{H}^i\big(\bigwedge^k(\mathrm{K}_{|V_{\mathrm{et}}})\big)\big)$: this proves that $\bigwedge^k(K)$ has cohomology sheaves which are cartesian. □

In order to show that we obtain power operations in K-theory, one needs to verify that certain conditions are satisfied by the exterior powers. These are the conditions denoted (E1) through (E5) in [Grayson 1989], [Gunnarsson and Schwänzl 2002, Section 2] and [Köck and Zanchetta 2021, Definition 1.1]. We summarize them here:

(E1) Given $V_1 \rightarrowtail V_2 \rightarrowtail \cdots \rightarrowtail V_k \rightarrowtail \cdots \rightarrowtail W_1 \rightarrowtail \cdots \rightarrowtail W_n \in w\mathrm{S}_{k+n} \operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$, there is a natural map

$$\textstyle\bigwedge^k(\langle V_i\rangle_{i=1}^k) \otimes \bigwedge^n(\langle W_i\rangle_{i=1}^n) \to \bigwedge^{k+n}(\langle V_i\rangle_{i=1}^k, \langle W_i\rangle_{i=1}^n).$$

These maps are associative in the obvious sense.

(E2) Given $V_1 \rightarrowtail V_2 \rightarrowtail \cdots \rightarrowtail V_k \rightarrowtail W_1 \rightarrowtail \cdots \rightarrowtail W_n \in w\mathrm{S}_{k+n}\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$, there is a natural map

$$\bigwedge^{k+n}(\langle V_i\rangle_{i=1}^k, \langle W_i\rangle_{i=1}^n) \to \bigwedge^k(\langle V_i\rangle_{i=1}^k) \otimes \bigwedge^n(\langle W_i/V_k\rangle_{i=1}^n).$$

These maps are associative in the obvious sense. The above conditions are for any choice of quotient objects $W_1/V_k, \ldots, W_n/V_k$.

(E3) Given $V_1 \rightarrowtail V_2 \rightarrowtail \cdots \rightarrowtail V_k \rightarrowtail W_1 \rightarrowtail \cdots \rightarrowtail W_n \rightarrowtail U_1 \rightarrowtail \cdots \rightarrowtail U_l \in w\mathrm{S}_{k+n+l}\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$, the following diagram commutes:

$$\begin{array}{ccc}
\bigwedge^{k+n}(\langle V_i\rangle_{i=1}^k, \langle W_i\rangle_{i=1}^n) \otimes \bigwedge^l(\langle U_i\rangle_{i=1}^l) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\bigwedge^k(\langle V_i\rangle_{i=1}^k) \otimes \bigwedge^n(\langle W_i/V_k\rangle_{i=1}^n) \otimes \bigwedge^l(\langle U_i/V_k\rangle_{i=1}^l) & \longrightarrow & Y
\end{array}$$

where

$$X = \bigwedge^{k+n+l}(\langle V_i\rangle_{i=1}^k, \langle W_i\rangle_{i=1}^n, \langle U_i\rangle_{i=1}^l),$$

and

$$Y = \bigwedge^k(\langle V_i\rangle_{i=1}^k) \otimes \bigwedge^{n+l}(\langle W_i/V_k\rangle_{i=1}^n, \langle U_i/V_k\rangle_{i=1}^l)$$

(E4) Given $V_1 \rightarrowtail V_2 \rightarrowtail \cdots \rightarrowtail V_k \rightarrowtail W_1 \rightarrowtail \cdots \rightarrowtail W_n \rightarrowtail U_1 \rightarrowtail \cdots \rightarrowtail U_l \in w\mathrm{S}_{k+n+l}\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$, the following diagram commutes:

$$\begin{array}{ccc}
\bigwedge^k(\langle V_i\rangle_{i=1}^k) \otimes \bigwedge^{n+l}(\langle W_i\rangle_{i=1}^n, \langle U_i\rangle_{i=1}^l) & \to & \bigwedge^{k+n+l}(\langle V_i\rangle_{i=1}^k, \langle W_i\rangle_{i=1}^n, \langle U_i\rangle_{i=1}^l) \\
\downarrow & & \downarrow \\
\bigwedge^k(\langle V_i\rangle_{i=1}^k) \otimes \bigwedge^n(\langle W_i\rangle_{i=1}^n) \otimes \bigwedge^l(\langle U_i/W_n\rangle_{i=1}^l) & \longrightarrow & Z
\end{array}$$

where

$$Z = \bigwedge^{k+n}(\langle V_i\rangle_{i=1}^k, \langle W_i\rangle_{i=1}^n) \otimes \bigwedge^l(\langle U_i/W_n\rangle).$$

(E5) Given $V_1 \rightarrowtail V_2 \rightarrowtail \cdots \rightarrowtail V_k \rightarrowtail W_1 \rightarrowtail \cdots \rightarrowtail W_n \rightarrowtail U_1 \rightarrowtail \cdots \rightarrowtail U_l \in w\mathrm{S}_{k+2+l}\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$, the following sequence of perfect complexes is an *exact* sequence:

$$\bigwedge^{k+l+1}(\langle V_i\rangle_{i=1}^k, W_1, \langle U_i\rangle_{i=1}^l) \to \bigwedge^{k+l+1}(\langle V_i\rangle_{i=1}^k, W_2, \langle U_i\rangle_{i=1}^l) \\ \to \bigwedge^k(\langle V_i\rangle_{i=1}^k) \otimes \bigwedge^{l+1}(W_2/W_1, \langle U_i/W_1\rangle_{i=1}^l),$$

that is, the first map is a cofibration and the second is its quotient.

$(\mathrm{E}5)_0$ We will also allow the case $k = 0$, which is the statement that we get an exact sequence

$$\bigwedge^{l+1}(W_1, \langle U_i\rangle_{i=1}^l) \to \bigwedge^{l+1}(W_2, \langle U_i\rangle_{i=1}^l) \to \bigwedge^{l+1}(W_2/W_1, \langle U_i/W_1\rangle_{i=1}^l).$$

Here is an outline of how to establish these properties.

5.5.1. Let $\mathrm{N} = \mathrm{N}^v \circ \mathrm{N}_h$ as before. Now we observe from (B.3) that the functor $\mathrm{Tot} \circ \mathrm{N}$ is compatible with the obvious tensor structures: that is, given $C, C' \in \mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S}))$, there are natural maps $\mathrm{Tot}(\mathrm{N}(C)) \otimes \mathrm{Tot}(\mathrm{N}(C')) \to \mathrm{Tot} \circ \mathrm{N}(C \otimes C')$, and $\mathrm{Tot} \circ \mathrm{N}(C \otimes C') \to \mathrm{Tot}(\mathrm{N}(C)) \otimes \mathrm{Tot}(\mathrm{N}(C'))$, that are associative (and are in fact quasiisomorphisms). (Observe that the tensor structure on $\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S}))$ is given by sending $C = \{C^i_j \mid i, j\}$ and $C' = \{C'^i_j \mid i, j\}$ to $C \otimes C' = \{C^i_j \otimes C'^i_j \mid i, j\}$. The tensor structure on complexes is the obvious one.)

We will now consider the statement in (E1). Since the functor $\bigwedge^k_{cs}$ is induced by the functor $\bigwedge^k$ (as in (5.1)), the existence of the corresponding map in (E1) when

$$V_1 \rightarrowtail V_2 \rightarrowtail \cdots \rightarrowtail V_k \rightarrowtail W_1 \rightarrowtail \cdots \rightarrowtail W_n$$

belongs to $w\mathrm{S}_{k+n}\,\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S}))$ is clear. (In fact, this follows readily from the case where each V_i and W_j is an $\mathcal{O}_S$-module in $\mathrm{P}_{fl}(\mathcal{S})$.) Now one applies the functor $\mathrm{Tot} \circ \mathrm{N}$ to both sides and makes use of the Section 5.5.1 above to obtain the map in (E1), when

$$V_1 \rightarrowtail V_2 \rightarrowtail \cdots \rightarrowtail V_k \rightarrowtail W_1 \rightarrowtail \cdots \rightarrowtail W_n \in w\mathrm{S}_{k+n}\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}).$$

One proves (E2) by first observing the corresponding statement is true when $\bigwedge^k$ is replaced by $\bigwedge^k_{cs}$ and for

$$V_1 \rightarrowtail V_2 \rightarrowtail \cdots \rightarrowtail V_k \rightarrowtail W_1 \rightarrowtail \cdots \rightarrowtail W_n \in w\mathrm{S}_{k+n}(\mathrm{Cos.mixt}(\mathrm{P}_{fl}(\mathcal{S}))).$$

This follows readily from the definition of the functor $\bigwedge^k$ in (5.1). Next apply $\mathrm{Tot} \circ \mathrm{N}$ and use the observation in 5.5.1 to obtain the associativity of the maps there. The remaining assertions (E3) through (E5) are established similarly: one observes these are true for the functor $\bigwedge^k_{cs}$, and then applies the functor $\mathrm{Tot} \circ \mathrm{N}$ along with the observation in 5.5.1.

5.5.2. Let $\otimes : w\mathrm{S}_\bullet\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \times w\mathrm{S}_\bullet\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \to w\mathrm{S}_\bullet\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ denote the functor taking two perfect complexes of flat $\mathcal{O}_S$ modules and sends it to their tensor product. Observe that this preserves cofibrations and weak equivalences in each argument.

5.5.3. Following [Grayson 1989, Sections 3 and 5] and [Gunnarsson and Schwänzl 2002, Sections 1 and 2], we next define for each $A \in \Delta$ and each integer $k \geq 1$, a category (actually a partially ordered set) $\Gamma^k(A)$. First one defines $\gamma(A)$ to be the partially ordered set $\{\mathrm{L}, \mathrm{R}\} \sqcup A$ with $\mathrm{L} < a$, $\mathrm{R} < a$ for all $a \in A$. For $c, d \in A$, we have $c < d$ if $c < d$ in the usual order in A. In fact, if A is the category

$0 \to 1 \to 2 \to \cdots \to n$, then $\gamma(A)$ is given by the diagram

$$\begin{tikzcd} L \arrow[dr] & & & & & \\ & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \\ R \arrow[ur] & & & & & \end{tikzcd}$$

$\Gamma(A)$ is the category whose objects are the morphisms in the category $\gamma(A)$, except for the identity morphisms $\mathrm{L} \to \mathrm{L}$ and $\mathrm{R} \to \mathrm{R}$. The morphism $j \to i$ will be denoted i/j. The morphisms in the category $\Gamma(A)$, $i'/j' \to i/j$ are the obvious commutative squares. We define a sequence of categories $\Gamma^k(A)$, for $k \geq 1$, with $\Gamma^1(A) = \Gamma(A)$. We take for the objects of $\Gamma^k(A)$, the collections $\alpha = (i_1/l_1, *_2, i_2/l_2, *_3, \ldots, *_k, i_k/l_k)$, where for each r the following conditions are satisfied:

(A1) $i_r \in \gamma(A)$, $l_r \in \gamma(A)$, and $*_r \in \{\wedge, \otimes\}$,

(A2) $l_r \leq i_r$, $i_r \in A$, and

(A3) if $*_r = \wedge$ and $r > 1$, then $l_{r-1} = l_r$ and $i_{r-1} \leq i_r$.

(Note: in [Gunnarsson and Schwänzl 2002, Section 2], $\wedge$ ($\otimes$) is replaced by $\diamond$ ($\boxtimes$, respectively).) One defines morphisms and *exact sequences* in the category $\Gamma^k(A)$ as in [Gunnarsson and Schwänzl 2002, Sections 1 and 2] or [Grayson 1989, Section 5]. One may call these exact sequences *cofibration sequences*. With this structure, the categories $\Gamma^k(A)$ may be viewed as categories with cofibrations. Moreover, one defines a functor

$$\Xi : \Gamma(A_1) \times \cdots \times \Gamma(A_k) \to \Gamma^k(A_1 \cdots A_k), \tag{5.6}$$

where $A_1 \cdots A_k$ is the concatenation; see [Gunnarsson and Schwänzl 2002, 2.4] or [Grayson 1989, Section 5].

Now one considers the categories $\mathrm{Exact}(\Gamma^k(A), \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))$ of exact functors (that is, functors preserving cofibrations) $F : \Gamma^k(A) \to \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$, for $k \geq 1$, $n \geq 0$. One may define the subcategory $w\,\mathrm{Exact}(\Gamma^k(A), \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))$ to have the same objects as $\mathrm{Exact}(\Gamma^k(A), \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))$, and where a morphism $\phi : F' \to F$ is a natural transformation, so that for each object $\gamma \in \Gamma^k(A)$, the induced map $\phi(\gamma) : F'(\gamma) \to F(\gamma)$ belongs to the subcategory $w\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$. As in [Gunnarsson and Schwänzl 2002, Section 1, page 5] one obtains the identification

$$w\mathrm{G}_A\,\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) = w\,\mathrm{Exact}(\Gamma(A), \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})),$$

for each $A \in \Delta$. Next one defines

$$\bigwedge^k : w\,\mathrm{Exact}(\Gamma(A), \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})) \to w\,\mathrm{Exact}(\Gamma^k(A), \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})) \tag{5.7}$$

by the same formula as in [Grayson 1989, Section 7]. We will recall this here: first, we denote

$$\bigwedge^k : \mathrm{Filt}_k(\mathrm{Perf}_{fl}(\mathcal{S})) \to \mathrm{Perf}_{fl}(\mathcal{S})$$

applied to an object $\mathrm{K}_{0,1} \rightarrowtail \mathrm{K}_{0,2} \rightarrowtail \cdots \rightarrowtail \mathrm{K}_{0,k} \in \mathrm{Filt}_k(\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))$ by $\mathrm{K}_{0,1} \wedge \mathrm{K}_{0,2} \wedge \cdots \wedge \mathrm{K}_{0,k}$. If $*_i$ denotes either $\wedge$ or $\otimes$ for each $1 \le i \le k$, we will let $\mathrm{K}_{0,1} *_1 \mathrm{K}_{0,2} *_2 \cdots *_k \mathrm{K}_{0,k}$ denote an iterated product involving $\wedge$ and $\otimes$ with $\wedge$ always having higher precedence than $\otimes$. Let $M \in w\,\mathrm{Exact}(\Gamma(A), \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))$. Now $\bigwedge^k(M)$ applied to the object $(i_1/l_1, *_2, i_2/l_2, \ldots, *_k, i_k/l_k) \in \Gamma^k(A)$ is given by

$$M(i_1/l_1) *_2 M(i_2/l_2) \cdots *_k M(i_k/l_k).$$

Observe that if $f : \mathcal{S}' \to \mathcal{S}$ is a map of algebraic stacks, the induced map

$$f^* : w\mathrm{S}_\bullet \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \to w\mathrm{S}_\bullet \mathrm{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}')$$

(where $\mathcal{S}'_0 = \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}_0$), commutes with $\bigwedge^k$. This follows from the above definition, the observation that f^* commutes with $\otimes$ and $\bigwedge^k$ (as in (5.1)), as well as from Proposition B.0.1 (in Appendix B) which shows it commutes with the functors N and DN.

Let $w\,\mathrm{Exact}(\Xi, \ )$ denote the functor obtained by precomposing $w\,\mathrm{Exact}(\ ,\ )$ and Ξ (with Ξ applied to the first factor of $w\,\mathrm{Exact}(\ ,\ )$). On replacing A by the concatenation $A_1 \cdots A_k$, and following $\bigwedge^k$ by the composition with $\mathrm{Exact}(\Xi, \ )$, we obtain

$$\begin{aligned}\lambda^k : w\,\mathrm{Exact}(\Gamma(A_1 \cdots A_k), \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))& \\ \to w\,\mathrm{Exact}(\Gamma(A_1) \times \cdots \times \Gamma(A_k), \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})).&\end{aligned} \tag{5.8}$$

(Observe that since all cofibrations are maps of complexes that are degree-wise split injective, the extension condition in [Gunnarsson and Schwänzl 2002, 4.3 Definition and 4.4 Remark] is satisfied. Therefore, the last term may be identified with $w\mathrm{G}^k_\bullet \mathrm{Perf}_{fl}(\mathcal{S})$, which is the k-th iterate of the construction in (3.1).) Therefore, identifying the first term with $\mathrm{sub}_k\, w\mathrm{G}_\bullet \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})$ (where sub_k denotes the k-th subdivision which produces a multisimplicial set of order k; see [Grayson 1989, Section 4]), one obtains the exterior power operation:

$$\lambda^k : \mathrm{sub}_k\, w\mathrm{G}_\bullet \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}) \to w\mathrm{G}^k_\bullet \mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}). \tag{5.9}$$

It is shown in [Grayson 1989, Section 4] that the realization of the first term is homeomorphic to $|w\mathrm{G}_\bullet \operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})| \simeq \mathrm{K}_{\mathcal{S}_0}(\mathcal{S})$. It is shown in [Gunnarsson et al. 1992, page 264] that the realization of the last is homeomorphic to $|w\mathrm{G}_\bullet \operatorname{Perf}_{\mathcal{S}_0}(\mathcal{S})| \simeq |\Omega w\mathrm{S}_\bullet \operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})| \simeq \mathrm{K}_{\mathcal{S}_0}(\mathcal{S})$. Therefore, (5.9) defines the exterior power operations on $\mathrm{K}_{\mathcal{S}_0}(\mathcal{S})$ as the map:

$$\lambda^k : |w\mathrm{G}_\bullet \operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})| \to |w\mathrm{G}_\bullet \operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S})|. \tag{5.10}$$

Moreover, the naturality of the above operations shows that they are (strictly) compatible with pull-back maps associated to morphisms $\mathrm{f} : \mathcal{S}' \to \mathcal{S}$ of algebraic stacks, that is, the operations λ^k are compatible with the induced map

$$|w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))| \xrightarrow{|w\mathrm{G}_\bullet(f^*)|} |w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))|,$$

where $\mathcal{S}'_0 = \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}_0$. Therefore, in view of the pull-back square

$$\begin{tikzcd}
w\mathrm{G}(f) \arrow[r] \arrow[d, "\pi'"] & \mathrm{P}(|w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))_*|) \arrow[d, "\pi"] \\
|w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))| \arrow[r, "|w\mathrm{G}_\bullet(f^*)|"] & |w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))|,
\end{tikzcd} \tag{5.11}$$

and the observation that the map on the path space $\mathrm{P}(|w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))_*|)$ induced by λ^k is compatible with the map π, one obtains induced maps

$$\lambda^k : w\mathrm{G}(f) \to w\mathrm{G}(f), \tag{5.12}$$

compatible under π' with the corresponding operation λ^k on $|w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))|$. Taking the map f to be the closed immersion $\mathcal{S}' \to \mathcal{S}$ of algebraic stacks, shows one may define exterior power operations in relative K-theory, that is, on $\mathrm{K}(\mathcal{S}, \mathcal{S}')$ which is defined as the canonical homotopy fiber of the restriction map $\mathrm{K}(\mathcal{S}) \to \mathrm{K}(\mathcal{S}')$.

