GENERALIZED MULTIPLICITY FORMULAE FOR MODULES OVER CONVOLUTION ALGEBRAS

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Abstract. Let G denote a complex reductive group acting on G-quasi-projective varieties $\overset{o}{U}$, U and $f : \overset{o}{U} \rightarrow U$ a G-equivariant *proper* map. We will further assume that $\overset{o}{U}$ is *smooth*. Under these assumptions we had provided, in our earlier work, functorial constructions of modules over the convolution algebra $H_*^{\mathbf{G}}(\overset{o}{U} \underset{U}{\times} \overset{o}{U}; \mathbb{Q})$ starting with the equivariant derived category of U. In the present paper we will provide a general multiplicity formula for the simple modules forming the composition series of these modules in terms of *equivariant intersection cohomology.* This generalizes (and is in fact inspired by Ginzburg's proof of) the multiplicity formulae for the simple modules in the composition series of the standard and co-standard modules over the affine Hecke-algebra associated to G. Moreover our constructions using equivariant perverse sheaves on the nilpotent variety provides a unification of the work of Kazhdan and Lusztig using equivariant homology and that of Ginzburg using non-equivariant homology.

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0. **Introduction**

This paper is a continuation of our earlier work where we provided general functorial constructions of modules over convolution algebras from suitable equivariant derived categories. In the present paper we will provide a general multiplicity formula for the simple modules forming the the composition series of these modules. We will show that these generalize the multiplicity formulae for the simple modules in the composition series of the standard and co-standard modules over affine Hecke-algebras. The Brylinski-Kashiwara proof of the Kazhdan-Lusztig conjecture (see [Bryl-K]) provides a multiplicity formula valid for a large class of modules, not just the Verma modules. In a sense this motivates our constructions.

We will assume the following basic conventions throughout the paper. A variety will mean a reduced scheme of finite type over Spec $\mathbb{C}$. Let G denote a linear algebraic group. A G-variety X is G-quasi-projective if it admits a G-equivariant locally closed immersion into a large projective space on which G acts linearly. (A basic theorem of Sumihiro (see [Sum]) shows that if G is connected, any normal quasi-projective variety with a G-action is G-equivariantly quasi-projective.) Now $D_b^{c,G}(X;\mathbb{C})$ will denote the G-equivariant derived category of complexes of $\mathbb{C}$-vector spaces with bounded constructible cohomology sheaves.

(0.1) We will assume the following basic situation throughout the paper. Let G denote a complex linear algebraic group acting on G-quasi-projective varieties $\overset{o}{U}$, U and $f : \overset{o}{U} \rightarrow U$ a G-equivariant *proper* map. We will further assume that $\overset{o}{U}$ is *smooth.*

Let V denote an open G-stable sub-variety of U, $\overset{o}{V} = f^{-1}(V)$ and $\overset{oo}{V} = \overset{o}{V} \underset{V}{\times} \overset{o}{V}$. Under these assumptions we had provided, in [J-1] and [J-2], functorial constructions of modules over the convolution algebra $H_*^G(\overset{oo}{V};\mathbb{C})$ starting with the equivariant derived category of V. (The results of the present paper are independent of those of [J-2] and depend only on those of [J-1].) The above convolution algebra is the G-equivariant (Borel-Moore) homology of $\overset{oo}{V}$ provided with an associative operation called convolution. The above convolution algebra $H_*^G(\overset{oo}{V};\mathbb{C})$ will be denoted $\mathbf{H}_{gr}$.

The following are *typical examples* of this set-up. The first two examples lead to Hecke algebras; the third seems to be an un-explored new situation and the fourth is closely related to affine quantum universal enveloping algebras of type A_n. (One may show readily that all varieties are in fact G-quasi-projective.) In the examples (0.2.1) through (0.2.2) we will let $\mathcal{B}$ denote the variety of all Borel-subgroups of a complex reductive (connected) group $\mathbf{G}$. Let $\mathcal{U}$ denote the variety of all unipotent elements in $\mathbf{G}$. Making use of the exponential mapping from the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$, one may observe that $\mathcal{U}$ is isomorphic to the variety $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$. Let $T^*\mathcal{B}$ denote the cotangent bundle to $\mathcal{B}$; using the above isomorphism of $\mathcal{U}$ with $\mathcal{N}$, one may identify $T^*\mathcal{B}$ with the desingularization of $\mathcal{U}$ (see [Stein-2]) given by $\Lambda = \{(u,B)|u \,\epsilon\, \mathcal{U},\ B \,\epsilon\, \mathcal{B} \text{ and } u \,\epsilon\, B\}$. (One may also identify the obvious map $\mu : \Lambda \rightarrow \mathcal{U}$ given by $(u,B) \rightarrow u$ with the moment-map $T^*\mathcal{B} \rightarrow \mathcal{N}$.) Now $\mathbf{G} \times \mathbb{C}^*$ acts on the right on Λ by

(0.2.0) $(B,u) \circ (g,q) = (g^{-1}Bg, g^{-1}u^q g)$.)

One may define a right-action of $\mathbf{G} \times \mathbb{C}^*$ on the unipotent variety $\mathcal{U}$ by $u.(g,q) = g^{-1}.u^q.g$, $u \,\epsilon\, \mathcal{U}$.

(0.2.1). $\overset{o}{U} = \mathcal{B} =$ the variety of all Borel subgroups (or the variety of all parabolic subgroups conjugate to a fixed parabolic subgroup P) of a complex reductive group $\mathbf{G}$, $U =$ Spec $\mathbb{C}$, $G = \mathbf{G}$ and $f =$ the obvious map.

(0.2.2). With $\mathbf{G}$, $\mathcal{U}$ and $\mathcal{B}$ as before, let $U =$ a $\mathbf{G}$-stable open sub-variety of $\mathcal{U}$ and $\overset{o}{U} = \Lambda_U = \{(u,B)|u \,\epsilon\, U \cap B, B \,\epsilon\, \mathcal{B}\}$. Let $G = \mathbf{G} \times \mathbb{C}^*$ and $f = \mu : \Lambda_U \rightarrow U$ the map sending $(u,B) \rightarrow u$. The action of G on U and on $\overset{o}{U}$ is described above.

(0.2.3) With $\mathbf{G}$, $\mathcal{U}$ and $\mathcal{B}$ as before, let P denote a fixed parabolic subgroup of $\mathbf{G}$, $\tilde{\mathcal{U}}^P =$ the partial desingularization of $\mathcal{U} =\{(x,P')|x \,\epsilon\, \mathcal{U} \cap P', P'=$ a parabolic subgroup conjugate to $P\}$. Now the variety Λ (= the Springer desingularization of $\mathcal{U}$) maps naturally onto $\tilde{\mathcal{U}}^P$. Let this map be denoted η. Let $U =$ a $\mathbf{G}$-stable open sub-variety of $\tilde{\mathcal{U}}^P$, $\overset{o}{U} = \eta^{-1}(U)$, $f =$ the obvious map induced by η and $G= \mathbf{G} \times \mathbb{C}^*$ with the actions as defined above.

(0.2.4). (See [G-V].) Let d denote an integer ≥ 1 and let $\mathfrak{F}$ = the set of all n-step flags in $\mathbb{C}^d$ of the form $F = (0 = F_0 \subseteq F_1 \subseteq ... \subseteq F_n = \mathbb{C}^d)$. Let $M = \{(F, x) \in \mathfrak{F} \times gl_d(\mathbb{C}^d) | x(F_i) \subseteq F_{i-1},\ i = 1, 2, ..., n\}$ and N= the variety of all $\mathbb{C}$-linear maps $x : \mathbb{C}^d \rightarrow \mathbb{C}^d$ so that $x^n = 0$. The group $G = GL(\mathbb{C}^d) \times \mathbb{C}^*$ acts on N and M as in (0.2.0). We let f denote the obvious projection to the second factor, U = any $GL(\mathbb{C}^d) \times \mathbb{C}^*$-stable open sub-variety of N and $\overset{o}{U} = f^{-1}(U)$. (Observe that M is smooth and f is proper.)

