MOTIVIC $E_{\infty}$-ALGEBRAS AND THE MOTIVIC DGA

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Abstract. In this paper we define an $E_{\infty}$-structure, i.e. a coherently homotopy associative and commutative product on chain complexes defining (integral and mod-$l$) motivic cohomology as well as mod-$l$ étale cohomology. We also discuss several applications.

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1. Introduction

The main result of this paper is the existence of explicit $E_{\infty}$-structures on mod-$l$ étale cohomology (with $l$ different from the residue characteristics), on the motives (in the sense of [VV1]) of all smooth schemes and on the motivic complexes of all smooth schemes. Such results have often been implicitly assumed in the past: we make them explicit by constructing explicit operad actions on these complexes.

The following are some of the applications. First of all, the $E_{\infty}$-structure on motives enables us to readily obtain an $E_{\infty}$-structure on a complex defining étale cohomology, thereby establishing cohomology operations in étale cohomology and a dga structure on continuous $l$-adic étale cohomology. This structure is shown to be independent of a different $E_{\infty}$-structure we put on the motivic complex. Moreover, the two separate constructions shed light on the fact that the source of cohomology operations in topology (along with a similar one on étale cohomology) and on motivic cohomology are somewhat distinct. In the first case the existence of cohomology operations is a consequence of the fact that the passage from simplicial abelian groups to associated chain complexes is only homotopy commutative. In this case, this pairing involves the Alexander-Whitney map (from the chain complex associated to the tensor product of two simplicial abelian groups to the tensor product of the corresponding chain complexes) which is only homotopy commutative. In the motivic case, the intersection pairing on the motivic complexes is obtained from an external pairing of chain complexes followed by a pairing that involves only the shuffle maps which are strictly commutative. Therefore the source of the lack of strict commutativity for the product on the motivic complex is the lack of strict commutativity of the corresponding external pairing. In other words, our results show that the $E_{\infty}$-structure on the singular co-chain complex and on complexes defining étale cohomology are strictly chain-theoretic phenomena whereas the $E_{\infty}$-structure on the motivic complex is not. The $E_{\infty}$-structure on the motivic complex also allows the construction of an $E_{\infty}$-dga structure on the integral motivic complex and on the motivic complex with finite coefficients as well as a dga-structure on the rational motivic complex associated to any smooth scheme over a field. The latter has applications to the construction of categories of mixed Tate motives associated to a large class of schemes.

The following is an outline of the paper. The second section is devoted to setting up the basic framework for the rest of the paper while the third section is devoted to a detailed study of what we call Eilenberg-Zilber operads. We begin by considering the endomorphism operad in a general setting. As a motivating example for what follows,
we next consider, as a special case of this operad, the Eilenberg-Zilber operad well-known in algebraic topology. We then consider the analogous operad in the motivic case which we call the motivic Eilenberg-Zilber operad. The main results in this section may be summarized as follows. (See sections 2 and 3 for details and terminology.)

**Theorem 1.1.** There exists an operad, \( End(M_\Delta) \), that is acyclic in the derived category \( D((\text{sm.schms})_{Nis})_{\mathbb{A}^1} \) = the \( \mathbb{A}^1 \)-localization of the derived category of complexes on the big Nisnevich site of smooth schemes over \( k \).

(i) The motive of any smooth scheme \( X \) is a co-algebra over this operad.

(ii) Let \( l \) denote a fixed prime different from \( \text{char}(k) = p \) and let \( \nu > 0 \) be an integer. There exists an operad \( End(M_\Delta/\nu) \) that is acyclic in \( D((\text{sm.schms})_{et}) \) which is the derived category of complexes on the big étale site of smooth schemes over \( k \). The complex \( \mu_r[0] \) is an algebra over this operad.

As a consequence we obtain cohomology operations in étale cohomology and an \( E_\infty \)-structure on étale cohomology. These results show that the \( E_\infty \)-structure on étale cohomology is obtained in the same manner as in the case of singular cohomology.

In the fourth section, we consider two other operads which are variants of the well-known Barratt-Eccles operad. Then, in the fifth section we show that these operads act on the motivic complexes, i.e. the motivic complexes of smooth schemes are algebras over both these operads. In fact a key result in this paper is the explicit construction of this operad. This leads to the construction of the motivic dga and motivic \( E_\infty \)-dga. We summarize some of these results in following theorem.

**Theorem 1.2.** Let \( (\text{schms}/k) \) denote the category of smooth schemes of finite type over a field \( k \). Let \( l \) be a prime different from \( \text{char}(k) = p \) and let \( \nu > 0 \).

1. There exists functors
   \[
   \mathbb{Q}^{mot} : (\text{schms}/k)^{op} \to (D\text{-algebras over } \mathbb{Q}),
   \]
   \[
   \mathbb{Q}^{et} : (\text{schms}/k)^{op} \to (D\text{-algebras over } \mathbb{Q}_l)
   \]
   so that \( \mathbb{Q}^{mot} = \bigoplus_{r \geq 0} \mathbb{Q}^{mot, \nu}(r) \) and \( \mathbb{Q}^{et} = \bigoplus_{r \geq 0} \mathbb{Q}^{et, \nu}(r) \). If \( X \) is a quasi-projective smooth scheme over \( k \) and \( \mathbb{Q}^{mot}_X = \Gamma(X, \mathbb{Q}^{mot}) \), \( \mathbb{Q}^{et}_X = R\Gamma(X, \mathbb{Q}^{et}) \), \( H^q(\mathbb{Q}^{mot}_X(r)) = H^q_{\mathbb{A}^1}(X, \mathbb{Q}(r)) \) = the (rational) motivic cohomology of \( X \) in degree \( q \) and weight \( r \) while \( H^q(\mathbb{Q}^{et}_X(r)) = H^q_{\mathbb{A}^1}(X, \mathbb{Q}_l(r)) \) = the \( l \)-adic étale cohomology of \( X \) in degree \( q \) and weight \( r \).

2. More generally there exists functors
   \[
   \mathbb{Z}^{mot} : (\text{schms}/k)^{op} \to (E_\infty \text{-DGA}),
   \]
   \[
   \mathbb{Z}^{et}/\nu : (\text{schms}/k)^{op} \to (E_\infty \text{-DGA})
   \]
   so that \( \mathbb{Z}^{mot} = \bigoplus_{r \geq 0} \mathbb{Z}^{mot, \nu}(r) \) and \( \mathbb{Z}^{et}/\nu = \bigoplus_{r \geq 0} \mathbb{Z}/\nu^{et, \nu}(r) \). If \( X \) is a quasi-projective smooth scheme over \( k \) and \( \mathbb{Z}^{mot}_X = \Gamma(X, \mathbb{Z}^{mot}) \), \( \mathbb{Z}/\nu^{et, \nu}_X = R\Gamma(X, \mathbb{Z}/\nu^{et}) \), \( H^q(\mathbb{Z}^{mot}_X(r)) = H^q_{\mathbb{A}^1}(X, \mathbb{Z}(r)) \) = the motivic cohomology of \( X \) in degree \( q \) and weight \( r \) while \( H^q(\mathbb{Z}/\nu^{et, \nu}_X(r)) = H^q_{\mathbb{A}^1}(X, \mathbb{Z}/\nu^{et})(r) \) = the étale cohomology of \( X \) in degree \( q \) and weight \( r \).

Moreover, all the above functors are additive presheaves satisfying cohomological descent (as in definition 9.2) on the big Zariski and Nisnevich sites of smooth quasi-projective schemes over \( k \). The functors providing the étale dga and étale \( E_\infty \)-dga are presheaves satisfying cohomological descent on the big étale site as well.

With this key result at our disposal, one may make use of abstract arguments, like the existence of a model structure on the category of operads (provided first by Hinich and then extended to more general situations by Spitzweck) to provide actions by other operads. This is considered in section 6.

The rest of the paper is devoted to applications. In the seventh section, making use of the existence of the motivic dga, we construct a category of Tate motives for a large class of varieties that includes all projective smooth linear varieties over a field. In the eighth section, we show that the operad actions defined earlier lead to classical cohomology operations on motivic cohomology. The co-algebra structure for the motives of smooth schemes considered in section 3, does not seem to directly lead to the motivic operations of Voevodsky on motivic cohomology: this is because of the rather complicated relation between motivic homology and cohomology.
Acknowledgments. This has been a rather long project for us, partly because the area of operads had been new to us when we embarked on this project. We first began work on this ourselves, then enlisted the help of Peter May with whom we had a great deal of correspondence for about a year on the subject matter of the paper. The expectation at that point was that the $E_{\infty}$-structures on the motivic complex and the singular chain complex would be entirely similar so that a suitable form of the Eilenberg-Zilber operad would provide the required $E_{\infty}$-structure. That this is not so became clear during this correspondence and we thank May for pointing that out. However, it also became clear during the correspondence that our styles and expertise are so disjoint that collaborating on this project would not be practical. We thank May for letting me finish this project as a single-authored paper. I would be happy to acknowledge that some of the insights and results, notably in sections 3.1, 3.2 and parts of 3.3 came from our correspondence of over a year on the subject matter of the paper.

In addition, we would like to thank Spencer Bloch, Patrick Brosnan, Zig Fiedorowicz, Herbert Gangl, Eric Friedlander, Markus Rost, Bertrand Toen and Vladimir Voevodsky for several helpful discussions/correspondence. The suggestion of Bloch to use the motivic complexes in the place of the higher cycle complex and the use of $\mathbb{A}^1$-homotopy theory of schemes as developed by Morel and Voevodsky (see [M-V]) have been crucial for us.

2. The basic framework

Throughout the paper $k$ will denote a fixed field of arbitrary characteristic $p \geq 0$. ($\text{smt.schms}$) will denote the category of smooth schemes of finite type over $k$. ($\text{smt.schms}_{zar}$, ($\text{smt.schms}_{et}$), ($\text{smt.schms}_{et}$)] will denote this category provided with the big Zariski, Nisnevich or étale topologies. Observe that these sites have enough points. We will let $S$ denote any one of these sites. $C(S)$ will denote the category of unbounded complexes of abelian sheaves on $S$; $C_{-}(S) (C_{0}(S))$ will denote the full sub-category of bounded above complexes (that are also trivial in positive degrees, respectively). For a fixed commutative ring $R$ with unit, $C(S, R) (C_{-}(S, R), C_{0}(S, R))$ will denote the corresponding categories of complexes of sheaves of $R$-modules. Observe that $C(S)$ and $C(S, R)$ are symmetric monoidal with product $\otimes$ and an internal Hom functor we denote by $\text{Hom}$. A map $f : K \to L$ in $C(S, R)$ is a quasi-isomorphism if the induced map by $f$ at each stalk is a quasi-isomorphism. $D(S, R) (D_{-}(S, R), D_{0}(S, R))$ will denote the corresponding derived categories obtained by inverting quasi-isomorphisms. When $R = \mathbb{Z}$, $D(S, R)$ will be denoted simply $D(S)$. Observe that $C_{-}(S, R)$ and $C_{0}(S, R)$ are not closed under the formation of the internal hom: this is the main reason for considering the category $C(S, R)$ of unbounded complexes in this paper.

It is well-known (see the appendix) that $C_{0}(S, R)$ is a cofibrantly generated closed model category, where a map $f : K \to L$ is a weak-equivalence (cofibration) if it induces an isomorphism on cohomology sheaves (is an injection stalk-wise in each degree, respectively). Proposition 9.1 in the appendix shows that an analogue of it defines the following cofibrantly generated model structure on $C(S, R)$:

cofibrations are injections, weak-equivalences are stalk-wise weak-equivalences and fibrations are defined by the right lifting property with respect to trivial cofibrations. The cofibrations (trivial cofibrations) $i : K \to L$ are generated by those cofibrations (trivial cofibrations, respectively) for which there exists a large enough cardinal $\alpha$ so that the cardinality of $\Gamma(U, L)$ is less than $\alpha$ for all smooth schemes $U$. One may then observe that $GK$ is a fibrant object for any $K \in C(S, R)$: here $G$ denotes the homotopy limit of the Godement resolution as in Proposition 9.1.

Let $SP_{\text{sh}}(S)$ denote the category of of all additive simplicial presheaves on the site $S$, i.e. presheaves of simplicial sets. The free $R$-module functor sends simplicial presheaves on $S$ to simplicial abelian presheaves and applying the normalization functor one obtains a complex of abelian presheaves. This defines a functor, we denote by $N \circ Z : SP_{\text{sh}}(S) \to C_{0}(S, R)$.

An $\mathbb{A}^1$-homotopy $H : P \otimes \mathbb{A}^1 \to Q$ between two simplicial presheaves $P$ and $Q$ induces a homotopy $NZ(H) : NZ(P) \otimes NZ(\mathbb{A}^1) \to NZ(Q)$. We say two maps $f, g : K \to L$ in $D(S, R)$ are $\mathbb{A}^1$-homotopic if there is map $H : K \otimes NZ(\mathbb{A}^1) \to L$ so that $H \circ i_{0} = f$ and $H \circ i_{1} = g$; here $i_{0}, i_{1} : K \otimes NZ(\text{Spec } k) \to K \otimes NZ(\mathbb{A}^1)$ are the obvious maps corresponding to 0 and 1 in $\mathbb{A}^1$. One may define a map $f : K \to L$ to be an $\mathbb{A}^1$-equivalence as in (9.7); $D(S, R)_{\mathbb{A}^1}$ will denote the localization of the derived category $D(S, R)$ with respect to morphisms that are $\mathbb{A}^1$-equivalences.

2.1. Convention. Given a product of $n$ complexes $K(1), \cdots, K(n)$ in $C(S, R)$, any permutation $\sigma \in \Sigma_{n}$ induces a map $K(1) \otimes \cdots \otimes K(n) \to K(1) \otimes \cdots \otimes K(n)$: this will be simply denoted $\sigma$ though in each degree, it involves also multiplication by an appropriate sign.

Remark 2.1. The motivic complexes are complexes of abelian sheaves which are only bounded above in general, and since the paper is centered around these, it seems preferable not to work with general simplicial presheaves.
3. Endomorphism operads

Let $K \in C(S, R)$. Then we define endomorphism operads $\mathcal{E}nd^K$ and $\mathcal{E}nd_K$ as follows. $\mathcal{E}nd^K(n) = \text{Hom}(K^{\otimes n}, K)$ and $\mathcal{E}nd_K(n) = \text{Hom}(K, K^{\otimes n})$. The structure morphism $\theta^n_K : \mathcal{E}nd^K(n) \otimes \mathcal{E}nd^K(k_1) \otimes \cdots \otimes \mathcal{E}nd^K(k_n) \to \mathcal{E}nd^K(\Sigma k_i)$ is defined as the composition of $\text{Hom}(K^{\otimes k_1}, K) \otimes \cdots \otimes \text{Hom}(K^{\otimes k_n}, K) \to \text{Hom}(K^{\otimes k_1}, K) \otimes \cdots \otimes \text{Hom}(K^{\otimes k_n}, K)$ where both maps are the obvious ones. Similarly the structure morphism $\theta^n_{K, n} : \mathcal{E}nd_K(n) \otimes \mathcal{E}nd_K(k_1) \otimes \cdots \otimes \mathcal{E}nd_K(k_n) \to \mathcal{E}nd_K(\Sigma _k k_i)$ are defined in the obvious manner.

Let $D$ be any small category. For a pair of covariant functors $F, G : D \to C(S, R)$, define $\text{Hom}_D(F, G)$ to be the equalizer in $C(S, R)$ displayed in the diagram

\[
\begin{array}{c}
\text{Hom}_D(F, G) \\
\overset{\alpha \in D}{\longrightarrow} \prod \text{Hom}_D(F_d, G_d) \\
\underset{\alpha \in D}{\longrightarrow} \prod \text{Hom}_D(F_d, G_e)
\end{array}
\]

where the second product runs over all morphisms $\alpha$ of $D$. We are writing $F_d$ for the object $F(d)$ and writing $F_{\alpha} : F_d \to F_e$ for the morphism $F(\alpha)$. The parallel arrows send $(f_d \circ f_{\alpha}, \epsilon, \alpha)$ and to $(G_{\alpha} \circ f_d, \alpha, \epsilon)$. Thus the $\alpha$-th components of the parallel arrows are the compositions of the projections to the $\varepsilon$-th or the $d$-th component followed by $\text{Hom}(F_{\alpha}, id) : \text{Hom}(F_{\varepsilon}, G_e) \to \text{Hom}(F_d, G_e)$ or $\text{Hom}(id, G_{\alpha}) : \text{Hom}(F_d, G_{\varepsilon}) \to \text{Hom}(F_d, G_e)$.

Fix a functor $\Lambda : D \to C(S, R)$. We have the diagonal power functor $\Lambda^{\otimes n}$. Its sends $d$ to the $n$-fold $\otimes$-power $(\Lambda d)^{\otimes n}$. By convention, the 0-fold power of $\Lambda$ is the constant functor at $R$. We define the endomorphism operad $\mathcal{E}nd_\Lambda$ of the functor $\Lambda$ by letting

\[
\mathcal{E}nd_\Lambda(n) = \text{Hom}_D(\Lambda, \Lambda^{\otimes n})
\]

with $\mathcal{E}nd_\Lambda(0) = R$. The structure morphisms are defined as before. Similarly one defines an operad $\mathcal{E}nd^\Lambda$ by letting $\mathcal{E}nd^\Lambda(n) = \text{Hom}_D(\Lambda^{\otimes n}, \Lambda)$.