We proceed to verify these satisfy the usual relations so that $\pi_*\mathrm{K}_{\mathcal{S}_0}(\mathcal{S})$ is a pre-λ-ring without unit when $\mathcal{S}$ is smooth. For this it is necessary to define the pullback squares, for each fixed $k \geq 1$:

$$\begin{tikzcd}
w\mathrm{G}_k(f) \arrow[r] \arrow[d, "\pi'"] & \mathrm{P}(|\mathrm{sub}_k\, w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))_*|) \arrow[d, "\pi"] \\
|\mathrm{sub}_k\, w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))| \arrow[r, "|\mathrm{sub}_k\, w\mathrm{G}_\bullet(f^*)|"] & |\mathrm{sub}_k\, w\mathrm{G}_\bullet(\operatorname{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))|
\end{tikzcd} \tag{5.13}$$

and

$$\begin{tikzcd}
w\mathrm{G}^k(f) \arrow[r] \arrow[d, "\pi'"] & \mathrm{P}(|w\mathrm{G}^k_\bullet(\operatorname{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))_*|) \arrow[d, "\pi"] \\
|w\mathrm{G}^k_\bullet(\operatorname{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))| \arrow[r, "|w\mathrm{G}^k_\bullet(f^*)|"] & |w\mathrm{G}^k_\bullet(\operatorname{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))|
\end{tikzcd} \tag{5.14}$$

Here $\mathrm{sub}_k\, w\mathrm{G}_\bullet(\mathrm{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))_*$ and $w\mathrm{G}^k_\bullet(\mathrm{Perf}_{fl,\mathcal{S}'_0}(\mathcal{S}'))_*$ denote the corresponding path component containing the base point. Then λ^k defines a map from the bottom two vertices and the top right vertex of the first square to the corresponding vertices of the second square, making the corresponding diagrams commute, so that one obtains induced maps

$$\lambda^k : w\mathrm{G}_k(f) \to w\mathrm{G}^k(f). \tag{5.15}$$

Making use of the observation that the vertices of the square (5.13) are homeomorphic to the corresponding vertices of the square (5.11) and further observing that the vertices of the square (5.14) are weakly equivalent to the corresponding vertices of the square (5.11), one sees that the maps λ^k in (5.15) are variants of the same maps λ^k considered in (5.12), but able to handle k-different arguments, $(\mathrm{K}_1, \dots, \mathrm{K}_k)$ of complexes in $\mathrm{Perf}_{fl,\mathcal{S}_0}(\mathcal{S}))$. Moreover the above extension of the operations λ^k enable us to verify these satisfy the usual relations, so that we will show $\pi_*\mathrm{K}_{\mathcal{S}_0}(\mathcal{S})$ is a pre-λ-ring without unit. Making use of the above observations and Example 3.6.1, we are able to reduce this to verifying the above relations hold in certain relative Grothendieck groups.

Let $\mathcal{S}$ denote a given smooth algebraic stack over the given base scheme S. Let

$$\Delta[n] = \mathrm{Spec}\big(\mathcal{O}_S[x_0, \dots, x_n]/\big(\sum_i x_i - 1\big)\big),$$

let $\delta^i\Delta[n]$ denote its i-th face, and let $\delta\Delta[n]$ denote its boundary that is, $\bigcup_{i=0}^n \delta^i\Delta[n]$. The relative K-theory space $\mathrm{K}_{\mathcal{S}'\times\Delta[n]}\big(\mathcal{S}\times\Delta[n], \mathcal{S}\times\bigcup_{i=0,\dots,k}\delta^i\Delta[n]\big)$ is defined as $w\mathrm{G}(i_k^*)$, where

$$i_k^* : \mathrm{Perf}_{fl,\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n]) \to \mathrm{Perf}_{fl,\mathcal{S}'\times\Delta[n]}\left(\mathcal{S}\times\bigcup_{i=0,\dots,k}\delta^i\Delta[n]\right) \tag{5.16}$$

is the functor of categories with cofibrations and weak equivalences induced by the closed immersion

$$\mathcal{S}\times\bigcup_{i=0,\dots,k}\delta^i\Delta[n] \to \mathcal{S}\times\Delta[n].$$

Lemma 5.16.1. $\mathrm{K}_{\mathcal{S}'\times\Delta[n]}\big(\mathcal{S}\times\Delta[n], \mathcal{S}\times\big(\bigcup_{i=0,\dots,k}\delta^i\Delta[n]\big)\big)$ *is contractible for all* n *and all* $0 \le k \le n-1$.

Proof. One proves this using ascending induction on k. Observe that $\Delta[n]$ is isomorphic to the affine space $\mathbb{A}^n_S$. The case $k=0$ follows from the fact $\mathcal{S}\times\Delta[n]$ and $\mathcal{S}\times\delta^0\Delta[n] \cong \mathcal{S}\times\Delta[n-1]$ are smooth, and therefore

$$\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n], \mathcal{S}\times\delta^0\Delta[n]) \simeq G(\mathcal{S}'\times\Delta[n], \mathcal{S}'\times\Delta[n-1]),$$

and because G-theory has been shown to have the homotopy property; see [Joshua 2012, Theorem 5.17]. To continue the induction, one uses the fibration sequence

$$\begin{aligned}
&\mathrm{K}_{\mathcal{S}'\times\Delta[n]}\Big(\mathcal{S}\times\Delta[n],\mathcal{S}\times\bigcup_{i=0,\dots,k}\delta^i\Delta[n]\Big)\\
&\qquad\to \mathrm{K}_{\mathcal{S}'\times\Delta[n]}\Big(\mathcal{S}\times\Delta[n],\mathcal{S}\times\bigcup_{i=0,\dots,k-1}\delta^i\Delta[n]\Big)\\
&\qquad\to \mathrm{K}_{\mathcal{S}'\times\delta^k\Delta[n]}(\mathcal{S}\times\delta^k\Delta[n],\mathcal{S}\times((\delta^0\Delta[n]\cup\cdots\cup\delta^{k-1}\Delta[n])\cap\delta^k\Delta[n]))\\
&\qquad\qquad\cong \mathrm{K}_{\mathcal{S}'\times\Delta[n-1]}\Big(\mathcal{S}\times\Delta[n-1],\mathcal{S}\times\Big(\bigcup_{i=0,\dots,k-1}\delta^i\Delta[n-1]\Big)\Big).
\end{aligned}$$

The last two terms are contractible by the inductive hypothesis, so that the first one is also. ☐

Let

$$i^*:\mathrm{Perf}_{fl,\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n])\to\mathrm{Perf}_{fl,\mathcal{S}'\times\delta\Delta[n]}(\mathcal{S}\times\delta\Delta[n]) \tag{5.17}$$

denote the functor induced by the closed immersion $\mathcal{S}\times\delta\Delta[n]\to\mathcal{S}\times\Delta[n]$.

Proposition 5.17.1. *One obtains the isomorphism*:

$$\pi_n\mathrm{K}_{\mathcal{S}'}(\mathcal{S})\cong\pi_0\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n])\cong\pi_0(w\mathrm{G}(i^*)).$$

Proof. The key idea is the observation that one obtains a fibration sequence:

$$\begin{aligned}
\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n])&\to\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\Sigma)\\
&\to\mathrm{K}_{\mathcal{S}'\times\Delta[n-1]}(\mathcal{S}\times\Delta[n-1],\mathcal{S}\times\delta\Delta[n-1]),
\end{aligned}$$

where $\Sigma=\bigcup_{i=0,\dots,n-1}\delta^i\Delta[n]$, the last map is the restriction to the last face of $\Delta[n]$, and the first map is the obvious inclusion of the fiber. The middle term is contractible by the above lemma, so that the long exact sequence associated to the above fibration provides us with an isomorphism

$$\pi_{k-1}(\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n]))\cong\pi_k(\mathrm{K}_{\mathcal{S}'\times\Delta[n-1]}(\mathcal{S}\times\Delta[n-1],\mathcal{S}\times\delta\Delta[n-1]).$$

Repeating this n-times, we obtain the first isomorphism in the proposition. The second isomorphism in the proposition follows from the fact $w\mathrm{G}(i^*)$ is the homotopy fiber of the G-construction applied to the functor i^* defined as in (3.2) and (5.11). ☐

Remarks 5.17.2. (i) Lemma 5.16.1 and Proposition 5.17.1 are clearly inspired by [Bloch and Lichtenbaum 1997, Lemmas (1.2.1) and (1.2.2)], which play a key role in the construction of the Bloch–Lichtenbaum spectral sequence.

(ii) The product structure on relative K-theory may now be viewed as the pairing:

$$\begin{aligned}&\pi_0\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n])\otimes\pi_0\mathrm{K}_{\mathcal{S}'\times\Delta[m]}(\mathcal{S}\times\Delta[m],\mathcal{S}\times\delta\Delta[m])\\&\to\pi_0\mathrm{K}_{\mathcal{S}'\times\Delta[n]\times\Delta[m]}(\mathcal{S}\times\Delta[n]\times\Delta[m],\mathcal{S}\times(\delta\Delta[n]\times\Delta[m]\cup\Delta[n]\times\delta\Delta[m]))\\&\qquad\qquad\cong\pi_0\mathrm{K}_{\mathcal{S}'\times\Delta[n+m]}(\mathcal{S}\times\Delta[n+m],\mathcal{S}\times\delta\Delta[n+m]).\end{aligned}\tag{5.18}$$

Proof of Theorem 1.1.1(i) *and* (ii). Recall $\mathcal{S}$ denotes a smooth algebraic stack with $\mathcal{S}'$ a closed algebraic substack.

Then we need to show $\pi_0(\mathrm{K}(\mathcal{S}))$ *is a pre-λ-ring*, *and that each* $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ *is a pre-λ-algebra over the pre-λ-ring* $\pi_0(\mathrm{K}(\mathcal{S}))$. Moreover, we need to show *the pre-λ-algebra structure is compatible with pull-backs*: that is, if $f:\tilde{\mathcal{S}}\to\mathcal{S}$ is a map of smooth algebraic stacks and $\tilde{\mathcal{S}}'=\tilde{\mathcal{S}}\times_{\mathcal{S}}\mathcal{S}'$, then the induced map $f^*:\pi_0(\mathrm{K}(\mathcal{S}))\to\pi_0(\mathrm{K}(\tilde{\mathcal{S}}))$ is a map of pre-λ-rings, and the induced map $f^*:\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))\to\pi_n(\mathrm{K}_{\tilde{\mathcal{S}}'}(\tilde{\mathcal{S}}))$ is a map of pre-λ-algebras over $\pi_0(\mathrm{K}(\mathcal{S}))$, for each fixed $n\geq 0$. In addition, we need to show *the λ-operations are homomorphisms on* $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ *for all* $n>0$.

Observe first that the isomorphisms in the last Proposition are compatible with respect to the λ-operations defined in (5.9): this follows from the naturality of these operations with respect to pull-backs. In fact, one may verify readily that one has the following homotopy commutative diagram of fibration sequences:

$$\begin{tikzcd}
\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n]) \arrow[r, "\lambda^k"] \arrow[d] & \mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n]) \arrow[d] \\
\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\Sigma) \arrow[r, "\lambda^k"] \arrow[d] & \mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\Sigma) \arrow[d] \\
\mathrm{K}_{\mathcal{S}'\times\Delta[n-1]}(\mathcal{S}\times\Delta[n-1],\mathcal{S}\times\delta\Delta[n-1]) \arrow[r, "\lambda^k"] & \mathrm{K}_{\mathcal{S}'\times\Delta[n-1]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n-1])
\end{tikzcd}$$

Therefore, it follows that the λ-operations are compatible with the boundary maps of the corresponding long exact sequence of homotopy groups. Using the isomorphism in Proposition 5.17.1, it suffices to show that $\pi_0\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n])$ is a pre-λ-ring without unit, and for $\mathcal{S}'=\mathcal{S}$ and $n=0$, it is a pre-λ-ring. This may be done as for vector bundles: that is, the proof of this theorem follows along the same lines as the proof in [Grayson 1989, Section 8]. Here are some details.

Using the identification of $\pi_0(\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n]))$ as $\pi_0(w\mathrm{G}(i^*))$, the definition of $w\mathrm{G}(i^*)$ as in (5.11) shows that a connected component of $w\mathrm{G}(i^*)$ corresponds to a pair of perfect complexes V and W on $S\times\Delta[n]$, with supports contained in $\mathcal{S}'\times\Delta[n]$ together with a zig-zag path (as in (3.6)) p joining the restriction $(i^*(V),i^*(W))$ to the base point, namely the pair $(0,0)$ in

$w\mathrm{G}(\mathrm{Perf}_f, \mathcal{S}' \times \delta\Delta[n](\mathcal{S} \times \delta\Delta[n])$. This pair corresponds to the difference $[V]-[W]$ in the above Grothendieck group. We will begin with the special case where $W = 0$.

Viewing λ^k as a map $w\mathrm{G}_k(i^*) \to w\mathrm{G}^k(i^*) \simeq w\mathrm{G}(i^*)$, we see that

$$[\lambda^k(V)] = \Big[\bigwedge^k(V)\Big], \tag{5.19}$$

where $\bigwedge^k$ denotes the functor defined in (5.4) and $\big[\bigwedge^k(V)\big]$, $[\lambda^k(V)]$ denote the corresponding classes in $\pi_0(w\mathrm{G}(i^*)) = \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$. This follows from the observations as in [Grayson 1989, Section 8], but we will provide the relevant details. The vertices of $\mathrm{sub}_k\, w\mathrm{G}(i^*)$ correspond to pairs (V, W) of perfect complexes on $\mathcal{S} \times \Delta[n]$ with supports contained in $\mathcal{S}' \times \Delta[n]$ (together with a zig-zag path p (as in the last paragraph) joining the restriction $(i^*(V), i^*(W))$ to (0, 0)), positioned at the vertices of a k-dimensional cube. Then the multisimplicial map λ^k sends such a vertex (V, W) to a sequence, each term of which is of the form $\bigwedge^a(V) \otimes \bigwedge^{b_1}(W) \otimes \cdots \otimes \bigwedge^{b_u}(W)$, for some choice of $a, b_1, \ldots, b_u \geq 0$, so that $a + b_1 + \cdots + b_u = k$. When $W = 0$ as we have chosen, then this has only one nonzero term, namely $\bigwedge^k(V)$.

Next let $K' \rightarrowtail K$ denote a cofibration in $\mathrm{Perf}_{fl,\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n])$, so that together with the choice of paths joining their restrictions to (0, 0), both belong to $w\mathrm{G}(i^*)$. Then one obtains the following formula in $\pi_0(w\mathrm{G}(i^*))$:

$$[\lambda^m(K)] = \sum_{k=0}^{m} [\lambda^k(K') \otimes \lambda^{m-k}(K/K')] = \sum_{k=0}^{m} [\lambda^k(K')]\cdot [\lambda^{m-k}(K/K')] \tag{5.20}$$

with the understanding that $\lambda^0(K')\cdot \lambda^m(K/K') = \lambda^m(K/K')$ and $\lambda^m(K')\cdot\lambda^0(K/K')$ $= \lambda^m(K')$. In view of (5.19) above, it suffices to prove this with $\lambda^k(K)$ replaced by $\bigwedge^k(K)$. This holds by repeatedly applying Corollaries 4.2.3, 4.6.3 and (E5) by taking, $m = k + l + 1$, first with $k = 0$, $W_1 = K'$, $W_2 = K = U_j$, $j = 1, \ldots, l$, which gives

$$\Big[\bigwedge^m(\overbrace{K, K, \ldots, K}^{m})\Big] = \Big[\bigwedge^m(\overbrace{K', K, \ldots, K}^{m})\Big] + \Big[\bigwedge^m(\overbrace{K/K', K/K', \ldots, K/K'}^{m})\Big].$$

Then with $k = 1$, $V_1 = K'$, $W_1 = K'$, $W_2 = K = U_j$, $j = 1, \ldots, l-1$, enables us to obtain

$$\Big[\bigwedge^m(\overbrace{K', K, \ldots, K}^{m})\Big]$$

$$= \Big[\bigwedge^1(K') \otimes \bigwedge^{m-1}(\overbrace{K/K', K/K', \ldots, K/K'}^{m-1})\Big] + \Big[\bigwedge^m(\overbrace{K', K', K, \ldots, K}^{m})\Big], \ldots,$$

ending with

$$\Big[\bigwedge^m(\overbrace{K',\dots,K'}^{m-1},K)\Big]=\Big[\bigwedge^{m-1}(\overbrace{K',\dots,K'}^{m-1})\otimes\bigwedge^1(K/K')\Big]+\Big[\bigwedge^m(\overbrace{K',\dots,K'}^{m})\Big].$$

Moreover, in view of (5.19), one observes readily that if $n=0$, $\bigwedge^0(K)=\mathcal{O}_{\mathcal{S}}$ and in general, $\bigwedge^1(K)=K$, $K\in\pi_0\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n])$.

5.20.4. At this point we make the following important observation. Given a perfect complex K on $S\times\Delta[n]$ and acyclic on $(\mathcal{S}-\mathcal{S}')\times\Delta[n]$, and so that the pair $(K,0)$ denotes a class in $\pi_0(w\mathrm{G}(i^*))$, the canonical construction of its cone (that is, the mapping cone of the identity map $K\to K$) along with Theorem A.0.6 (see also Corollary 4.6.3) shows that the class of $K[1]$ denotes the additive inverse of the class of K in the above Grothendieck group. (Given a perfect complex K in $\mathrm{Perf}_{fl,\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n])$, so that $(K,0)$ represents a class in $\pi_0(w\mathrm{G}(i^*))$, we will let $[K]$ denote its class in the above Grothendieck group.) Therefore, given two perfect complexes K', K in $\mathrm{Perf}_{fl,\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n])$, so that $(K,0)$ and $(0,K')$ represent classes in $\pi_0(w\mathrm{G}(i^*))$, the class $[K]-[K']$ in the above Grothendieck group is represented by the class of the perfect complex $K\oplus K'[1]$. It follows that the identity in (5.20) suffices to prove the identity

$$\lambda^n(r+s)=\sum_{i=0}^{n}\lambda^i(r)\cdot\lambda^{n-i}(s)$$

holds for all r, s in the group $\pi_0\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n])\cong\pi_n\mathrm{K}_{\mathcal{S}'}(\mathcal{S})$, with the understanding that $\lambda^0(r)\cdot\lambda^n(s)=\lambda^n(s)$ and $\lambda^n(r)\cdot\lambda^0(s)=\lambda^n(r)$. These observations prove that there is the structure of a pre-λ-ring without unit on each $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))\cong\pi_0(\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n]))$ and the structure of a pre-λ-ring on $\pi_0(\mathrm{K}(\mathcal{S}))$ (that is, on $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))\cong\pi_0(\mathrm{K}_{\mathcal{S}'\times\Delta[n]}(\mathcal{S}\times\Delta[n],\mathcal{S}\times\delta\Delta[n]))$, when $n=0$ and $\mathcal{S}'=\mathcal{S}$).