One may generalize the set-up in (0.1) as follows. Let $s \in G$ denote a fixed semi-simple element so that the fixed-point schemes U^s and $(\overset{o}{U})^s$ are *non-empty.* Now one may replace $f : \overset{o}{U} \rightarrow U$ by the map $f^s : (\overset{o}{U})^s \rightarrow U^s$ and G by $Z_G(s)$. Now $(0.2.1)_s$ $((0.2.2)_s$, $(0.2.3)_s$ and $(0.2.4)_s)$ will denote (0.2.1), ((0.2.2), (0.2.3) and (0.2.4) respectively) with G $(f : \overset{o}{U} \rightarrow U)$ replaced by $Z_G(s)$ $(f^s : \overset{o}{U}{}^s \rightarrow U^s$, respectively).

(0.3.1)Let V denote a G-stable *open* sub-variety of U, let $s \in G$ denote a fixed semi-simple element and let W denote a $Z_G(s)$-stable *locally-closed* sub-variety of V^s. Let $\overset{o}{W} = (f^s)^{-1}(W)$ and $\overset{oo}{W} = \overset{o}{W} \underset{W}{\times} \overset{o}{W}$. For each $v \in W$, we will let $M(v, s)$ denote the stabilizer of v in $Z_G(s)$ and let $M^o(v, s)$ = its connected component containing the identity.

Let $u \in W$ denote a fixed element so that u belongs to *a locally closed orbit* O of $Z_G(s)$ on W. Let $k_O : O \rightarrow W$ and $\kappa : W \rightarrow V$ denote the obvious maps. Let $i_u^s : u \rightarrow O$ denote the inclusion of u in O and let $\mathbb{C}_{(s)}$ denote the residue fields of $H^*(BZ_G(s); \mathbb{C})$ and $H^*(BG; \mathbb{C})$ corresponding to the maximal ideal associated to s. Now we obtain the following result.

(0.3.2) **Theorem**. (See (2.3) and (2.6.2).) Assume that in the above situation, the hypotheses (1.2.1) and (1.2.5) hold. (i) Now there exist two functors:

$$\bar{\mathfrak{M}}^*_{u,s}, \bar{\mathfrak{M}}^!_{u,s} : D_b^{c, Z_G(s)}(W; \mathbb{C}) \rightarrow (\text{finitely generated left and right modules over } \mathbb{C}_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H_*^{Z_G(s)}(\overset{oo}{W}; \mathbb{C}))$$

(ii) If in addition to the above hypotheses, $W = V^s$ and that $\mathbf{H}_{gr} = H_*^G(\overset{oo}{V}; \mathbb{C})$ is a projective $H^*(BG; \mathbb{C})$-module one obtains an isomorphism of convolution algebras: $\mathbf{H}_{gr} \simeq \mathbb{C}_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H_*^{Z_G(s)}(\overset{oo}{W}; \mathbb{C})$. Moreover, under the additional hypotheses (1.2.2) and (1.2.4), the composite functors $\bar{\mathfrak{M}}^*_{u,s} \circ \kappa^*$ and $\bar{\mathfrak{M}}^!_{u,s} \circ R\kappa^!$ are exact functors on restriction to $C^G(V)$ = the abelian category of G-equivariant perverse sheaves on V. □

The main result of this paper is the following theorem which provides a multiplicity formula for the simple $\mathbb{C}_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H_*^{Z_G(s)}(\overset{oo}{W}; \mathbb{C})$-modules forming the composition series of the modules obtained in (0.3.2). We need to first assume that $W \subseteq V^s$ is a $Z_G(s)$-stable *open* subvariety of V^s and that the given orbit O of $Z_G(s)$ in W is *closed.* Now observe that if $\mathcal{O}$ is a $Z_G(s)$-orbit on W whose closure contains the given $Z_G(s)$-orbit O, the $Z_G(s)$-equivariant irreducible locally constant sheaves on $\mathcal{O}$ correspond one-to-one with the irreducible representations of the finite group $\bar{M}(v, s) = M(v, s)/M^o(v, s)$ for any fixed point $v \in \mathcal{O}$. Assume we have chosen a point $v \in \mathcal{O}$ for each such $Z_G(s)$-orbit whose closure contains O.

(0.4) **Theorem** (See (3.11).). Assume in addition to the hypotheses in (0.1) and (0.3.1) the hypotheses (1.2.1), (1.2.5) and (1.2.6) below. (i) Now the simple modules forming the composition series of the modules in (0.3.2) are parameterized by pairs $(\mathcal{O}, \sigma)$ where $\mathcal{O}$ is a $Z_G(s)$-orbit on W so that its closure contains the given $Z_G(s)$-orbit O and σ is an irreducible representation of the finite group $\bar{M}(v, s)$. (These simple modules are denoted $L_{\mathcal{O},\sigma}$.)

(ii) Let $\mathcal{O}$ denote a $Z_G(s)$-orbit on V and for each irreducible representation σ of the finite group $\bar{M}(v, s)$, let $\mathcal{L}_\sigma$ denote the corresponding $Z_G(s)$-equivariant locally constant sheaf on $\mathcal{O}$. Let $K \in D_b^{c, Z_G(s)}(W; \mathbb{C})$. Now the multiplicity of the simple module $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^!_{u,s}(K^\cdot)$ is given by

$$dim(\mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{M^o(u,s)}(u;\ Ri_u^{s!}(Rk_O^!(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri_u^{s!}Rk_O^!(K^\cdot)))$$

Similarly the multiplicity of the simple module $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^*_{u,s}(K^\cdot)$ is given by

$$dim(\mathbb{C}_{(s)} \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{M^o(u,s)}(u;\, i_u^{s*}(k_O^*(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes i_u^{s*}k_O^*(K^\cdot)))$$

Here $IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma})$ denotes the equivariant intersection cohomology complex with the middle perversity (see (1.2.2) below) and obtained by starting with the local system $\mathcal{L}_{\mathcal{O}_\sigma}$ on $\mathcal{O}$.)

(iii) Assume in addition to the above hypotheses, the hypotheses (1.2.2) as well as (1.2.4). Let $P^\cdot \,\epsilon\, C^G(V)$ = the category of equivariant perverse sheaves on V. Now the multiplicity of the simple module $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^!_{u,s}(R\kappa^! P^\cdot)$ is given by

$$\sum_i dim H^i(Ri_u^{s!}(Rk_O^!(IC(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri_u^{s!}Rk_O^!(R\kappa^! P^\cdot))$$

Similarly the multiplicity of the simple module $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^*_{u,s}(\kappa^* P^\cdot)$ is given by

$$\sum_i dim H^i(i_u^{s*}(k_O^*(IC(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes i_u^{s*}k_O^*(\kappa^* P^\cdot)) \quad \square$$

Remarks i). The last result of Theorem (0.3.2) is particularly useful in relating the constructions of Kazdhan and Lustig with those of Ginzburg. Observe that the constructions of Kazdhan and Lusztig (see [K-L] and [Lusz-2]) are done almost exclusively at the level of equivariant homology while those of Ginzburg (see [C-G] and [G-V]) are done exclusively at the level of non-equivariant homology. These two different approaches give the same candidates for the standard and co-standard modules only because the modules that are constructed by Kazhdan and Lusztig in terms of equivariant homology are in fact projective modules over the cohomology ring of an appropriate group. The second statement of theorem (0.3.2) is a generalization of this fact to modules over convolution algebras constructed from equivariant perverse sheaves.

ii) The proof of the general multiplicity formula is *formally similar to and in fact inspired by* Ginzburg's proof of the multiplicty formula for standard and co-standard modules for Hecke algebras. It is shown in ([C-G](7.6.8)) that $\{L_{\mathcal{O},\sigma}|\mathcal{O},\sigma\}$ forms a complete collection of all the simple modules, some of them possibly zero. (Note: as remarked in [C-G] (7.6) whether or not a particular $L_{\mathcal{O},\sigma}$ is non-zero is rather a delicate question depending on the particular situation.) *Given this identification of the simple modules, we show that a careful analysis of the group action on the transverse slices as in (3.4.*) along with decomposition theorems in equivariant intersection cohomology and various projection formulae, enables one to compute the multiplicity formulae by an argument formally similar to that of [C-G].*

The above theorem specializes to provide the same multiplicity formulae for the simple terms of the composition series for the standard and co-standard modules over affine (and graded) Hecke-algebras as in [C-G]. It also applies to a wide variety of other situations as shown. In fact, we axiomatize the basic frame-work so that our results hold in as much generality as possible.