3.1. Co-algebras and algebras over operads. Let $\{O(n)|n\}$ denote an operad in $C(S, R)$. If $K \in C(S, R)$, a co-algebra structure (algebra structure) for $K$ over $\{O(n)|n\}$ is given by a morphism of operads $\{O(n)|n\} \to \{\mathcal{E}nd_K(n)|n\}$ ( $\{O(n)|n\} \to \{\mathcal{E}nd_K(n)|n\}$, respectively). Similarly, a co-algebra structure for a functor $X : D \to C(S, R)$ over the given endomorphism operad $\{\mathcal{E}nd_\Lambda(n)|n\}$ is given by a map of operads $\{\mathcal{E}nd_\Lambda(n)|n\} \to \{\mathcal{E}nd_\Lambda(n)|n\}$. Let $Y : D \to C(S, R)$ be a functor. An algebra structure for functors $Y$ over the operad $\{\mathcal{E}nd^\Lambda(n)|n\}$ is given by a map of operads $\{\mathcal{E}nd^\Lambda(n)|n\} \to \{\mathcal{E}nd^\Lambda(n)|n\}$.

3.2. The (topological) Eilenberg-Zilber operad. As motivation for our constructions on the motives of schemes, we show how the abstract theory of the last section specializes to give the standard Eilenberg-Zilber operad $\mathcal{E}Z$ that acts on the chain complex giving simplicial homology in algebraic topology (see [Sm]) making the latter a co-algebra over the Eilenberg-Zilber operad.

Fix a commutative ring $R$ and let $\mathcal{M}R$ be the category of $R$-modules and let $F : (\text{sets}) \to \mathcal{M}R$ be the free $R$-module functor. Let $\mathcal{C}M_R$ be the category of chain complexes of $R$-modules or $R$-co-chain complexes, with differentials of degree +1. The category $\mathcal{C}M_R$ will play the role of the category $C(S, R)$ as in the above discussion. Now we take $D = \Delta$. Now the functor $\Lambda$ will be the functor $n \mapsto C_n(\Delta[n], R)$ which is the singular complex on the standard topological $n$-simplex $|\Delta[n]|$. This will be denoted $C_n(\Delta[n], R)$.

Let $X$ denote a simplicial set and let $C_n(X, R)$ denote the chain complex obtained from the simplicial $R$-module $R(X)$ obtained by applying the free $R$-module functor degree-wise to the simplicial set $X$. (We view this now as co-chain complex trivial in positive degrees.) For each $n$-simplex $\gamma_n \in C_n(X, R)$, we obtain a map $\tilde{x}_n : C_n(\Delta[n], R) \to C_n(X, R)$ by sending $u_n \cdot C_n(\Delta[n], R)$ (which is the canonical generator) to $x_n$. Now we define a map

\[
\mu : \mathcal{E}nd_{C_n(\Delta[n], R)} \to \mathcal{E}nd_{C_n(X, R)}
\]

by $\mu(f \otimes x_n) = \tilde{x}_n \circ f \otimes u_n$, where $f \in \mathcal{E}nd_{C_n(\Delta[n], R)}(\gamma)$ and $\mu(f \otimes u_n)$ utilizes the obvious co-algebra structure of $C_n(\Delta[n], R)$ over $\mathcal{E}nd_{C_n(\Delta[n], R)}$. On taking $C^*(X) = \text{Hom}_R(C_n(X, R))$ where $\text{Hom}_R$ denotes the internal Hom in the category $\mathcal{C}M_R$, we obtain the structure of an algebra for $C^*(X, R)$ over the endomorphism operad $\mathcal{E}nd_{C_n(\Delta[n], R)}$.

Since each $C_n(\Delta[n], R)$ is acyclic, one may easily show that each $\mathcal{E}nd_{C_n(\Delta[n], R)}(n)$ is acyclic as well, for each $n \geq 0$. This is the (topological or classical) Eilenberg-Zilber operad. One may utilize the general discussion in
section 6 to define a map of operads from an $E_\infty$-operad to the Eilenberg-Zilber operad: this will put an $E_\infty$ structure on the singular chain complex $C_*(X, R)$ and the singular cochain complex $C^*(X, R)$.

**Remark 3.1.** It may be worth pointing out that, the operad action we construct in this paper, when applied to smooth complex algebraic varieties, provides an $E_\infty$-operad that acts on a chain complex also defining singular cohomology with finite coefficients. This uses the correspondence between singular cohomology and étale cohomology with finite coefficients and the correspondence between motivic and étale cohomology. More details on this may be found in the last section, including a construction of the usual cohomology operations in the singular cohomology of smooth complex algebraic varieties from our operad action.

### 3.3. The motivic Eilenberg-Zilber operad

Given an Abelian sheaf $P$ on $(sschms/k)$, we extend $P$ to a simplicial sheaf (i.e. a sheaf of simplicial Abelian groups) as is done for the simplicial case in [F-S]: we let $C_*(P)$ denote the simplicial sheaf defined by $\Gamma(U, C_n(P)) = \Gamma(U \times \Delta[n], P)$. This simplicial Abelian sheaf may be replaced by the corresponding chain complex (i.e. with a differential of degree $-1$): this complex will be denoted $N(C_*(P))$. (Now we may re-index $N(C_*(P)) = N(C_*(P))_1$ to obtain a co-chain complex of sheaves that is trivial in positive degrees.)

**3.3.1. Finite correspondences.** Given $X, Y \in (snt.schemes/k)$, we define $Cor_{q,f}(Y, X)$ $(Cor_f(Y, X))$ to be the free Abelian group generated by closed integral sub-schemes $Z \subseteq Y \times X$ that are quasi-finite (finite, respectively) and dominant over some component of $Y$. Clearly $Cor_{q,f}(\cdot, X)$ and $Cor_f(\cdot, X)$ are presheaves on $(sschms/k)$.

We define the **motive** of a given smooth scheme $X$ as $M(X) = C_*(Cor_f(\cdot, X))$ which is a chain complex of Abelian sheaves on the big Nisnevich site. If $R$ is a commutative ring with unit, $M(X, R)$ will denote $C_*(Cor_f(\cdot, X)) \otimes R$. Observe that $n \mapsto M(\Delta[n], R)$ defines a functor $M : \Delta \to C(S,R)$. We define the **motivic Eilenberg-Zilber operad** to be the endomorphism operad of this functor in the sense of 3.0.2. Since each $\Delta[n]$ is $k^1$-acyclic, this defines an **acyclic operad** in the category $C(S,R)_{k^1}$. The following is one of the main results of this section.

**Theorem 3.2.** For each smooth scheme $X$, the motive $M(X)$ is a co-algebra over the motivic Eilenberg-Zilber operad. If $f : X \to Y$ is a map of smooth schemes, then the pushforward $f_* : M(X) \to M(Y)$ is compatible with above co-algebra structure.

**Proof.** Step 1. We begin with the basic observation that there is a natural pairing (see [MVW] Lemma 1.7):

\[(3.3.2) \quad \circ : Cor_f(X,Y) \otimes Cor_f(Y,Z) \to Cor_f(X,Z)\]

defined as follows. Given correspondences $\Gamma_1 \in Cor_f(X,Y)$, $\Gamma_2 \in Cor_f(Y,Z)$, one defines $\Gamma_1 \circ \Gamma_2$ to be the direct image under the projection $X \times Y \times Z \to Y \times Z$ of the correspondence $(\Gamma_1 \times Z) \cap (X \times \Gamma_2)$. This pairing has the property that if $g : Y \to Y'$ is any map of smooth schemes,

\[(3.3.3) \quad g_* (\Gamma_1) \circ \Gamma_2 = \Gamma_1 \circ g^*(\Gamma_2), \quad \Gamma_1 \in Cor_f(X,Y), \quad \Gamma_2 \in Cor_f(Y',Z)\]

It follows that one has pairings:

\[(3.3.4) \quad \bullet : Cor_f(U \times \Delta[k], \Delta[n]) \otimes Cor_f(U \times \Delta[n], X) \to Cor_f(U \times \Delta[k], X)\]

where $U$ and $X$ are smooth schemes. The above pairing is defined by first using the map $\Delta_* : Cor_f(U \times \Delta[k], \Delta[n]) \to Cor_f(U \times \Delta[k], U \times \Delta[n])$ sending a correspondence on $U \times \Delta[k] \times \Delta[n]$ by the diagonal closed immersion $\Delta$ to a correspondence on $U \times \Delta[k] \times \Delta[n] \times U$ and then applying the pairing in (3.3.2) with $X = U \times \Delta[k]$, $Y = U \times \Delta[n]$ and $Z = X$.

**Step 2.** Let $x_n \in \Gamma(U, M_n(X)) = Cor_f(U \times \Delta[n], X)$. Then there is an obvious correspondence $[\Delta]_n \in Cor_f(U \times \Delta[n], \Delta[n]) = \Gamma(U, M_n(\Delta[n]))$ so that $x_n = [\Delta]_n \circ x_n$. In fact one may take $[\Delta]_n$ to be the class of $U \times \Delta[n]$. In particular, there is a collection of maps, compatible as $n$ varies. Let $x_n \in \Gamma(U, M_n(X)) = Cor_f(U \times \Delta[n], X)$ be given. Then $\alpha_{n,m}(\Delta[n])$ is a class in $\Gamma(U, M(\Delta[n]))^{\otimes k} = Cor_f(U \times \Delta[l], \Delta[n]) \otimes \cdots \otimes Cor_f(U \times \Delta[k], \Delta[n])$ so that $\Sigma l = m + n$. Now $x_n^k \in Cor_f(U \times \Delta[n], X) \otimes \cdots \otimes Cor_f(U \times \Delta[k], X) = \Gamma(U, M_n(X))^{\otimes k}$. Therefore one may compose each factor of $x_n$ above with a factor of $\alpha(\Delta[n])$ in $Cor_f(U \times \Delta[l], \Delta[n])$ to obtain a class in $Cor_f(U \times \Delta[l], X)$. The product of these classes obviously defines an element in $Cor_f(U \times \Delta[l], X, \otimes \cdots \otimes Cor_f(U \times \Delta[k], X)$ with $\Sigma l = m + n$. Therefore, this defines a pairing,

\[(3.3.5) \quad \bullet : \Gamma(U, M_n(X))^{\otimes k} \to \Gamma(U, M_n(X))^{\otimes k}\]
It is clear from the above description that this pairing is compatible with push-forwards as in the last statement of the theorem.

Finally one needs to verify that the above pairing is, in fact, compatible with the differentials on either side. For this we will use the following conventions. Given two chain complexes $A$ and $B$ the differential on the tensor product $A \otimes B$ is defined by $\delta(a \otimes b) = \delta(a) \otimes b + (-1)^n a \otimes \delta(b)$. The internal hom $\mathcal{H}om(A, B)$ is the chain complex given in degree $n$ by $\Pi_n \mathcal{H}om(A_i, B_{n-i})$ and with differential $\delta(f) = (\delta \circ f_1 + (-1)^{n-1} f_1 \circ \delta)$, where $f = (f_i)$.

Next observe the following: given an $\alpha \in \Gamma(U, \mathcal{H}om(M_*(\Delta))(k)_n)$, $\delta_\alpha = \delta(\alpha) + (-1)^{i+m} \alpha \circ \delta$ where $\circ$ denotes the obvious composition of maps and $\alpha = (\alpha_i)$, with $\alpha_i : M(\Delta[n]) \rightarrow M(\Delta[n])^{\otimes i}_{CM}$ being the corresponding component of $\alpha$. Now consider the differential on the tensor product on the left in (3.3.5). Observe that

$$
\delta(\alpha \otimes x_n) = \delta(\alpha) \otimes x_n + (-1)^{n+m} \alpha \circ \delta(x_n)
$$

Now observe that $(\alpha_{n-1-n} \circ \delta) \cdot x_n$ is defined by pairing a factor of $\alpha_{n-1-n}(\Sigma_i(-1)^i(\Delta[n])) \in \mathcal{E}or_f(U \times \Delta[l_i], \Delta[n])$ (where $\Sigma_i l_i = m-1+n$ with a copy of $x_n \in \mathcal{E}or_f(U \times \Delta[n], X)$). Since $\delta_1 \circ \alpha_{n-1,n-1} = \alpha_{n-1,n-1} \circ \delta_1$ and $\delta_1(\Sigma_i(\Delta[n])) \in \mathcal{E}or_f(U \times \Delta[n-1], \Delta[n])$ identifies with $\delta_1(\Sigma_i(\Delta[n-1]))$, one may identify $(\alpha_{n-1,n} \circ \delta) \cdot x_n$ with $\alpha_{n-1,n} \cdot (\delta(x_n))$. (See the property (3.3.3) of the pairing $\circ$ considered in Step 1.) Therefore, the last terms in the formula (3.3.6) above cancel out. Moreover, one sees readily by inspection of the pairing $\circ$ above that the class $(\delta \circ \alpha_{n,n}) \cdot x_n$ identifies with $\delta(\alpha_{n,n} \cdot x_n)$. Therefore the class $\delta(\alpha_{n,n} \cdot x_n)$ maps to $\delta(\alpha_{n,n} \cdot x_n)$. This proves the pairing $\circ$ defined above is in fact a pairing of complexes and completes the proof of the theorem.

**Corollary 3.3.** The motivic complexes $\mathbb{Z} = \oplus \mathbb{Z}(r)$ are co-algebras over the motivic Eilenberg-Zilber operad.

**Proof.** This follows readily from the last theorem once we recall that $\mathbb{Z}(r) = \mathbb{Z}(G_{\mathbb{R}^r}[-r])$ and $\mathbb{Z}(G_{\mathbb{R}^n}) = \text{kern}(\oplus M(G_m \times \cdots G_m \times \cdots G_m) \rightarrow M(G_m \times \cdots G_m \times \cdots G_m))$.

: see [MVW] Definition 2.8. □

3.4. The étale Eilenberg-Zilber operad. Let $l$ denote a fixed prime different from the characteristic of the base field $k$ and let $\nu > 0$ be a fixed integer. Let $(\text{sschms}/k)_{et}$ denote the category of smooth schemes over $k$ provided with the big étale topology. One considers the motivic operad $\mathcal{E}nd(M_*(\Delta)/l^\nu)$ sheafified on this big site: this operad will be called the mod-$l^\nu$ étale Eilenberg-Zilber operad and will be denoted $\mathcal{E}nd(M_*(\Delta)/l^\nu)$. Similarly, if $X$ denotes a smooth scheme over $k$, one lets $M_*(\Delta)/l^\nu$ denote the sheafification of $M_*(\Delta)/l^\nu$ on the site $(\text{sschms}/k)_{et}$.

**Proposition 3.4.** There is a natural quasi-isomorphism $\mathcal{R}Hom(M_*(\Delta)/l^\nu, \mathbb{Z}/l(n)) \simeq \mathcal{R}Hom(M_*(\Delta)/l^\nu, \mu_1(n)) \simeq \mu_1(n)$ for all $n \geq 1$.

**Proof.** Assume that $X$ is of pure dimension $d$. We proceed to show that there is a quasi-isomorphism of the presheaf $U \mapsto \Gamma(U, \mathcal{R}Hom(M(-)/l, \mathbb{Z}/l(n)))$ with the presheaf $\mu_1(n)$. First observe that the stalk at a given geometric point $x$ of $X$ is given by $\mathcal{R}Hom(M(\text{Spec } O_x^{\text{sh}})/l, \mathbb{Z}/l(n))$. Since the presheaf $U \mapsto \Gamma(U, \mathcal{R}Hom(M(-)/l, \mathbb{Z}/l(n)))$ is a presheaf with transfers, it follows that one may identify the last stalk with $\mathcal{R}Hom(M(k_x)/l, \mathbb{Z}/l(n)) \simeq \mathcal{R}Hom(\mu_1(0), \mu_1(n)) \simeq \mu_1(n)$. □

**Theorem 3.5.** $\mathcal{R}Hom(M_*(\Delta)/l^\nu, \mu_1)$ is an algebra over the operad $\mathcal{E}nd(M_*(\Delta)/l^\nu)$. The operad $\mathcal{E}nd(M_*(\Delta)/l^\nu)$ is acyclic.

**Proof.** From the results of the last section, it follows that $M_*(\Delta)/l^\nu$ is a co-algebra over the operad $\mathcal{E}nd(M_*(\Delta)/l^\nu)$. Since the obvious pairing $\mu_1 \otimes \mu_1 \rightarrow \mu_1$ is obviously strictly associative and commutative, it follows readily that $\mathcal{H}om(M_*(\Delta)/l^\nu, \mu_1)$ is an algebra over the same operad. One may define the derived functor $\mathcal{R}Hom(M_*(\Delta)/l^\nu, \mu_1)$ using the canonical Godement resolution applied to the sheaf $\mu_1$.

To see that the operad $\mathcal{E}nd(M_*(\Delta)/l^\nu)$ is acyclic one may proceed as follows. For this one observes (see 9.2 in the appendix) that $M_*(\Delta)/l^\nu$ is a cofibrant approximation to $M_*(\text{Spec } k)/l^\nu$ in the Reedy model structure considered there. Moreover on taking the Godement resolution of this we obtain a fibrant object in the same model structure as observed there. Therefore, one may identify $\mathcal{E}nd(M_*(\Delta)/l^\nu)(k)$ with a homotopy function complex as in [Hir] Chapter 19. It follows one may replace $M_*(\Delta)/l^\nu$ with the constant cosimplicial object $M_*(\text{Spec } k)/l^\nu$ and obtain the same $\mathcal{E}nd$ up to weak-equivalence. However $M_*(\text{Spec } k)/l^\nu$ is the motive of $\text{Spec } k$ and is
therefore $\simeq \mathbb{Z}/p^n[0]$. This proves the acyclicity of the above operad. (Observe that the above proof is entirely similar to the proof of the acyclicity of the topological Eilenberg-Zilber operad.)