In view of these observations, one may define a pre-λ algebra structure on $\pi_0(\mathrm{K}(\mathcal{S}))\oplus\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ by defining λ^m on $\pi_0(\mathrm{K}(\mathcal{S}))\oplus\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ by $\lambda^m(r,s)=\big(\lambda^m(r),\sum_{i=0}^{m-1}\lambda^i(r)\cdot\lambda^{m-i}(s)\big)$. (See Lemma 5.20.1 below.) Observe that each $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ has the structure of a module over $\pi_0(\mathrm{K}(\mathcal{S}))$ in the obvious manner using the tensor product of perfect complexes.

The naturality with respect to pull-back is clear from the construction. It may be also important to point out the following: the product structure on each $\pi_n\mathrm{K}_{\mathcal{S}'}(\mathcal{S})$ is trivial for all $n>0$. This is because this product structure is defined making use of the pull-back by the diagonal map $\Delta_1:\mathcal{S}\times\Delta[n]\to\mathcal{S}\times\Delta[n]\times\mathcal{S}\times\Delta[n]$, and involves the co-H-space structure on $S^n\simeq\Delta[n]/(\delta\Delta[n])$: see [Kratzer 1980a, Lemme 5.2]. (This is distinct from the product $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))\otimes\pi_m(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))\to\pi_{n+m}(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$, which makes use of pull-back by the diagonal, $\Delta_2 : \mathcal{S} \times \Delta[n] \times \Delta[m] \to (\mathcal{S} \times \Delta[n]) \times (\mathcal{S} \times \Delta[m])$.) In view of this observation, the λ-operations are all *homomorphisms* on $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$, for all $n > 0$.

In view of the following lemma, these observations prove the first two statements of Theorem 1.1.1.

Lemma 5.20.1. *Let* R *denote a prelambda ring and* S *an* R*-module, so that it is also provided with the structure of a prelambda ring without unit. Then* $\mathrm{R} \oplus \mathrm{S}$ *has the structure of a prelambda ring, where*

$$\begin{aligned}(r, s) + (r', t) &= (r + r', s + t),\\ (r, s) \circ (r', t) &= (r \cdot r', r \cdot t + r' \cdot s + s \cdot t), \quad \textit{and}\\ \lambda^n(r, s) &= \left(\lambda^n_{\mathrm{R}}(r), \sum_{i=0}^{n-1} \lambda^i_{\mathrm{R}}(r).\lambda^{n-i}_{\mathrm{S}}(s)\right),\end{aligned} \tag{5.21}$$

and where λ^i_{R} (λ^j_{S}) *denote the prelambda operations of* R (S, *respectively*).

Proof. We define $\lambda^0(r, s) = 1$, where 1 denotes the multiplicative unit in R. We also let $\lambda^i(r, 0) = \lambda^i_{\mathrm{R}}(r)$ and $\lambda^j(0, s) = \lambda^j_{\mathrm{S}}(s)$, for all $r \in \mathrm{R}$, $s \in \mathrm{S}$ and $i \geq 0$, $j > 0$. We also let $\lambda^i(r, 0) \cdot \lambda^0(0, s) = \lambda^i(r, 0)$ for all $i \geq 0$ and $r \in \mathrm{R}$, $s \in \mathrm{S}$. Next we let $\lambda^1(r, s) = (\lambda^1_{\mathrm{R}}(r), \lambda^1_{\mathrm{S}}(s))$. In general, we define λ^n on $\mathrm{R} \oplus \mathrm{S}$, by

$$\begin{aligned}\lambda^n(r, s) &= (\lambda^n_{\mathrm{R}}(r), \lambda^0_{\mathrm{R}}(r) \cdot \lambda^n_{\mathrm{S}}(s) + \cdots + \lambda^{n-1}_{\mathrm{R}}(r) \cdot \lambda^1_{\mathrm{S}}(s))\\ &= \left(\lambda^n_{\mathrm{R}}(r), \sum_{i=0}^{n-1} \lambda^i_{\mathrm{R}}(r).\lambda^{n-i}_{\mathrm{S}}(s)\right).\end{aligned}$$

In view of the above definitions, clearly we may identify the right-hand side above with $\sum_{i=0}^{n} \lambda^i(r) \cdot \lambda^{n-i}(s)$. One may also verify that if $r, r' \in \mathrm{R}$ and $s, s' \in \mathrm{S}$, then

$$\lambda^n(r + r') = \sum_{i=0}^{n} \lambda^i(r) \cdot \lambda^{n-i}(r')$$

and

$$\lambda^n(s + s') = \sum_{i=0}^{n} \lambda^i(s) \cdot \lambda^{n-i}(s')$$

since both R and S are assumed to be prelambda rings. In view of these observations, it suffices to check that

$$\lambda^n((r, s) + (r', s')) = \sum_{i=0}^{n} \lambda^i(r, s) \cdot \lambda^{n-i}(r', s'). \tag{5.22}$$

In fact the term on the left side is given by:

$$\begin{aligned}
&\lambda^n(r+r', s+s')\\
&\quad=\big(\lambda^n(r+r'), \lambda^{n-1}(r+r')\cdot\lambda^1(s+s'), \lambda^{n-2}(r+r')\cdot\lambda^2(s+s'), \dots,\\
&\qquad\qquad\qquad\qquad\lambda^1(r+r')\cdot\lambda^{n-1}(s+s'), \lambda^n(s+s')\big)\\
&\quad=\Big(\sum_{i=0}^{n}\lambda^i(r)\cdot\lambda^{n-i}(r'), \sum_{j=0}^{n-1}\lambda^j(r)\cdot\lambda^{n-1-j}(r')[\lambda^1(s)+\lambda^1(s')],\\
&\qquad\sum_{j=0}^{n-2}\lambda^j(r)\cdot\lambda^{n-2-j}(r')[\lambda^2(s)+\lambda^1(s)\cdot\lambda^1(s')+\lambda^2(s')], \dots,\\
&\qquad\quad[\lambda^1(r)+\lambda^1(r')]\cdot(\lambda^{n-1}(s')+\lambda^1(s)\lambda^{n-2}(s')+\dots+\lambda^{n-1}(s)),\\
&\qquad\qquad\lambda^n(s')+\lambda^1(s)\cdot\lambda^{n-1}(s')+\dots+\lambda^{n-1}(s)\cdot\lambda^1(s')+\lambda^n(s)\Big).
\end{aligned}\tag{5.23}$$

The term on the right side of (5.22) is given by

$$\begin{aligned}
&\sum_{i=0}^{n}\lambda^i(r,s)\cdot\lambda^{n-i}(r',s')\\
&\quad=\sum_{i=0}^{n}\big(\lambda^i(r), \lambda^{i-1}(r)\cdot\lambda^1(s)+\dots+\lambda^1(r).\lambda^{i-1}(s)+\lambda^i(s)\big)\\
&\qquad\times(\lambda^{n-i}(r'), \lambda^{n-i-1}(r')\cdot\lambda^1(s')+\dots+\lambda^1(r')\cdot\lambda^{n-i-1}(s')+\lambda^{n-i}(s')).
\end{aligned}\tag{5.24}$$

Now it is straightforward to check that we obtain equality in (5.22). (Moreover, in case the multiplication on S is trivial, $\lambda^n(s+s') = \lambda^n(s)+\lambda^n(s')$ as well.) □

Remark 5.24.1. In dealing with the K-theory of exact categories, there is no analogue of the suspension functor $K \mapsto K[1]$ (used in 5.20.4 above), and as a result it takes much more effort to deduce that the λ-operations defined as in (5.8) and (5.9) define a pre-λ-ring structure even on the Grothendieck groups; see [Grayson 1989, Section 8].

6. The lambda-ring structure on higher K-theory: proof of Theorem 1.1.1(iii)

In this section, we consider statement (iii) in Theorem 1.1.1. Recall that this is the following statement: *in case every coherent sheaf on the smooth stack $\mathcal{S}$ is the quotient of a vector bundle, then each $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ is a λ-algebra over $\pi_0(\mathrm{K}(\mathcal{S}))$ in the sense of Definition 1.0.1.*

The additional assumption that every coherent sheaf is the quotient of a vector bundle first enables one to restrict to strictly perfect complexes. This observation, together with an adaptation of some arguments of Gillet, Soulé and Deligne [Gillet and Soulé 1987, Section 4] enable us to show that the λ operations we define satisfy all the expected relations, so that we obtain the structure of a λ-ring on the higher K-groups of all smooth stacks satisfying this hypothesis.

We begin by recalling the framework from [Kratzer 1980b] and [Gillet and Soulé 1987]; see also [Soulé 1992, page 22–27] for a somewhat simplified account of this, as well as [Serre 1968]. Let $\mathrm{H} = \mathbb{Z}[\mathrm{M_N} \times \mathrm{M_N}]$ denote the bialgebra $\mathbb{Z}[\mathrm{M_N} \times \mathrm{M_N}]$ of the multiplicative monoid of pairs of $\mathrm{N} \times \mathrm{N}$ matrices for some fixed integer $N \geq 1$, that is, it is the algebra of polynomials $\mathbb{Z}[X_{11}, X_{12}, \dots, X_{\mathrm{NN}}; Y_{11}, Y_{12}, \dots, Y_{\mathrm{NN}}]$ with the coproduct $\mu : \mathrm{H} \to \mathrm{H} \otimes \mathrm{H}$ satisfying $\mu(X_{ij}) = \sum_{k=1}^{\mathrm{N}} X_{ik} \otimes X_{kj}$ and $\mu(Y_{ij}) = \sum_{k=1}^{\mathrm{N}} Y_{ik} \otimes Y_{kj}$. Let $\mathrm{P}_{\mathbb{Z}}(\mathrm{M_N} \times \mathrm{M_N})$ denote the exact category of left-$\mathbb{Z}[\mathrm{M}_N \times \mathrm{M}_N]$ comodules that are free and finitely generated over $\mathbb{Z}$: an element of this category is called *a representation* of $\mathrm{M}_N \times \mathrm{M}_N$. We let $\mathrm{R}_{\mathbb{Z}}(\mathrm{M_N} \times \mathrm{M_N})$ denote the Grothendieck group of $\mathrm{P}_{\mathbb{Z}}(\mathrm{M_N} \times \mathrm{M_N})$, which is a ring via the tensor product of comodules.

Let p_i, $i = 1, 2$ denote the representation of $\mathrm{M}_N \times \mathrm{M}_N$ corresponding to the projection to the i-th factor. It is shown in [Gillet and Soulé 1987, Theorem 4.2] (see also [Kratzer 1980b, Proposition 4.3]) that the following are true:

Proposition 6.0.1. *The ring* $\mathrm{R}_{\mathbb{Z}}(\mathrm{M}_N \times \mathrm{M}_N)$ *is isomorphic to the polynomial ring*

$$\mathbb{Z}[\lambda^1(p_1), \dots, \lambda^N(p_1); \lambda^1(p_2), \dots, \lambda^N(p_2)].$$

Exterior powers make it a λ-ring.

It is shown in [Gillet and Soulé 1987, 4.3] that the center of the monoid $\mathrm{M_N} \times \mathrm{M_N}$ is $\mathrm{M_1} \times \mathrm{M_1}$ imbedded diagonally, and that the category of representations of $\mathrm{M_1} \times \mathrm{M_1}$ is equivalent to the category of positively bigraded $\mathbb{Z}$-modules. For any representation E of $\mathrm{M_N} \times \mathrm{M_N}$, the decomposition $\mathrm{E} = \bigoplus_{p,q} \mathrm{E}^{p,q}$ over the center of $\mathrm{M_N} \times \mathrm{M_N}$ is stable under the action of $\mathrm{M}_N \times \mathrm{M_N}$. E has *degree at most d* if $\mathrm{E}^{p,q} = 0$ unless $p + q \leq d$. Let $\mathrm{R}_{\mathbb{Z}}(\mathrm{M_N} \times \mathrm{M_N})^{\mathrm{d}}$ denote the Grothendieck group of the category of representations of $\mathrm{M_N} \times \mathrm{M_N}$ of degree at most d. The following is also known (see [Gillet and Soulé 1987, Lemma 4.3]):

Proposition 6.0.2. *The group* $\mathrm{R}_{\mathbb{Z}}(\mathrm{M}_N \times M_{\mathrm{N}})^{\mathrm{d}}$ *maps injectively into* $\mathrm{R}_{\mathbb{Z}}(\mathrm{M}_N \times \mathrm{M}_N)$ *and the image of this map consists of the elements*

$$\mathrm{R}(\lambda^1(p_1), \dots, \lambda^N(p_1); \lambda^1(p_2), \dots, \lambda^N(p_2)),$$

where R *runs over all polynomials of weight at most* d.

6.1. ***The functor*** $\mathbf{T_E}$ (see [Gillet and Soulé 1987, 4.4, 4.5 and Lemma 4.5])[7]. Next, for any representation E of $\mathrm{M}_N \times \mathrm{M}_N$, a scheme X and two locally free coherent

[7] It needs to be pointed out that this functor does not extend to one on pairs of all perfect complexes, but only on to pairs of strictly perfect complexes. This is the reason the results of this section hold only under the strong assumption that every coherent sheaf is a quotient of a vector bundle.

sheaves P, Q of rank at most N on X, a vector bundle $T_E(P, Q)$ on X is defined. It is shown that for fixed P and Q, the functor $E \mapsto T_E(P, Q)$ has the following properties:

(i) If the representation E has degree at most d, the functor $E \mapsto T_E(P, Q)$ has degree at most d (that is, the cross-effect functor $E \mapsto T_E(P, Q)_s = 0$ for $s > d$).

(ii) The above functor is exact in E and it commutes with direct sum, tensor product and exterior powers in E for a fixed P and Q.

(iii) $T_{p_1}(P, Q) = P$, $T_{p_2}(P, Q) = Q$, and $T_E(0, 0) = 0$.

(iv) Commutes with base-change of the scheme X, that is, the following holds: if $p : Y \to X$ is a map of schemes, $T_E(p^*(P), p^*(Q))$ is canonically isomorphic to $p^*(T_E(P, Q))$.

(v) $T_E(P, Q)$ is functorial in P and Q for a fixed E.

Proposition 6.1.3. *The functor* T_E *extends to algebraic stacks with the same properties. Given two bounded complexes of vector bundles* P, Q *on an algebraic stack* $\mathcal{S}$ *the functor* $E \mapsto T_E(P, Q)$ *defines a bounded complex of vector bundles on the stack* $\mathcal{S}$. *The functor* T_E *preserves quasiisomorphisms in either argument. If* P *and* Q *have cartesian cohomology sheaves, then* $T_E(P, Q)$ *also has cartesian cohomology sheaves.*

Proof. One may first consider bounded complexes of vector bundles on a scheme. Then one may show that the functor $T_E(\, , \,)$ preserves degree-wise split short-exact sequences of bounded complexes of vector bundles in both arguments and that if P or Q is acyclic, then $T_E(P, Q)$ is also acyclic. The latter follows by working locally on a given scheme X, where we may assume that there is a null chain-homotopy for both P or Q and by using the observation that the functor preserves chain homotopies. Since the functor $T_E(\, , \,)$ preserves degree-wise split short exact sequences in each argument, it follows that it preserves quasiisomorphisms in either argument.

Next we show this functor extends to algebraic stacks. Let $x : X \to \mathcal{S}$ denote an atlas for the stack $\mathcal{S}$ with X a scheme and let P, Q denote two bounded complexes of vector bundles on the stack $\mathcal{S}$. Then $P_0 = x^*(P)$ and $Q_0 = x^*(Q)$ define two bounded complexes of vector bundles on the scheme X. The property that the functor $E \mapsto T_E(P, Q)$ commutes with respect to base-change on schemes, shows that

$$\begin{aligned} pr_1^* T_E(P_0, Q_0) &\cong T_E(pr_1^* x^*(P), pr_1^* x^*(Q)) \\ &= T_E(pr_2^* x^*(P), pr_2^* x^*(Q)) \\ &\cong pr_2^* T_E(P, Q), \end{aligned}$$

where $pr_i : X \times_{\mathcal{S}} X \to X$, $i = 1, 2$ are the two projections. (We skip the verification that the above isomorphism satisfies a cocycle condition on further pull-back to

$\mathrm{X} \times_{\mathcal{S}} \mathrm{X} \times_{\mathcal{S}} \mathrm{X}$.) Therefore, it follows readily that $\mathrm{T_E(P, Q)}$ defines a bounded complex of vector bundles on the stack $\mathcal{S}$. The properties (i) through (v) may be checked by pulling back to

$$\mathrm{X} \times_{\mathcal{S}} \mathrm{X} \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \mathrm{X},$$

where $x : \mathrm{X} \to \mathcal{S}$ is an atlas for the stack.

Next we consider the last statement in the Proposition. Assume P and Q have cartesian cohomology sheaves. As in the proof of Proposition 5.5.2, it suffices to show that for a smooth map $f : \mathrm{U} \to \mathrm{V}$ in $\mathcal{S}_{\mathrm{lis-et}}$, one obtains isomorphisms $f^*\mathcal{H}^i(\mathrm{T_E(P, Q)}) \cong \mathcal{H}^i(\mathrm{T_E}(f^*(\mathrm{P}), f^*(\mathrm{Q})))$, for all i. The base-change property shows $f^*\mathrm{T_E(P, Q)} \cong \mathrm{T_E}(f^*(\mathrm{P}), f^*(\mathrm{Q}))$. Since f is a smooth map, f^* commutes with taking cohomology sheaves, as it is an exact functor. This completes the proof. □

Throughout the following discussion we will fix a smooth algebraic stack $\mathcal{S}$ with $\mathcal{S}'$ a closed substack. *We will assume throughout the rest of the discussion that every coherent sheaf on $\mathcal{S}$ is the quotient of a vector bundle.* Let $\mathrm{Vect}(\mathcal{S})$ ($\mathrm{Vect}_N(\mathcal{S})$) denote the category of vector bundles on the stack $\mathcal{S}$ (vector bundles on the stack $\mathcal{S}$ with rank $\leq N$, respectively). Let t_k denote the *naive truncation* functor that sends a simplicial object to the corresponding truncated simplicial object, truncated in degree $\leq k$. If $\boldsymbol{E}$ is an exact category and C denotes a chain complex with differentials of degree -1 in $\boldsymbol{E}$ and trivial in negative degrees, we will also use $t_k(\mathrm{C})$ to denote the corresponding truncated chain complex, truncated to degrees $\leq k$. We let $\mathrm{Simp}(\boldsymbol{E})$ ($\mathrm{Simp}_k(\boldsymbol{E})$) denote the category of all simplicial objects in $\boldsymbol{E}$ (simplicial objects truncated to degrees $\leq k$, respectively). Similarly, $\mathrm{C}(\boldsymbol{E})$ ($\mathrm{C}_k(\boldsymbol{E})$) will denote the category of complexes in $\boldsymbol{E}$ that are trivial in negative degrees and with differentials of degree -1 (the category of complexes in $\boldsymbol{E}$ that are trivial in negative degrees and in degrees $> k$ and with differentials of degree -1, respectively). We let $e_k : \mathrm{C}_k(\boldsymbol{E}) \to \mathrm{C}(\boldsymbol{E})$ denote the obvious inclusion functor.