In the first section we axiomatize the correct framework for the rest of the paper; we also recall the basic results in our earlier work. In the second section we define the fibered functors $\bar{\mathfrak{M}}^*_{u,s}$ and $\bar{\mathfrak{M}}^!_{u,s}$ considered above. The main result in this section is theorem (2.3). We also show that under the hypothesis that $W = V^s$ and $H^G_*(\overset{oo}{V};\mathbb{C})$ is a projective module over $H^*(BG;\mathbb{C})$, we obtain an isomorphism of convolution algebras: $\mathbb{C}_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H_*^{Z_G(s)}(\overset{oo}{W};\mathbb{C}) \simeq \mathbb{C}_{(s)} \underset{H^*(BG;\mathbb{C})}{\otimes} H^G_*(\overset{oo}{V};\mathbb{C})$.

In the third section, we obtain the general multiplicity formula. We briefly consider some examples in the fourth section. The fifth section considers some supplementary results on equivariant derived categories.

1. The axiomatic framework.

In addition to the hypotheses in (0.1), we often impose various conditions on the varieties U and $\overset{o}{U}$. These are axiomatized so as to apply to as wide a context as possible.

(1.2.1) G acts with finitely many orbits on U.

(1.2.2) Each G-orbit on U is of *even* dimension (i.e. over $\mathbb{C}$). If $\mathcal{C}$ is such an orbit, $\mathcal{L}$ is a G-equivariant local system on $\mathcal{C}$ and $IC(\mathcal{L})$ is the intersection cohomology complex with the middle perversity so that $IC(\mathcal{L})_{|\mathcal{C}} \cong \mathcal{L}[\dim \ \mathcal{C}]$, then $\mathcal{H}^i(IC(\mathcal{L})) = 0$ for all *odd* i. (Observe as an immediate consequence that if $IC^G(\mathcal{L})$ is the corresponding *equivariant intersection cohomology complex* (see [Bryl] or [J-0]), $\mathcal{H}^i(IC^G(\mathcal{L})) = 0$ for all *odd* i. We will identify $IC(\mathcal{L})$ $(IC^G(\mathcal{L}))$ with its extension by zero from the closure of $\mathcal{C}$ to all of U (from the closure of $EG\underset{G}{\times}\mathcal{C}$ to all of $EG\underset{G}{\times}U$, respectively .) (Now $IC^G(\mathcal{L})$ is a complex on $EG\underset{G}{\times}U$ whose restriction to $EG\underset{G}{\times}\mathcal{C}$ is isomorphic to $\mathcal{L}[dim \ \ \mathcal{C}]$ viewed as a sheaf on the latter.)

(1.2.3) U is *rationally smooth* i.e. if $\mathbb{D}_{\mathbb{C}}$ is the dualizing complex on U, $\mathcal{H}^i(\mathbb{D}_{\mathbb{C}}) \cong \underline{\mathbb{C}}$ if $i = -2.\dim \ U$ and trivial otherwise.

(1.2.4) Let $s \in G$ denote a fixed semi-simple element and let $u \in U$ denote a fixed element. Now

$H_i(f^{-1}(u)^s;\, \mathbb{C}) = H^i(f^{-1}(u)^s;\, \mathbb{C}) = 0$ for all *odd* i.

(Here $H_i(f^{-1}(u)^s;\, \mathbb{C})$ denotes the Borel-Moore homology of the fixed point scheme of s on $f^{-1}(u)$; since the latter is projective, the above Borel-Moore homology may be identified with singular homology.)

(1.2.5) Let $s \in G$ denote a fixed semi-simple element and let $u \in U$ denote a point fixed by s. Let $\mathcal{C}_u$ denote the G-orbit of u in U. Now the fixed point scheme $\mathcal{C}_u^s$ is the disjoint union of closed subvarieties $\mathcal{C}_{u\,i}^s$, $i = 1, ..., n$, each of which is an orbit of $Z_G(s)$ on $\mathcal{C}_u^s$.

(1.2.6) *Existence of transverse slices.* Assume in addition to (0.1) that $s \in G$ and W is an open $Z_G(s)$-stable subvariety of V^s. For each $u \in W$ a fixed element, there exists a locally closed subvariety S_u of W so that the following hold:

S_u is stable by a maximal reductive subgroup $M^o(u,s)_{red}$ of the group $M^o(u,s)$

The obvious map $Z_G(s) \times S_u \to W$, $(g,v) \mapsto g.v$, is smooth and the dimension of S_u is the codimension of the $Z_G(s)$-orbit of u (i.e. S_u is a *transverse slice* to the $Z_G(s)$-orbit at u - see [Sl] pp. 60 -62.)

One can find a transverse slice satisfying the last condition alone for any algebraic group G acting on a G-quasi-projective variety in characteristic 0. The first condition is also satisfied at least in the case where the variety W admits a $Z_G(s)$-equivariant locally closed immersion into a smooth affine variety onto which the $Z_G(s)$-action extends. (For example one can find a transverse slice satisfying both conditions for the variety U in any one of the situations (0.2.2) or (0.2.4) as well as for the variety U^s in the situations $(0.2.2)_s$ or $(0.2.4)_s$.)

Notation. Let $S \subseteq U$ denote a G-stable locally closed subvariety. Now the obvious map $f^{-1}(S) \to S$ induced by f will be denoted f_S. If $s \in G$ is a fixed semi-simple element and $T \subseteq U^s$ is a $Z_G(s)$-stable locally-closed sub-variety, $f_T^s : f^{-1}(T)^s \to T$ will denote the map induced by f^s.

We conclude this section by recalling the following theorem from ([J-1](4.7) and (4.9)).

(1.3)**Theorem.** (i) Let $K \in D_b^{c,G}(\overset{oo}{V};\, \mathbb{C})$. Then $\mathbb{H}_G^*(\overset{oo}{V};\, K)$ has the structure of a left as well as right module over the convolution algebra $\mathbf{H}_{gr}$. Moreover if $K' \to K$ is a map of complexes in $D_b^{c,G}(\overset{oo}{V};\, \mathbb{C})$, one obtains an induced map $\mathbb{H}_G^*(\overset{oo}{V};\, K') \to \mathbb{H}_G^*(\overset{oo}{V};\, K)$ of modules over $\mathbf{H}_{gr}$.