Remark 3.6. In the motivic situation, $\Gamma(X, \mathbb{Z}) = RHom(M_n(X), \mathbb{Z})$ where $\mathbb{Z}$ denotes the motivic complex. In this case, though $M_n(X)$ is a coalgebra over the motivic Eilenberg-Zilber operad, $\Gamma(X, \mathbb{Z})$ fails to be an algebra over the same operad, the key difficulty here being that the obvious pairing $\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ is not strictly commutative, since $\mathbb{Z}$ is no longer a commutative ring, but instead a complex of sheaves. This makes it necessary to construct explicitly another operad that acts on the motivic complexes: this is the content of the next two sections.

We conclude this section by recalling the definition of the motivic complexes.

Definition 3.7. Given a fixed $X \in (smt.schemes/k)$, one defines presheaves with transfers, $Z_{eq}(X)$, $Z_{tr}(X)$ by $\Gamma(U, Z_{eq}(X)) = Cor_{et}(U, X)$ and $\Gamma(U, Z_{tr}(X)) = Corf(U, X)$. It is observed in [F-S-V], that this defines a sheaf with transfer on $(smt.schemes/k)_{Nis}$ and $(smt.schemes/k)_{zar}$.

3.5. The motivic complexes $\mathbb{Z}(n)$.

Definition 3.8. We let $\mathbb{Z}(n)$ be the presheaf $C^*(Z_{eq}(Ang))[-2n]$ on either of the sites $(sschms/k)_{Nis}$ or $(sschms/k)_{Zar}$. Alternatively we may also use the presheaf $C^*(Z_{tr}(\mathbb{Z}_m(1)^{sm}))[-2n]$; it is shown in [MVW] that both definitions are equivalent up to natural quasi-isomorphism. Therefore, we will use whichever definition is more convenient. If $X$ is a smooth scheme over $k$, we also let $X(n)$ denote the restriction of $\mathbb{Z}(n)$ to the Zariski or Nisnevich site of $X$. $X(n) = RF(X, \mathbb{Z}_X(n))$. We define $\mathbb{Z} = \bigoplus_n \mathbb{Z}(n)$ and $\mathbb{Z} = RF(X, \mathbb{Z}_X) \cdot \mathbb{Z}/l(n)$ will denote $\mathbb{Z}(n) \otimes \mathbb{Z}/l$.

4. The Barratt-Eccles operads

One of the standard examples of $E_\infty$-operads are the ones commonly called the Barratt-Eccles operads. We will consider two variants of these in this section.

Given a discrete group $G$, one forms the simplicial group $EG$ given in degree $n$ by $EG_n = G^{n+1}$. One may verify readily that if $G$ and $H$ are two groups, $E(G \times H)$ is naturally isomorphic to $E(G) \times E(H)$. One may also observe the isomorphism $EG \cong \cosk_0(G)$. Therefore, taking $\cosk_0$, we see that the map $\gamma_k : \Sigma_1 \times \Sigma_n_1 \times \Sigma_n_2 \to \Sigma_n$ defined by $(\sigma_k, \sigma_{n_1}, \cdots, \sigma_{n_2}) \mapsto (\sigma_{n_1}(1), \cdots, \sigma_{n_2}(k))$ induces the structure map $\gamma : E(\Sigma_k) \times E(\Sigma_{n_1}) \times E(\Sigma_{n_2}) \to E(\Sigma_n)$. Henceforth we will denote the base scheme Spec $k$ by $S$. Given a simplicial set $X$, one lets $X \otimes S$ denote the simplicial scheme defined by $(X \otimes S)_n = \coprod X_n$ and with the structure maps induced by the structure maps of the simplicial set $S$. Thus $E\Sigma_m \otimes S$ is a simplicial scheme over $S$ for each $m$. Observe that there is a principal fibration $E\Sigma_m \to B\Sigma_m$ and $B\Sigma_m$ denotes the classifying simplicial sheaf associated to the group $\Sigma_m$.

Definition 4.1. (The simplicial Barratt-Eccles operad) Henceforth we will let $E\Sigma_m$, denote the sheaf on the site $(sschms/k)_{Nis}$ represented by this simplicial scheme. We will apply the free abelian group functor $N$ followed by the normalization functor $\mathbb{Z}$ to produce an operad in $C((smt.schms)_{Nis})$ and in $C((smt.schms)_{et})$ from this. This defines the simplicial Barratt-Eccles operad on the above site; this will be denoted $NZ\Sigma = (\mathbb{Z}(E\Sigma_m))[n]$.

Remark 4.2. Assume the base scheme $S$ is the spectrum of a field Then $M(Spec \ k) = \mathbb{Z}[0]$ so that $M(E\Sigma_m) \cong N(Z(E\Sigma_m))$.

While the above operad seems sufficient for producing a motivic dga, the applications to motivic cohomology operations make it necessary for us to consider a second variant of this. We begin by recalling briefly the construction of the geometric classifying space of an algebraic group: originally this is due to Totaro and Eddidin-Graham - see [Tot] and [EG]. Let G be a linear algebraic group over $k$ i.e. a closed subgroup in $GL_n$ over $S$ for some n. For a (closed) subset $i : G \to GL_n$ the geometric classifying space $BG_i$ is defined with respect to $i$ as follows. For $m \geq 1$ let $U_m$ be the open subScheme of $A^{nm}_n$ where the diagonal action of $G$ determined by $i$ is free. Let $A^{nm}_n/G$ be the quotient $S$-scheme of the (diagonal) action of $G$ on $A^{nm}_n/V_m$ be the image of $U_m$ in $A^{nm}_n/G$, an open subScheme : the projection $U_m \to V_m$ defines $V_m$, as the quotient scheme of $U_m$ by the free action of $G$, and $V_m$ is thus a smooth $S$-scheme. We have closed embeddings $U_m \to U_{m+1}$ and $V_m \to V_{m+1}$ corresponding to the embeddings $Id \times \{0\} : A^{nm}_n \hookrightarrow A^{nm}_n \times A^n$ and we set $E\Sigma_m = \lim_{m \to \infty} U_m$ and $B\Sigma_m = \lim_{m \to \infty} V_m$ where the colimit is taken in the category of sheaves on $(sschms/k)_{Nis}$ or on $(sschms/k)_{et}$. Observe that if $G = \Sigma_n$ acting on $A^n$ by permuting the n-coordinates, we may take $U_m = \{ \{x(1), \cdots, x(n)\} | x(i,j) \neq x(i,k), \text{ for all } i, j \neq k \}$.

The motivic cohomology of these geometric classifying spaces enter into the definition of motivic cohomology operations. We proceed to make an operad out of the $(E\Sigma^m_m)[n]$. We first define a map

$$\gamma_k : (A^{nm}_n)^k \times (A^n)^j_1 \times \cdots \times (A^n)^j_k \to (A^{nm+i-1}_n)^{\Sigma^j_k},$$
as follows.
\[(4.0.2)\quad \gamma_k((y_1, \ldots, y_k), (x(1), \ldots, x(1)_{j_1}), \ldots, (x(k), \ldots, x(k)_{j_k}))
= ((x(1), y_1), \ldots, (x(1)_{j_1}, y_1), \ldots, (x(k), y_k), \ldots, (x(k)_{j_k}, y_k))\]

For each \((\mathcal{A}^P)\) let \(U(\mathcal{A}^P)(p) = \{(u_1, \ldots, u_p)\in(\mathcal{A})^p| u_i \neq u_j, i \neq j\}.\) The symmetric group \(\Sigma_p\) acts on (the right on) \(U(\mathcal{A}^P)(q)\) by permuting the \(p\)-factors. One may then see that \(\gamma_k\) induces a map
\[(4.0.3)\quad \gamma_k: U(\mathcal{A}^m)(k) \times U(\mathcal{A}^m)(j_1) \times \cdots U(\mathcal{A}^m)(j_k) \to U(\mathcal{A}^{m+m'})(\Sigma j_i)\]

One may verify that the above map \(\gamma_k\) has the following behavior with respect to the action of the symmetric groups:
\[(4.0.4)\quad \gamma_k((u_1, \ldots, u_k)\sigma, (x(1), \ldots, x(1)_{j_1}), \ldots, (x(k), \ldots, x(k)_{j_k}))
= \gamma_k((u_1, \ldots, u_k), (x(\sigma^{-1}(1)), \ldots, x(\sigma^{-1}(k))_{j_{\sigma^{-1}(i)}}), \ldots, (x(\sigma^{-1}(k))_{j_{\sigma^{-1}(i)}})) \circ \sigma(j_1, \ldots, j_k)\]

where \(\sigma\in\Sigma_k\) and \(\sigma(j_1, \ldots, j_k)\) denotes the permutation of \(j = \Sigma j_i\)-letters that permutes the \(k\)-blocks of letters determined by the given partition of \(j\) as \(\sigma\) permutes \(k\)-letters. Similarly one may verify that
\[(4.0.5)\quad \gamma_k((u_1, \ldots, u_k), (x(1), \ldots, x(1)_{j_1})\tau_1, \ldots, (x(k), \ldots, x(k)_{j_k})\tau_k)
= \gamma_k((u_1, \ldots, u_k), (x(1), \ldots, x(1)_{j_1}), \ldots, (x(k), \ldots, x(k)_{j_k}) \circ (\tau_1 \oplus \cdots \oplus \tau_k)\]

where \(\tau_k\in\Sigma_{j_k}\) and \(\tau_1 \oplus \cdots \oplus \tau_k\) denotes the image of \(\tau_1 \times \cdots \times \tau_k\) under the obvious inclusion of \(\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \Sigma_j, j = \Sigma j_i\). Moreover the pairings \(\{\gamma_k|k\}\) are associative in the sense that the diagram below commutes (where \(j_\alpha = \Sigma_{k\alpha}^{j_\alpha}, \alpha = 1, \ldots, l\)):
\[(4.0.6)\quad U(\mathcal{A}^m)(l) \times (\Pi^{k_{\alpha}}_{\alpha=1} U(\mathcal{A}^m)(k_{\alpha})) \times (\Pi^{l_{\alpha}}_{\alpha=1} U(\mathcal{A}^m)(j_{\alpha})) \overset{id \times \Pi^{\gamma_{k_{\alpha}}}_{\alpha}}{\longrightarrow} U(\mathcal{A}^m)(l) \times (\Pi^{l_{\alpha}}_{\alpha=1} U(\mathcal{A}^{m+m'})(j_{\alpha})) \overset{\gamma_k \times id}{\longrightarrow} U(\mathcal{A}^{m+m'})(\Sigma_{k_{\alpha}}k_{\alpha}) \times (\Pi^{l_{\alpha}}_{\alpha=1} U(\mathcal{A}^{m+m'})(j_{\alpha})) \overset{\mu_{k_{\alpha}} l_{\alpha}}{\longrightarrow} U(\mathcal{A}^{m+m+m'})(\Sigma_{k_{\alpha}l_{\alpha}})\]

Now one may take the direct limit \(\lim_{m\to\infty} U(\mathcal{A}(m))\) to define \(U(\mathcal{A})\): this is a simplicial sheaf on \((smtschms)_{Nis}\) and on \((smtschms)_{et}\) and will be also denoted \(E\Sigma^m_q\). The commutative diagrams in (4.0.4) through (4.0.6) and the arguments above show \(\{E\Sigma^m_q|q\}\) defines an operad in the category of (simplicial) sheaves on \((smtschms)_{Nis}\) and on \((smtschms)_{et}\).

We define the motive \(M(E\Sigma^m_q) = \lim_{m\to\infty} M(U(\mathcal{A}(m)))\). The commutative diagrams in (4.0.4) through (4.0.6) and the arguments above show that the resulting object is an operad in \(C((smtschms)_{Nis})\) and in \(C((smtschms)_{et})\).

**Definition 4.3.** The above operad is called the geometric Barratt-Eccles operad and will be denoted \(M(E\Sigma^m_q) = \{M(E\Sigma^m_q)|q\}\).

**Proposition 4.4.** The simplicial Barratt-Eccles operad is an acyclic operad in the category \(C((smtschms)_{Nis})\) and \(C((smtschms)_{et})\). The geometric Barratt-Eccles operad is an acyclic operad in the category \(C((smtschms)_{Nis})\) and \(C((smtschms)_{et})\) which are localizations of \(C((smtschms)_{Nis})\) and \(C((smtschms)_{et})\) by inverting \(A^1\)-equivalences.

5. **Action of the Barratt-Eccles operads on the motivic complex**

After considerable experimentation, we have come to realize that one has to first define the action of an acyclic operad explicitly on the motivic complexes. After wards, one may use model structures on the category of operads to provide an action by better operads, for example the geometric Barratt-Eccles operad in the last section.

We will accomplish this by defining explicitly an action by the (first) Barratt-Eccles operad \(\{E\Sigma_n|n\}\) on the motivic complexes. To motivate our construction, we will consider first explicitly how one proves (first order)
homotopy commutativity of the product on the motivic complexes. Recall \( \mathbb{Z}(n) = C^*\text{Cor}_{q.f}(\cdot, \mathbb{A}^n)[-2n] \). Therefore, the product is defined by the pairing:

\[
(5.0.7) \quad \text{Cor}_{q.f}(\times \Delta[\cdot], \mathbb{A}^l) \otimes \text{Cor}_{q.f}(\times \Delta[\cdot], \mathbb{A}^m) \rightarrow \text{Cor}_{q.f}(\times \Delta[\cdot], \mathbb{A}^{l+m})
\]

of simplicial abelian sheaves. Here \( \Delta[\cdot] \) denotes the cosimplicial space \( \{\Delta[n]/n\} \) with the obvious structure maps. i.e. we start with correspondences \( \Gamma_0 \) on \( U \times \Delta[n] \times \mathbb{A}^l \) and \( \Gamma_2 \) on \( U \times \Delta[n] \times \mathbb{A}^m \) and take their external product to define a correspondence \( \Gamma_1 \times \Gamma_2 \) on \( U \times U \times \Delta[p] \times \Delta[n] \times \mathbb{A}^{l+m} \). Next we pull-back by the diagonal \( \Delta : U \times \Delta[n] \rightarrow U \times \Delta[n] \times U \times \Delta[n] \). Finally we apply the normalization functor to pass from a simplicial abelian group to a chain complex. Since the shuffle maps from the product of the resulting chain complexes to the chain complex associated to the product of the simplicial abelian groups strictly commutes with the action of the symmetric group \( \Sigma_2 \) permuting the two factors, the need for homotopy commutativity arises only from the switching of the two weight-factors \( \mathbb{A}^l \) and \( \mathbb{A}^m \).

Assume that we are given 2 cycles \( Z_1 \in \text{Cor}_{q.f}(U \times \Delta[n], \mathbb{A}^l) \) and \( Z_2 \in \text{Cor}_{q.f}(U \times \Delta[n], \mathbb{A}^m) \). Now after pull-back by the diagonal \( \Delta : U \times \Delta[n] \rightarrow U \times \Delta[n] \times U \times \Delta[n] \), \( Z_1 \times Z_2 = (Z_2 \times Z_1) \) defines the cycle \( \Delta^*(Z_1 \times Z_2) \in \text{Cor}_{q.f}(U \times \Delta[n], \mathbb{A}^l \times \mathbb{A}^m) \) and the cycle \( \Delta^*(Z_2 \times Z_1) \in \text{Cor}_{q.f}(U \times \Delta[n], \mathbb{A}^m \times \mathbb{A}^l) \). At this point we may assume that the cycles \( Z_1, Z_2 \) are in fact closed subschemes of \( U \times \Delta[n] \times \mathbb{A}^l \) and \( U \times \Delta[n] \times \mathbb{A}^m \) and further that the scheme \( U \) is affine. For each point \( a \) of the variety \( \Delta^*(Z_1 \times Z_2) \), let \( J_a \) denote the maximal ideal in \( U \times \Delta[n] \times \mathbb{A}^{l+m} \). Let \( a = (a(1), a(2)) \) with \( a(1) \in \Gamma(U \times \Delta[n], \mathbb{O}_U \times \Delta[n]) \) and \( a(2) \in \mathbb{A}^{l+m} \). Since the base field is not assumed to be algebraically closed, one may see that the residue-field at the point \( a(2) \), \( k(a(2)) \) is a finite extension of \( k \). Observe that if \( J_{\Delta^*(Z_1 \times Z_2)} \) is the ideal defining the variety \( Z_1 \times Z_2 \), then \( J_{\Delta^*(Z_1 \times Z_2)} = \bigcap_a J_a \). Since \( U \) is affine, we may observe that \( J_a \) is defined by polynomials \( p_a \) in the \( l + m \) variables \( (x_1 - a(2)_1, \ldots, x_{l+m} - a(2)_{l+m}) \) with coefficients in the ring \( R = \Gamma(U \times \Delta[n], \mathbb{O}_U \times \Delta[n]) \).