Next let N(1) denote the normalized chain complex of the standard 1-simplex: $\mathrm{N}(1)_n = 0$ if $n > 1$, $\mathrm{N}(1)_1 = \mathbb{Z}[e]$, $\mathrm{N}(1)_0 = \mathbb{Z}[e_0] \oplus \mathbb{Z}[e_1]$ with $\delta(e) = e_0 - e_1$. Let $\mathrm{K}(1) = \mathrm{DN}_h(\mathrm{N}(1))$, where DN_h denotes the de-normalizing functor as in Appendix B; this is a simplicial abelian group. Given an object $\mathrm{S}_\bullet \in \mathrm{Simp}(\mathrm{Vect}(\mathcal{S}))$, $\mathrm{K}(1) \otimes S_\bullet$ will denote the obvious simplicial object: $(\mathrm{K}(1) \otimes S_\bullet)_n = \mathrm{K}(1)_n \otimes S_n = \bigoplus_{k_n \in \mathrm{K}(1)_n} S_n$ and with the obvious structure maps. Observe that $\mathrm{K}(1) \otimes S_\bullet \in \mathrm{Simp}(\mathrm{Vect}(\mathcal{S}))$.

Let A, B denote two strictly perfect complexes on $\mathcal{S} \times \Delta[n]$ and let C, D denote two strictly perfect complexes on $\mathcal{S} \times \Delta[n]$ acyclic on $(\mathcal{S} - \mathcal{S}') \times \Delta[n]$. Choose m so that $\mathrm{A}^n = \mathrm{B}^n = 0 = \mathrm{C}^n = \mathrm{D}^n$ if $n > m$. We will first apply the shift $[m]$ so that $\mathrm{A}[m]^n = \mathrm{B}[m]^n = \mathrm{C}[m]^n = \mathrm{D}[m]^n = 0$ for all $n > 0$. Therefore we may view $\mathrm{A}[m]$, $\mathrm{B}[m]$, $\mathrm{C}[m]$ and $\mathrm{D}[m]$ as complexes in nonnegative degrees with differentials of degree -1. We denote these by A', B', C' and D' respectively. Choose the integer k so that

$k > md$. Choose the integer N so that all the components of $t_k(\mathrm{K}(1) \otimes (\mathrm{DN}(\mathrm{A}')))$, $t_k(\mathrm{K}(1) \otimes \mathrm{DN}(\mathrm{B}'))$ lie in $\mathrm{Simp}_k(\mathrm{Vect}_N(\mathcal{S}))$ and all the components of $t_k(\mathrm{K}(1) \otimes (\mathrm{DN}(\mathrm{C}')))$, $t_k(\mathrm{K}(1) \otimes (\mathrm{DN}(\mathrm{D}')))$ lie in $\mathrm{Simp}_k(\mathrm{Vect}_N(\mathcal{S} \times \Delta[n], (\mathcal{S} - \mathcal{S}') \times \Delta[n])$: the latter denotes the full subcategory of $\mathrm{Simp}_k(\mathrm{Vect}_N(\mathcal{S} \times \Delta[n]))$ with supports contained in $\mathcal{S}' \times \Delta[n]$. For any representation E of $\mathrm{M}_N \times \mathrm{M}_N$ of degree at most d, we define

$$\mathrm{S_E}(k;\, \mathrm{A}', \mathrm{B}') = e_k \mathrm{N}_k \mathrm{T_E} t_k \Delta(\mathrm{DN}(\mathrm{A}'), \mathrm{DN}(\mathrm{B}')), \quad \text{and} \tag{6.2}$$

$$\mathrm{S_E}(k;\, \mathrm{C}', \mathrm{D}') = e_k \mathrm{N}_k \mathrm{T_E} t_k \Delta((\mathrm{DN}(\mathrm{C}'), \mathrm{DN}(\mathrm{D}')), \tag{6.3}$$

where Δ denotes the diagonal of the bisimplicial object appearing there, and N_k denotes the normalization functor for truncated simplicial objects in an abelian category: this is defined as the normalization functor for simplicial objects in Appendix B.

Throughout the following discussion we will let $i : \mathcal{S} \times \delta\Delta[n] \to \mathcal{S} \times \Delta[n]$ denote the obvious closed immersion. Let $\mathcal{S}'$ denote a closed algebraic substack of $\mathcal{S}$. Then we let

$$i^* : w\mathrm{G}_\bullet\, \mathrm{StPerf}_{fl, \mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n]) \to w\mathrm{G}_\bullet\, \mathrm{StPerf}_{fl, \mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \delta\Delta[n])$$

denote the corresponding pull-back functor, and let $w\mathrm{G}(i^*)$ denote its homotopy fiber defined as in (5.11) or (3.2).

Proposition 6.3.1 (see also [Gillet and Soulé 1987, Lemma 4.8]). *Let* C, D *denote two strictly perfect complexes on* $\mathcal{S} \times \Delta[n]$ *acyclic on* $(\mathcal{S} - \mathcal{S}') \times \Delta[n]$, *and provided with an explicit zig-zag path* p *as in* (3.6) *running from the restriction of the pair* (C, D) *to* $\mathcal{S} \times \delta\Delta[n]$, *to the base point* $(0, 0)$ *in* $w\mathrm{G}(i^*)$. *Let* $d \geq 1$ *be an integer. Let* A, B *denote two strictly perfect complexes on* $\mathcal{S}$. *Then there exists an integer* $N \geq 1$ *and homomorphisms*

$$\alpha : R_{\mathbb{Z}}(\mathrm{M}_N \times \mathrm{M}_N)^d \to \pi_0(\mathrm{K}(\mathcal{S})),$$
$$\beta : R_{\mathbb{Z}}(\mathrm{M}_N \times \mathrm{M}_N)^d \to \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n])),$$

which preserve the additive, multiplicative and pre-λ-ring structures. Moreover, the following holds:

(i) $[\mathrm{A}] = \alpha(p_1)$, $[\mathrm{B}] = \alpha(p_2)$, $[\mathrm{C}] = \beta(p_1)$, $[\mathrm{D}] = \beta(p_2)$

(ii) $\alpha(xy) = \alpha(x)\alpha(y)$ *and* $\beta(xy) = \beta(x)\beta(y)$ *if* x, y *and* xy *are in* $\mathrm{R}_{\mathbb{Z}}(\mathrm{M}_N \times \mathrm{M}_N)^d$ (*where the product on* $\pi_0(\mathrm{K}(\mathcal{S}))$ $(\pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ *is given by the tensor products of perfect complexes.*

(iii) $\alpha(\lambda^k(x)) = \lambda^k(\alpha(x))$ *and* $\beta(\lambda^k(x)) = \lambda^k(\beta(x))$ *if* x *and* $\lambda^k(x)$ *are both in* $\mathrm{R}_{\mathbb{Z}}(\mathrm{M}_N \times \mathrm{M}_N)^d$.

Proof. In [Gillet and Soulé 1987, Lemma 4.8] they consider a similar result for complexes C and D that are acyclic *off* of a closed subscheme of the given scheme: therefore the above result does not follow by simply extending their result to stacks. Instead one needs to argue as follows. Choose m so that $\mathrm{C}^n = \mathrm{D}^n = \mathrm{A}^n = \mathrm{B}^n = 0$ if $n > m$, and the integer k so that $k > md$. We will first apply the shift $[m]$ so that $\mathrm{C}[m]^n = \mathrm{D}[m]^n = \mathrm{A}[m]^n = \mathrm{B}[m]^n = 0$ for all $n > 0$. Therefore we may view $\mathrm{C}[m]$, $\mathrm{D}[m]$, $\mathrm{A}[m]$ and $\mathrm{B}[m]$ as complexes in nonnegative degrees with differentials of degree -1. We denote these by C', D', A' and B' respectively. Choose the integer N, and for any representation E of $\mathrm{M}_N \times \mathrm{M}_N$ of degree at most d, we define the functors $\mathrm{E} \mapsto \mathrm{S}_\mathrm{E}(k, \mathrm{A}', \mathrm{B}')$ and $\mathrm{E} \mapsto \mathrm{S}_\mathrm{E}(k, \mathrm{C}', \mathrm{D}')$ as in (6.2) and (6.3).

That $\mathrm{E} \mapsto \mathrm{T}_\mathrm{E}(k; \mathrm{A}', \mathrm{B}')$ defines a map $\mathrm{R}_\mathbb{Z}(\mathrm{M}_N \times \mathrm{M}_N)^d \to \mathrm{Simp}_k(\mathrm{Vect}_N(\mathcal{S}))$ is clear from the definition. The property that the functor T_E commutes with base-change shows there is a natural isomorphism $i^* \circ \mathrm{S}_\mathrm{E}(k; \mathrm{C}', \mathrm{D}') \cong \mathrm{S}_\mathrm{E}(k; i^*(\mathrm{C}'), i^*(\mathrm{D}'))$. Next assume that C' and D' are *acyclic* on $\mathcal{S}' \times \Delta[n]$. Since they are both complexes of locally free coherent sheaves, locally on the stack $\mathcal{S}' \times \Delta[n]$, one may find a contracting homotopy for the restriction of C' and D' to $\mathcal{S}' \times \Delta[n]$. Therefore, again the same argument applied to a presentation of the stack $\mathcal{S}$ (by an affine scheme) proves that $\mathrm{S}_\mathrm{E}(k; \mathrm{C}', \mathrm{D}')$ is *acyclic* on restriction $\mathcal{S}' \times \Delta[n]$. (In more detail: let $x : X \to \mathcal{S}$ be a presentation of the stack $\mathcal{S}$ with X affine, $y : Y \to \mathcal{S}'$ be a presentation of $\mathcal{S}'$. Now apply [Gillet and Soulé 1987, Lemma 3.5(iii)] to $\mathrm{X} - Y$.) It follows that $\mathrm{S}_\mathrm{E}(k; \mathrm{C}', \mathrm{D}')$ is a strictly perfect complex on $\mathcal{S} \times \Delta[n]$ so that it is acyclic on $\mathcal{S}' \times \Delta[n]$. It follows that $\mathrm{E} \mapsto \mathrm{S}_\mathrm{E}(k; \mathrm{C}', \mathrm{D}')$ defines a map $\mathrm{R}_\mathbb{Z}(\mathrm{M}_N \times \mathrm{M}_N)^d \to \mathrm{Simp}_k(\mathrm{Vect}_N(\mathcal{S} \times \Delta[n], (\mathcal{S} - \mathcal{S}') \times \Delta[n]))$.

Recall that $\pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ has been proven to be a prelambda ring without unit and that $\pi_0(\mathrm{K}(\mathcal{S}))$ has been proven to be a prelambda ring by the first two statements in Theorem 1.1.1. At this point we recall from Proposition 6.0.1 that the ring $\mathrm{R}_\mathbb{Z}(\mathrm{M}_N \times \mathrm{M}_N)$ is isomorphic to the polynomial ring

$$\mathbb{Z}[\lambda^1(p_1), \ldots, \lambda^N(p_1); \lambda^1(p_2), \ldots, \lambda^N(p_2)].$$

One may also recall from Proposition 6.0.2 that the group $\mathrm{R}_\mathbb{Z}(\mathrm{M_N} \times \mathrm{M_N})^\mathrm{d}$ maps injectively into $\mathrm{R}_\mathbb{Z}(\mathrm{M}_N \times \mathrm{M}_N)$ and the image of this map consists of the elements

$$\mathrm{R}(\lambda^1(p_1), \ldots, \lambda^N(p_1); \lambda^1(p_2), \ldots, \lambda^N(p_2)),$$

where R runs over all polynomials of weight at most d. Therefore, it should be clear now that the maps α and β *are completely determined by their values on the representations* p_1 *and* p_2, that is, assuming both α and β commute with λ-operations.

Therefore, for a representation E of $\mathrm{M_N} \times \mathrm{M_N}$ of degree at most d, we define

$$\alpha(\mathrm{E}) \in \pi_0(\mathrm{K}(\mathcal{S})) \tag{6.4}$$

to be the class of $S_E(k; A', B')[-m]$ and

$$\beta(E) \in \pi_0(K_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$$

to be the class of $S_E(k; C', D')[-m]$, respectively. The property that $T_{p_1}(A', B') = A'$ and $T_{p_2}(A', B') = B'$ shows that $\alpha(p_1) = [A]$ and $\alpha(p_2) = [B]$. Similarly, $\beta(p_1) = [C]$ and $\beta(p_2) = [D]$. This proves (i).

By Proposition 6.0.2, any element in $R_{\mathbb{Z}}(M_N \times M_N)^d$ is a polynomial of weight at most d in the exterior powers of p_1 and p_2. Since the functor $E \mapsto T_E(P, Q)$ (for a fixed P and Q) is exact in E and preserves sums, products and exterior powers in E as already observed (see Section 6.1 and Proposition 6.1.3), and both $\pi_0(K(\mathcal{S}))$ and $\pi_0(K_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ are pre-λ-rings, we see that we obtain additive homomorphisms

$$\alpha : R_{\mathbb{Z}}(M_N \times M_N)^d \to \pi_0(K(\mathcal{S})),$$
$$\beta : R_{\mathbb{Z}}(M_N \times M_N)^d \to \pi_0(K_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n])),$$

which are also multiplicative, and preserve the λ-operations. Moreover, each element in the image of α (β) can be written as $T_E(k; A, B)$ ($S_E(k; C, D)$, respectively) for some $E \in R_{\mathbb{Z}}(M_N \times M_N)^d$.

By the properties of the functor T_E discussed before, it follows that if $R(X_1, \dots, X_N; Y_1, \dots, Y_N)$ is a polynomial with integral coefficients and of weight at most d, we obtain

$$\alpha(R(\lambda^1(p_1), \dots, \lambda^N(p_1); \lambda^1(p_2), \dots, \lambda^N(p_2))) \\ = R(\lambda^1(x), \dots, \lambda^N(x); \lambda^1(y), \dots, \lambda^N(y)) \quad (6.5)$$

where $x = \alpha([A])$ and $y = \alpha([B])$. Similarly,

$$\beta(R(\lambda^1(p_1), \dots, \lambda^N(p_1); \lambda^1(p_2), \dots, \lambda^N(p_2))) \\ = R(\lambda^1(x), \dots, \lambda^N(x); \lambda^1(y), \dots, \lambda^N(y)) \quad (6.6)$$

where $x = \beta([C])$ and $y = \beta([D])$. Since the functors $E \mapsto T_E(k; A, B)$ and $E \mapsto S_E(k; C, D)$ are compatible with tensor products and exterior powers, (ii) and (iii) of the proposition follow readily. □

First we draw the following consequences of the last proposition

Corollary 6.6.1. $\pi_0(K(\mathcal{S}))$ *is a λ-ring and, for each* $n \geq 0$, $\pi_n(K_{\mathcal{S}'}(\mathcal{S})) \cong \pi_0(K_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ *is a λ-ring without a unit element.*

Proof. We already know from Theorem 1.1.1(i) that $\pi_0(K(\mathcal{S}))$ is a pre-λ-ring with unit and that $\pi_n(K_{\mathcal{S}'}(\mathcal{S})) \cong \pi_0(K_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ is a pre-λ-ring (without a unit): the λ-operations in both cases are defined by the exterior powers of perfect complexes. Therefore, what remains to be shown is that they satisfy the relations in (1.1). This is a formal consequence of the last proposition. Let C, D denote two strictly perfect complexes on $\mathcal{S} \times \Delta[n]$ acyclic on $(\mathcal{S} - \mathcal{S}') \times \Delta[n]$ and provided with an explicit zig-zag path p as in (3.6) running from their restriction to $\mathcal{S} \times \delta\Delta[n]$, to the base point $(0, 0)$ in $w\mathrm{G}(i^*)$. Let A, B denote two strictly perfect complexes on $\mathcal{S}$ and let $x = [\mathrm{A}]$, $y = [\mathrm{B}]$.

To check the identity $\lambda^k(\lambda^l(x)) = \mathrm{P}_{k,l}(\lambda^1(x), \dots, \lambda^{kl}(x))$ for a certain universal polynomial $\mathrm{P}_{k,l}$, let $d = kl$ and choose N as in the last proposition. Then

$$\lambda^k(\lambda^l(x)) = \alpha(\lambda^k(\lambda^l(\mathrm{p}_1)))$$

and

$$\mathrm{P}_{k,l}(\lambda^1(x), \dots, \lambda^{kl}(x)) = \alpha(\mathrm{P}_{k,l}(\lambda^1(\mathrm{p}_1), \dots, \lambda^{kl}(\mathrm{p}_1))).$$

Since $\mathrm{R}_{\mathbb{Z}}(\mathrm{M}_N \times \mathrm{M}_N)^d$ is contained in the λ-ring $\mathrm{R}_{\mathbb{Z}}(\mathrm{M}_N \times \mathrm{M}_N)$, we obtain the equality

$$\lambda^k(\lambda^l(\mathrm{p}_1)) = \mathrm{P}_{k,l}(\lambda^1(\mathrm{p}_1), \dots, \lambda^{kl}(\mathrm{p}_1)).$$

In view of properties of the functor $\mathrm{E} \mapsto \mathrm{T}_{\mathrm{E}}$ as discussed above, it follows that α is an additive homomorphism that commutes with products and exterior powers. Therefore, we obtain the formula

$$\lambda^k(\lambda^l(x)) = \mathrm{P}_{k,l}(\lambda^1(x), \dots, \lambda^{kl}(x)).$$

Similarly, one checks the identity $\lambda^k(xy) = \mathrm{P}_k(\lambda^1(x), \dots, \lambda^k(x); \lambda^1(y), \dots, \lambda^k(y))$ for a certain universal polynomial P_k. These prove that $\pi_0(\mathrm{K}(\mathcal{S}))$ is a λ-ring.