(ii) Let $\pi : EG\underset{G}{\times}V \to BG$ denote the obvious map and let $f^* : H_G^*(V;\mathbb{C}) \to H_G^*(\overset{o}{V};\mathbb{C}) \to H_*^G(\overset{o}{V};\mathbb{C})$ denote the obvious map. Under the above hypotheses, the induced map $\Delta_* \circ f^* \circ \pi^* : H^*(BG;\, \mathbb{C}) \to H_*^G(\overset{oo}{V};\, \mathbb{C}) = \mathbf{H}_{gr}$ sends the cup-product on $H^*(BG;\, \mathbb{C})$ to the center of the convolution algebra $\mathbf{H}_{gr}$. (Here $\Delta : \overset{o}{V} \to \overset{oo}{V}$ is the obvious diagonal map.) □

Remark. Observe that, since the convolution algebra is not commutative, the left and right module structures on $\mathbb{H}_G^*(\overset{oo}{V};K)$ do *not* make it a bi-module. Moreover, the map $\Delta_* \circ f^* \circ \pi^*$ in (1.3)(ii) need *not* be an injective homomorphism. (For it to be injective certain additional conditions also need to be satisfied.)

(1.4) Assume the situation of (0.1). Now one may readily observe that the restriction $H^G_*(\overset{oo}{U};\mathbb{C}) \to H^G_*(\overset{oo}{V};\mathbb{C})$ is a homomorphism of convolution algebras. It follows that all left (right) modules over $H^G_*(\overset{oo}{V};\mathbb{C})$ inherit the structure of left (right, respectively) modules over $H^G_*(\overset{oo}{U};\mathbb{C})$.

2. The fibered functors

We begin with the following proposition.

(2.1)**Proposition**. Let G denote a linear algebraic group and let X denote a G-quasi-projective variety. Let $K^\cdot = \{K_n^\cdot|n\} \epsilon D_b^{c,G}(X;\mathbb{C})$ so that $\mathbb{H}^i(X; K_0) = 0$ for all *odd* i.

(i) Now $\mathbb{H}^n_{G'}(X; K) = 0$ for all *odd* n and for all closed subgroups G' of G.

(ii)If G' is *connected*, $\mathbb{H}^*_{G'}(X; K)$ is the finitely generated projective module over $H^*(BG'; \mathbb{C})$ given by $H^*(BG'; \mathbb{C}) \otimes \mathbb{H}^*(X; K_0)$.

Proof. Let G' denote a closed subgroup of G. We consider the spectral sequence:

$E_2^{u,v} = H^u(BG'.; R^v\pi_*(K)) \Rightarrow \mathbb{H}^{u+v}_{G'}(X; K)$

where $\pi : EG'\underset{G'}{\times}X \to BG'$ is the obvious map. Let $i : X \to EG'\underset{G'}{\times}X$ denote the obvious closed immersion. Now observe that the stalks $(R^v\pi_*(K))_{\bar{x}} \cong \mathbb{H}^v(X; i^*(K)) \cong \mathbb{H}^v(X; K_0)$ for each point $\bar{x} \epsilon BG'$. Moreover, for each fixed v, $R^v\pi_*(K)$ is a locally constant sheaf on BG' and the cohomology of BG' with respect to any locally constant sheaf is trivial in odd degrees. Therefore, the hypothesis implies that $E_2^{u,v} = 0$ for all u *odd* or v *odd*. Moreover the boundedness on K implies that $E_2^{u,v} = 0$ for all v sufficiently large. It follows that the above spectral sequence degenerates and $E_2^{u,v} = E_\infty^{u,v}$ for all u and v. Since the abutment $\mathbb{H}^n_{G'}(X;K)$ has a finite filtration whose associated graded terms are isomorphic to $E_\infty^{u,v}$, with $u+v=n$, it follows that $\mathbb{H}^n_{G'}(X;K) = 0$ is trivial if n is *odd*. This proves (i).

Assume G' is connected. Now BG' is simply-connected and therefore the G'-equivariant locally constant sheaf $R^v\pi_*(K)$ on BG' is constant. Therefore

(2.1.1) $E_2^{u,v} = H^u(BG'; R^v\pi_*(K)) \cong H^u(BG';\mathbb{C}) \otimes \mathbb{H}^v(X;K_0)$.

For each integer n, let $\sigma_{\leq n}$ denote the functor that kills the cohomology in degrees greater than n. Now we obtain a distinguished triangle : $\sigma_{\leq n-1}(R\pi_*(K)) \to \sigma_{\leq n}(R\pi_*(K)) \to \mathcal{H}^n(R\pi_*(K))[-n]$. Using the identification of the stalks of $R^v\pi_*(K)$ as above, the hypothesis that K is bounded and an ascending induction on n, *one may observe that the cohomology groups of BG' with respect to any of the above complexes vanish in odd degrees.* Now the distinguished triangle above provides a finite increasing filtration:

$\{F_n = H^*(BG';\sigma_{\leq n}(R\pi_*(K)))|n\}$ of $H^*(BG';R\pi_*(K)) \cong \mathbb{H}^*_{G'}(X;K)$

The associated graded term $F_n/F_{n-2} \cong H^*(BG';R^n\pi_*(K)) \cong \underset{m}{\oplus} E_2^{m,n}$ for any n *even*. Let $n \geq 0$ denote a fixed *even* integer. We will assume, using ascending induction on n, that F_k is a finitely generated projective module over $H^*(BG';\mathbb{C})$ for all k. Now observe the following: (i) if d is a sufficiently large positive integer, $F_{2d} \cong \mathbb{H}^*(BG';R\pi_*(K)) \cong \mathbb{H}^*_{G'}(X;K)$ and (ii) each F_n/F_{n-2} is clearly a finitely generated projective module over $H^*(BG';\mathbb{C})$. It follows that each F_n is also a finitely generated projective module over $H^*(BG';\mathbb{C})$ and that the short-exact sequences $0 \to F_{n-2} \to F_n \to F_n/F_{n-2} \to 0$ are split. It follows that $\mathbb{H}^*_{G'}(X;K) \cong H^*(BG';\mathbb{C}) \otimes \mathbb{H}^*(X;K_0)$. $\square$

(2.2) Let $s \epsilon G$ denote a fixed semi-simple element, fixed throughout the rest of this section. We will assume the hypotheses (1.2.1) and (1.2.5) throughout this section. Let $W \subseteq V^s$ denote a $Z_G(s)$-stable *locally closed* sub-variety; let $\kappa : W \to V$ denote the corresponding immersion. Let $k_O : O \to W$ denote the immersion of a *locally-closed* $Z_G(s)$-orbit into W and let $f^s_O : f^{-1}(O)^s \to O$ denote the induced map. Let $u \epsilon O$ denote a fixed point, let $M(u,s) \subseteq Z_G(s)$ denote the stabilizer of u and let $M^o(u,s)$ denote its connected component containing the identity element. Let $\bar{i}^s_u : f^{-1}(u)^s \to f^{-1}(O)^s$ denote the obvious closed immersion. Now the element s corresponds to a maximal ideal in $H^*(BM^o(u,s); \mathbb{C})$; let $\mathbb{C}_{(s)}$ denote the corresponding residue field.

(2.3) **Theorem**. Assume the above situation. Now the following hold. (i) There exist two functors:

$\bar{\mathfrak{M}}^*_{u,s,+},\ \bar{\mathfrak{M}}^!_{u,s,+} : D^{c,Z_G(s)}_b(W;\ \mathbb{C})$

$\rightarrow$(finitely generated left and right modules over $H^*(BM^o(u,s);\ \mathbb{C}) \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H^{Z_G(s)}_*(\overset{oo}{W};\mathbb{C})$

defined by

$\bar{\mathfrak{M}}^*_{u,s,+}(K) = \mathbb{H}^*_{M^o(u,s)}(f^{-1}(u)^s;\ \bar{i}^{s*}_u f^{s*}_O k^*_O(K^\cdot))$ and $\bar{\mathfrak{M}}^!_{u,s,+}(K) = \mathbb{H}^*_{M^o(u,s)}(f^{-1}(u)^s;\ R\bar{i}^{s!}_u Rf^{s!}_O Rk^{s!}_O(K^\cdot))$

(ii) Let $C^G(V)$ denote the category of G-equivariant perverse sheaves on V. Under the additional hypotheses (1.2.2) and (1.2.4), the composite functors $\bar{\mathfrak{M}}^*_{u,s,+} \circ \kappa^*$ and $\bar{\mathfrak{M}}^!_{u,s,+} \circ R\kappa^!$ *are exact* on restriction to $C^G(V)$ and take values in the category of *finitely generated projective modules* over $H^*(BM^o(u,s);\ \mathbb{C})$.