There is an obvious action by the group \( \Sigma_2 \) on \( \mathbb{A}^{l+m} = \mathbb{A}^l \times \mathbb{A}^m \) switching the two factors \( \mathbb{A}^l \) and \( \mathbb{A}^m \). This action is algebraic. Therefore, if \( \sigma \in \Sigma_2 \), its action on \( \mathbb{A}^{l+m} \) is given by polynomials: we will denote the corresponding polynomial also by \( \sigma \). Moreover let \( s_1 = (id, \sigma) \) denote the obvious 1-simplex of \( E \Sigma_2 \). Next \( t.id + (1 - t) \sigma \) is a polynomial in the \( l + m \) variables corresponding to \( \mathbb{A}^{l+m} \). For this we may assume without loss of generality that \( \Delta^*(Z_1 \times Z_2) \) itself is irreducible; if this is not the case, one may replace this scheme by one of its irreducible components. By our assumption this is also affine. Now the map \( p \mapsto p \circ (t.id + (1 - t) \sigma) \) defines a projection \( \Delta^*(Z_1 \times Z_2) \rightarrow \Delta^*(Z_1 \times Z_2) \). The obvious closed immersion of the latter in the former is a section to this projection. Therefore the required result follows from the lemma 5.1 below.

At this point we invoke the isomorphism of schemes \( \Delta[p+q] \rightarrow \Delta[p] \times \Delta[q] \) as provided by each shuffle map. Therefore the shuffle map induces an isomorphism of complexes \( \nabla : T_{q.f}(\{\text{Cor}_{q.f}(U \times \Delta[p] \times \Delta[q], \mathbb{A}^{l+m})\}[p, q]) \rightarrow \text{Cor}_{q.f}(U \times \Delta[p+q], \mathbb{A}^{l+m}) \). Therefore the shuffle map provides a map \( \nabla : \text{Cor}_{q.f}(U \times \Delta[n], \mathbb{A}^m) \rightarrow \text{Cor}_{q.f}(U \times \Delta[n+1], \mathbb{A}^{l+m}) \). Therefore, \( \nabla((Z_1 \times Z_2)(\Delta[1])) \) defines the homotopy between the two products \( \Delta^*(Z_1 \times Z_2) \) and \( \Delta^*(Z_1 \times Z_1) \). This completes the proof of first order homotopy commutativity of the product on the motivic complexes.

We will presently extend this to provide higher order homotopies for the product structure on the motivic complexes. We will again assume that \( U \) is affine. Observe that for \( q \geq 1 \), a \( q \)-simplex, \( s_q \), of \( E \Sigma_k \) is given by a sequence \( (\sigma_0, \ldots, \sigma_q) \) with each \( \sigma_i \in \Sigma_k \).

5.0.9. Assume that we are given a closed irreducible sub-variety \( V \subset U \times \Delta[n] \times \Delta[p_1] \times \cdots \times \Delta[p_k] \times \mathbb{A}^{\ell_1} \times \cdots \times \mathbb{A}^{\ell_k} \). For example, we are given cycles \( V_\ell \in \text{Cor}_{q.f}(U \times \Delta[n] \times \Delta[p_\ell], \mathbb{A}^l) \), \( \ell = 1, \ldots, k \) and that these are in fact closed subschemes of \( U \times \Delta[n] \times \mathbb{A}^l \). We let \( V = \Delta^*(\Pi_{i=1}^k V_\ell) \) and for each \( \sigma \in \Sigma_k \), \( V(\sigma) = \Delta^*(\Pi_{i=1}^k V(\sigma(i))) \), where
$\Delta : U \times \Delta[n] \to (U \times \Delta[n]) \times \cdots \times (U \times \Delta[n])$ is the obvious diagonal. For each point $a$ of the variety $V(\sigma_0)$, let $J_a$ denote the maximal ideal in $U \times \Delta[n] \times \Delta[p_1] \times \cdots \times \Delta[p_k] \times \mathbb{A}^{\Sigma_i t_i}$. Let $a = (a(1), a(2))$ with $a(1) \in \Gamma(U \times \Delta[n] \times \mathcal{O}_U \times \Delta[n])$ and $a(2) \in \mathbb{A}^{\Sigma_i t_i}$. Since the base field is not assumed to be algebraically closed, one may see that the residue-field at the point $a(2)$, $k(a(2))$ is a finite extension of $k$. Observe that if $J_V(\sigma_0)$ is the ideal defining the variety $V(\sigma_0)$, then $J_V(\sigma_0) = \bigcap_a J_a$. Since $U$ is affine, we may observe that $J_a$ is defined by polynomials $p_a$ the in $\Sigma_i t_i$ variables $(x_1 - a(2)_{t_1}, \ldots, x_k - a(2)_{t_k}, x_{t_1+1} - a(2)_{t_1+1}, \ldots, x_{t_k+1} - a(2)_{t_k+1}, \ldots, x_{\Sigma_i t_i} - a(2)_{\Sigma_i t_i})$ with coefficients in the ring $R = \Gamma(U \times \Delta[n] \times \times_{i=1}^{\Sigma_i t_i} \Delta[p_i] \times \mathcal{O}_U \times \Delta[n] \times \times_{i=1}^{\Sigma_i t_i} \Delta[p_i])$.

We let $J_{(\sigma(a))}$ be the ideal of functions obtained by substituting $(t_0 \sigma_0(a_2) + t_1 \sigma_1(a_2) + \cdots + (1 - t_1 - \cdots - t_{q-1}) \sigma_q(a_2))$ in the place of $a_2$ in the polynomials $p \in J_a, t_i \in A^1$. Let

$$J_{(\sigma(a))}(V) = \bigcap_{a \in V(\sigma)} J_{\sigma(a)}$$

This defines a closed subscheme $V(\sigma) \subseteq U \times \Delta[n] \times \Delta[p_1] \times \cdots \times \Delta[p_k] \times \Delta[q]$ for every $\sigma$. We will denote this by

$$|\sigma|(V(\sigma))$$

In the situation of 5.0.9, one defines $|\sigma|(V)$ similarly. Observe that this is a closed subscheme of $U \times \Delta[n] \times \Delta[p_1] \times \cdots \times \Delta[p_k] \times \Delta[q] \times A^{\Sigma_i t_i}$. More generally if $\sigma_q = (\sigma_0, \ldots, \sigma_q), s_p = (s_0, \ldots, s_p)$ are $q$ and $p$ simplices of $E \Sigma_k$, we may define

$$|\sigma_q \times s_p|(V)$$

similarly by replacing $J_a$ by the ideal of functions obtained by substituting $(t_0 \sigma_0 + t_1 \sigma_1 + \cdots + (1 - t_0 - \cdots - t_{q-1}) \sigma_q) \circ (s_0 \sigma_0 + s_1 \sigma_1 + \cdots + (1 - s_0 - \cdots - s_{p-1}) \sigma_p)$ for $a_2 = (s_0 \sigma_0(a_2) + s_1 \sigma_1(a_2) + \cdots + (1 - s_0 - \cdots - s_{p-1}) \sigma_p(a_2))$ in the place of $a_2, p \in J_a, t_i, s_j \in A^1$. One may now also consider $|\sigma_q|(V)$.

Moreover for each increasing map $(\phi, \psi) : [p+q] \to [p] \times [q]$, one may see readily from the above definition that

$$|\phi, \psi|^{*}(|\sigma_q \times s_p)|(V) = |\phi, \psi|^{*}(|\sigma_q \times s_p|(V))$$

Here $(\phi, \psi)^{*}(|\sigma_q \times s_p|) \in E \Sigma_k$ denotes the $q+p$-simplex defined as in (9.3.4). One may extend the above identification to products of several simplices: i.e. if $s_{p_1}, \ldots, s_{p_t}$ are simplices of $E \Sigma_k$ of dimensions $p_1, \ldots, p_t$, and $V$ is as above, then

$$|\sigma_q \times s_{p_1} \times \cdots \times s_{p_t}|(V) = |\sigma_q|(|\sigma_{p_1} \times \cdots \times s_{p_t}|(V))$$

Next assume that $V = V(\sigma_0)$ as above. Then it is clear that the projection $V(\sigma_q) \to U \times \Delta[n] \times \Delta[q]$ is quasi-finite and dominant. Now an argument as in the case of the 1-simplex above (using the lemma 5.1 below) will show the scheme $V(\sigma_q)$ defines a class in $Cor_{q,f}(U \times \Delta[n] \times \Delta[q], A^{\Sigma_i t_i})$. This defines a pairing

$$\mu' : Z(\sigma_q) \otimes Cor_{q,f}(U \times \Delta[n] \times \Delta[p_1], A^{\Sigma_i t_i}) \otimes \cdots \otimes Cor_{q,f}(U \times \Delta[n] \times \Delta[p_k], A^{\Sigma_i t_i}) \to Cor_{q,f}(U \times \Delta[n] \times \Delta[p_1] \times \cdots \times \Delta[p_k] \times \Delta[q], A^{\Sigma_i t_i})$$

where $Z(\sigma_q)$ denotes the sub-complex of $ZNE \Sigma_k$ generated by the $q$-simplex $\sigma_q$. Clearly the action of the symmetric group $\Sigma_k$ on the simplex $\sigma_q, (\sigma', \sigma_1, \cdots, \sigma_p) \mapsto (\sigma' \circ \sigma_1 \cdots \sigma' \circ \sigma_p)$ corresponds to the action of $\sigma' \in \Sigma_k$ on $Cor_{q,f}(U \times \Delta[n] \times \Delta[p_1] \times \cdots \times \Delta[p_k] \times \Delta[q], A^{\Sigma_i t_i})$ permuting the weight-factors $A^{\Sigma_i t_i}$.

5.0.17. One may observe readily that the above pairing is compatible with pull-backs in the argument $U$. This, in turn, implies that the above pairing commutes with shuffle maps in the following sense so that the composition (where $s_q$ denotes a $q$-simplex of $NZNE \Sigma_k$)
\[ Z(s_q) \otimes \text{Cor}_{q,f}(U \times \Delta[n] \times \Delta[p_1], \mathbb{A}^{l_1}) \otimes \cdots \otimes \text{Cor}_{q,f}(U \times \Delta[n] \times \Delta[p_k], \mathbb{A}^{l_k}) \rightarrow \text{Cor}_{q,f}(U \times \Delta[n + \Sigma_i p_i], \mathbb{A}^{l_i}) \otimes \cdots \otimes \text{Cor}_{q,f}(U \times \Delta[n + \Sigma_i p_i] \times \Delta[q], \mathbb{A}^{l_i}) \]

Composing the pairing \( \mu' \) in (5.0.16) with shuffle maps, one maps this to a cycle in \( \text{Cor}_{q,f}(U \times \Delta[n + \Sigma_i p_i + q], \mathbb{A}^{l_i}) \).

We may also start with cycles \( V \otimes \text{Cor}_{q,f}(U \times \Delta[n_i], \mathbb{A}^{l_i}) \), \( i = 1, \cdots, k \); using shuffle maps one first produces corresponding cycles in \( \text{Cor}_{q,f}(U \times \Delta[n], \mathbb{A}^{l_i'}) \) where \( n = \Sigma_i n_i \) and then one applies the above construction. Moreover, the above description shows the action is compatible with restriction to the faces of the simplex \( s_q \).

Therefore, the above construction defines a pairing of complexes:

\[ (5.0.18) \quad NZ(s_q) \otimes Z(l_1) \otimes \cdots \otimes Z(l_k) \rightarrow Z(\Sigma_i l_i) \]

where \( Z(m) \) denotes the motivic complex of weight \( m \) restricted to smooth affine schemes. \( Z(s_q) \) denotes the sub-chain complex of \( NZ(E(\Sigma_k)) \) generated by the \( q \) simplex \( s_q \). Let \( \Delta[n] \) denote the usual simplicial set given by \( \Delta[n] = \text{Hom}_{\Delta}([k], [n]) \). By composing with the obvious map \( \Delta[s_q] \rightarrow E(\Sigma_k) \) sending the generator \( i_{s_q} \) to the simplex \( s_q \), we also obtain a pairing

\[ (5.0.19) \quad NZ(\Delta[q]) \otimes Z(l_1) \otimes \cdots \otimes Z(l_k) \rightarrow Z(\Sigma_i l_i) \]

where \( NZ(\Delta[q]) \) is the obvious chain complex obtained from the simplicial abelian group \( Z(\Delta[q]) \).

By ascending induction on \( p \geq 1 \), we may now assume that we have already established a pairing \( NZ(s_{k-1})(E(\Sigma_k)) \otimes Z \otimes \cdots \otimes Z \rightarrow Z \), where \( Z \) denotes the motivic complex and there are \( k \)-factors of this on the left. \( NZ(s_{k-1})(E(\Sigma_k)) \) denotes the chain complex obtained by normalizing the simplicial abelian group obtained by applying the free abelian group functor \( Z \) dimension-wise to \( s_{k-1} \text{-factors of } E(\Sigma_k) \). Observe the \( q \)-skeleton of \( E(\Sigma_k) \) is the filtered colimit of all its \( m \)-cells, \( m \leq q \). Now observe the co-cartesian square:

\[
\begin{array}{c}
\oplus NZ(\Delta'q) \\
\downarrow \\
NZ(s_{k-1}(E(\Sigma_k))) \quad NZ(s_{l-1}(E(\Sigma_k)))
\end{array}
\]

Clearly one may start the induction when \( q = 1 \); then one uses the above co-cartesian square and the pairing (5.0.19) to define the pairing

\[ (5.0.20) \quad NZ(E(s_{k-1} \Sigma_k)) \otimes Z(l_1) \otimes \cdots \otimes Z(l_k) \rightarrow Z(\Sigma_i l_i) \]

where there are \( k \)-factors of \( Z \) on the left. Finally take the colimit over \( q \rightarrow \infty \) to get the pairing

\[ (5.0.21) \quad \mu_k : NZ(E(\Sigma_k)) \otimes Z(l_1) \otimes \cdots \otimes Z(l_k) \rightarrow Z(\Sigma_i l_i) \]

The construction of the above pairing shows that the action of the symmetric group \( \Sigma_k \) by an element \( \sigma \in \Sigma_k \) on \( NZ(E(\Sigma_k)) \) cancels out with the action by the element \( \sigma^{-1} \) on the \( k \)-factors \( Z(l_1), \cdots, Z(l_k) \). Abbreviating \( \oplus_i Z(l) \) to \( Z \), the above pairing may be shortened to \( \mu_k : NZ(E(\Sigma_k)) \otimes Z^{\otimes k} \rightarrow Z \).

In order to extend the above pairing to all smooth schemes of finite type over \( k \), one observes that the pairing is contravariantly functorial in the argument \( U \). Therefore, one may use any Zariski open cover of the given scheme \( X \) and Zariski cohomological descent for the motivic complexes to extend the above pairing to all smooth schemes \( X \).

**Lemma 5.1.** Let \( n \) denote a fixed positive integer and \( p : E \rightarrow B \) denote a surjective map of schemes with fibers all \( \mathbb{A}^n \). Assume \( s : B \rightarrow E \) is a section to \( p \) and defines \( B \) as a closed sub-scheme of \( E \). If \( B \) is an irreducible scheme of finite type over \( k \), then so is \( E \).

**Proof.** One may use ascending induction on the dimension of \( B \) and generic flatness of the map \( p \) to obtain a proof. The case \( \text{dim}(B) = 0 \) is clear. \( \square \)
Theorem 5.2. (Associativity of \( \mu \)) Let \( \gamma_k : NZ(E(\Sigma_k)) \otimes NZ(E(\Sigma_{k-1})) \otimes \cdots \otimes NZ(E(\Sigma_1)) \to NZ(E(\Sigma_k)) \) denote the pairing defined by the operad-structure on \( \{NZ(E(\Sigma_k))\}_k \). Then the following diagram commutes strictly:

\[
\begin{array}{ccc}
NZ(E(\Sigma_k)) \otimes NZ(E(\Sigma_{j_1})) \otimes \cdots \otimes NZ(E(\Sigma_{j_k})) & \overset{shuffle}{\longrightarrow} & NZ(E(\Sigma_j)) \otimes Z^k \\
(5.0.22) & & \\
& \overset{\gamma_k \otimes id}{\longrightarrow} & \\
& & NZ(E(\Sigma_j)) \otimes Z^k
\end{array}
\]

Proof. The key observation here is that the pairing \( \mu' \) as in 5.0.16 defined above commutes with shuffle maps. The commutativity of the above diagram follows essentially from this observation. However we provide some details below, mainly for the sake of completeness.

Let \( s_{j_1}, \ldots, s_{j_k} \) denote simplices in \( NZ(E(\Sigma_{j_1})), \ldots, NZ(E(\Sigma_{j_k})) \) of dimensions \( p_1, \ldots, p_k \) respectively. Let \( t_k \) denote a q-simplex of \( NZ(E(\Sigma_k)) \).