Next one lets $x = [\mathrm{C}]$ and $y = [\mathrm{D}]$, and repeats the above argument with β in the place of α to prove $\pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ is a λ-ring without a unit element. $\square$

Let $\mathbb{Z}$ denote the ring of integers with its canonical λ-ring structure; see [Atiyah and Tall 1969, Section 1].

Lemma 6.6.2. *Assume the above framework. Then* $\mathbb{Z} \oplus \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ *is a λ-ring where the operations are defined as follows. If $n, m \in \mathbb{Z}$ and $s, t \in \pi_0(\mathrm{K}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$,*

$$(n, s) + (m, t) = (n + m, s + t),$$
$$(n, s) \circ (m, t) = (n.m, n.t + m.s + s.t),$$
$$\lambda^n(k, s) = \left(\lambda^n(k), \sum_{i=0}^{n-1} \lambda^i(k).\lambda^{n-i}(s)\right), \quad n > 0.$$

Here $\circ$ denotes the multiplication in the graded ring

$$\mathbb{Z} \oplus \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n])).$$

Proof. It is straightforward to verify that these define a pre-λ-ring structure, where $(1, 0)$ is the unit element: see Lemma 5.20.1. We proceed to verify the relations in (1.1) are satisfied. For these the following observations will be helpful.

For any ring R with unit, let $\hat{G}(\mathrm{R}) = 1 + \mathrm{R}[\![t]\!]^+$ = the power series in t with coefficients in R and with the starting term 1. This is a λ ring with the addition (which will be denoted by $\boxplus$) being the product of power series, and multiplication (denoted $\bullet$) and exterior power operations defined as in [Atiyah and Tall 1969, page 258]. If R is also a pre-λ ring, then the map $r \mapsto \lambda_t(r) = \sum_i \lambda^i(r)t^i$ is an *additive homomorphism of abelian groups* from R to $\hat{G}(\mathrm{R})$. The same map is a ring homomorphism (a map of pre-λ-rings) if and only if the first relation in (1.1) is satisfied (both the relations in (1.1), respectively are satisfied).

Therefore, to prove the first relation in (1.1), it suffices to show that

$$\lambda_t((n, x) \circ (m, y)) = \lambda_t(n, x) \bullet \lambda_t(m, y), \tag{6.7}$$

for $n, m \in \mathbb{Z}$, and $x, y \in \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$. Using the product on the ring $\mathbb{Z} \oplus \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$, the left-hand side identifies with

$$\lambda_t((n, 0) \circ (m, 0) + (n, 0) \circ (0, y) + (0, x) \circ (m, 0) + (0, x) \circ (0, y)).$$

Since λ_t is an additive homomorphism, this identifies with

$$\lambda_t((n, 0) \circ (m, 0)) \boxplus \lambda_t((n, 0) \circ (0, y)) \boxplus \lambda_t((0, x) \circ (m, 0)) \boxplus \lambda_t((0, x) \circ (0, y)).$$

The term on the right-hand side of (6.7) identifies with

$$\begin{aligned}(\lambda_t(n, 0)) \boxplus (\lambda_t(0, x))] \bullet [(\lambda_t(m, 0)) \boxplus (\lambda_t(0, y))] \\ = (\lambda_t(n, 0) \bullet \lambda_t(m, 0)) \boxplus (\lambda_t(n, 0) \bullet (\lambda_t(0, y)) \boxplus ((\lambda_t(0, x) \bullet (\lambda_t(m, 0)) \\ \boxplus (\lambda_t(0, x)) \bullet (\lambda_t(0, y)).\end{aligned}$$

Since $\mathbb{Z}$ and $\pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ are λ-rings, $\lambda_t((n, 0) \circ (m, 0)) = \lambda_t(n, 0) \bullet \lambda_t(m, 0)$ and $\lambda_t((0, x) \circ (0, y)) = \lambda_t(0, x) \bullet \lambda_t(0, y)$. Moreover, observe that $(0, x) \circ (m, 0) = (0, mx) = (m, 0) \circ (0, x)$. Therefore, it suffices to show that $\lambda_t((m, 0) \circ (0, x)) = \lambda_t(m, 0) \bullet \lambda_t(0, x)$ for any positive integer m. However, $(m, 0) \circ (0, x) = (0, mx)$. Since $\lambda_t(1, 0) = 1 + (1, 0)t$ is the multiplicative unit in $\hat{G}(\mathbb{Z} \oplus \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n])))$, it follows that $\lambda_t(0, 1.x) = \lambda_t(1, 0) \bullet \lambda_t(0, x)$.

Assuming that $\lambda_t(0, nx) = \lambda_t(n, 0) \bullet \lambda_t(0, x)$ for all $n < m$, we observe that

$$\begin{aligned}
\lambda_t(0, mx) &= \lambda_t(0, x + (m-1)x) \\
&= \lambda_t(0, x) \boxplus \lambda_t(0, (m-1)x) \\
&= \lambda_t(0, x) \boxplus \lambda_t((m-1, 0) \circ (0, x)) \\
&= \lambda_t(1, 0) \bullet \lambda_t(0, x) \boxplus \lambda_t((m-1), 0) \bullet \lambda_t(0, x) \\
&= (\lambda_t(1, 0) \boxplus \lambda_t((m-1), 0)) \bullet \lambda_t(0, x) \\
&= \lambda_t(m, 0) \bullet \lambda_t(0, x).
\end{aligned}$$

This completes the proof of the first relation in (1.1).

To prove the second, we observe the square in [SGA 6 1971, Exposé V, (3.7.1)]

$$\begin{tikzcd}
S \arrow[r, "\lambda_u"] \arrow[d, "\lambda_t"'] & \hat{G}^u(S) \arrow[d, "\hat{G}^u(\lambda_t)"] \\
\hat{G}^t(S) \arrow[r, "\lambda_u"] & \hat{G}^u(\hat{G}^t(S)),
\end{tikzcd}$$

where $S = \mathbb{Z} \oplus \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$. Given a pre-$\lambda$-ring R, $\hat{G}^u(R)$ ($\hat{G}^t(R)$) denotes the power series ring considered above in the variable u (t, respectively). The second relation in (1.1) holds if and only if the above square *commutes*; see [SGA 6 1971, Exposé V, 3.7]. Since all the maps in the above diagram are group homomorphisms, it suffices to show the square above commutes separately for elements of the form $(n, 0)$ and $(0, x)$ with $n \in \mathbb{Z}$ and $x \in \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$. But this is equivalent to showing the required relations hold separately for elements of the form $(n, 0)$ and $(0, x)$ with $n \in \mathbb{Z}$ and $x \in \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$. This is clear since we already know from Corollary 6.6.1 that the elements in $\pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ satisfy the second relation in (1.1). (Clearly the elements in $\mathbb{Z}$ also satisfy this relation since $\mathbb{Z}$ is a λ-ring with its canonical structure.) This completes the proof of the lemma. □

Proposition 6.7.1. *Assume the above framework. Then*

$$\pi_0(\mathrm{K}(\mathcal{S})) \oplus \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$$

is a λ-ring where the operations are defined as follows. If $u, v \in \pi_0(\mathrm{K}(\mathcal{S}))$ and $s, t \in \pi_0(\mathrm{K}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$,

$$\begin{aligned}
(u, s) + (v, t) &= (u + v, s + t), \\
(u, s) \circ (v, t) &= (u.v, u.t + v.s + s.t) \\
\lambda^n(u, s) &= \left(\lambda^n(u), \sum_{i=0}^{n-1} \lambda^i(u).\lambda^{n-i}(s)\right), \quad n > 0.
\end{aligned}$$

Here $\circ$ denotes the multiplication in the graded ring

$$\pi_0(\mathrm{K}(\mathcal{S})) \oplus \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n])).$$

Proof. The proof of this proposition will be very similar to the proof of Lemma 6.6.2. Observe first that the proof of the second relation in (1.1) given in the proof of Lemma 6.6.2 carries over verbatim with the ring $\mathbb{Z}$ replaced by $\pi_0(\mathrm{K}(\mathcal{S}))$. Therefore, it suffices to consider the proof of the first relation in (1.1), that is, it suffices to prove

$$\lambda_t((u, x) \circ (v, y)) = \lambda_t(u, x) \bullet \lambda_t(v, y), \tag{6.8}$$

for $u, v \in \pi_0(\mathrm{K}(\mathcal{S}))$ and $x, y \in \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$. Using the product on the ring $\pi_0(\mathrm{K}(\mathcal{S})) \oplus \pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$, the left-hand side identifies with

$$\lambda_t((u, 0) \circ (v, 0) + (u, 0) \circ (0, y) + (v, 0) \circ (0, x) + (0, x) \circ (0, y)).$$

Since λ_t is an additive homomorphism, this identifies with

$$\lambda_t((u, 0) \circ (v, 0)) \boxplus \lambda_t((u, 0) \circ (0, y)) \boxplus \lambda_t((v, 0) \circ (0, x)) \boxplus \lambda_t((0, x) \circ (0, y)).$$

The term on the right-hand side of (6.8) identifies with

$$\begin{aligned} &[(\lambda_t(u, 0)) \boxplus (\lambda_t(0, x))] \bullet [(\lambda_t(v, 0)) \boxplus (\lambda_t(0, y))] \\ &\quad = (\lambda_t(u, 0) \bullet \lambda_t(v, 0)) \boxplus (\lambda_t(u, 0) \bullet (\lambda_t(0, y)) \boxplus ((\lambda_t(v, 0) \bullet (\lambda_t(0, x)) \\ &\qquad \boxplus (\lambda_t(0, x)) \bullet (\lambda_t(0, y)). \end{aligned}$$

Since $\pi_0(\mathrm{K}(\mathcal{S}))$ and $\pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$ are λ-rings, $\lambda_t((u, 0) \circ (v, 0)) = \lambda_t(u, 0) \bullet \lambda_t(v, 0)$ and $\lambda_t((0, x) \circ (0, y)) = \lambda_t(0, x) \bullet \lambda_t(0, y)$. Moreover, observe that $(0, x) \circ (v, 0) = (0, vx) = (v, 0) \circ (0, x)$. Therefore, it suffices to show that $\lambda_t((v, 0) \circ (0, x)) = \lambda_t(v, 0) \bullet \lambda_t(0, x)$ for any class $v \in \pi_0(\mathrm{K}(\mathcal{S}))$.

At this point, one may apply the splitting principle to elements of $\pi_0(\mathrm{K}(\mathcal{S}))$ (by taking the projective space bundle associated to a given vector bundle on $\mathcal{S}$), so that we may assume the class v breaks up into a finite sum of the classes of line bundles: $v = \sum_{i=1}^{m} [\mathcal{L}_i]$, where each $\mathcal{L}_i$ is a line bundle on $\mathcal{S}$.

Assuming that $\lambda_t\big(0, \big(\sum_{i=1}^{n}[\mathcal{L}_i]\big)x\big) = \lambda_t\big(\sum_{i=1}^{n}[\mathcal{L}_i], 0\big) \bullet \lambda_t(0, x)$ for all $n < m$, we observe that

$$\begin{aligned}
\lambda_t\left(0, \left(\sum_{i=1}^{m}[\mathcal{L}_i]\right)x\right) &= \lambda_t\left(0, [\mathcal{L}_m]x + \left(\sum_{i=1}^{m-1}[\mathcal{L}_i]\right)x\right)\\
&= \lambda_t(0, [\mathcal{L}_m]x) \boxplus \lambda_t\left(0, \left(\sum_{i=1}^{m-1}[\mathcal{L}_i]\right)x\right)\\
&= \lambda_t(0, [\mathcal{L}_m]x) \boxplus \lambda_t\left(\left(\sum_{i=1}^{m-1}[\mathcal{L}_i]\right), 0\right) \circ (0, x))\\
&= \lambda_t([\mathcal{L}_m], 0) \bullet \lambda_t(0, x) \boxplus \lambda_t\left(\left(\sum_{i=1}^{m-1}[\mathcal{L}_i]\right), 0\right) \bullet \lambda_t(0, x)\\
&= \left(\lambda_t([\mathcal{L}_m], 0) \boxplus \lambda_t\left(\left(\sum_{i=1}^{m-1}[\mathcal{L}_i]\right), 0\right)\right) \bullet \lambda_t(0, x)\\
&= \lambda_t\left(\left(\sum_{i=1}^{m}[\mathcal{L}_i]\right), 0\right) \bullet \lambda_t(0, x).
\end{aligned}$$

Therefore, it suffices to prove that if $v = [\mathcal{L}]$ is the class of a line bundle on $\mathcal{S}$, and x denotes a class in $\pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$, then one obtains

$$\lambda_t(0, [\mathcal{L}]x) = \lambda_t([\mathcal{L}], 0) \bullet \lambda_t(0, x).$$

We may assume the class x is represented by the class of a perfect complex P on $\mathcal{S} \times \Delta[n]$ acyclic on $(\mathcal{S} - \mathcal{S}') \times \Delta[n]$, provided with a zig-zag path (as in (3.6)) p joining the restriction $(i^*(\mathrm{P}), 0)$ to the base point, namely the pair $(0, 0)$ in $w\mathrm{G}(i^*)$. Now verifying the above relation amounts to verifying the first relation in (1.1): as is well known, since $\mathcal{L}$ is a line bundle on $\mathcal{S}$, this amounts to observing the (functorial) isomorphism

$$\left[\bigwedge^n(\mathcal{L} \otimes \mathrm{P})\right] = \left[\mathcal{L}^{\otimes n} \otimes \bigwedge^n(\mathrm{P})\right], n \geq 0.$$

as classes in $\pi_0(\mathrm{K}_{\mathcal{S}' \times \Delta[n]}(\mathcal{S} \times \Delta[n], \mathcal{S} \times \delta\Delta[n]))$. This is clear since there is a functorial isomorphism $\bigwedge^n(\mathcal{L} \otimes \mathrm{P}) \cong \mathcal{L}^{\otimes n} \otimes \bigwedge^n(\mathrm{P})$. This completes the proof of the first relation in (1.1) and hence the proof of the proposition. □

This concludes the proof of Theorem 1.1.1. □

Remark 6.8.1. Observe also that the restriction to smooth stacks becomes necessary so that one has the homotopy property for *K*-theory. (This fails, in general, even for nonregular schemes.)

7. γ-operations and absolute cohomology

7.1. ***Standing hypothesis.*** For the rest of the paper we will assume that all algebraic stacks $\mathcal{S}$ we consider are smooth, and every coherent sheaf on $\mathcal{S}$ is the quotient of a vector bundle. If $\mathcal{S}'$ is a closed substack of $\mathcal{S}$, Theorem 1.1.1(iii) shows that there is the structure of a λ-algebra (in the sense of Definition 1.0.1) on each $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ over $\pi_0(\mathrm{K}(\mathcal{S}))$.

Definition 7.1.1. (a) Recall that each $\pi_0(\mathrm{K}(\mathcal{S})) \oplus \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ is a λ-ring with the operations defined above. Therefore, one may define the operations γ^n on $\pi_0(\mathrm{K}(\mathcal{S})) \oplus \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ as follows:

$$\gamma^n(\alpha, \beta) = \lambda^n((\alpha + (n-1).\mathcal{O}_{\mathcal{S}}), \beta), \quad \alpha \in \pi_0\mathrm{K}(\mathcal{S}),\ \beta \in \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})). \tag{7.2}$$

One may observe that if $\alpha = 0$, then $\gamma^n(0, \beta) = \big(0, \sum_{i=0}^{n-1} \lambda^i((n-1).\mathcal{O}_{\mathcal{S}}).\lambda^{n-i}(\beta)\big)$ (see (5.21)), so that each γ^n induces a map on $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ which we will also denote by γ^n.

(b) One defines the γ-filtration on each $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ as follows. Let $\epsilon : \pi_0(\mathrm{K}(\mathcal{S})) \to \mathbb{Z}$ denote the augmentation given by the *rank*-map: the function ϵ is the rank of a strictly perfect complex defined as an obvious Euler characteristic involving the ranks of the constituent terms of the complex. Then we define $\mathrm{F}^m(\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})))$ to be generated by $\gamma^{i_1}a_1 \cdots \gamma^{i_k}a_{i_k}\gamma^{j_1}x_1 \cdots \gamma^{j_p}x_p$, where $a_i \in \pi_0(\mathrm{K}(\mathcal{S}))$ with $\epsilon(a_i) = 0$, for all $i = 1, \dots, k$, and $x_{j_i} \in \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$, so that $i_1 + \cdots + i_k + j_1 + \cdots + j_p \geq m$; see [Kratzer 1980a, Section 6] or [Weibel 2013, page 105].

(c) One may define the Adams operations ψ^k using ascending induction on k and the formula: $\psi^k = \psi^{k-1}\lambda^1 - \cdots + (-1)^k\psi^1\lambda^{k-1} + (-1)^{k+1}k\lambda^k$; see [Weibel 2013, page 102].

Then one may readily verify the following properties of the γ-filtration for each n:

(i) $\mathrm{F}^{m+1}(\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))) \subseteq \mathrm{F}^m(\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})))$, for each $m \geq 0$.

(ii) $\mathrm{F}^1(\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))) \subseteq \mathrm{F}^0(\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))) = \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$.

Since the product on each $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ is trivial for all $n > 0$, one may observe that the γ-filtration $\mathrm{F}^m(\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})))$, for $n > 0$, is generated by $\gamma^{i_1}a_1 \cdots \gamma^{i_k}a_{i_k}\gamma^{j_1}x_{j_1}$, where $a_i \in \pi_0(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ with $\epsilon(a_i) = 0$, for all $i = 1, \dots, k$, and $x_{j_1} \in \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$, so that $i_1 + \cdots + i_k + j_1 + \geq m$. Now one may readily verify the following additional properties for each $m, m' \geq 0$:

(1) One has a pairing $\mathrm{F}^m(\pi_0(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))) \otimes \mathrm{F}^{m'}(\pi_0(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))) \to \mathrm{F}^{m+m'}(\pi_0(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})))$.

(2) $\mathrm{F}^m(\pi_0(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})))$ is a λ-ideal in $\pi_0(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$.

Proof of Theorem 1.1.3. The properties of the γ-operations follow from the observation that $\pi_0(\mathrm{K}(\mathcal{S})) \oplus \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ is a λ-ring. Again one observes that each ψ^k induces a self-map of $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ for each closed substack $\mathcal{S}'$ of $\mathcal{S}$. The last but one statement in (i) follows from the functoriality of the λ and γ-operations with respect to pull-back. The last statement in (i) is a pure consequence of the λ-ring structure on $\pi_0(\mathrm{K}(\mathcal{S})) \oplus \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$.