Proof. Let $\pi : \tilde{O} = M^o(u,s)\backslash Z_G(s) \rightarrow O = M(u,s)\backslash Z_G(s)$ denote the *universal covering.* Let $\widetilde{f^{-1}(O)}^s$ be defined by the cartesian square:

$$(2.3.1)\qquad \begin{array}{ccc} \widetilde{f^{-1}(O)}^s & \xrightarrow{\bar{\pi}} & f^{-1}(O)^s \\ \tilde{f}^s_O \downarrow & & \downarrow f^s_O \\ \tilde{O} & \xrightarrow{\pi} & O \end{array}$$

where $f^s_O : f^{-1}(O)^s \rightarrow O$ denotes the map induced by $f : \overset{o}{U} \rightarrow U$. Let $u \in O$. Consider

(2.3.2) $\mathbb{H}^*_{Z_G(s)}(\widetilde{f^{-1}(O)}^s;\ \tilde{f}^{s*}_O \pi^* k^*_O(K))$

Let $\tilde{i}^s_u : f^{-1}(u)^s \rightarrow \widetilde{f^{-1}(O)}^s$ denote the obvious closed immersion. Observe that $\widetilde{f^{-1}(O)}^s \cong M^o(u,s)\backslash(Z_G(s) \times f^{-1}(u)^s)$ where $M^o(u,s)$ acts on $(Z_G(s) \times f^{-1}(u)^s)$ by $m \circ (g,y) = (g.m^{-1}, m.y)$. Therefore [J-1] (2.P.3)' with $X = f^{-1}(u)^s$, $H = M^o(u,s)$ and $\bar{H} = Z_G(s)$ provides the isomorphism of $H^*(BZ_G(s);\mathbb{C})$-modules:

(2.3.3) $\mathbb{H}^*_{Z_G(s)}(\widetilde{f^{-1}(O)}^s;\ \tilde{f}^{s*}_O \pi^* k^*_O(K)) \cong \mathbb{H}^*_{M^o(u,s)}(f^{-1}(u)^s;\ \tilde{i}^{s*}_u \tilde{f}^{s*}_O \pi^* k^*_O(K)),\ K^\cdot \in D^{c,Z_G(s)}_b(W;\mathbb{C})$.

Observe that the composition $\pi \circ \tilde{f}^s_O \circ \tilde{i}^s_u = f^s_O \circ \bar{i}^s_u$. Therefore the term in (2.3.3) identifies with $\bar{\mathfrak{M}}^*_{u,s,+}(K)$.

Similarly one obtains the isomorphism of $H^*(BZ_G(s);\mathbb{C})$-modules:

(2.3.4) $\mathbb{H}^*_{Z_G(s)}(\widetilde{f^{-1}(O)}^s;\ R\tilde{f}^{s!}_O R\pi^! Rk^!_O(K)) \cong \mathbb{H}^*_{M^o(u,s)}(f^{-1}(u)^s; \tilde{i}^{s*}_u R\tilde{f}^{s!}_O R\pi^! Rk^!_O(K)),\ K \in D^{c,Z_G(s)}_b(W;\ \mathbb{C})$.

Next observe that $R\tilde{i}^{s!}_u \circ R\tilde{f}^{s!}_O \circ R\pi^! \circ Rk^!_O(K) \simeq \tilde{i}^{s*}_u \circ R\tilde{f}^{s!}_O \circ R\pi^! \circ Rk^!_O(K)$ (modulo an even dimensional shift). The last identification follows from (5.3.4) with $H = M^o(u,s)$, $\bar{H} = Z_G(s)$ and $X = f^{-1}(u)^s$. Now the observation that $\pi \circ \tilde{f}^s_O \circ \tilde{i}^s_u = f^s_O \circ \bar{i}^s_u$ shows the right hand side of (2.3.4) identifies with $\bar{\mathfrak{M}}^!_{u,s,+}(K)$.

Let $\bar{k}_O : f^{-1}(O)^s \rightarrow f^{-1}(W)^s$, $\overset{o}{W} = f^{-1}(W)^s$, $\overset{oo}{W} = \overset{o}{W} \underset{W}{\times} \overset{o}{W}$ and $\Delta : f^{-1}(W)^s = \overset{o}{W} \rightarrow \overset{oo}{W} = \overset{o}{W} \underset{W}{\times} \overset{o}{W}$ denote the diagonal immersion. If $L \in D^{c,Z_G(s)}_b(\widetilde{f^{-1}(O)}^s;\mathbb{C})$, $\Delta_* R\bar{k}_{O*} \bar{\pi}_*(L) \in D^{c,Z_G(s)}_b(\overset{oo}{W};\mathbb{C})$. Moreover

$\mathbb{H}^*_{Z_G(s)}(\widetilde{f^{-1}(O)}^s; L) \cong \mathbb{H}^*_{Z_G(s)}(\overset{oo}{W}; \Delta_* R\bar{k}_{O*} \bar{\pi}_*(L))$

Therefore, (1.3) with V replaced by W (G replaced by $Z_G(s)$ and $\overset{oo}{V}$ replaced by $\overset{oo}{W}$) shows that the latter has the structure of a left as well as right module over the convolution algebra $H^{Z_G(s)}_*(\overset{oo}{W};\mathbb{C})$. Taking $L = \tilde{f}^{s^*}_O \pi^* k^*_O K$ or $R\tilde{f}^{s^!}_O R\pi^! Rk^!_O K$, the identifications in (2.3.3) and (2.3.4) above show the action of $H^*(BZ_G(s);\mathbb{C})$ is through the action of $H^*(BM^o(u,s);\mathbb{C})$; therefore, it follows that the above functors take values in the category of left and right modules over the algebra $H^*(BM^o(u,s);\mathbb{C}) \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H^{Z_G(s)}_*(\overset{oo}{W};\mathbb{C})$. Moreover these are finitely generated since they are finitely generated over $H^*(BM^o(u,s);\mathbb{C})$ which maps into the center of the convolution algebra $H^*(BM^o(u,s) : \mathbb{C}) \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H^{Z_G(s)}_*(\overset{oo}{W};\mathbb{C})$ - see (1.3)(ii). This proves (i).