We will first consider the \( q + p_1 + \cdots + p_k \)-simplex of \( NZ(E(\Sigma_{j_1} + \cdots + j_k)) \) defined by \( \gamma(t_k, s_{j_1}, \ldots, s_{j_k}) \). The associativity of the shuffle maps entering into its definition shows that this may be obtained as follows. One first applies a \( (p_1, \ldots, p_k) \)-shuffle \( (\sigma_{j_1}, \ldots, \sigma_{j_k}) \) to the simplex \( (s_{j_1}, \ldots, s_{j_k}) \) to obtain the \( p_1 + \cdots + p_k \)-simplex \((\sigma_{j_1}(s_{j_1}), \ldots, \sigma_{j_k}(s_{j_k}))\) of \( E(\Sigma_{j_1}) \otimes \cdots \otimes E(\Sigma_{j_k}) \). Next one applies a \((q + \Sigma_{p_i})\)-shuffle \((\sigma_{j_i}, \sigma_{j_k})\) to \((t_k, (\sigma_{j_1}(s_{j_1}), \ldots, \sigma_{j_k}(s_{j_k})))\) to obtain a \( q + \Sigma_{p_i} \)-simplex of \( E(\Sigma_j) \otimes E(\Sigma_{j_i}) \). Here \( \sigma_{j_i} : [q + \Sigma_{p_i}] \to q \), \( \sigma_{j_k} : [q + \Sigma_{p_i}] \to [q] \). Therefore, one may define a \( q + p \)-simplex \( \gamma_k(t_k, s_{j_1}, \ldots, s_{j_k}) \) as follows: \( \gamma_k(t_k, s_{j_1}, \ldots, s_{j_k}) \) is obtained as follows: \( \sigma_{j_{i_1}}(t_k) \) is a \( q + \Sigma_{p_i} \)-simplex of \( E(\Sigma_j) \) while \( \sigma_{j_{i_2}}(s_{j_1}, \ldots, s_{j_k}) \) is \( (\sigma_{j_{i_1}}(s_{j_1}), \ldots, \sigma_{j_{i_2}}(s_{j_k})) \) by the \( q + \Sigma_{p_i} \)-simplices \( E(\Sigma_j) \otimes \cdots \otimes E(\Sigma_{j_i}) \) i.e. \( \sigma_{j_{i_1}}(t_k) \) is a sequence of length \( q + \Sigma_{p_i} \). Let \( \gamma_k(t_k, s_{j_1}, \ldots, s_{j_k}) \) be a permutation of \( \sigma_{j_{i_1}}(t_k) \). Therefore, one lets the \( i \)-th element of \( \sigma_{j_{i_1}}(t_k) \) act by permuting the \( i \)-th elements of \( \sigma_{j_{i_1}}(t_k), \ldots, \sigma_{j_{i_2}}(s_{j_k}) \). This defines \( \gamma(t_k, s_{j_1}, \ldots, s_{j_k}) \).

For comparison with the other product, it is necessary to obtain a slightly different definition of the same product \( \gamma_k(t_k, s_{j_1}, \ldots, s_{j_k}) \). Each \( (q, \Sigma_{p_i}) \)-shuffle of \( q + \Sigma_{p_i} \) corresponds to a pair of non-decreasing maps \( \hat{\sigma}_k : [q + \Sigma_{p_i}] \to [q] \) and \( \hat{\sigma}_j : [q + \Sigma_{p_i}] \to [q] \) so that the pair \( (\hat{\sigma}_k, \hat{\sigma}_j) : [q + \Sigma_{p_i}] \to [q] \) is strictly increasing. Therefore, one may define a \( q + p \)-simplex \( t_k \) \( \bullet \) \( (\sigma_{j_1}(s_{j_1}), \ldots, \sigma_{j_k}(s_{j_k})) \) as follows. First one considers the \( q + p \)-simplex of \( NZ(E(\Sigma_k)) \otimes NZ(E(\Sigma_j)) \) defined by the product \( (9.3.4) \) in appendix. Observe that this consists of a sequence \((g_{\gamma_{j_1}}, g_{\gamma_{j_2}}, \ldots), (g_{\gamma_{j_1}(p_1+q)}, g_{\gamma_{j_2}(p_1+q)}, \ldots)\) of pairs with \( g_{\gamma_{j_1}} \in E(\Sigma_k) \) and \( g_{\gamma_{j_2}} \in E(\Sigma_k) \). In our case, for each \( l = 0, \ldots, p_1 + q \), \( g_{\gamma_{j_1}}(l) = (\gamma_{j_1}(l), \ldots, \gamma_{j_1}(l)) \) with each \( \gamma_{j_1}(l) \in E(\Sigma_{j_1}) \) for \( l = 1, \ldots, k \). Let each \( g_{\gamma_{j_1}}(l) \) act on the corresponding term \( g_{\gamma_{j_2}}(l) \) by permuting the \( k \)-factors. One readily sees that this product is the same as the one defined in the last paragraph.

Next one starts with cycles \( Z(\j_1), \ldots, Z(\j_k) \) in \( Z_n(l(\j_1)), \ldots, Z_n(l(\j_k)) \), \( i = 1, \ldots, k \). The cycle \( \mu(\gamma(t_k, s_{j_1}, \ldots, s_{j_k})) \) is obtained by applying the pairing of \( (5.0.20) \). Since the shuffle maps commute with the pairing \( \mu' \) (see 5.0.17 and \( (5.0.14) \)), it follows that the last pairing may be obtained as

\[
(5.0.22) \quad \text{shuffle} \circ \mu'(t_k \times s_{j_1} \times \cdots \times s_{j_k}) \times Z(\j_1), \ldots, Z(\j_1), \ldots, Z(\j_k), \ldots, Z(\j_k): Z(n+\mu_1+p) \to Z(n) \times Z(n+\mu_1+p).
\]

Here \( \text{shuffle} \) denotes the composition of all the shuffle maps for passage from \( \Delta[n] \times \Delta[p_1] \times \cdots \Delta[p_k] \times [q] \) to \( \Delta[n+\mu_1+p] \times [q] \).

Next consider the pairing defined by the composition of maps in the left-most column, the bottom row and the bottom part of the right column applied to the same cycles as above. One may see that the resulting cycle may be obtained from the cycles \( Z(\j_1) \circ \text{Cor}_{\kappa_1}(U \times \Delta[n]) \times s_{j_1} \) given by \( \mu'(s_{j_1}, Z(\j_1), \ldots, Z(\j_k)) \) as follows. (Here \( l(j) = l(j_1) + \cdots + l(j_k) \) where \( Z(\j_1) \circ \text{Cor}_{\kappa_1}(U \times \Delta[n]) \times l(\j_1) \). One composes this with a suitable shuffle map to obtain the class \( \mu(s_{j_1}, Z(\j_1), \ldots, Z(\j_k)) \circ \text{Cor}_{\kappa_1}(U \times \Delta[n] + p), l(\j_1) \). Next one applies shuffle maps corresponding to \( (\sigma_{j_1}, \ldots, \sigma_{j_k}) \) to obtain classes \( (\sigma_{j_1})^*Z(\j_1) \circ \text{Cor}_{\kappa_1}(U \times \Delta[n] + p), l(\j_1) \). (Recall \( p = \Sigma_{p_i} \)). Next one applies the pairing in \( (5.0.20) \) to these \( k+1 \) simplices as \( \sigma \) varies among the vertices of the simplex \( t_k \) to obtain the class \( \mu(t_k, s_{j_1}, Z(\j_1), \ldots, Z(\j_k), \ldots, Z(\j_k)) = \text{shuffle} \circ \mu'(t_k, \text{shuffle} \circ \text{shuffle}) \).
\( \mu'(s_{j_1}, Z(j_1), \ldots, Z(j_1)), \ldots, shuffle \circ shuffle \mu'(s_{j_k}, Z(j_k), \ldots, Z(j_k))) \). (Here the shuffle map marked \( shuffle \circ shuffle \) corresponds to the passage from \( \Delta[n] \times \Delta[p] \) to \( \Delta[n+p] \) (the passage from \( \Pi_i \Delta[n+p_i] \) to \( \Delta[n+p] \), the passage from \( \Delta[q] \times \Delta[n+p] \) to \( \Delta[n+p+q] \), respectively.)

In view of the observation in 5.0.17, one may postpone applying the shuffle maps so that one may identify the last product with

\[
(5.0.23) \quad shuffle \mu'(t_k, shuffle \mu'(s_{j_1}, Z(j_1)), \ldots, Z(j_1)), \ldots, shuffle \mu'(s_{j_k}, Z(j_k), \ldots, Z(j_k)))
\]

Here \( shuffle \mu' \) denotes the passage from \( \Pi_i \Delta[p_i] \) to \( \Delta[p] \), the last shuffle maps are the obvious remaining shuffle maps and \( \mu'(t_k, shuffle \mu'(s_{j_1}, Z(j_1)), \ldots, Z(j_1)), \ldots, shuffle \mu'(s_{j_k}, Z(j_k), \ldots, Z(j_k)) \) is a Corollary 5.5. Theorem 5.3. □

### Remark 5.6

A key observation now is that the action of \( \sigma \Sigma_k \) on the k-cycles \( Z(j_i), i = 1, \ldots, k \), corresponds to an action on the p-simplex \( \sigma_{j_1}(s_{j_1}) \cdots \sigma_{j_k}(s_{j_k}) \) of \( Z \Sigma_{j_1} \otimes \cdots \otimes Z \Sigma_{j_k} \) permuting the k-factors.

5.0.24. In view of the above observations the shuffle maps all commute with \( \mu' \) so that the cycle in (5.0.23) may be identified with

\[
(5.0.25) \quad \mu'(t_k \times (s_{j_1} \times \cdots \times s_{j_k}), \Pi_{i=1}^{j_1} Z(j_1), \ldots, \Pi_{i=1}^{j_k} Z(j_k))
\]

followed by appropriate shuffle maps. In view of the identification in (5.0.15) this may be identified with the term in (5.0.22) thereby proving the theorem. □

Clearly this also proves the following theorem

**Theorem 5.3.** \( \mathbb{Z} \) is an \( E^\infty \)-algebra over the \( E^\infty \)-operad \( \{NZ E \Sigma_k | k \} \) in the category \( C_\ast(S) \).

**Definition 5.4.** (i) \( \mathbb{Z}^{mot} \) will denote the motivic complex viewed as an algebra over the operad \( \{NZ E \Sigma_k | k \} \).

(ii) Recall from [K-M] Theorem 1.4, Chapter II, that there exists a functor \( W \) that converts any \( E^\infty \)-algebra tensored with \( \mathbb{Q} \) to a quasi-isomorphic strictly commutative differential graded algebra. We will let \( \mathbb{Q}^{mot}_X \) denote \( W(\mathbb{Z}^{mot} \otimes \mathbb{Q}) \). We also let \( \mathbb{Q}^{mot}_X(n) \) the corresponding piece of weight \( n \). If \( X \) is a smooth quasi-projective scheme over \( k \), \( \mathbb{Q}^{mot}_X = RF(X, \mathbb{Q}^{mot}_X) \) will be called the **motivic DGA** associated to \( X \).

Next we obtain the following corollary.

**Corollary 5.5.** \( \mathbb{Q}^{mot}_X \) is quasi-isomorphic to \( \mathbb{Z}_X \otimes \mathbb{Q} \) as a presheaf of \( E^\infty \)-differential graded algebras. Moreover, \( \mathbb{Q}^{mot}_X(n) \) is quasi-isomorphic to \( \mathbb{Z}_X(n) \otimes \mathbb{Q} \) and if \( X \) is a smooth quasi-projective variety, \( \mathbb{Q}^{mot}_X \) is quasi-isomorphic to \( \mathbb{Z}^{mot}_X \).

**Proof.** It follows from the proof [K-M] Corollary (1.5), Part II that there is a natural map \( \mathbb{Z}_X(n) \otimes \mathbb{Q} \to \mathbb{Q}^{mot}_X(n) \) which is a quasi-isomorphism compatible with the pairings. The quasi-isomorphism in the last statement now follows from the observation that the integral motivic complex, being quasi-isomorphic to the higher cycle complex, satisfies the localization property and hence has cohomological descent on the Zariski site of smooth quasi-projective schemes. □

**Remark 5.6.** It will be convenient and often necessary to obtain actions by other operads, (for example, the geometric Barratt-Eccles operad) on the motivic complex. These actions will be induced by the above action of the simplicial Barratt-Eccles operad and will be produced by defining a map from the new operad to the above Barratt-Eccles operad. It is convenient to invoke model structures on the category of operads (provided for example by Hinich and improved upon by Spitzweck: see [Hi] and [Sp]) to define such maps. We devote the next section to consider this technique.

### 6. Model structure on the category of operads and actions by other operads on the motivic complex

Now we proceed to define a model structure on the category of operads in \( C(S, R) \). Recall an operad \( O \) in \( C(S, R) \) is given by a sequence \( \{O(k) | k \geq 0\} \) of objects in \( C(S, R) \) satisfying the hypotheses in (9.4.1) through (9.4.3).
Next let $Op(S, R)$ denote the category of operads in the category $C(S, R)$. We will presently recall results from [Hin] and [Sp] that show the existence of a cofibrantly generated model category structure on $Op(S, R)$. First one shows that if $C(S, R)^\Sigma$ denotes the category of symmetric sequences in $C(S, R)$, i.e. sequences $\{K(i) | i \in \mathbb{Z}\}$ together with an action of $\Sigma_n$ on $K(n)$, then there is a cofibrantly generated model structure on $C(S, R)^\Sigma$, where the generating cofibrations (generating acyclic fibrations) are $I(\Sigma)$ ($J(\Sigma)$, respectively); here $I$ ($J$) denotes the generating set of cofibrations (acyclic cofibrations, respectively) in the cofibrantly generated model structure on $C(S, R)$ considered above. $I(\Sigma)$ denotes morphisms $f : K \to L$ of symmetric sequences given by sequences of maps $f(n) : K(n) \to L(n)$ that belong to $I$. $J(\Sigma)$ is defined similarly. Next one needs a free functor, $F : C(S, R)^\Sigma \to Op(S, R)$ left-adjoint to the obvious forgetful functor $U$ sending an operad $\{O(n)|n\}$ to the same sequence of complexes with the given action by the symmetric groups. In case $R$ is a field, for example, $\mathbb{Z}/p$ for $p$ a prime or $\mathbb{Q}$, every module over $R$ is flat and therefore, one may readily see from the definition of the free functor $F$ that it preserves weak-equivalences. Therefore, the following general result implies the existence of the required cofibrantly generated model structure on $Op(S, R)$:

**Proposition 6.1.** (See [SS] and [Sp]) Let $T$ be a monad in a cofibrantly generated model category $C$ with generating cofibrations $I$ and generating acyclic cofibrations $J$. Assume that $C[T]$ is the subcategory of algebras over $T$ in $C$ has coequalizers and suppose that every map in $TJ$-cell, where the cell complex is built in $C[T]$, is a weak equivalence in $C$. Then there is a cofibrantly generated model structure on $C[T]$, where a map is a weak equivalence or fibration if and only if it is a weak equivalence or fibration in $C$.

In the general case, for example, when $R = \mathbb{Z}$, the hypothesis that every map in $FJ$-cell is a weak-equivalence is often not satisfied. In this case, one has a slightly weaker structure, namely that of a cofibrantly generated semi-model category structure on $Op(S, R)$ provided by the following result of Spitzweck.

**Definition 6.2.** Let $C$ denote a cofibrantly generated model category with $I$ ($J$) the set of generating cofibrations (acyclic cofibrations, res). (i) A $J$-semi model category over $C$ is a left adjunction $F : C \to D$ and subcategories of weak equivalences, fibrations and cofibrations in $D$ such that the following axioms are fulfilled: (a) The adjoint of $F$ preserves fibrations and trivial fibrations. (b) $D$ is bicomplete and the two out of three and retract axioms hold in $D$ (c) Cofibrations in $D$ have the left lifting property with respect to trivial fibrations, and trivial cofibrations whose domain are cofibrant in $C$ have the left lifting property with respect to fibrations. (d) Every map in $D$ can be functorially factored into a cofibration followed by a trivial fibration, and every map in $D$ whose domain is cofibrant in $C$ can be functorially factored into a trivial cofibration followed by a fibration. (e) Cofibrations in $D$ whose domain is cofibrant in $C$ become cofibrations in $C$, and the initial object in $D$ is mapped to a cofibrant object in $C$. (f) Fibrations and trivial fibrations are closed under pullback.

We say that $D$ is cofibrantly generated if there are sets of morphisms $I$ and $J$ in $D$ such that $I$-inj is the class of trivial fibrations and $J$-inj the class of fibrations in $D$ and if the domains of $I$ are small relative to $I$-cell and the domains of $J$ are small relative to maps from $J$-cell whose domain becomes cofibrant in $C$.

**Proposition 6.3.** (Spitzweck) Let $T$ be a monad in a cofibrantly generated model category $C$ with generating cofibrations $I$ and generating acyclic cofibrations $J$. Assume that $C[T]$ has coequalizers. Suppose that every map in $TJ$-cell whose domain is cofibrant in $C$ is a weak equivalence in $C$ and every map in $T1$-cell whose domain is cofibrant in $C$ is a cofibration in $C$ (here in both cases the cell complexes are built in $C[T]$). Assume furthermore that the initial object in $C[T]$ is cofibrant in $C$. Then there is a cofibrantly generated $J$-semi model structure on $C[T]$ over $C$, where a map is a weak equivalence or fibration if and only if it is a weak equivalence or fibration in $C$.

**Corollary 6.4.** (i) The category $Op(S, R)$ is a cofibrantly generated $J$-semi-model category over $C(S, R)$ where $J$ denotes the generating acyclic cofibrations of $C(S, R)$. A map of operads $\{O(n)|n\} \to \{O(n)|n\}$ is a fibration if (weak-equivalence) if and only if each $O(n) \to O(n)$ is a fibration (quasi-isomorphism, respectively) in $C(S, R)$.

(ii) For any operad $\{O(n)|n\}$ in $Op(S, R)$, there exists an operad $C(\{O(n)|n\})$ cofibrant in the $J$-semi model structure on $Op(S, R)$ along with a natural map $C(\{O(n)|n\}) \to \{O(n)|n\}$ in $Op(S, R)$ which is a weak-equivalence, i.e. the induced map $C(\{O(n)|n\})(n) \to O(n)$ is a quasi-isomorphism. Moreover the action of $\Sigma_n$ on $C(\{O(n)|n\})(n)$ is free.

(iii) Given any acyclic operad, $\{O(n)|n\}$ (i.e. one for which the augmentation $\epsilon$ to the trivial operad $\mathbb{Z}$ is a quasi-isomorphism), and any operad $\{O(n)|n\}$ cofibrant in the above $J$-semi-model structure on $Op(S, R)$, there exists a map $\{O(n)|n\} \to \{O(n)|n\}$ of operads.