These prove the statements in (i); the proof of statements in (ii) are clear since the λ-operations are compatible with respect to pull-backs. □

Remarks 7.2.1. (1) It is important to point out that the action of ψ_k above is *not* locally nilpotent, which is necessary to conclude that $\pi_*\mathrm{K}(\mathcal{S}) \otimes \mathbb{Q}$ is isomorphic to the sum of the associated graded terms of the γ-filtration. This is false in general as may be seen from the following simple counterexample: consider $\mathcal{S} = [(\operatorname{Spec} k)/\mathrm{G}]$ where G is a finite group and k is a field. In this case, it is shown in [Atiyah 1961, Proposition (6.13)] that the γ-filtration has just two terms modulo torsion.

(2) Observe also that the γ-operations on $\pi_0(\mathrm{K}(\mathcal{S}))$ are compatible with the γ-operations on $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ in the following sense. Let $\alpha \in \pi_0(\mathrm{K}(\mathcal{S}))$ and $\beta \in \pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$. Then $(\alpha, 0).(0, \beta) = (0, \alpha.\beta)$ using the module structure of $\pi_n(\mathrm{K}_{\mathcal{S}'}(\mathcal{S}))$ over $\pi_0(\mathrm{K}(\mathcal{S}))$. Now

$$\gamma^i(\alpha, 0).\gamma^j(0, \beta) = (\gamma^i(\alpha), 0).(0, \gamma^j(\beta)) = (0, \gamma^i(\alpha).\gamma^j(\beta)).$$

Moreover, since $(0, \alpha.\beta) = (\alpha, 0).(0, \beta)$, it follows that

$$\begin{aligned}
(0, \gamma^k(\alpha.\beta)) &= \gamma^k(0, \alpha.\beta) \\
&= \gamma^k((\alpha, 0).(0, \beta)) \\
&= Q_k((\gamma^1(\alpha), 0), \dots, (\gamma^k(\alpha), 0); (0, \gamma^1(\beta)), \dots, (0, \gamma^k(\beta))) \\
&= (0, \mathrm{Q}_k(\gamma^1(\alpha), \dots, \gamma^k(\alpha); \gamma^1(\beta), \dots, \gamma^k(\beta))).
\end{aligned} \tag{7.3}$$

Definition 7.3.1. Let $\mathrm{gr}^n(\pi_j\mathrm{K}_{\mathcal{S}'}(\mathcal{S}) \otimes \mathbb{Q})$ denote the n-th graded piece of the γ-filtration. We let $\mathrm{H}^i_{\mathcal{S}',\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(j)) = \mathrm{gr}^j(\pi_{2j-i}(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})) \otimes \mathbb{Q})$. We define the i-th Chern class

$$c_i(j) : \pi_0(\mathrm{K}(\mathcal{S})) \oplus \pi_i(\mathrm{K}(\mathcal{S})) \to \mathrm{H}^{2j}_{\mathrm{abs}}(\mathcal{S}; \mathbb{Q}(j)) \oplus \mathrm{H}^{2j-i}_{\mathrm{abs},\mathcal{S}'}(\mathcal{S}; \mathbb{Q}(j))$$

by $c_i(j)(\alpha, \beta) = \gamma^j(\alpha - rk(\alpha).\mathcal{O}_{\mathcal{S}}, \beta)$ where γ^j is the j-th γ-operation on $\pi_0(\mathrm{K}(\mathcal{S})) \otimes \mathbb{Q} \oplus \pi_i(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})) \otimes \mathbb{Q}$. If $i = 0$ and $\beta = 0$, we let the Chern class $c_i(j)$ be denoted $C(j)$. If $\alpha = 0$, we obtain Chern classes $c_i(j) : \pi_i(\mathrm{K}_{\mathcal{S}'}(\mathcal{S})) \to \mathrm{H}^{2j-i}_{\mathrm{abs},\mathcal{S}'}(\mathcal{S}; \mathbb{Q}(j))$. We define the Chern-character into $\prod_i \mathrm{H}^{2j-i}_{\mathrm{abs}}(\mathcal{S}; \mathbb{Q}(j))$ by the usual formula; see [SGA 6 1971, Exposé 0: Appendix]. (Observe we are taking the product in the last expression and not the sum, only because the γ-filtration is not locally nilpotent.) For a vector bundle $\mathcal{E}$, one may define its Todd class by the usual Todd polynomial in the Chern classes; see [Fulton and Lang 1985, Chapter 1, Section 4].

Proof of Theorem 1.1.4. Recall that the statement we want to prove is the existence of the long exact sequence of absolute cohomology groups

$$\cdots \to \mathrm{H}^n_{\mathcal{S}_0',\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(i)) \to \mathrm{H}^n_{\mathcal{S}_1',\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(i)) \to \mathrm{H}^n_{\mathcal{S}_1'-\mathcal{S}_0',\mathrm{abs}}(\mathcal{S}-\mathcal{S}_0', \mathbb{Q}(i)) \\ \to \mathrm{H}^{n+1}_{\mathcal{S}_0',\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(i)) \to \cdots,$$

where $\mathcal{S}$ is a smooth algebraic stack with the property that every coherent sheaf is the quotient of a vector bundle and that $\mathcal{S}_0' \subseteq \mathcal{S}_1'$ are two closed algebraic substacks.

We begin with the fibration sequence (localized at $\mathbb{Q}$)

$$\Omega(\mathrm{K}_{\mathcal{S}_1'-\mathcal{S}_0'}(\mathcal{S}-\mathcal{S}_0')_{\mathbb{Q}}) \to \mathrm{K}_{\mathcal{S}_0'}(\mathcal{S})_{\mathbb{Q}} \to \mathrm{K}_{\mathcal{S}_1'}(\mathcal{S})_{\mathbb{Q}} \to \mathrm{K}_{\mathcal{S}_1'-\mathcal{S}_0'}(\mathcal{S}-\mathcal{S}_0')_{\mathbb{Q}}.$$

On taking the associated homotopy groups one obtains a long exact sequence

$$\cdots \to \pi_k(\mathrm{K}_{\mathcal{S}_0'}(\mathcal{S}))\otimes\mathbb{Q} \xrightarrow{\alpha} \pi_k(\mathrm{K}_{\mathcal{S}_1'}(\mathcal{S}))\otimes\mathbb{Q} \xrightarrow{\beta} \pi_k(\mathrm{K}_{\mathcal{S}_1'-\mathcal{S}_0'}(\mathcal{S}-\mathcal{S}_0'))\otimes\mathbb{Q} \xrightarrow{\gamma} \cdots \tag{7.4}$$

Since the γ-filtration is compatible with respect to pull-backs, one obtains the commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i+1}_{n+2} \arrow[r, "\alpha^{i+1}"] \arrow[d, "f^{i+1}_{n+2}"] & B^{i+1}_{n+2} \arrow[r, "\beta^{i+1}"] \arrow[d, "g^{i+1}_{n+2}"] & C^{i+1}_{n+2} \arrow[r, "\gamma^{i+1}"] \arrow[d, "h^{i+1}_{n+2}"] & A^{i+1}_{n+3} \arrow[r] \arrow[d, "f^{i+1}_{n+3}"] & \cdots \\
\cdots \arrow[r] & A^i_n \arrow[r, "\alpha^i"] & B^i_n \arrow[r, "\beta^i"] & C^i_n \arrow[r, "\gamma^i"] & A^i_{n+1} \arrow[r] & \cdots
\end{tikzcd} \tag{7.5}$$

where $A^i_n = \mathrm{F}^i(\pi_{2i-n}(\mathrm{K}_{\mathcal{S}_0'}(\mathcal{S}))\otimes\mathbb{Q})$, $B^i_n = \mathrm{F}^i(\pi_{2i-n}(\mathrm{K}_{\mathcal{S}_1'}(\mathcal{S}))\otimes\mathbb{Q})$ and $C^i_n = \mathrm{F}^i(\pi_{2i-n}(\mathrm{K}_{\mathcal{S}_1'-\mathcal{S}_0'}(\mathcal{S}-\mathcal{S}_0'))\otimes\mathbb{Q})$. The maps α^i (β^i, γ^i) are the maps induced by α (β, γ, respectively). Observe that *all the vertical maps are given by the inclusion of* F^{i+1} *into* F^i, *and are therefore injective* and that

$$\mathrm{H}^n_{\mathcal{S}_0',\mathrm{abs}}(\mathcal{S},\mathbb{Q}(i)) = \mathrm{coker}(f^{i+1}_{n+2}), \quad \mathrm{H}^n_{\mathcal{S}_1',\mathrm{abs}}(\mathcal{S},\mathbb{Q}(i)) = \mathrm{coker}(g^{i+1}_{n+2}) \quad \text{and} \\ \mathrm{H}^n_{\mathcal{S}_1'-\mathcal{S}_0',\mathrm{abs}}(\mathcal{S}-\mathcal{S}_0',\mathbb{Q}(i)) = \mathrm{coker}(h^{i+1}_{n+2}).$$

We proceed to show that both rows in the diagram (7.5) are *exact*. For example, we will show that $\ker(\beta^i) = \mathrm{Im}(\alpha^i)$. Let $b \in B^i_n$ so that $\beta^i(b) = 0$. Then the exactness of the long exact sequence of homotopy groups in (7.4) shows that there is a class $a \in \pi_{2i-n}(\mathrm{K}_{\mathcal{S}_0'}(\mathcal{S}))\otimes\mathbb{Q}$ so that $\alpha(a) = b$. Now A^i_n is a direct factor of $\pi_{2i-n}(\mathrm{K}_{\mathcal{S}_0'}(\mathcal{S}))\otimes\mathbb{Q}$. We let a' denote the projection of a to the factor A^i_n. Now, both $b = \alpha(a)$ and $\alpha(a')$ belong to B^i_n. It suffices to show $\alpha(a) - \alpha(a') = \alpha(a-a') = 0$. Observe that the associated graded terms in the γ-filtration of $a - a'$ are of weight *strictly lower* than i. In particular, when one breaks $a - a'$ into the sum of terms a_j

belonging to eigenspaces for the Adams operations ψ^k, the eigenvalues will all be of the form k^j, $0 \le j < i$. (Observe that this also means $a - a'$ breaks up into a finite sum $\sum a_j$, with a_j belonging to the eigenspace for ψ^k with eigenvalue k^j, $0 \le j < i$.)

Since α preserves the γ-filtrations, the Adams operations act on $\alpha(a_j)$ with eigenvalue k^j, $j < i$. The eigenvalues of ψ^k on $B^i_n = F^i(\pi_{2i-n}(\mathrm{K}_{\mathcal{S}'_1}(\mathcal{S})) \otimes \mathbb{Q})$ are all of the form k^j, $j \ge i$. Therefore, the projection of $\alpha(a) - \alpha(a')$ to B^i_n is zero and $b = \alpha(a) = \alpha(a') = \alpha^i(a')$ as classes in B^i_n. A similar argument shows the exactness of both rows. Now a diagram-chase shows that the sequence

$$\cdots \to \mathrm{H}^n_{\mathcal{S}'_0,\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(m)) \to \mathrm{H}^n_{\mathcal{S}'_1,\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(m)) \to \mathrm{H}^n_{\mathcal{S}'_1-\mathcal{S}'_0,\mathrm{abs}}(\mathcal{S} - \mathcal{S}'_0, \mathbb{Q}(m)) \\ \to \mathrm{H}^{n+1}_{\mathcal{S}'_0,\mathrm{abs}}(\mathcal{S}, \mathbb{Q}(m)) \to \cdots$$

is exact at the second term. See, for example, [Iversen 1986, Proposition 1.4]: observe that the sequence of absolute cohomology groups above is obtained by taking the cokernels of each column in the diagram (7.5). The exactness at the remaining terms may be proved similarly. □

Remark 7.5.1. Assume that the stack $\mathcal{S}$ has a coarse moduli space $\mathfrak{M}$. In this case, the observation that the γ-filtration on $\pi_*\mathrm{K}(\mathcal{S}) \otimes \mathbb{Q}$ is compatible with the γ-filtration on $\pi_*\mathrm{K}(\mathfrak{M}) \otimes \mathbb{Q}$ shows that the absolute cohomology of the stack we have defined is an algebra over the (usual) absolute cohomology of the moduli space when the latter is defined.

8. Examples

We begin with the following theorem of Thomason as a source of several examples.

Theorem 8.0.1 [Thomason 1987b, Lemmas 2.4, 2.6, 2.10 and 2.14]. *Let k denote a field*, X *a normal Noetherian scheme over k with an ample family of line bundles* (*for example*, *a smooth separated Noetherian scheme*). *Let* G *denote an affine flat group scheme of finite type over k which is an extension of a finite flat group scheme by a smooth connected group-scheme*; *let* G *act on* X. *Then the quotient stack* [X/G] *has the resolution property.*

To keep things simple, we will restrict to Noetherian schemes defined over a field k.

Examples 8.0.2. (i) Let D denote a diagonalizable group scheme acting *trivially* on a smooth scheme X. Then any D-equivariant vector bundle on X corresponds to giving a grading by the characters of D on the vector bundle obtained by forgetting the action. It follows readily that $\pi_*\mathrm{K}([\mathrm{X}/\mathrm{D}]) \cong \mathrm{R}(\mathrm{D}) \otimes \pi_*\mathrm{K}(\mathrm{X})$. This is an isomorphism of λ-rings. Moreover on computing the absolute cohomology, we obtain

$$\begin{aligned}\mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{D}], \mathbb{Q}(\bullet)) &\cong \mathrm{grd}(\mathrm{R}(\mathrm{D}) \otimes \mathbb{Q}) \otimes \mathrm{H}^*_{\mathrm{abs}}(\mathrm{X}, \mathbb{Q}(\bullet)) \\ &= \mathrm{grd}(\pi_0(K([\operatorname{Spec} k/\mathrm{D}]) \otimes \mathbb{Q})) \otimes \mathrm{H}^*_{\mathrm{abs}}(\mathrm{X}, \mathbb{Q}(\bullet)),\end{aligned}$$

where $\mathrm{grd}(\mathrm{R}(\mathrm{D}) \otimes \mathbb{Q})$ denotes the associated graded terms with respect to the γ-filtration.

Clearly the graded ring $\mathrm{grd}(\mathrm{R}(\mathrm{D}) \otimes \mathbb{Q}) = \mathrm{grd}(\pi_0(K([\operatorname{Spec} k/\mathrm{D}])) \otimes \mathbb{Q})$ has a natural decreasing filtration, and completing it with respect to this filtration we obtain

$$\prod_{n=0}^{\infty} \mathrm{grd}_n(\pi_0(K([\operatorname{Spec} k/\mathrm{D}])) \otimes \mathbb{Q}).$$

It is shown in [Köck 1998, (5.1) Proposition and (5.3) Proposition] that the latter is isomorphic to the completion $(\pi_0(K([\operatorname{Spec} k/\mathrm{D}])) \otimes \mathbb{Q})\widehat{\ }_{I_D}$, where $\widehat{\ }_{I_{\mathrm{D}}}$ denotes completion at the augmentation ideal. Moreover, by [Edidin and Graham 2000], the latter is isomorphic to $\prod_{i=0}^{\infty} \mathrm{CH}^i(\mathrm{BD}, \mathbb{Q})$, where BD denotes the *classifying space* for D defined as in [Totaro 1999] or [Morel and Voevodsky 1999]. Thus we see that on completing the graded ring $\mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{D}], \mathbb{Q}(\bullet)) \cong \mathrm{grd}(\pi_0(K([\operatorname{Spec} k/\mathrm{D}])) \otimes \mathbb{Q}) \otimes \mathrm{H}^*_{\mathrm{abs}}(\mathrm{X}, \mathbb{Q}(\bullet))$ with respect to the natural decreasing filtration induced from the one on $\mathrm{grd}(\pi_0(K([\operatorname{Spec} k/\mathrm{D}]) \otimes \mathbb{Q})$, we obtain the isomorphism:

$$\mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{D}], \mathbb{Q}(\bullet))\widehat{\ } \cong \left(\prod_{i=0}^{\infty} C\mathrm{H}^i(BD, \mathbb{Q})\right) \otimes \mathrm{H}^*_{\mathrm{abs}}(X, \mathbb{Q}(\bullet)).$$

(ii) Let T denote a split torus acting on a smooth scheme X. Assume further that there is a T-*stable stratification of* X *by strata which are all affine spaces*. In this case one obtains the isomorphism of λ-rings: $\pi_*(K[\mathrm{X}/\mathrm{T}]) \cong \mathrm{R}(\mathrm{T}) \otimes \pi_*(\mathrm{K}(\mathrm{X}))$. One may obtain this isomorphism as follows; see [Joshua 2001] for related results. One shows the obvious map of spectra $\mathrm{K}([\operatorname{Spec} k/\mathrm{T}]) \bigotimes^L_{K(\operatorname{Spec} k)} \mathrm{K}(\mathrm{X}) \to \mathrm{K}([\mathrm{X}/\mathrm{T}])$ is a weak equivalence. Here one needs to use the framework of [Joshua 2001] of ring and module-spectra to be able to define the derived tensor product. Since X is smooth, its K-theory identifies with G-theory and one uses the localization sequence associated to the stratification of X to show the above map is a weak equivalence. Now one obtains an associated spectral sequence with E_2-terms given by

$$\mathrm{Tor}^{\pi_*(\mathrm{K}(\operatorname{Spec} k))}(\pi_*(\mathrm{K}([\operatorname{Spec} k/\mathrm{T}])), \pi_*\mathrm{K}(\mathrm{X})) \Rightarrow \pi_*(\mathrm{K}([\mathrm{X}/\mathrm{T}])).$$

This spectral sequence degenerates at the E_2-terms in view of the isomorphism

$$\pi_*(\mathrm{K}([\operatorname{Spec} k/\mathrm{T}])) \cong \mathrm{R}(\mathrm{T}) \otimes \pi_*(\mathrm{K}(\operatorname{Spec} k))$$

and provides the isomorphism $\pi_*\mathrm{K}([\mathrm{X}/\mathrm{T}]) \cong R(T) \otimes \pi_*\mathrm{K}(X)$. This result applies to the case when X is a flag variety or a smooth projective variety on which T acts with finitely many fixed points. One also obtains the isomorphism of absolute cohomology

$$\mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{T}], \mathbb{Q}(\bullet)) \cong (\mathrm{grd}(\pi_0(K([\mathrm{Spec}\, k/\mathrm{T}]) \otimes \mathbb{Q}) \otimes \mathrm{H}^*_{\mathrm{abs}}(\mathrm{X}, \mathbb{Q}(\bullet))$$

and therefore,

$$\begin{aligned}\mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{T}], \mathbb{Q}(\bullet))\widehat{\ } &\cong \left(\prod_{n=0}^{\infty} \mathrm{grd}_n(\pi_0(K([\mathrm{Spec}\, k/\mathrm{T}]) \otimes \mathbb{Q}))\right) \otimes \mathrm{H}^*_{\mathrm{abs}}(\mathrm{X}, \mathbb{Q}(\bullet)) \\ &= \left(\prod_{n=0}^{\infty} \mathrm{CH}^n(\mathrm{BT}, \mathbb{Q})\right) \otimes \mathrm{H}^*_{\mathrm{abs}}(\mathrm{X}, \mathbb{Q}(\bullet)).\end{aligned}$$

where $\widehat{\ }$ denotes completion with respect to the decreasing filtration on the graded ring $(\mathrm{grd}(\pi_0(K([\mathrm{Spec}\, k/\mathrm{T}]) \otimes \mathbb{Q}) \otimes \mathrm{H}^*_{\mathrm{abs}}(\mathrm{X}, \mathbb{Q}(\bullet))$ and BT again denotes the classifying space of T in the sense of [Totaro 1999] or [Morel and Voevodsky 1999]. These isomorphisms follow along the same lines as in (i).