Next we consider (ii) for the functor $\bar{\mathfrak{M}}^!_{u,s,+}$ assuming that the conditions in (1.2.2) and (1.2.4) also hold. Let $P \in C^G(V)$. Now observe that $R(\kappa \circ k_O)^!(P) = D(\kappa \circ k_O)^* D(P) \cong (\kappa \circ k_O)^* D(P)^\vee[2n-2c]$ if $n =$ the dimension of V and $c =$ the codimension of O in V. (Here $(\kappa \circ k_O)^* D(P)^\vee = Hom_{\mathbb{C}}((\kappa \circ k_O)^* D(P), \underline{\mathbb{C}})$. Next observe that the category $C^G(V)$ is Artinian and Nöetherian with every object having a finite filtration whose simple quotients are the equivariant intersection cohomology complexes on the orbit closures in V. Moreover if a diagram

$$0 \to P' \xrightarrow{\alpha} P \to P'' \to 0$$

of objects in $C^G(V)$ is *exact*, $P'' \simeq Cone(\alpha) =$ the mapping cone of α. Now the hypothesis in (1.2.2) implies that $\mathcal{H}^i(P) = \mathcal{H}^i(D(P)) = 0$ for all *odd i*. Therefore, it follows from the hypothesis (1.2.2) that the cohomology sheaves of the complexes $(\kappa \circ k_O)^*(P)$ and $R(\kappa \circ k_O)^!(P)$ vanish in all *odd degrees.*

Now consider the distinguished triangle:

$$\sigma_{\le m-2} R(\kappa \circ k_O)^!(P) \simeq \sigma_{\le m-1} R(\kappa \circ k_O)^!(P) \to \sigma_{\le m} R(\kappa \circ k_O)^!(P) \to \mathcal{H}^m(R(\kappa \circ k_O)^!(P))[-m]$$

(Here m is assumed to be even.) Observe that the composite map $\pi \circ \tilde{f}^s_O \circ \tilde{i}^s_u : f^{-1}(u)^s \to O$ factors also as $i^s_u \circ p^s_u$, where $p^s_u : f^{-1}(u)^s \to u$ is the obvious projection and $i^s_u : u \to O$ is the obvious map. Apply the functor $Rp^{s!}_u Ri^{s!}_u = R\tilde{i}^{s!}_u \circ R\tilde{f}^{s!}_O \circ R\pi^! \simeq \tilde{i}^{s*}_u \circ R\tilde{f}^{s!}_O \circ R\pi^!$ (modulo an even dimensional shift) to the above distinguished triangle. (The last identification once again follows from (5.3.4).) Now the hypothesis (1.2.4) shows $\mathbb{H}^n(f^{-1}(u)^s; Rp^{s}_u{}^! Ri^s_u{}^! \mathcal{H}^m(R(\kappa \circ k_O)^!(P))[-m]) = 0$ for all *odd n* and all m even. By ascending induction on m we may assume that $\mathbb{H}^n(f^{-1}(u)^s; Rp^s_u{}^! Ri^s_u{}^! \sigma_{\le m-1}(R(\kappa \circ k_O)^!(P))) = 0$ for all *odd n* and all m even. Now the long-exact sequence in hyper-cohomology:

$$\dots \to \mathbb{H}^n(f^{-1}(u)^s; Rp^s_u{}^! Ri^s_u{}^! \sigma_{\le m-1}(R(\kappa \circ k_O)^!(P))) \to \mathbb{H}^n(f^{-1}(u)^s; Rp^s_u{}^! Ri^s_u{}^! \sigma_{\le m}(R(\kappa \circ k_O)^!(P)))$$
$$\to \mathbb{H}^n(f^{-1}(u)^s; Rp^s_u{}^! Ri^s_u{}^! \mathcal{H}^m(R(\kappa \circ k_O)^!(P))[-m]) \to \dots$$

shows that

(2.3.6) $\mathbb{H}^n(f^{-1}(u)^s; Rp^s_u{}^! Ri^s_u{}^! \sigma_{\le m}(R(\kappa \circ k_O)^!(P))) = 0$ for all *odd n.*

Therefore the hypotheses of (2.1) are satisfied with $X = f^{-1}(u)^s$, $G' = M^o(u,s)$ and the complex $K = Rp^s_u{}^! Ri^s_u{}^! \sigma_{\le m}(R(\kappa \circ k_O)^!(P))$. Taking m large enough, this proves that $\bar{\mathfrak{M}}^!_{u,s,+}(R\kappa^! P)$ is trivial in *odd* degrees and is a projective module over $H^*(BM^o(u,s); \mathbb{C})$.

To see the functor $\bar{\mathfrak{M}}^!_{u,s,+} \circ R\kappa^!$ is exact on restriction to $C^G(V)$ one may argue as follows. Let

$$0 \to P' \to P \to P'' \to 0$$

denote a short-exact sequence in the abelian category $C^G(V)$. As observed earlier, P'' is quasi-isomorphic to the mapping cone of the map $P' \to P$. Therefore one obtains a long-exact sequence:

$$\dots \to \mathbb{H}^i_{M^o(u,s)}(f^{-1}(u)^s; Rp^s_u{}^! Ri^s_u{}^!(R(\kappa \circ k_O)^!(P'))) \to \mathbb{H}^i_{M^o(u,s)}(f^{-1}(u)^s; Rp^s_u{}^! Ri^s_u{}^!(R(\kappa \circ k_O)^!(P)))$$
$$\to \mathbb{H}^i_{M^o(u,s)}(f^{-1}(u)^s; Rp^s_u{}^! Ri^s_u{}^!(R(\kappa \circ k_O)^!(P''))) \to \mathbb{H}^{i+1}_{M^o(u,s)}(f^{-1}(u)^s; Rp^s_u{}^! Ri^s_u{}^!(R(\kappa \circ k_O)^!(P))) \to \dots$$

The results of the last paragraph show that this breaks up into short-exact sequences. This proves the exactness of the functor $\bar{\mathfrak{M}}^!_{u,s,+} \circ R\kappa^!$ restricted to $C^G(V)$. The proof of the corresponding assertions for the functor $\bar{\mathfrak{M}}^*_{u,s,+} \circ \kappa^*$ are similar. $\square$

The *goal* of the remainder of this section is to establish the assertion in (2.5.1) below. In preparation for this, we will first prove the following results of a general nature.

(2.4.0) If D is a diagonalizable algebraic group acting on a variety X, we will let $K^0_D(X)$ denote the Grothendieck group of D-equivariant locally free coherent sheaves on X. We will assume that X is D-quasi-projective. In this case, one can readily find a closed D-equivariant immersion of X into an ambient smooth D-variety $\tilde{X}$. We define $K^0_{D,X}(\tilde{X}) =$ the Grothendieck group of D-equivariant coherent sheaves on $\tilde{X}$ with supports in X and $H^*_{D,X}(\tilde{X}) =$ the D-equivariant cohomology of $\tilde{X}$ with supports in X. We will let the isomorphisms

$K^0_{D,X}(\tilde{X}) \overset{\simeq}{\to} K_0^D(X)$ and $H^*_{D,X}(\tilde{X}) \overset{\simeq}{\to} H_*^D(X)$

provided by Poincare-Lefschetz duality be denoted PL. (Here $H_*^D(X)$ = the D-equivariant Borel-Moore homology of X.)

(2.4.1) If s is a semi-simple element in the group D, it corresponds to a maximal ideal in the cohomology ring $H^*(BD;\mathbb{C})$. We will let $H_*^D(X;\mathbb{C})_{(s)}$ denote the localization of $H_*^D(X;\mathbb{C})$ at this maximal ideal. The corresponding residue field will be denoted $\mathbb{C}_{(s)}$. Assume that $\tilde{X}$ is an ambient *smooth* D-variety containing X (as in (2.4.0)) and that $i : \tilde{Y} = \tilde{X}^s \to \tilde{X}$ is the obvious closed immersion. Let N ($N^\vee$) denote the normal (co-normal, respectively) bundle associated to the closed immersion i and let $\lambda_{-1}(N^\vee) = \Sigma(-1)^i \wedge^i N^\vee$: this is a class in the Grothendieck group $K^0_D(\tilde{Y})$. Assume henceforth that s is a *regular semi-simple* element in D (i.e. $\tilde{X}^s = \tilde{X}^D$). Now $\lambda_{-1}(N^\vee)$ is a unit in the localized ring $K^0_D(\tilde{Y})_{(s)}$. Therefore $ch(\lambda_{-1}(N^\vee)) \,\epsilon\, H^*_D(\tilde{Y};\mathbb{C})_{(s)}$ is a unit where ch denotes the equivariant Chern-character $ch : K^0_D(Y) \to H^*_D(Y;\mathbb{C})$. Observe that $i_* : K_0^D(Y)_{(s)} \cong K^0_{D,Y}(\tilde{Y})_{(s)} \to K_{D,X}(\tilde{X})_{(s)} \cong K_0^D(X)_{(s)}$ is now an isomorphism with the inverse provided by the map $\alpha \to \lambda_{-1}(N^\vee)^{-1} \cup i^*(\alpha)$. Now we will define

(2.4.2) $res^* : H_*^D(X;\mathbb{C})_{(s)} \to H_*^D(Y;\mathbb{C})_{(s)}$ by

$$res^*(\alpha) = [ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td(N)^{-1}] \cap PL(i^*(PL^{-1}(\alpha)))$$
$$= PL([ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td(N)^{-1}] \cup i^*(PL^{-1}(\alpha)))$$

where $Td(N)$ denotes the equivariant Todd class of the normal bundle N. (The last equality follows from the usual relations between cup and cap products since the isomorphism denoted PL is given by taking the cap-product with respect to a *fundamental class.*)

(2.4.3) **Lemma**. Assume the above situation. Now the map res^* is an isomorphism with inverse given by i_*.