**Proof.** It follows readily from the definition of the free functor $F$ that the action of $\Sigma_n$ on $F(\{O(n)|n\})(n)$ is free. Since $F$ along with the underlying functor $U$ provides a monad in the above sense, one readily obtains (i). One
Theorem 7.1. The triangulated category $(\text{Linear schemes over } \text{Spec } k)$ is of finite dimension.

Definition 7.3. The full sub-categories of finite dimensional objects in the categories (i), (ii) and (iii) and of finite matrices in the category (iv) are also equivalent.

Proof. The proof is essentially in [K-M] Theorem 1.1, Part IV. (The key idea here is to use the theory of minimal Hopf-algebras.)

□

Corollary 6.5. The geometric Barratt-Eccles operad $M(\Sigma^n) = \{M(\Sigma^n)|n\}$ acts on the motivic complex $\text{Sym}_n^{mot}$.

Proof. We first replace the geometric Barratt-Eccles operads by cofibrant ones up to quasi-isomorphism. By the above corollary, these cofibrant operads then map to the simplicial Barratt-Eccles operads. Therefore, the required action is provided by this induced action through the action of the simplicial Barratt-Eccles operad on the motivic complexes considered in the last section.

□

7. Mixed Tate motives for smooth schemes over a field $k$

The results of this section generalize the constructions of [Bl-3], [Bl-K] and [K-M] for the category of mixed Tate motives over a field. We fix a smooth quasi-projective scheme $X$ over a field $k$. We let $A = \mathcal{Q}^{mot}_X$. We may assume therefore that $A$ is bi-graded via $k$-modules $A(r)$, where $q \in \mathbb{Z}$ and $r \geq 0$. Let $\mathcal{D}_A$ denote the derived category of co-modules over $A$-modules, i.e., differential bi-graded $A$-modules $M = \bigoplus M^{i,q}(r)$ where $M^{i,q}(r)$ may be non-zero for any pair of integers so that $\mathcal{H}^q(M)(r) = 0$ for all sufficiently small $q$. One may define a functor $Q : \mathcal{D}_A \to D(\mathbb{Q} - \text{vector spaces})$ by $Q(M) = \mathbb{Q}^{L_A} M$. Here $D(\mathbb{Q} - \text{vector spaces})$ denotes the derived category of bounded below complexes of $\mathbb{Q}$-vector spaces. Observe that this category has a natural $t$-structure, the heart of which is given by the complexes that have cohomology trivial in all degrees except 0. We let $\mathcal{H}_A$ denote the full sub-category of $\mathcal{D}_A$ consisting of complexes $K$ so that $\mathcal{H}^q(Q(K)) = 0$ for all $q \neq 0$. Let $\mathcal{F}\mathcal{H}_A$ denote the full sub-category of $\mathcal{H}_A$ consisting of complexes $K$ so that $\mathcal{H}^0(Q(K))$ is a finite dimensional $\mathbb{Q}$-vector space. We will make the following assumption throughout:

7.0.26. the DGA $A$ is connected in the following sense: $H^i(A)(r) = 0$ for $i < 0$, $H^0(A)(r) = 0$ if $r \neq 0$ and $H^0(A)(0) = \mathcal{Q}$.

Now we obtain the following theorem as in [K-M].

Theorem 7.1. The triangulated category $\mathcal{D}_A$ admits a $t$-structure whose heart is $\mathcal{H}_A$. Moreover $\mathcal{F}\mathcal{H}_A$ is a graded neutral Tannakian category over $\mathbb{Q}$ with fiber functor $w = \mathcal{H}^0 \circ Q$.

Proof. The proof is essentially in [K-M] Theorem 1.1, Part IV. (The key idea here is to use the theory of minimal models.)

□

One may apply the bar construction (see [K-M] p. 76) to the algebra $A$: we will denote this by $\overline{B}A$. Let $IA$ denote the augmentation ideal of $A$. We let $\chi_A = H^0(\overline{B}A)$. This is a commutative Hopf-algebra and, as in [K-M] p. 76, is a polynomial algebra with $k$-module of indecomposable elements a co-Lie algebra which is denoted $\gamma_A$. Now we obtain the following result.

Theorem 7.2. (See [K-M] p. 77) Assume the hypothesis (7.0.26). Then the following categories are equivalent:

(i) The heart $\mathcal{H}_A$ of $\mathcal{D}_A$

(ii) The category of generalized nilpotent representations of the co-Lie algebra $\gamma_A$

(iii) The category of co-modules over the Hopf-algebra $\chi_A$

(iv) The category $\mathcal{T}_A$ of generalized nilpotent twisting matrices in $A$

The full sub-categories of finite dimensional objects in the categories (i), (ii) and (iii) of finite matrices in the category (iv) are also equivalent.

Definition 7.3. (Linear schemes over $k$) (i) A scheme over $\text{Spec } k$ is 0-linear if it is either empty or isomorphic to any affine space $A^n_{\text{Spec } k}$.

(ii) Let $n > 0$ be an integer. A scheme $Z$, over $\text{Spec } k$, is $n$-linear, if there exists a triple $(U, X, Y)$ of schemes over $\text{Spec } k$ so that $Y \subseteq X$ is an $S$-closed immersion with $U$ its complement, $Y$ and one of the schemes $U$ or $X$ is $(n-1)$-linear and $Z$ is the other member in $\{U, X\}$. We say $Z$ is linear if it is $n$-linear for some $n \geq 0$.

(iii) Any reduced scheme $X$ of finite type over $\text{Spec } k$ will be called a variety. Linear varieties over $k$ are varieties over $\text{Spec } k$ that are linear schemes.
Example 7.4. The following are common examples of linear varieties. In these examples we fix a separably closed base field \( k \) and consider only varieties over \( k \).

- All toric varieties
- All spherical varieties (A variety \( X \) is spherical if there exists a reductive group \( G \) acting on \( X \) so that there exists a Borel subgroup having a dense orbit.)
- Any variety on which a connected solvable group acts with finitely many orbits. (For example projective spaces and flag varieties.)
- Any variety that has a stratification into strata each of which is the product of a torus with an affine space.

Remark 7.5. If the field is not separably closed, not all tori are split; therefore the varieties appearing above need not be linear in the sense of the definition 7.3. Over non-separably closed fields, any of the examples will be linear if and only if the tori appearing in the field are all split.

Corollary 7.6. Let \( X \) denote a smooth (connected) projective linear variety over a field \( k \) (not necessarily separably closed), or any one of the schemes appearing in the examples above which are also connected, projective and smooth. Assume that the Beilinson-Soulé conjecture holds for the rational motivic cohomology of \( \text{Spec} \ k \), i.e. \( H^*_S(\text{Spec} \ k; \mathbb{Q}(r)) = 0 \) if \( r < 0 \), \( H^*_S(\text{Spec} \ k; \mathbb{Q}(r)) = 0 \) if \( r \neq 0 \), \( H^*_S(\text{Spec} \ k; \mathbb{Q}(0)) \cong \mathbb{Q} \) (For example, \( k \) is a number field). Then the conclusions of theorem (7.2) hold for \( X \).

Proof. It suffices to show that the DGA, \( A \), appearing in the theorem is connected. If the field \( k \) is not separably closed, one may find a finite separable extension \( k' \) of \( k \) so that all the tori in the stratification of \( X \) split. By a transfer argument, one may therefore readily reduce to the case where \( k \) is separably closed. In this case, the variety in question is linear; therefore we may invoke the strong Kunneth decomposition for the class of the diagonal \( \Delta \) in \( CH^*(X \times X) \). (See [J-1] or [Tot-1] i.e.

\[(7.0.27) \Delta = \Sigma_i \alpha_i \times \beta_i = \Sigma p_{1i}^*(\alpha_i) \circ p_2^*(\beta_i) \] where \( p_i : X \times X \to X \) is the projection to the \( i \)-factor and \( \circ \) denotes the intersection product. Now we proceed to show that any class \( x \in CH^*(X, n) \) may be written as a linear combination

\[(7.0.28) x = \Sigma_i \alpha_i \circ p_1^* (p_2^*(\beta_i \circ x)) = \Sigma_i \alpha_i \circ p_2^*(p_1^*(\beta_i \circ x)) \]

Here \( p_i : X \to \text{Spec} \ k \) is the obvious projection. To obtain (7.0.28), first observe that \( x = p_i^* (\Delta) \circ p_2^*(\beta_i) \). By the projection formula and the observation that the class \( [\Delta] = [\Delta](1) = [X] \circ CH^*(X) \), we obtain equality of the classes \( [\Delta] \circ p_2^*(\beta_i) = \Delta_* (\Delta^*(p_2^*(\beta_i))) \). Therefore \( p_i^* (\Delta) \circ p_2^*(\beta_i) = p_i^*(\Delta) (\Delta^*(p_2^*(\beta_i))) = (p_1 \circ \Delta) \circ ((p_2 \circ \Delta)^*(\beta)) \).

Now substitute the formula for \( [\Delta] \) from (7.0.27) and use the projection formula to obtain the first equality in (7.0.28). The equality of this with the right-hand-side follows by flat-base-change. Observe that \( \alpha_i \circ CH^*(X, 0) \).

Therefore, the hypothesis that \( H^*_S(\text{Spec} \ k; \mathbb{Q}(r)) = 0 \) for \( r < 0 \) shows readily that \( H^*_S(X; \mathbb{Q}(r)) = CH^*(X, 2r - q) = 0 \) for \( q < 0 \). (In more detail: let \( \alpha_i \circ CH^*(X, 0; \mathbb{Q}) = H^{2i}_M(X; \mathbb{Q}(s)) \) and let \( p_i^*(\beta) \circ x \circ CH^*(X, 0; \mathbb{Q}) \).

The DGA \( A \) has a 1-minimal model, \( i : A < 1 \mapsto A \). (Recall a connected DGA \( B \) is said to be minimal if it is a free commutative algebra with decomposable differential : \( d(B) \subseteq (I(B))^2 \) where \( I(B) \) is the augmentation ideal of \( B \). \( B < 1 > \) is the sub-DGA of \( B \) generated by the elements of degree \( \leq 1 \) and their differentials. The 1-minimal model of a DGA \( A \) is a composite map \( B < 1 > \subseteq A \wedge A \) with the last map a quasi-isomorphism and with \( B \) minimal.) The map \( i \) induces an isomorphism on \( H^1 \) and is injective on \( H^2 \). We say \( A \) is a \( K(\pi, 1) \) if \( i \) is a quasi-isomorphism.

Theorem 7.7. (See [K-M] p. 77.) The derived category of bounded below chain complexes in \( \mathcal{H}_A \) is equivalent to the derived category \( D_{\mathcal{H}_A} \).

Definition 7.8. (The category of mixed Tate motives over \( X \).) Let \( \chi^{mot}_X \) denote the Hopf algebra \( H^0(\mathcal{B}A) \). The category of (rational) relative mixed Tate motives over \( X \), denoted \( \mathcal{MT}_F(X) \), will be defined to be the category of finite dimensional co-modules over \( \chi^{mot}_X \).

Theorem 7.9. If the DGA \( A \) is connected (in the sense of 7.0.26), \( \mathcal{MT}_F(X) \) is equivalent to the category \( 
\mathcal{F}_A \).

In particular, this holds for all smooth (connected) projective linear varieties over \( k \) or for any of the varieties appearing in the list 7.4 which are also connected, projective and smooth assuming the Beilinson-Soulé conjecture (see above) holds for the motivic cohomology of \( \text{Spec} \ k \), for example if \( k \) is a number field.
Let $Q(r)$ be the copy of $Q$ concentrated in bi-degree $(0, r)$ and regarded as a representation of $\gamma_A$ in the obvious manner.

**Corollary 7.10.** If $A$ is a $K(\pi, 1)$, then $\text{Ext}^q_{\mathcal{MTF}(X)}(Q, Q(r)) = \text{Ext}^q_{\mathcal{H}(A)}(Q, Q(r)) \cong H^q(A(r)) = H^q_{\mathcal{M}}(X, r) = CH^q(X, 2r - q; \mathbb{Q})$

7.1. $l$-adic realization: DGA structure for étale cohomology. We will first sketch an outline of the corresponding $l$-adic realization. First, following the approach in [BGSV] and [BMS], one defines a Hopf algebra $\chi_X^{et}$ so that the category of co-modules over it is equivalent to the category of mixed Tate $l$-adic representations of the (algebraic) fundamental group $\pi_1(X)$. (Here $l$ is different from the characteristic of the ground field $k$.) (The elements of the Hopf algebra $\chi_X^{et}$ are framed mixed Tate $l$-adic representations.) Using the cycle map and a variant of the bar construction, one applies the constructions of [Bl-K] to produce the $l$-adic realization functor. The relevant arguments are entirely similar to those in [Bl-3], [Bl-K] and are therefore omitted.

Let $k$ denote a field of characteristic $p \geq 0$ and let $l$ denote a fixed prime different from $p$, $\nu > 0$ an integer. Let $\epsilon : (\text{schms}/k)^{et} \to (\text{schms}/k)^{zar}$ denote the obvious map of sites associated to the obvious functor sending a Zariski open $U \to X$ to the same map but now viewed as an étale map. Let $\mathbb{Z}/l^{\nu \text{mot}} = \epsilon^*(\mathbb{Z}/l^{\nu \text{mot}})$ be the mod-$l^{\nu}$ motivic complex sheafified on the big étale site of all smooth schemes of finite type over $k$. We already showed in section 3, that this is an algebra over the étale Eilenberg-Zilber operad which is an acyclic operad in $C_{\nu}(S, \mathbb{Z}/l^{\nu})$. The acyclicity of this operad implies, as argued in section 6, that there exists a map of operads from the Barratt-Eccles operad to this. Using this map, $\mathbb{Z}/l^{\nu \text{mot}}$ gets the structure of an algebra over the Barratt-Eccles operad. Alternatively let $[NZ E\Sigma_n^\nu] = \epsilon^*(NZ E\Sigma_n^\nu[n \geq 0])$ denote the Barratt-Eccles operad sheafified on the big étale site; this is a sheaf of $E^{\infty}$-operads on the big étale site. Clearly $\mathbb{Z}/l^{\nu \text{mot}}$ is an algebra over this operad.

Moreover (see [MVW] chapter 10) by the rigidity theorem, one knows that $\mathbb{Z}/l^{\nu \text{mot}}$ identifies with the sheaf $\mu_\nu$ and that the (étale) hyper-cohomology of a smooth scheme $X$ over $k$ with respect to the former complex identifies with the étale cohomology with respect to $\mu_\nu$. Therefore $\mathbb{Z}/l^{\nu \text{mot}}$ provides an $E^{\infty}$-algebra structure for étale cohomology. Taking the sections over a given smooth scheme $X$, then the homotopy inverse limit over $\nu \to \infty$, tensoring with $\mathbb{Q}$ and applying a functor $W$ as in 5.4, one obtains a strict DGA-structure for continuous $l$-adic étale cohomology.

Observe as a consequence that the cycle map from the motivic complex to mod-$l^{\nu}$-étale cohomology may now be identified with the map induced by the obvious map: $\mathbb{Z}^{\nu \text{mot}} \to R_\epsilon(\mathbb{Z}/l^{\nu \text{mot}})$ where $\epsilon$ is the obvious map of sites from the big étale site to the big Nisnevich site of smooth schemes of finite type over $k$. (Observe that the obvious map $NZ E\Sigma_n^\nu \to R_\epsilon(NZ E\Sigma_n^{et})$ makes the latter a sheaf of $E^{\infty}$-algebras over the former.) It follows that the above cycle map is compatible with $E^{\infty}$-DGA structures. (In fact this is needed in the $l$-adic realization as outlined in the first paragraph above.)

8. Classical cohomology operations

The results of this section follow readily by invoking standard results (see for example, [May] or [H-Sch]) which deduces the existence of cohomological operations from the existence of an $E^{\infty}$-structure on cohomology. Let $X$ denote a smooth scheme of finite type over a field $k$. (We may assume this is quasi-projective for the sake of simplicity.) Let $l$ denote a prime different from the characteristic of $k$ and let $\mathbb{Z}^{\nu \text{mot}}$ denote the motivic complex associated to $X$.

8.1. Let $A_X^* = \mathbb{Z}^{\nu \text{mot}} \otimes \mathbb{Z}/l, \mathbb{Z}$ and let $A_X^{\nu \text{et}} = R^\Gamma(X, A_X)$ where the derived functor is taken on the Zariski (or Nisnevich site). We let $A_{Spec}^k$ and $A_{Spec}^{et} \otimes k$ the corresponding objects when $X = Spec \ k$. Recall that $A_X$ and $A_{Spec}^{et} \otimes k$ are sheaves of bi-graded $E^{\infty}$-algebras over the $E^{\infty}$-operad defined in the last section. Therefore, $A_X$ and $A_{Spec}^{et} \otimes k$ are now bi-graded $E^{\infty}$-differentiable graded algebras, so that $A_X = \oplus_r A_X^r$. Moreover $H^q_{\mathcal{M}}(X; \mathbb{Z}/l(r)) = H^q(A_X^r(r))$ which will be isomorphic to $CH^q(X, 2r - i; \mathbb{Z}/l)$ when $X$ is assumed to be quasi-projective as well.

8.2. Let $A_X^{\nu \text{et}, \nu} = \mathbb{Z}^{\nu \text{mot}} \otimes \mathbb{Z}/l, \mathbb{Z}$ and let $A_X^{\nu \text{et}, \nu} = R^\Gamma(X, A_X^{\nu \text{et}})$ where the derived functor is taken on the étale site. Moreover $H^q_{\mathcal{M}}(X; \mathbb{Z}/l(r)) = H^q(A_X^{\nu \text{et}, \nu}(r))$.