(iii) Next, let G denote any split reductive group over k with $\pi_1(\mathrm{G})$ torsion free. Let T denote a fixed maximal torus in G. Let X denote a smooth G-scheme. Then [Merkurjev 1997, Proposition 4.1] shows the isomorphism (of λ-rings):

$$\pi_*\mathrm{K}([\mathrm{X}/\mathrm{T}]) \cong \mathrm{R}(\mathrm{T}) \bigotimes_{\mathrm{R}(\mathrm{G})} \pi_*\mathrm{K}([\mathrm{X}/\mathrm{G}]), \quad \text{and therefore,}$$

$$\mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{T}], \mathbb{Q}(\bullet)) \cong \mathrm{grd}(\mathrm{R}(\mathrm{T}) \otimes \mathbb{Q}) \bigotimes_{\mathrm{grd}(\mathrm{R}(\mathrm{G}) \otimes \mathbb{Q})} \mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{G}], \mathbb{Q}(\bullet)).$$

Observe that there is a natural conjugation action by $\mathrm{N}(T)$ on T, which induces a $\mathrm{W} = \mathrm{N}(T)/T$-action on $\pi_*\mathrm{K}([\mathrm{X}/\mathrm{T}])$ and on $\mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{T}], \mathbb{Q}(\bullet))$. Moreover $\mathrm{R}(\mathrm{T})^{\mathrm{W}} \cong \mathrm{R}(\mathrm{G})$. Therefore, taking the W-invariants of both sides, one obtains the isomorphism $\pi_*\mathrm{K}([\mathrm{X}/\mathrm{T}])^{\mathrm{W}} \cong \pi_*(\mathrm{K}([\mathrm{X}/\mathrm{G}]))$. At the level of absolute cohomology one obtains

$$\mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{T}], \mathbb{Q}(\bullet))^W \cong \mathrm{H}^*_{\mathrm{abs}}([\mathrm{X}/\mathrm{G}], \mathbb{Q}(\bullet)).$$

Example 8.0.3 (Hironaka's example). Here is a well-known example due to Hironaka; see [Knutson 1971, page 15]. Assume the base field is algebraically closed. (We may also assume the characteristic is 0 as in the original example of Hironaka.) Let V_0 be the projective 3-space and γ_1 and γ_2 two conics intersecting normally in exactly two points P_1 and P_2. For $i = 1, 2$, we construct $\overline{V}_i$ by blowing up first γ_i and then γ_{3-i} in the result. Let V_i be the open set in $\overline{V}_i$ of points lying over $(V_0 - P_{3-i})$. Let U be obtained by patching V_1 and V_2 together along the common open subset. Now U is a nonsingular variety and over P_1 and P_2 the curves γ_1 and

γ_2 have been blown up in opposite order. Let $\sigma_0 : V_0 \to V_0$ denote the projective transformation of order 2 that permutes P_1 and P_2 and γ_1 and γ_2. σ_0 induces an automorphism $\sigma : U \to U$ of order 2. Therefore we may take the finite group $G = \mathbb{Z}/2$ and let it act on U by the action of σ. In this case the geometric quotient U/G fails to exist in the category of schemes, but exists only in the category of algebraic spaces. Nevertheless Theorem 8.0.1 shows that the quotient stack [U/G] has the resolution property so that for each $n \geq 0$, $\pi_n(\mathrm{K}([\mathrm{U}/\mathrm{G}]))$ is a λ-ring.

Example 8.0.4. For the next example let $\mathcal{E}$ denote an elliptic curve. Then there are no nontrivial representations of $\mathcal{E}$ so that $\pi_*\mathrm{K}([\operatorname{Spec} k/\mathcal{E}]) \cong \pi_*\mathrm{K}(\operatorname{Spec} k)$. It follows that $\mathrm{H}^*_{\mathrm{abs}}([\operatorname{Spec} k/\mathcal{E}], \mathbb{Q}(\bullet)) \cong \mathrm{H}^*_{\mathrm{abs}}(\operatorname{Spec} k, \mathbb{Q}(\bullet))$.

8.1. ***Comparison with the higher equivariant Chow groups and further examples.*** The comparison with the higher equivariant Chow groups (in the sense of [Edidin and Graham 2000] or [Totaro 1999]) is much more involved in *general* than is possible in the examples considered above. This is due to the fact that the absolute cohomology for algebraic stacks obtained above is a *Bredon-style* cohomology theory in the sense of [Joshua 2007]. In the case of quotient stacks this is related to the more familiar equivariant higher Chow groups defined by making use of a Borel construction (as in [Edidin and Graham 2000; Totaro 1999]) by a completion at the augmentation ideal of the representation ring of the given linear algebraic group. However, such a completion is not an exact functor in general, unless the modules that one considers are finitely generated over the representation ring. In fact it is shown in [Carlsson and Joshua 2023] that one needs to apply a *derived completion* to pass from the Algebraic K-theory of smooth quotient stacks to the algebraic K-theory of the corresponding Borel construction. One may apply results of [Levine 1997] to the latter to define γ-operations and a form of absolute cohomology theory, which will then identify with the equivariant higher Chow groups with rational coefficients, as in [Edidin and Graham 2000] or [Totaro 1999].

Appendix A. Key theorems of Waldhausen K-theory

Definition A.0.1 [Thomason and Trobaugh 1990, 1.2.1]. A *category with cofibrations* $\boldsymbol{A}$ is a category with a zero object 0, together with a chosen subcategory $\mathrm{co}(\boldsymbol{A})$ satisfying the following axioms: (i) any isomorphism in $\boldsymbol{A}$ is a morphism in $\mathrm{co}(\boldsymbol{A})$, (ii) for every object $A \in \boldsymbol{A}$, the unique map $0 \to A$ belongs to $\mathrm{co}(\boldsymbol{A})$ and (iii) morphisms in $\mathrm{co}(\boldsymbol{A})$ are closed under cobase change by arbitrary maps in $\boldsymbol{A}$. The morphisms of $\mathrm{co}(\boldsymbol{A})$ are *cofibrations*. A category with fibrations is a category with a zero -object so that the dual category $\boldsymbol{A}^o$ is a category with cofibrations. A category with cofibrations and weak equivalences (or a *Waldhausen category*) is a category with cofibrations, $\mathrm{co}(\boldsymbol{A})$ together with a subcategory $w(\boldsymbol{A})$ so that the following conditions are satisfied: (i) any isomorphism in $\boldsymbol{A}$ belongs to $w(\boldsymbol{A})$, (ii) if

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

is a commutative diagram with the vertical maps all weak equivalences and the horizontal maps in the left square are cofibrations, then the induced map $B\bigsqcup_A C \to B'\bigsqcup_{A'} C'$ is also a weak equivalence. (iii) If f, g are two composable morphisms in $w(\boldsymbol{A})$ and two of the three f, g and $f \circ g$ are in $w(\boldsymbol{A})$, then so is the third. A functor $F : \boldsymbol{A} \to \boldsymbol{B}$ between categories with cofibrations and weak equivalences is *exact* if it preserves cofibrations and weak equivalences.

Given a Waldhausen category $(\boldsymbol{A}, \mathrm{co}(\boldsymbol{A}), w(\boldsymbol{A}))$, one associates to it the following simplicial category denoted $wS_\bullet \boldsymbol{A}$; see [Thomason and Trobaugh 1990, 1.5.1 Definition]. The objects of the category $w\mathrm{S}_n\boldsymbol{A}$ are sequences of cofibrations $A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$ in $\mathrm{co}(\boldsymbol{A})$ together with the choice of a quotient $A_{i,j} = A_j/A_i$ for each $i < j$ above. (The understanding is that $w\mathrm{S}_0\boldsymbol{A}$ is the category consisting of just the zero object 0.) The morphisms between two such objects $A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$ and $B_1 \rightarrowtail B_2 \rightarrowtail \cdots \rightarrowtail B_n$ are compatible collections of maps $A_{i,j} \to B_{i,j}$ in $w\boldsymbol{A}$. Varying n, one obtains the simplicial category $w\mathrm{S}_\bullet\boldsymbol{A}$ as discussed in [loc. cit., 1.5.1 Definition].

Such a Waldhausen category is *pseudoadditive* (see [Gunnarsson et al. 1992, Definition 2.3]) if for each cofibration $A \rightarrowtail C$, the induced maps $C \oplus_A C \to C \times C/A \leftarrow C \oplus C/A$ are weak equivalences. As pointed out earlier when $A \rightarrowtail C$ is a degree-wise split injective map of complexes of sheaves of $\mathcal{O}$-modules, for a sheaf of commutative Noetherian rings with 1 (on any site with enough points), it is easy to see that the maps $C \oplus_A C \to C \times C/A \leftarrow C \oplus C/A$ are isomorphisms in each degree.

The only categories with cofibrations and weak equivalences considered in this paper are *complicial* Waldhausen categories in the sense of [Thomason and Trobaugh 1990, 1.2.11]: in this situation the category $\boldsymbol{A}$ will be a full additive subcategory of the category of chain complexes with values in some abelian category. The cofibrations will be assumed to be maps of chain complexes that split degree-wise and weak equivalences will contain all quasiisomorphisms. All the complicial Waldhausen categories we consider will be closed under the formation of the canonical homotopy pushouts and homotopy pull-backs as in [loc. cit., 1.9.6, 1.2.11]. Therefore, all such categories with cofibrations and weak equivalences are pseudoadditive.

Definition A.0.2. Given a category $\boldsymbol{A}$ with cofibrations and weak equivalences that is pseudoadditive, we define its K-theory space to be given by the simplicial set $w\mathrm{G}_\bullet(\boldsymbol{A})$, where $\mathrm{G}_\bullet$ denotes the G-construction discussed in [Gunnarsson et al. 1992, Definition 2.2]; see also [Gillet and Grayson 1987, Section 3].

Then one of the main results of [Gunnarsson et al. 1992] is the following:

Theorem A.0.3 [Gunnarsson et al. 1992, Theorem 2.6]. *If $\boldsymbol{A}$ is pseudoadditive, there exists a natural map $w\mathrm{G}_\bullet(\boldsymbol{A}) \to \Omega w\mathrm{S}_\bullet \boldsymbol{A}$ that is a weak equivalence, where $w\mathrm{S}_\bullet \boldsymbol{A}$ is the simplicial set defined by the Waldhausen $\mathrm{S}_\bullet$-construction as in* [Waldhausen 1985, 1.3].

In view of the above weak equivalence, various key results proved in [Waldhausen 1985] extend readily to the K-theory spaces of complicial Waldhausen categories. We state these below.

Theorem A.0.4 (the Waldhausen approximation theorem; see [Thomason and Trobaugh 1990, 1.9.8]). *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ denote an exact functor between two complicial Waldhausen categories. Suppose F induces an equivalence of the derived categories $w^{-1}(\boldsymbol{A})$ and $w^{-1}(\boldsymbol{B})$. Then F induces a weak-homotopy equivalence of the associated K-theory spaces*, $\mathrm{K}(\boldsymbol{A})$ *and* $\mathrm{K}(\boldsymbol{B})$.

Theorem A.0.5 (localization theorem; see [Thomason and Trobaugh 1990, 1.8.2] and [Waldhausen 1985, 1.6.4]). *Let $\boldsymbol{A}$ be a small category with cofibrations and provided with two subcategories of weak equivalences $v(\boldsymbol{A}) \subseteq w(\boldsymbol{A})$ so that both $(\boldsymbol{A}, \mathrm{co}(\boldsymbol{A}), v(\boldsymbol{A}))$ and $(\boldsymbol{A}, \mathrm{co}(\boldsymbol{A}), w(\boldsymbol{A}))$ are complicial Waldhausen categories (as in* [Thomason and Trobaugh 1990, Section 1].) *Let $\boldsymbol{A}^w$ denote the full subcategory of $\boldsymbol{A}$ of objects A for which $0 \to A$ is in $w(\boldsymbol{A})$, that is, are w-acyclic. This is a Waldhausen category with $\mathrm{co}(\boldsymbol{A}^w) = \mathrm{co}(\boldsymbol{A}) \cap \boldsymbol{A}^w$ and $v(\boldsymbol{A}^w) = v(\boldsymbol{A}) \cap \boldsymbol{A}^w$. Then one obtains the fibration sequence of K-theory spaces*: $\mathrm{K}(v\boldsymbol{A}^w) \to \mathrm{K}(v(\boldsymbol{A})) \to \mathrm{K}(w(\boldsymbol{A}))$.

Theorem A.0.6 (additivity theorem; see [Waldhausen 1985, 1.3.2, 1.4.2] and [Gunnarsson et al. 1992, Theorem 2.10]). *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be small complicial Waldhausen categories. Let $F, F', F'' : \boldsymbol{A} \to \boldsymbol{B}$ be three exact functors so that there are natural transformations $F' \to F$ and $F \to F''$ so that* (*i*) *for all A in $\boldsymbol{A}$, $F'(A) \to F(A)$ is a cofibration with its cofiber $\cong F''(A)$ and* (*ii*) *for any cofibration $A' \to A$ in $\boldsymbol{A}$, the induced map $F'(A) \bigsqcup_{F'(A')} F(A') \to F(A)$ is a cofibration. Then the induced maps KF, KF' and KF'' on K-theory spaces have the property that $KF \simeq KF' + KF''$.*

Proof. As we make strong use of the above additivity theorem, we will explain how to deduce the above form of the additivity theorem from the form of the additivity theorem proven in [Gunnarsson et al. 1992, Theorem 2.10]. Recall that [loc. cit., Theorem 2.10] says the following: given a complicial Waldhausen category $\boldsymbol{A}$, let $\mathrm{E}(\boldsymbol{A})$ denote the Waldhausen category whose objects are short exact sequences

$$A \rightarrowtail B \rightarrowtail C$$

which are degree-wise split.

A cofibration from $A' \rightarrowtail B' \twoheadrightarrow C'$ to $A \rightarrowtail B \twoheadrightarrow C$ will be a commutative diagram:

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail] \\ A \arrow[r, tail] & B \arrow[r, two heads] & C \end{tikzcd}$$

so that the vertical maps are all cofibrations in $\boldsymbol{A}$ and the induced map $A \bigsqcup_{A'} B' \to B$ is also a cofibration in $\boldsymbol{A}$. Then, we have a functor

$$\mathrm{E}(\boldsymbol{A}) \to \boldsymbol{A} \times \boldsymbol{A}, \quad A \rightarrowtail B \twoheadrightarrow C \mapsto A \oplus C. \tag{A.1}$$

Then [Gunnarsson et al. 1992, Theorem 21.0] shows that the map $w\mathrm{G}_\bullet(E(\boldsymbol{A})) \to w\mathrm{G}_\bullet(\boldsymbol{A}) \times w\mathrm{G}_\bullet(\boldsymbol{A})$ is a weak equivalence. Now, one may deduce Theorem A.0.6 from the above form of the additivity theorem, by the same strategy adopted in [Waldhausen 1985, Proposition 1.3.2]: giving three exact functors F', F and F'' as in Theorem A.0.6 is equivalent to giving an exact functor $\tilde{F} : \boldsymbol{A} \to E(\boldsymbol{B})$. Therefore, Theorem A.0.6 above follows from the additivity theorem [Gunnarsson et al. 1992, Theorem 2.10] by naturality. □

Appendix B. Simplicial objects, cosimplicial objects and chain complexes

In this section a chain complex (a cochain complex) will denote a complex trivial in negative degrees and where the differentials are all of degree -1 ($+1$, respectively). Let $\boldsymbol{A} = \mathrm{Mod}(\mathcal{S}, \mathcal{O}_\mathcal{S})$ denote the Abelian category of all modules over $\mathcal{O}_\mathcal{S}$ where $\mathcal{S}$ is a given algebraic stack (which, as always in this paper, is assumed to be Noetherian). Let $\mathrm{Mod}_{fl}(\mathcal{S}, \mathcal{O}_\mathcal{S})$ denote the full subcategory of flat modules with finitely generated stalks. Recall that one has normalization functors N_h: (Simplicial objects in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_\mathcal{S})$) $\to$(Chain complexes in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_\mathcal{S})$) and its inverse DN_h:((Chain complexes in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_\mathcal{S})$) $\to$(Simplicial objects in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_\mathcal{S})$).[8] Recall N_h is defined by sending the simplicial object $\mathrm{S}_\bullet$ to the chain complex $\mathrm{K}_\bullet = \mathrm{N}(S_\bullet)$ defined by $\mathrm{K}_n = \bigcap_{0 \le i \le n-1}(\ker(d_i : S_n \to S_{n-1}))$. The differential $\delta : \mathrm{K}_n \to \mathrm{K}_{n-1}$ is defined by $\delta = (-1)^n d_n$. The functor DN_h is defined by $\mathrm{DN}_h(\mathrm{K}_\bullet)_n = \bigoplus_{0 \le m \le n} \bigoplus_{s_\alpha : [n] \to [m]} \mathrm{K}_m$ where the s_α range over all iterated degeneracies $[n] \to [m]$ in the category Δ; see [Curtis 1971] for more details. There are corresponding functors defined between the categories of cosimplicial objects in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_\mathcal{S})$ and cochain complexes in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_\mathcal{S})$. These will be denoted N^v and DN^v. Given a double complex $\mathrm{K}_\bullet^\bullet$ trivial everywhere except the second quadrant (that is, we assume $\mathrm{K}_i^j = 0$ for $i > 0$ or $j < 0$), we let $\mathrm{Tot}(\mathrm{K}_\bullet^\bullet)$ denote the complex defined by $\mathrm{Tot}(\mathrm{K}_\bullet^\bullet)^n = \bigoplus_{i+j=n} \mathrm{K}_i^j$. We

[8] In the literature, the inverse functor DN_h is often denoted K. However, as we have reserved K to denote complexes, our choice of DN_h seems preferable.

will often use N (DN) to denote either one of N_h or N^v (DN_h or DN^v, respectively) when there is no chance for confusion.