Proof. Now one obtains a commutative diagram:

$$\begin{array}{ccccccc}
K_0^D(X)_{(s)} & \xrightarrow{PL^{-1}} & K^0_{D,X}(\tilde{X})_{(s)} & \xrightarrow{i^*()\cup\lambda_{-1}(N^\vee)^{-1}} & K^0_{D,Y}(\tilde{Y})_{(s)} & \xrightarrow{PL} & K_0^D(Y)_{(s)} \\
\downarrow \tau_X & & \downarrow ch()\cup Td_{\tilde{X}} & & \downarrow ch()\cup Td_{\tilde{Y}} & & \downarrow \tau_Y \\
H_*^D(X)_{(s)} & \xrightarrow{PL^{-1}} & H^*_{D,X}(\tilde{X})_{(s)} & \xrightarrow{i^*()\cup ch(\lambda_{-1}(N^\vee)^{-1})\cup Td(N)^{-1}} & H^*_{D,Y}(\tilde{Y})_{(s)} & \xrightarrow{PL} & H_*^D(Y)_{(s)}
\end{array}$$

where the maps τ_X and τ_Y are defined so as to make the first and last squares commute and $Td_{\tilde{X}}$ ($Td_{\tilde{Y}}$) denotes the equivariant Todd class of $\tilde{X}$ ($\tilde{Y}$, respectively). (ch denotes the equivariant local chern character.) Observe that the top row is inverse to i_* and that the first and last vertical maps become isomorphisms on tensoring the first row with $\mathbb{C}$. Now the middle square commutes since $Td_{\tilde{Y}} = i^*(Td_{\tilde{X}}) \cup Td(N)^{-1}$. Since the top row and the vertical maps become isomorphisms on tensoring the top row with $\mathbb{C}$, it follows that the bottom row is also an isomorphism. Now the definition of the map res^* shows that it is the same as the bottom row. □

(2.4.4) **Corollary**. Let $f : X \to Y$ denote a D-equivariant proper map of smooth D-varieties, where D is a given diagonalizable group. Let s denote a regular semi-simple element of D and let $f^s : X^s \to Y^s$ denote the map induced by f. Now one obtains the commutative diagram:

$$\begin{array}{ccc}
H_*^D(X;\mathbb{C})_{(s)} & \xrightarrow{f_*} & H_*^D(Y;\mathbb{C})_{(s)} \\
res^* \downarrow & & \downarrow res^* \\
H_*^D(X^s;\mathbb{C})_{(s)} & \xrightarrow{f_*^s} & H_*^D(Y^s;\mathbb{C})_{(s)}
\end{array}$$

Proof. Consider the diagram when the vertical maps are replaced by i_{X*} and i_{Y*} where $i_X : X^s \to X$ and $i_Y : Y^s \to Y$ are the obvious closed immersions. These are inverses to the vertical maps above by (2.4.3). The functoriality of push-forward shows that this new diagram commutes and therefore so does the original one. □

(2.5.1) Once again assume that we are in the situation of (0.1). Let $s \,\epsilon\, G$ denote a fixed semi-simple element and let $D(s)$ denote the diagonalizable sub-group scheme generated by s. (Observe that s is a regular semi-simple element of $D(s)$.) Let V denote a locally closed G-stable sub-variety of U and $W = V^s$. Let $\overset{o}{W} = (\overset{o}{V})^s$

and $\overset{oo}{W} = \overset{o}{W} \underset{W}{\times} \overset{o}{W} \cong (\overset{oo}{V})^s$. If G' denotes a closed algebraic subgroup of G and $s \in G'$ denotes a semi-simple element, we will let (s) denote the corresponding maximal ideal in $H^*(BG';\mathbb{C})$ as well as in $H^*(BG;\mathbb{C})$ while $\mathbb{C}_{(s)}$ will denote the corresponding residue fields. *In the remainder of this section, we will prove that, if in addition* $H^G_*(\overset{oo}{V};\mathbb{C})$ *is a projective module over* $H^*(BG;\mathbb{C})$, *one obtains an isomorphism of convolution algebras*:

$$\mathbb{C}_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H^{Z_G(s)}_*(\overset{oo}{W};\mathbb{C}) \cong \mathbb{C}_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H^G_*(\overset{oo}{V};\mathbb{C})$$

Let $i : \overset{o}{W} \to \overset{o}{V}$ denote the obvious closed immersion and let $N^\vee$ denote the co-normal bundle associated to i. Let $\lambda_{-1}(N^\vee) = \Sigma(-1)^i \wedge^i N^\vee$ denote the class in $K^0_{D(s)}(\overset{o}{W})$ as above. Now $1 \boxtimes N^\vee$ $(N^\vee \boxtimes 1)$ denotes the co-normal bundle associated to the closed immersion $\overset{o}{V} \times \overset{o}{W} \to \overset{o}{V} \times \overset{o}{V}$ ($\overset{o}{W} \times \overset{o}{V} \to \overset{o}{V} \times \overset{o}{V}$, respectively). If $\bar{N}^\vee$ denotes the co-normal bundle associated to the closed immersion $\overset{o}{W} \times \overset{o}{W} \to \overset{o}{V} \times \overset{o}{V}$, one observes that $\lambda_{-1}(\bar{N}^\vee) = \lambda_{-1}(N^\vee) \boxtimes \lambda_{-1}(N^\vee) = (\lambda_{-1}(N^\vee)\boxtimes 1) \circ (1 \boxtimes \lambda_{-1}(N^\vee))$. (Here $\circ$ denotes the product in $K^0_{D(s)}(\overset{o}{W} \times \overset{o}{W})$. Since $\lambda_{-1}(\bar{N}^\vee)$ is a unit in $K^0_{D(s)}(\overset{o}{W} \times \overset{o}{W})_{(s)}$, it follows that both $\lambda_{-1}(N^\vee) \boxtimes 1$ and $1 \boxtimes \lambda_{-1}(N^\vee)$ are also units in $K^0_{D(s)}(\overset{o}{W} \times \overset{o}{W})_{(s)}$.

(2.5.2) It follows that $1 \times ch(\lambda_{-1}(N^\vee)) = ch(1 \boxtimes (\lambda_{-1}(N^\vee)))$ and $ch(\lambda_{-1}(N^\vee)) \times 1 = ch(\lambda_{-1}(N) \boxtimes 1)$ in $H^*_{D(s)}(\overset{o}{W} \times \overset{o}{W};\mathbb{C})_{(s)}$ are units where $ch : K^0_{D(s)}(\overset{o}{W} \times \overset{o}{W})_{(s)} \to H^*_{D(s)}(\overset{o}{W} \times \overset{o}{W};\mathbb{C})_{(s)}$ denotes the equivariant chern-character and the $\times$ denotes the external product. Let $Td(N)$ denote the $D(s)$-equivariant Todd-class of N and let $1 \times Td(N) \in H^*_{D(s)}(\overset{o}{W} \times \overset{o}{W};\mathbb{C})$ denote the external product of the class 1 and $Td(N)$.