8.3. The motivic and étale derived categories associated to a scheme. For the purposes of this section we will define this as follows. Let $(smt\text{.schms})^{Nis}_X$ ((smt\text{.schms})^{et}_X) denote the big Nisnevich (étale, respectively) site of all smooth schemes of finite type over the given field $k$. For the most part, we will restrict to a fixed smooth scheme $X$. We consider for each smooth scheme $X$ of finite type over $k$, the small Nisnevich site $X^{Nis}_X$ and the corresponding small étale site $X^{et}_X$; we may denote either of these generically by $X^{et}_X$. Unless the distinction is
important we will continue to denote both the complexes $A_X$ and $A_{X_l}$ by $A_X$ itself. We consider unbounded (co-chain) complexes of sheaves $M$ of $A_X$-modules on the site $X$: this forms a category where the morphisms are module maps. We consider the corresponding homotopy category and the mod-$l$ motivic derived category will be the localization of this homotopy category by inverting maps that are quasi-isomorphisms. This category will be denoted $D(X)$. The external hom (internal hom) in this category will be denoted $\text{Ext}(-, -)$ ($\text{Hom}(-, -)$, respectively). (Observe that $\text{Ext}(M, N) = \text{Rf}(X, \text{RHom}(M, N))$.) We define the mod-$l$ cohomology of an object $M \in D(X)$ with weight $r$ to be $\text{Ext}^*(M; A_X(r))$. This will be denoted $H^*(M; \mathbb{Z}/l(r)) = H^{*+}(M; \mathbb{Z}/l)$. Let $D^-(X)$ denote the full sub-category of $D(X)$ consisting of (co-chain) complexes $K$ that are trivial in positive degrees. By identifying such co-chain complexes with chain complexes that are trivial in negative degrees, one may see that that the derived category $D^-(X)$ is equivalent to the derived category of simplicial Abelian presheaves on $X$. Recall that for any simplicial Abelian sheaf $F$ there is a diagonal map $\Delta : F \to F \otimes F$: this (together with the Alexander-Whitney map and the equivalence between simplicial Abelian sheaves and complexes of sheaves trivial in negative degrees) induces a diagonal map $\Delta : F \to F \otimes F$, $F \in D^-(X)$.

8.3.1. Observe (making use of the above diagonal map) that if $\text{MeD}^-(X)$, $\text{RHom}(M, A_X)$ has the obvious induced structure of a sheaf of $E^\infty$-algebras over the operad $\mathcal{O}(n)[n \geq 0]$ the operad $\mathcal{O}(n)[n \geq 0]$ in the étale case, respectively. (The required pairings are defined as the composition $\mathcal{O}(n) \otimes \text{RHom}(M, A_X) \otimes \mathcal{d}[\text{Hom}(\mathbb{Z}/l(r))] \to \text{RHom}(M, A_X)$.) The last map is defined by its adjoint: $\mathcal{O}(n) \otimes M \otimes \text{RHom}(M, A_X) \otimes \mathcal{d}[\text{Hom}(\mathbb{Z}/l(r))] \to \text{RHom}(M, A_X)$. Hence one obtains a graded ring structure on $\otimes H^*(M; \mathbb{Z}/l(r))$. Moreover, if $\mathbb{Z}/l(0)$ denotes the mod-$l$ motivic complex of weight 0, $\text{RHom}(\mathbb{Z}/l(0), A_X) \simeq A_X$ and there is a natural pairing $\text{RHom}(M, A_X) \otimes \text{RHom}(\mathbb{Z}/l(0), A_X) \to \text{RHom}(M, A_X)$ that is compatible with the above algebra structure on $\text{RHom}(M, A_X)$.

Recall the complex $\mathbb{Z}/l(0)$ is the complex with the constant sheaf $\mathbb{Z}/l$ in degree 0 and trivial elsewhere in both the motivic and the étale cases. Therefore, $\mathbb{Z}/l(0)[i]$ is the complex concentrated in degree $-i$ where it is the constant sheaf $\mathbb{Z}/l$: tensoring with this complex defines the degree-suspension $S_{deg}^i$. One may also obtain the following characterization of the degree-suspension (or the simplicial suspension): $\text{Ext}^*(S^i_{deg} M, K) \cong \text{Ext}^*(M, K[l])$, for $M, \text{MeD}(X)$. We define the Tate suspension (in the motivic case), $S^1_{T} M$ by $Z_{\text{et}}(A^1) \otimes M$. More precisely, we may make use of the pairing $\text{RHom}(\mathbb{Z}_{\text{et}}(A^1) \otimes M, \mathbb{Z}(1)) \to \text{RHom}(\mathbb{Z}_{\text{et}}(A^1) \otimes M, \mathbb{A}_X(r))$ to define the Tate suspension of the motivic cohomology of $M$. The composition of these two suspensions may be effected by tensoring with the canonical class $\text{TeH}^2(\mathbb{P}^1; \mathbb{Z}/l(1))$. We denote the composite suspension of $M$ by $S^1_{T} M$. Now we obtain the natural isomorphisms for any $\text{MeD}^-(X)$:

$$H^n(M; \mathbb{Z}/l(r)) \cong H^{n+1}(S^1_{T} M; \mathbb{Z}/l(r)) (8.3.2)$$

$H^n(M; \mathbb{Z}/l(r)) \cong H^{n+1}(S^1_{T} M; \mathbb{Z}/l(r + 1))$ and

$H^n(M; \mathbb{Z}/l(r)) \cong H^{n+2}(S^1_{T} M; \mathbb{Z}/l(r + 1))$

Remarks 8.1. 1. Observe that the second isomorphism in the motivic case shows $H^1(\text{RHom}(\mathbb{Z}_{\text{et}}(A^1) \otimes M, \mathbb{Z}(1))) \cong \mathbb{Z}$. Let $\tau$ denote the canonical class corresponding to $1 \in \mathbb{Z}$: clearly the Tate suspension in the motivic case may be effected by tensoring with this class.

2. All of the above discussion in the motivic case applies equally well when the Nisnevich site is replaced by the Zariski site.

Throughout the following discussion $H^*$ will denote either motivic or étale cohomology. We define bistable mod-$l$ cohomology operations of bi-degree $(i, j)$ to be sequences of natural transformations $H^n \to H^{n+i}(\mathbb{Z}/l(r))$ on $D^-(X)$ and which are contravariantly functorial in $X \in \text{shemes}$. In view of the suspension-isomorphisms above, these are determined by their restrictions to $H^{2n}(\mathbb{Z}/l(n))[n]$.

Recall there are Bockstein homomorphisms $\beta : H^n(M; \mathbb{Z}/l(r)) \to H^{n+1}(M; \mathbb{Z}/l(r))$ which are defined in the usual manner as the boundary homomorphism associated to the short-exact sequence: $0 \to \mathbb{Z}/l^2(r) \to \mathbb{Z}/l^2(r) \to \mathbb{Z}/l(r) \to 0$. These are clearly bi-stable cohomology operations.

One of the main results in this section is the following theorem, which shows the existence of classical motivic and étale cohomology operations for all primes $l$. 

Theorem 8.2. There exist operations $Q^a : H^q(X, \mathbb{Z}/l(t)) \to H^{q+2s(l-1)}(X, \mathbb{Z}/l(l.t))$ and 
$\beta Q^a : H^q(X, \mathbb{Z}/l(t)) \to H^{q+2s(l-1)+1}(X, \mathbb{Z}/l(l.t))$.

These operations satisfy the following properties:

(i) Contravariant functoriality: if $f : X \to Y$ is a map between smooth schemes over $k$, $f^* \circ Q^a = Q^a \circ f^*$
(ii) They commute with the simplicial suspension isomorphism in (8.3.2).
(iii) Let $x \in H^q(X, \mathbb{Z}/l(t))$. $Q^a(x) = 0$ if $2s > q$, $\beta Q^a(x) = 0$ if $2s \geq q$ and if $(q = 2s)$, then $Q^a(x) = x^l$.
(iv) If $\beta$ is the Bockstein, $\beta \circ Q^a = \beta Q^a$.
(v) Cartan formulae: For all primes $l$, $Q^a(x \otimes y) = \sum_{i+j=s} Q^i(x) \otimes Q^j(y)$ and
$\beta Q^a(x \otimes y) = \sum_{i+j=s} \beta Q^i(x) \otimes Q^j(y) + Q^i(x) \otimes \beta Q^j(y)$
(vi) Adem relations: (See [May] p. 183.) With the same terminology as above:
If $a < lb$, and $\epsilon = 0, 1$ one has

(8.3.2) \[ \beta^a Q^a = \Sigma_i (-1)^{ai+i}(a-i, (l-1)b-a+i-1)\beta^{a+i}Q^i \]

where $\beta^a Q^a = Q^a$ while $\beta^1 Q^a = \beta Q^a$. One also has

(8.3.3) \[ \beta^a Q^a \beta^b = (1-\epsilon)\Sigma_i (-1)^{ai+i}(a-i, (l-1)b-a+i-1)\beta^{a+b-i}Q^i = (-\epsilon)\Sigma_i (-1)^{ai+i}(a-i, (l-1)b-a+i)\beta^{a+b-i}Q^i \]

(vii) More generally, for any $M \in D^-(X)$, there exist cohomology operations

$Q^a : H^q(M; \mathbb{Z}/l(t)) \to H^{q+2s(l-1)}(M; \mathbb{Z}/l(l.t))$ and

$\beta Q^a : H^q(M; \mathbb{Z}/l(t)) \to H^{q+2s(l-1)+1}(M; \mathbb{Z}/l(l.t))$

satisfying the properties (ii) through (vi) as above. Moreover, if $f : M' \to M$ is a map in $D^-(X)$, the operations $Q^a$ commute with pull-back by $f$.

(viii) The operation $Q^a$ commutes with change of base fields and also with the higher cycle map into mod-$l \acute{e}tale$ cohomology.

Proof. Recall that $A_X$ is an $E^\infty$-dga over the endomorphism operad constructed in the last section, which was observed to be an $E^\infty$-operad. Therefore one obtains the existence of cohomology operations $Q^a$ which are defined as follows (see [May] p. 161):

(8.3.4) \[ Q^a(x) = (-1)^s \nu(q) \tilde{\theta}_* (e_{(2s-q)(l-1)} \otimes x^l), \]

$\beta Q^a(x) = (-1)^s \nu(q) \tilde{\theta}_* (e_{(2s-q)(l-1)+1} \otimes x^l)$

where $\nu(q) = (-1)^{(q-1)m/2}(ml)^q$, where $m = (l-1)/2$. (Observe that $e_{(2s-q)(l-1)}$ has degree $2s - q)(l-1)$ so that the total degree of $e_{(2s-q)(l-1)} \otimes x^l$ is $q + 2s(l-1)$. Since the weight of $x^l$ is $t.l$ and $\theta_* (e_{(2s-q)(l-1)} \otimes x^l)$ leaves the weight unchanged, $Q^a(x)$ has weight $t.l$.) The map $\tilde{\theta}_*$ is the map

(8.3.5) \[ \tilde{\theta}_* : H_*(B\pi; \mathbb{Z}/l) \otimes H^*(M; \mathbb{Z}/l(r))^t \otimes H^*(M; \mathbb{Z}/l(l(r)), \quad \tilde{\theta}_* (\tilde{e} \otimes \tilde{x}^l) = [\theta(e \otimes x^l)] \]

where $\pi$ denotes the group $\mathbb{Z}/l$, $e(x)$ denotes a cycle representing the cohomology class $\tilde{e}$ (respectively) and $[z]$ denotes the cohomology class represented by the cycle $z$.

Now all assertions except for the last assertion are immediate consequences of standard results on cohomology operations on algebras over $E^\infty$-operads: see [May] or [H-Sch]. (Recall that $e_i$ is defined only for $i \leq 0$; therefore we let $e_i = 0$ for $i > 0$ so that $Q^a(x) = 0$ if $q < 2s$.) The action of the endomorphism operad $\{O(p)|p\}$ on the complex $A_X$ was shown to be functorial in the base field $k$; therefore the operation $Q^a$ commutes with respect to
change of base fields. Moreover, as observed in Remark 8.1, the action of the operads \( \{O(p)|p\} \) and \( \{O(p)_{\text{et}}|p\} \) are compatible on the complexes \( \mathcal{A}_X \) and \( \mathcal{A}_{X_{\text{et}}} \); therefore, the operation \( Q^* \) is also compatible with respect to the higher cycle map as defined above.

**Remark 8.3.** The above operations cannot commute with the Tate suspension as one may see from elementary weight considerations. Therefore they are *not* bi-stable cohomology operations. Moreover, observe that \( Q^0 \) is not the identity and therefore \( \beta Q^0 \) is not \( \beta \). This difficulty will be removed once we define better operations \( P^* \) taking into account the weights properly: we leave this and other related issues to a sequel. Observe also that some of the above cohomology operations may be trivial owing to the fact that the weight may be high enough. (Recall: \( H^i_M(X;\mathbb{Z}/(j)) \cong CH^j(X,\mathbb{Z}/l;2j-i) = 0 \) if \( j > \text{dim}(X) + 2j - i \).) This shows that the motivic cohomology operations of Voevodsky cannot be deduced from the classical operations considered above.

We will denote the operations \( Q^* \) considered above on étale cohomology as \( Q^*_\text{et} \). When the base field is separably closed, one may identify \( \mu_l(r) \) with the constant sheaf \( \mathbb{Z}/l \); in this case, therefore, the weights are irrelevant, and we obtain cohomology operations in the usual sense, once it is shown that \( Q^0_{\text{et}} = \text{id} \). (This is proved below.) For example, if the base field is the field of complex numbers, the operations we obtain identify with the usual cohomology operations in mod-\( l \) singular cohomology (once the latter is identified with mod-\( l \) étale cohomology).

**Proposition 8.4.** The operation \( Q^0_{\text{et}} = \text{id} \).

**Proof.** Since the Tate suspension is irrelevant now, \( Q^0 \) commutes with the simplicial suspension and is contravariantly functorial we may reduce to checking this when \( M = \) the constant sheaf \( \mathbb{Z}/l \). In this case, \( Q^0_{\text{et}}(\alpha) = \alpha^l = \alpha \), for any \( \alpha \in \mathbb{Z}/l \). This proves the proposition. □

**Remark 8.5.** Another way to define cohomology operations in étale cohomology is to make use of étale homotopy theory, as in [A-M] or [Fr].
9. Appendix

9.1. Localization of simplicial presheaves (after Hirschorn and Morel-Voevodsky). We begin with the following two results, which are, by now, well-known in the literature. (See for example, [Jar].)

**Proposition 9.1.** Let $\mathcal{S}$ denote a small site with enough points. Then the following structure defines the structure of a simplicial model category on $\text{Simpl.Presh}(\mathcal{S})$ (or on the category $\text{Simpl.Sh}(\mathcal{S})$ of simplicial sheaves on the site $\mathcal{S}$):

- cofibrations are monomorphisms of presheaves
- a map $f : P' \to P$ is a weak-equivalence if it induces a weak-equivalence of the simplicial sets forming the stalks
- fibrations are defined by the right lifting property with respect to maps that are trivial cofibrations (i.e. cofibrations and weak-equivalences)

Moreover the following additional properties hold:

- this model category structure is cofibrantly generated where the generating trivial cofibrations $I$ (generating trivial cofibrations $J$) is the set of cofibrations (trivial cofibrations, respectively) $i : U \to V$ so that there exists a large enough cardinal number $\alpha$ with the cardinality of each $V_n(X)$ smaller than $\alpha$, $X \in \mathcal{S}$, $n \geq 0$.
- Let $\mathcal{G}P = \text{holim}(\mathcal{G}P[n])$ denote the Godement resolution (as in [J-2]). Then $\mathcal{G}Ex^\infty P$ is a fibrant object in $\text{Simpl.Presh}(\mathcal{S})$ so that the natural map $P \to \mathcal{G}Ex^\infty P$ is a weak-equivalence. (Here $Ex^\infty$ is a functor which produces a functorial fibrant approximation to a simplicial set.)

**Definition 9.2.** (i) We say a presheaf $P$ has cohomological descent on the site $\mathcal{S}$, if for each $U \in \mathcal{S}$, the natural map $\Gamma(U, P) \to \text{holim}(\Gamma(U, G^n P)[n])$ is a weak-equivalence.

(ii) Let $\mathcal{S}$ denote the Zariski site of a Noetherian scheme $X$. We say that a presheaf $P$ has the Mayer-Vietoris property on $X$, if for any two Zariski open sub-schemes $U$, $V$ of $X$, the diagram

$$
\Gamma(U \cup V, P) \to \Gamma(U, P) \times \Gamma(V, P) \to \Gamma(U \cap V, P)
$$

is a fibration sequence of simplicial sets.

**Proposition 9.3.** Suppose $X$ is a Noetherian scheme and $P$ is a simplicial presheaf on the Zariski site of $X$ having the Mayer-Vietoris property. Then $P$ has cohomological descent on the Zariski site of $X$.

For the rest of this section we will assume that $\mathcal{S}$ is either the big Nisnevich or Zariski site of all schemes of finite type over a given base scheme $S$. In order to consider the $A^1$-homotopy theory on $\text{Simpl.Presh}(\mathcal{S})$, we will first recall a few basic facts on localization of model categories, from [Hir].

**Definition 9.4.** (See [Hir] chapters 12 and 15.) (i) A model category is left proper if every pushout of a weak-equivalence along a cofibration is also a weak-equivalence.