Proposition B.0.1. (i) *The functors* N_h *and* DN_h *are strict inverses of each other, that is,* $\mathrm{N}_h \circ \mathrm{DN}_h = \mathrm{id}$ *and* $\mathrm{DN}_h \circ N_h = \mathrm{id}$. *Similarly,* $\mathrm{N}^v \circ \mathrm{DN}^v = \mathrm{id}$ *and* $\mathrm{DN}^v \circ \mathrm{N}^v = \mathrm{id}$.

(ii) *The functors* N *and* DN *associated to both simplicial and cosimplicial objects in* $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ *preserve degree-wise flatness and the property of having finitely generated stalks. They also commute with filtered colimits.*

(iii) *The functors* N *and* DN *associated to both simplicial and cosimplicial objects in* $\mathrm{Mod}_{fl}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ *commute with the pull-back* $f^* : \mathrm{Mod}_{fl}(\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \to \mathrm{Mod}_{fl}(\mathcal{S}', \mathcal{O}_{\mathcal{S}'})$ *associated to a map* $f : \mathcal{S}' \to \mathcal{S}$ *of algebraic stacks.*

Proof. (i) Is a standard result and is therefore skipped; see [Curtis 1971] for the simplicial case. We prove (ii) first in the simplicial case. Let $\mathrm{S}_\bullet$ denote a simplicial object in $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ where each S_n is a flat $\mathcal{O}_{\mathcal{S}}$-module with finitely generated stalks. We will now prove, using ascending induction on n that each $\mathrm{K}_n = \mathrm{N}(S_\bullet)_n$ is a flat $\mathcal{O}_{\mathcal{S}}$-module. Since $\mathrm{K}_0 = \mathrm{N}(\mathrm{S}_\bullet)_0 = S_0$ this is clear for $n = 0$. The general case follows from Lemma B.0.2 below. The definition of the functor DN as a sum in each degree shows that it preserves flatness. The situation for the cosimplicial objects and cochain complexes is entirely similar and is therefore skipped.

Since the functors DN for simplicial and cosimplicial objects are defined as iterated sums, it is clear f^* commutes with DN. The functor N for cosimplicial objects is defined as an iterated cokernel and therefore it commutes with f^*. The corresponding assertion for simplicial objects follows from the lemma below. □

Lemma B.0.2. (i) *Let* $\mathrm{S}_\bullet$ *denote a simplicial object in* $\mathrm{Mod}(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ *that is flat (with finitely generated stalks) in each degree. Then for each integer* $n \geq 1$, *and* $0 \leq m \leq n-1$, $\bigcap_{0 \leq i \leq m} (\ker d_i : \mathrm{S}_n \to \mathrm{S}_{n-1})$ *is a flat* $\mathcal{O}_{\mathcal{S}}$*-module (with finitely generated stalks).*

(ii) *Let* $f : \mathcal{S}' \to \mathcal{S}$ *denote a map of algebraic stacks, let* $x' : X' \to \mathcal{S}'$ ($x : X \to \mathcal{S}$) *denote an atlas with* $B_{x'}\mathcal{S}'$ ($B_x\mathcal{S}$, *respectively*) *denoting the associated simplicial classifying space. Assume the atlases are chosen so that there is an induced map of simplicial algebraic spaces* $Bf : B_{x'}\mathcal{S}' \to B_x\mathcal{S}$. *Let* $\mathrm{S}_\bullet$ *denote a simplicial object in* $\mathrm{Mod}(B_x\mathcal{S}, \mathcal{O}_{B_x\mathcal{S}})$ *that is flat in each degree. Then for each integer* $n \geq 1$, *and* $0 \leq m \leq n-1$, $f^*\big(\bigcap_{0 \leq i \leq m} (\ker d_i : \mathrm{S}_n \to \mathrm{S}_{n-1})\big) \cong \bigcap_{0 \leq i \leq m} (\ker d_i : f^*(\mathrm{S}_n) \to f^*(\mathrm{S}_{n-1}))$.

Proof. As observed by B. Koeck, this Lemma may be readily proven by observing that $\mathrm{N}(\mathrm{S}.)_n$ is a direct summand of S_n and that $\mathrm{N}(\mathrm{S}.)_n$ arises by taking the quotient of S_n by the image of the degeneracy maps. We may also prove (i) and (ii) simultaneously using ascending induction on m making use of the above observation. We skip the remaining details. □

B.1. ***The Eilenberg–Zilber and Alexander–Whitney pairings.*** These are well-known between Chain complexes in any abelian category and the corresponding simplicial objects; see [May 1967, page 129 and page 133]. These readily extend to similar pairings for cochain complexes and cosimplicial objects in any abelian category: for example, one may interpret cosimplicial objects in an abelian category as simplicial objects in the dual abelian category and make use of the well-known pairings for simplicial objects and chain complexes. Therefore, such pairings extend to similar pairings between cosimplicial-simplicial objects in an abelian category $\boldsymbol{A}$ and the corresponding category of cochain complexes in $\boldsymbol{A}$, in the setting of Section 5. In more detail, we obtain the following.

Let $\boldsymbol{A}$ denote an abelian category and let Double($\boldsymbol{A}$) denote the category of double cochain complexes in $\boldsymbol{A}$ concentrated in the second quadrant. Given such a double cochain complex K, and applying the composite functor $\mathrm{DN}^v \circ \mathrm{DN}_h$ produces a cosimplicial-simplicial object. The category of such objects will be denoted Cos.mixt($\boldsymbol{A}$). The inverse functor,

$$\mathrm{N} = \mathrm{N}^v \circ \mathrm{N}_h = \mathrm{N}_h \circ \mathrm{N}^v \tag{B.2}$$

sends such a cosimplicial simplicial object to a double cochain complex concentrated in the second quadrant. Then we obtain associative pairings

$$\begin{gathered} \mathrm{Tot}(\mathrm{N}(P)) \otimes \mathrm{Tot}(\mathrm{N}(Q)) \to \mathrm{Tot}(\mathrm{N}(P \otimes Q)) \\ \mathrm{Tot}(\mathrm{N}(P \otimes Q)) \to \mathrm{Tot}(\mathrm{N}(P)) \otimes \mathrm{Tot}(\mathrm{N}(Q)) \end{gathered} \tag{B.3}$$

for any two objects P, Q $\in$ Cos.mixt($\boldsymbol{A}$) which are both functorial in P and Q. Here Tot denotes the total complex: for a double cochain complex $\mathrm{K} = \{\mathrm{K}^i_j \mid i \geq 0,\ j \leq 0\}$ concentrated in the second quadrant,

$$\mathrm{Tot}(\mathrm{K})^n = \bigoplus_{i+j=n} \mathrm{K}^i_j. \tag{B.4}$$

Acknowledgments

We would like to thank Zig Fiedorowicz, (the late) Rainer Vogt and an anonymous reviewer for various discussions related to a much earlier draft of the paper. We also owe an enormous thanks to Dan Grayson for undertaking a critical reading of the paper, suggesting several improvements in exposition, and also for several e-mail exchanges that clarified the Gillet–Grayson $G_\bullet$-construction to us. We also thank Bernhard Koeck for undertaking a critical reading and providing several helpful comments which we have incorporated into the current version. Finally the authors are very grateful to the referee for undertaking a careful and detailed reading of the manuscript and for making a number of very helpful comments and suggestions which surely have improved the paper.

References

[Atiyah 1961] M. F. Atiyah, "Characters and cohomology of finite groups", *Inst. Hautes Études Sci. Publ. Math.* 9 (1961), 23–64. MR Zbl

[Atiyah and Tall 1969] M. F. Atiyah and D. O. Tall, "Group representations, λ-rings and the J-homomorphism", *Topology* **8** (1969), 253–297. MR Zbl

[Bloch and Lichtenbaum 1997] S. Bloch and S. Lichtenbaum, "A spectral sequence for motivic cohomology", preprint, 1997, https://faculty.math.illinois.edu/K-theory/0062/motspseq.pdf. Zbl

[Brenner and Schröer 2003] H. Brenner and S. Schröer, "Ample families, multihomogeneous spectra, and algebraization of formal schemes", *Pacific J. Math.* **208**:2 (2003), 209–230. MR Zbl

[Carlsson and Joshua 2023] G. Carlsson and R. Joshua, "Equivariant algebraic K-theory, G-theory and derived completions", *Adv. Math.* **430** (2023), art. id. 109194. MR Zbl

[Curtis 1971] E. B. Curtis, "Simplicial homotopy theory", *Advances in Math.* **6** (1971), 107–209. MR Zbl

[Dold and Puppe 1961] A. Dold and D. Puppe, "Homologie nicht-additiver Funktoren: Anwendungen", *Ann. Inst. Fourier* (*Grenoble*) **11** (1961), 201–312. MR Zbl

[Edidin and Graham 2000] D. Edidin and W. Graham, "Riemann–Roch for equivariant Chow groups", *Duke Math. J.* **102**:3 (2000), 567–594. MR Zbl

[Edidin et al. 2001] D. Edidin, B. Hassett, A. Kresch, and A. Vistoli, "Brauer groups and quotient stacks", *Amer. J. Math.* **123**:4 (2001), 761–777. MR Zbl

[Edidin et al. 2017] D. Edidin, T. J. Jarvis, and T. Kimura, "Chern classes and compatible power operations in inertial K-theory", *Ann. K-Theory* **2**:1 (2017), 73–130. MR Zbl

[Fulton and Lang 1985] W. Fulton and S. Lang, *Riemann–Roch algebra*, Grundl. Math. Wissen. **277**, Springer, 1985. MR Zbl

[Gabriel and Zisman 1967] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas] **Band 35**, Springer, 1967. MR Zbl

[Gillet and Grayson 1987] H. Gillet and D. R. Grayson, "The loop space of the Q-construction", *Illinois J. Math.* **31**:4 (1987), 574–597. MR Zbl

[Gillet and Soulé 1987] H. Gillet and C. Soulé, "Intersection theory using Adams operations", *Invent. Math.* **90**:2 (1987), 243–277. MR Zbl

[Gillet and Soulé 1999] H. Gillet and C. Soulé, "Filtrations on higher algebraic K-theory", pp. 89–148 in *Algebraic K-theory* (Seattle, WA, 1997), edited by W. Raskind and C. Weibel, Proc. Sympos. Pure Math. **67**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Grayson 1989] D. R. Grayson, "Exterior power operations on higher K-theory", *K-Theory* **3**:3 (1989), 247–260. MR Zbl

[Grayson 2012] D. R. Grayson, "Algebraic K-theory via binary complexes", *J. Amer. Math. Soc.* **25**:4 (2012), 1149–1167. MR Zbl

[Gunnarsson and Schwänzl 2002] T. Gunnarsson and R. Schwänzl, "Operations in A-theory", *J. Pure Appl. Algebra* **174**:3 (2002), 263–301. MR Zbl

[Gunnarsson et al. 1992] T. Gunnarsson, R. Schwänzl, R. M. Vogt, and F. Waldhausen, "An undelooped version of algebraic K-theory", *J. Pure Appl. Algebra* **79**:3 (1992), 255–270. MR Zbl

[Harris et al. 2017] T. Harris, B. Köck, and L. Taelman, "Exterior power operations on higher K-groups via binary complexes", *Ann. K-Theory* **2**:3 (2017), 409–449. MR Zbl

[Illusie 1971] L. Illusie, *Complexe cotangent et déformations*, *I*, Lecture Notes in Mathematics **239**, Springer, 1971. MR Zbl

[Iversen 1986] B. Iversen, *Cohomology of sheaves*, Springer, 1986. MR Zbl

[Joshua 2001] R. Joshua, "Algebraic K-theory and higher Chow groups of linear varieties", *Math. Proc. Cambridge Philos. Soc.* **130**:1 (2001), 37–60. MR Zbl

[Joshua 2003] R. Joshua, "Riemann–Roch for algebraic stacks, I", *Compositio Math.* **136**:2 (2003), 117–169. MR Zbl

[Joshua 2007] R. Joshua, "Bredon-style homology, cohomology and Riemann–Roch for algebraic stacks", *Adv. Math.* **209**:1 (2007), 1–68. MR Zbl

[Joshua 2012] R. Joshua, "K-theory and G-theory of DG-stacks", pp. 175–217 in *Regulators*, edited by J. I. Burgos Gil et al., Contemp. Math. **571**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Knutson 1971] D. Knutson, *Algebraic spaces*, Lecture Notes in Mathematics **203**, Springer, 1971. MR Zbl

[Köck 1998] B. Köck, "The Grothendieck–Riemann–Roch theorem for group scheme actions", *Ann. Sci. École Norm. Sup.* (4) **31**:3 (1998), 415–458. MR Zbl

[Köck and Zanchetta 2021] B. Köck and F. Zanchetta, "Comparison of exterior power operations on higher K-theory of schemes", 2021. arXiv 2110.06599v1

[Kratzer 1980a] C. Kratzer, "λ-structure en K-théorie algébrique", *Comment. Math. Helv.* **55**:2 (1980), 233–254. MR Zbl

[Kratzer 1980b] C. Kratzer, "Opérations d'Adams et représentations de groupes", *Enseign. Math.* (2) **26**:1-2 (1980), 141–154. MR Zbl

[Laumon and Moret-Bailly 2000] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Math. (3) **39**, Springer, 2000. MR Zbl

[Lecomte 1998] F. Lecomte, "Opérations d'Adams en K-théorie algébrique", *K-Theory* **13**:2 (1998), 179–207. MR Zbl

[Levine 1997] M. Levine, "Lambda-operations, K-theory and motivic cohomology", pp. 131–184 in *Algebraic K-theory* (Toronto, ON, 1996), edited by V. P. Snaith, Fields Inst. Commun. **16**, Amer. Math. Soc., Providence, RI, 1997. MR Zbl

[May 1967] J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand Mathematical Studies **No. 11**, Van Nostrand, Princeton, NJ, 1967. MR Zbl

[Merkurjev 1997] A. S. Merkurjev, "Comparison of the equivariant and the standard K-theory of algebraic varieties", *Algebra i Analiz* **9**:4 (1997), 175–214. In Russian. MR

[Morel and Voevodsky 1999] F. Morel and V. Voevodsky, "$\mathbf{A}^1$-homotopy theory of schemes", *Inst. Hautes Études Sci. Publ. Math.* 90 (1999), 45–143. MR Zbl

[Olsson 2007] M. Olsson, "Sheaves on Artin stacks", *J. Reine Angew. Math.* **603** (2007), 55–112. MR Zbl

[Riou 2010] J. Riou, "Algebraic K-theory, $\mathbf{A}^1$-homotopy and Riemann–Roch theorems", *J. Topol.* **3**:2 (2010), 229–264. MR Zbl

[Serre 1968] J.-P. Serre, "Groupes de Grothendieck des schémas en groupes réductifs déployés", *Inst. Hautes Études Sci. Publ. Math.* 34 (1968), 37–52. MR Zbl

[SGA 6 1971] A. Grothendieck, P. Berthelot, and L. Illusie, *Théorie des intersections et théorème de Riemann–Roch* (Séminaire de Géométrie Algébrique du Bois Marie 1966–1967), Lecture Notes in Math. **225**, Springer, 1971. MR Zbl

[Soulé 1992] C. Soulé, *Lectures on Arakelov geometry*, Cambridge Studies in Advanced Mathematics **33**, Cambridge University Press, 1992. MR Zbl

[Stacks 2005–] "The Stacks project", electronic reference, 2005–, http://stacks.math.columbia.edu. Zbl

[Thomason 1986] R. W. Thomason, "Comparison of equivariant algebraic and topological K-theory", *Duke Math. J.* **53**:3 (1986), 795–825. MR Zbl

[Thomason 1987a] R. W. Thomason, "Algebraic K-theory of group scheme actions", pp. 539–563 in *Algebraic topology and algebraic K-theory* (Princeton, NJ, 1983), edited by W. Browder, Ann. of Math. Stud. **113**, Princeton Univ. Press, 1987. MR Zbl

[Thomason 1987b] R. W. Thomason, "Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes", *Adv. in Math.* **65**:1 (1987), 16–34. MR Zbl

[Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, "Higher algebraic K-theory of schemes and of derived categories", pp. 247–435 in *The Grothendieck Festschrift*, *III*, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

[Totaro 1999] B. Totaro, "The Chow ring of a classifying space", pp. 249–281 in *Algebraic K-theory* (Seattle, WA, 1997), edited by W. Raskind and C. Weibel, Proc. Sympos. Pure Math. **67**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Totaro 2004] B. Totaro, "The resolution property for schemes and stacks", *J. Reine Angew. Math.* **577** (2004), 1–22. MR Zbl

[Waldhausen 1985] F. Waldhausen, "Algebraic K-theory of spaces", pp. 318–419 in *Algebraic and geometric topology* (New Brunswick, NJ, 1983), edited by A. Ranicki et al., Lecture Notes in Math. **1126**, Springer, 1985. MR Zbl

[Weibel 2013] C. A. Weibel, *The K-book*, Graduate Studies in Mathematics **145**, American Mathematical Society, Providence, RI, 2013. MR Zbl

[Zanchetta 2021] F. Zanchetta, "Unstable operations on K-theory for singular schemes", *Adv. Math.* **384** (2021), art. id. 107716. MR Zbl

Received 3 Sep 2023. Revised 15 Apr 2024. Accepted 14 Jul 2024.

ROY JOSHUA: joshua.1@math.osu.edu
Department of Mathematics, Ohio State University, Columbus, OH, United States

PABLO PELAEZ: pablo.pelaez@im.unam.mx
Instituto de Matemáticas, IMATE, UNAM, Ciudad Universitaria, Ciudad de México, Mexico

ANNALS OF K-THEORY

msp.org/akt

Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

See inside back cover or msp.org/akt for submission instructions.

The subscription price for 2024 is US $605/year for the electronic version, and $685/year (+$30, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

AKT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANNALS OF K-THEORY

2024 vol. 9 no. 3

Uniqueness of real ring spectra up to higher homotopy 447
JACK MORGAN DAVIES

Hochschild cohomology for functors on linear symmetric monoidal categories 475
NADIA ROMERO

An additive variant of the differential symbol maps 499
TOSHIRO HIRANOUCHI

λ-ring structures on the K-theory of algebraic stacks 519
ROY JOSHUA and PABLO PELAEZ