(ii) A cellular model category is a cofibrantly generated model category for which there exists a set $I$ ($J$) of generating cofibrations (generating trivial cofibrations, respectively) so that both the domains and the co-domains of the elements of $I$ are compact, the domains of the elements of $J$ are small relative to $I$ and the cofibrations are effective monomorphisms.

**Proposition 9.5.** (See [Hir] chapters 12 and 14.) (i) Every pushout of a weak-equivalence between cofibrant objects is a weak-equivalence in any model category. (ii) If $\mathcal{M}$ is a model category where every object is cofibrant, $\mathcal{M}$ is a left proper model category. In particular, the categories, $\text{Simpl.Presh}(\mathcal{S})$ and $\text{Simpl.Sh}(\mathcal{S})$ on any small site $\mathcal{S}$ are left-proper as well as cellular.

**Proof.** All the assertions are clear, except the one claiming the categories $\text{Simpl.Presh}(\mathcal{S})$ and $\text{Simpl.Sh}(\mathcal{S})$ on any small site $\mathcal{S}$ are cellular. To see this, recall that the generating cofibrations $I$ (generating trivial cofibrations $J$) are those cofibrations $i : U \to V$ as in Proposition 9.1. Therefore, the cellularity of the above categories is also clear. (One may readily show that these cofibrations are in fact effective monomorphisms.)

9.2. **Reedy model structures on $C(S, R)^\Delta$.** The Reedy model structure on cosimplicial objects in a model category is discussed in [Hir] Chapters 17 -20. See also [B-K] for a discussion of the Reedy model structure for cosimplicial simplicial sets and cosimplicial simplicial groups. These define a Reedy model structure on $\text{Simpl.Presh}(\mathcal{S})^\Delta$ and on $\text{Simpl.Sh}(\mathcal{S})^\Delta$. These extend to a Reedy model structure on $C(S, R)^\Delta$. In particular any object $K = \{K(n)[n]\}$ of $C(S, R)^\Delta$ with each $K(n)[n]C(S, R)$ fibrant is a fibrant object of $C(S, R)^\Delta$. It follows from this that if $M$ is a cofibrant object and $N$ is a fibrant object of $C(S, R)^\Delta$, then $\text{Hom}_\Delta(M, N)$ defined
9.3. Left Bousfield localization of simplicial model categories. Let $\mathcal{M}$ denote a simplicial model category and let $\text{Map} : \mathcal{M}^{op} \times \mathcal{M} \rightarrow (\text{simplicial sets})$ denote the bi-functor defined by the simplicial structure. Assume that every object of $\mathcal{M}$ is cofibrant.

**Definition 9.6.** Let $S$ denote a set of morphisms in $\mathcal{M}$. An object $W$ of $\mathcal{M}$ is $S$-local if $W$ is fibrant and for every $s : S_1 \rightarrow S_2$ in $S$ and any $Z$ in $\mathcal{M}$, the induced map

$$\text{Map}(Z \times S_2, W) \rightarrow \text{Map}(Z \times S_1, W)$$

is a weak-equivalence. A morphism $f : X \rightarrow Y$ in $\mathcal{M}$ is an $S$-weak-equivalence, if for any $S$-local object $S_0$ of $\mathcal{M}$, the map $\text{Hom}_{\mathcal{M}}(Y, S_0) \rightarrow \text{Hom}_{\mathcal{M}}(X, S_0)$ induced by $f$ is an isomorphism, where $\text{HM}$ denotes the homotopy category associated to $\mathcal{M}$. The $S$-cofibrations are defined to be the same as the cofibrations of $\mathcal{M}$ and the $S$-fibrations are defined by right-lifting property with respect to all maps that are cofibrations and $S$-weak-equivalences.

**Definition 9.7.** ($A^1$-equivalences) Let $\mathcal{M} = \text{Presh}(S)$ or $\text{Sh}(S)$ with $S$ denoting the two maps $i_0 : \{0\} \rightarrow A^1$ and $i_1 : \{1\} \rightarrow A^1$. Then an $A^1$-equivalence corresponds to an $S$-equivalence.

The left Bousfield localization of $\mathcal{M}$ with respect to $S$ is a model category structure on the same category underlying $\mathcal{M}$ where the cofibrations (fibrations, weak-equivalences) are the $S$-cofibrations ($S$-fibrations, $S$-weak-equivalences, respectively).

**Remark 9.8.** One may verify readily that $W$ is $S$-local if and only if $W$ is fibrant and for every map $s : S_1 \rightarrow S_2$, the induced map $s^* : \text{Map}(S_2, W) \rightarrow \text{Map}(S_1, W)$ is a weak-equivalence.

**Theorem 9.9.** (See [Hir] chapter 4.) Let $\mathcal{M}$ denote a cellular left proper simplicial model category and $S$ a set of morphisms in $\mathcal{M}$. Then the left Bousfield localization of $\mathcal{M}$ with respect to $S$ exists. If $L_S(\mathcal{M})$ denotes this localization, then this is a left proper cellular simplicial model category.

**Theorem 9.10.** ($S$-local Whitehead theorem)(See [Hir] (3.3.7).) Assume the situation of the above theorem. Let $f : P \rightarrow P'$ denote a map that is an $S$-weak-equivalence with $P$ and $P'$ both $S$-local. Then $f$ is a weak-equivalence.

Let $A$ denote an Abelian category; a chain complex $K$ in $A$ will denote a sequence $K_\cdot \in A$ provided with maps $d : K_i \rightarrow K_{i-1}$ so that $d^2 = 0$. Let $C_0(A)$ denote the category of chain complexes in $A$ that are trivial in negative degrees. One defines the de-normalizing functor $D_N : C_0(A) \rightarrow (\text{simplicial objects in } A)$ as in [Ill] pp. 8-9. $D_N$ will be inverse to the functor $N : (\text{simplicial objects in } A) \rightarrow C_0(A)$ defined by $(NK)^n = \bigcap \ker(d_i : (NK)_n \rightarrow (NK)_{n-1})$ induced by $d_0$. We will also often consider the functor $C : (\text{simplicial objects in } A) \rightarrow C_0(A)$ defined by $K \mapsto$ the chain complex which in degree $n$ is $K_n$ and where the differential $d : C(K)_n \rightarrow C(K)_{n-1}$ is given by $d = \Sigma_i (-1)^id_i$. Given a chain complex trivial $K$ in negative degrees with differentials of degree $-1$, one may form the corresponding (co-)chain complex trivial in positive degrees with differentials of degree $+1$ by re-indexing $K$ in the obvious manner. This functor composed with the functor $C$ above will be denoted $C^* : ((\text{simplicial objects in } A) \rightarrow C_0(A)$ where the last is the category of (co-)chain complexes in the Abelian category $A$ that are trivial in negative degrees.

Given two positive integers $p$ and $q$, a $(p, q)$-shuffle $\pi$ is a permutation of $(1, \ldots, p+q)$ so that $\pi(i) < \pi(j)$ for $1 \leq i < j \leq p$ and for $p+1 \leq i < j \leq p+q$. We let $\mu$ be the restriction of $\pi$ to $(1, \ldots, p)$ and $\nu(j) = \pi(j+p)$, $1 \leq j \leq q$. Clearly $\pi$ is determined by $(\mu, \nu)$ and therefore, we will identify $\pi$ with the pair $(\mu, \nu)$. The set of all $(p, q)$-shuffles is in one-one correspondence with the set of all strictly increasing maps $(\phi, \psi) : [p+q] \rightarrow [p] \times [q]$; the correspondence is given by sending a shuffle $(\mu, \nu)$ to $(\phi, \psi)$ where $\phi(x)$ is the cardinality of the set $\{1 \leq i \leq p| (\mu, \nu)(i) \leq x\}$ and $\psi(x)$ is the cardinality of the set $\{p+1 \leq i \leq p+q| (\mu, \nu)(i) \leq x\}$. Each such map $(\phi, \psi)$ (and hence each shuffle map) defines an isomorphism of schemes

$$\Delta[p + q] \rightarrow \Delta[p] \times \Delta[q]$$

by the formula: $(t_0, \ldots, t_{p+q}) \mapsto t_0(\phi(0), \nu(0)) + \cdots + t_{p+q}(\phi(p+q), \psi(0)(p+q)).$
Given two simplicial objects \( K \) and \( K' \), one obtains a map called the shuffle map (where \( C \) denotes the functor \( 9.3 \) considered in the appendix)

\[
(9.3.4) \quad (\phi(0), \cdots, \phi(p), \psi(0), \cdots, \psi(q)) \quad \mapsto \quad (\phi(0), \cdots, \phi(p) \circ \psi(0), \cdots, \psi(q))
\]

which is defined by

\[
(9.3.5) \quad s = \text{shuffle} : K_p \otimes K'_q = C(K)_p \otimes C(K')_q \to C(K \otimes K')_{p+q}
\]

where the sum is over all \((p, q)\)-shuffles \((\mu, \nu)\) and where \(\sigma(\mu)\) is the signature of the permutation \(\mu\).

**Proposition 9.11.** Assume the above situation. Then the functor \(C\) is compatible with pairings. Moreover, the shuffle map above is strictly associative and strictly commutative.

We will define the Godement resolution for chain complexes of Abelian sheaves the following way. Given a chain complex, \( P \), of Abelian presheaves on the site \( S \), we let \( G(P) = N(\text{holim}_{\Delta}[G^n(DN(P))[n]] \). (One may observe that \(\text{holim}_{\Delta}[G^n(DN(P))[n] \) is a simplicial Abelian presheaf so that one may apply the normalization to it to produce a chain-complex.)

**9.4. Operads.** Basic definitions of operads and algebras over operads for topological spaces may be found in [?]. The same definitions apply with minor modifications to the unital symmetric monoidal category \( \text{Presh}(\mathcal{S}) \) as in section 1. Recall the main result of section 3 is the construction of an \( E^\infty \)-operad in the category \( \text{Co} - \text{chain}(\text{Ab} \cdot \text{Presh}(\mathcal{S})) \) by first constructing an operad in the category \( \text{Simpl} \cdot \text{Presh}(\mathcal{S}) \). The following detailed definition of operads in the general setting of \( \text{Presh}(\mathcal{S}) \) is provided so as to clarify the details of this construction. (Recall \( \text{Presh}(\mathcal{S}) \) is assumed to be provided with the structure of a unital symmetric monoidal structure as in section 1.)

An associative operad (or simply operad) \( O \) in \( \text{Presh}(\mathcal{S}) \) is given by a sequence \( \{O(k) | k \geq 0\} \) of objects in \( \text{Presh}(\mathcal{S}) \) along with the following data:

for every integer \( k \geq 1 \) and every sequence \( (j_1, \ldots, j_k) \) of non-negative integers so that \( \sum_{t=1}^{k} j_t = j \) there is given a map \( \gamma_k : O(k) \otimes (O(j_1) \otimes \cdots \otimes O(j_k)) \to O(j) \) so that the following associativity diagrams commute, where \( \sum_{t=1}^{k} j_t = j \) and \( \sum_{t=1}^{k} i_t = i \); we set \( g_s = j_1 + \cdots + j_s \) and \( h_s = i_{g_{s-1}+1} + \cdots + i_{g_s} \) for \( 1 \leq s \leq k \):

\[
(9.4.1) \quad \begin{array}{c}
O(k) \otimes (\bigotimes_{s=1}^{k} O(j_s)) \otimes (\bigotimes_{r=1}^{j_s} O(i_r)) \\
\downarrow \text{shuffle} \\
O(k) \otimes (\bigotimes_{s=1}^{k} O(j_s)) \otimes (\bigotimes_{q=1}^{i_{g_{s-1}+q}} O(i_{g_{s-1}+q})) \\
\downarrow id \otimes \gamma \\
O(k) \otimes (\bigotimes_{s=1}^{k} O(j_s)) \otimes (\bigotimes_{s=1}^{k} O(h_s))
\end{array}
\]

In addition one is provided with a unit map \( \eta : u \to O(1) \) so that the diagrams

\[
\begin{array}{ccc}
O(k) \otimes (u^\otimes k) & \xrightarrow{\eta \otimes \eta} & O(k) \\
\downarrow id \otimes \eta & & \downarrow \gamma \\
O(k) \otimes O(1)^\otimes k & \xrightarrow{id \otimes \eta} & O(1) \otimes O(j)
\end{array}
\]

and

\[
\begin{array}{ccc}
O(k) \otimes O(j) & \xrightarrow{\eta \otimes \eta} & O(j) \\
\downarrow id \otimes \eta & & \downarrow \gamma \\
O(1) \otimes O(j) & \xrightarrow{id \otimes \eta} & O(1) \otimes O(j)
\end{array}
\]
A commutative operad is an operad as above provided with an action by the symmetric group \( \Sigma_k \) on each \( \mathcal{O}(k) \) so that the diagrams commute.

\[
\begin{align*}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) & \xrightarrow{\sigma \otimes 1} \mathcal{O}(k) \otimes \mathcal{O}(j_{\sigma(1)}) \otimes \cdots \otimes \mathcal{O}(j_{\sigma(k)}) \\
\mathcal{O}(j) & \xrightarrow{\gamma} \mathcal{O}(j)
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) & \xrightarrow{\tau_1 \otimes \cdots \otimes \tau_k} \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \\
\mathcal{O}(j) & \xrightarrow{\gamma} \mathcal{O}(j)
\end{align*}
\]

commute. Here \( \sigma \in \Sigma_k \), \( \tau_1 \in \Sigma_{j_1} \), the permutation \( \sigma(j_{\sigma(1)}, \ldots, j_{\sigma(k)}) \) permutes the \( k \)-blocks of letters of length \( j_{\sigma(1)} \cdots j_{\sigma(k)} \) as \( \sigma \) permutes \( k \) letters and \( \tau_1 \cdots \tau_k \in \Sigma_j \) is the block sum.

For the remaining statements, we will assume that the category \( \text{Presh}(\mathcal{G}) \) has a model structure, so that it makes sense to consider objects that are acyclic, i.e. weakly-equivalent to the unit object \( u \). An operad is an \( A^\infty \)-operad (or acyclic operad) if each \( \mathcal{O}(k) \) is acyclic. It is an \( E^\infty \)-operad, if in addition, it is commutative and the given action of \( \Sigma_k \) on \( \mathcal{O}(k) \) is free.

Remark 9.12. The free Abelian group functor \( Z : (\text{Simpl.Presh}(\mathcal{G})) \to (\text{Simpl.ab.Presh}(\mathcal{G})) \) will provide a means of constructing operads in the category of simplicial Abelian presheaves by starting with an operad of simplicial presheaves. This has the following property: given two simplicial presheaves \( \mathcal{P} \) and \( \mathcal{P}' \), there exists a natural map \( Z(\mathcal{P}) \otimes Z(\mathcal{P}') \to Z(\mathcal{P} \times \mathcal{P}') \). i.e. the functor \( Z : \text{Simpl.Presh}(\mathcal{G}) \to \text{Simpl.ab.Presh}(\mathcal{G}) \) is compatible with the tensor structures.

Definition 9.13. An algebra \( A \) over an operad \( \mathcal{O} \) is an object in \( \text{Presh}(\mathcal{G}) \) provided with maps \( \theta : \mathcal{O}(j) \otimes A^\otimes j \to A \) for all \( j \geq 0 \) that are associative and unital in the sense that the following diagrams commute:

\[
\begin{align*}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) & \xrightarrow{\gamma \otimes 1} \mathcal{O}(j) \otimes A^\otimes \mathcal{O}(j_k) \\
\mathcal{O}(j) & \xrightarrow{\theta} A
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes A^\otimes \cdots \otimes \mathcal{O}(j_k) & \xrightarrow{\nu \otimes \gamma} \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \otimes A^\otimes \mathcal{O}(j_k) \\
\mathcal{O}(1) & \xrightarrow{\theta} A
\end{align*}
\]

If the operad is \( A^\infty \), we will refer to the algebra \( A \) as an \( A^\infty \)-algebra. If \( A \) is an algebra over an operad \( \mathcal{O} \) as above one defines a left \( A \)-module \( M \) to be an object in \( \text{Presh}(\mathcal{G}) \) provided with maps \( \lambda : \mathcal{O}(j) \otimes A^{\otimes j-1} \otimes M \to M \) satisfying similar associativity and unital conditions. Right-modules are defined similarly.

A commutative algebra over a commutative operad \( \mathcal{O} \) is an \( A^\infty \) algebra over the operad \( \mathcal{O} \) so that the following diagrams commute:
If, in addition, the operad is $E^\infty$, we will refer to the algebra $A$ as an $E^\infty$-algebra.

One may now observe the following. For each integer $n \geq 0$, let $u[\Sigma_n] = \bigsqcup_{\Sigma_n} u$ denote the sum of $u$ indexed by the symmetric group $\Sigma_n$. One may define the structure of a monoid on $u[\Sigma_n]$ as follows:

let $u_g$ denote the copy of $u$ indexed by $g \in \Sigma_n$. Now we map $u_g \otimes u_h$ to $u_{g,h}$ by the given map $\mu : u \otimes u \to u$.

If $O$ is a commutative operad in $Presh(\mathfrak{S})$, one may now observe that each $O(k)$ is a right-module over the monoid $u[\Sigma_k]$. (Observe that $O(k) \otimes u[\Sigma_k] \cong \bigoplus_{g \in \Sigma_k} O(k) \otimes u_g$. We map $O(k) \otimes u_g$ to $O(k) \otimes u$ by the map $g \otimes \text{id}$. Now apply the given map $O(k) \otimes u \to O(k)$.)
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