(2.5.3) **Proposition**. (See [C-G] (4.10.12).) Assume the above situation. Now the map $\widehat{res} : H^{D(s)}_*(\overset{oo}{V};\mathbb{C})_{(s)} \to H^{D(s)}_*(\overset{oo}{W};\mathbb{C})_{(s)}$ defined by $\widehat{res}(\alpha) = [1 \times ch(\lambda_{-1}(N^\vee)^{-1}) \cup (1 \times Td(N)^{-1})] \cap PL(\bar{i}^* PL^{-1}(\alpha))$ is an isomorphism of convolution algebras.

Proof. The proof is essentially in [C-G] (4.10.12) and (4.10.13). Throughout the proof, we will abbreviate $Td(N)$ to Td. Let $\hat{p}_{1,3}$ denote the projections to the first and third factor $\overset{ooo}{V} = \overset{o}{V} \underset{V}{\times} \overset{o}{V} \underset{V}{\times} \overset{o}{V} \to \overset{oo}{V}$ as well as $\overset{ooo}{W} = \overset{o}{W} \underset{W}{\times} \overset{o}{W} \underset{W}{\times} \overset{o}{W} \to \overset{oo}{W}$. Let α, $\beta \in H^{D(s)}_*(\overset{oo}{V};\mathbb{C})_{(s)}$. Now

$(\widehat{res}(\alpha)) * (\widehat{res}(\beta)) = \hat{p}_{1,3*}(p^*_{1,2}(\widehat{res}(\alpha)) \cup p^*_{2,3}(\widehat{res}(\beta)))$

$= \hat{p}_{1,3*}(p^*_{1,2}([1 \times ch(\lambda_{-1}(N^\vee)^{-1}) \cup (1 \times Td(N)^{-1})] \cap PL(\bar{i}^* PL^{-1}(\alpha))) \cup p^*_{2,3}([(1 \times ch(\lambda_{-1}(N^\vee)^{-1})) \cup (1 \times Td(N)^{-1})] \cap PL(\bar{i}^* PL^{-1}(\beta))))$

$= \hat{p}_{1,3*}([1 \times (ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1}) \times (ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1})] \cap PL(p^*_{1,2}(\bar{i}^* PL^{-1}(\alpha)) \cup p^*_{2,3}(\bar{i}^* PL^{-1}(\beta))))$

$= \hat{p}_{1,3*}(p^*_{1,3}((ch(\lambda_{-1}(N^\vee)) \cup Td) \times 1) \cup [(ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1}) \times (ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1}) \times (ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1})]$

$\cap PL(p^*_{1,2}(\bar{i}^* PL^{-1}(\alpha)) \cup p^*_{2,3}(\bar{i}^* PL^{-1}(\beta))))$

$= (ch(\lambda_{-1}(N^\vee) \cup Td) \times 1) \cap \hat{p}_{1,3*}([(ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1}) \times (ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1}) \times (ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1})]$

$\cap PL(p^*_{1,2}(\bar{i}^* PL^{-1}(\alpha) \cup p^*_{2,3}(\bar{i}^* PL^{-1}(\beta))))$

The last equality follows from the projection formula - see for example [J-1](2.P.10). Now observe that the last term may be identified with

$((ch(\lambda_{-1}(N)) \cup Td) \times 1) \cap \hat{p}_{1,3*}(res^*(PL(p^*_{1,2}(PL^{-1}(\alpha)) \cup p^*_{2,3} PL^{-1}(\beta))))$

where res^* denote the homomorphism defined in (2.4.2) associated to the closed immersion $\overset{o}{W} \times \overset{o}{W} \times \overset{o}{W} \to \overset{o}{V} \times \overset{o}{V} \times \overset{o}{V}$. One may now use (2.4.4) to identify the last term with

$((ch(\lambda_{-1}(N^\vee)) \cup Td) \times 1) \cap res^*(\hat{p}_{1,3*}((PL(p_{1,2}^*(PL^{-1}(\alpha)) \cup p_{2,3}^* PL^{-1}(\beta))))$

where the last res^* denotes the homomorphism in (2.4.2) associated to the closed immersion $\bar{i} : \overset{o}{W} \times \overset{o}{W} \to \overset{o}{V} \times \overset{o}{V}$. Now the definition of res^* in (2.4.2) shows that the last term may be identified with:

$$= [((ch(\lambda_{-1}(N^\vee)) \cup Td) \times 1) \cup ((ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1}) \times (ch(\lambda_{-1}(N^\vee)^{-1}) \cup Td^{-1}))]$$
$$\cap PL\bar{i}^* PL^{-1}(\hat{p}_{1,3*}(PL(p_{1,2}^*(PL^{-1}(\alpha)) \cup p_{2,3}^* PL^{-1}(\beta))))$$
$$= [(1 \times ch(\lambda_{-1}(N^\vee)^{-1})) \cup (1 \times Td^{-1})] \cap PL\bar{i}^* PL^{-1}(\alpha * \beta) = \widehat{res}(\alpha * \beta).$$

These prove that the map $\widehat{res}$ is a homomorphism of convolution algebras. In order to see it is an isomorphism, observe that the map $\alpha \to \widehat{res}(\alpha) \cap [(ch(\lambda_{-1}(N^\vee))^{-1} \times 1) \cup (Td^{-1} \times 1)] = res^*(\alpha)$ where res^* is the homomorphism defined in (2.4.2) associated to the closed immersion $\bar{i} : \overset{o}{W} \times \overset{o}{W} \to \overset{o}{V} \times \overset{o}{V}$. Since res^* is an isomorphism by (2.4.3) and since $[(ch(\lambda_{-1}(N^\vee)) \times 1) \cup (Td^{-1} \times 1)]$ is a unit in $H^*_D(\overset{o}{W} \times \overset{o}{W}; \mathbb{C})_{(s)}$, it follows that $\widehat{res}$ is also an isomorphism. □

(2.5.4) Assume the situation of (2.5.1). We will further assume that $H^G_*(\overset{oo}{V}; \mathbb{C})$ is a *projective module* over $H^*(BG; \mathbb{C})$. It will now follow from the Kunneth spectral sequence in (5.1.1) that

(2.5.4.1) $H^{Z_G(s)}_*(\overset{oo}{V}; \mathbb{C}) \cong H^*(BZ_G(s); \mathbb{C}) \underset{H^*(BG;\mathbb{C})}{\otimes} H^G_*(\overset{oo}{V}; \mathbb{C})$.

Moreover, it follows by the same arguments that, if $D(s)$ denotes the diagonalizable sub-group of G generated by s,

(2.5.4.2) $H^{D(s)}_*(\overset{oo}{V}; \mathbb{C}) \cong H^*(BD(s); \mathbb{C}) \underset{H^*(BG;\mathbb{C})}{\otimes} H^G_*(\overset{oo}{V}; \mathbb{C})$.

(2.5.5) **Proposition**. Assume the hypotheses of (2.5.4). Now one obtains an isomorphism

$$\mathbb{C}_{(s)} \underset{H^*(BG;\mathbb{C})}{\otimes} H^G_*(\overset{oo}{V}; \mathbb{C}) \overset{\cong}{\to} \mathbb{C}_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H^{Z_G(s)}_*(\overset{oo}{W}; \mathbb{C})$$

of convolution algebras.

Proof. Under the hypotheses of (2.5.4) we obtain the isomorphisms of convolution algebras:

(2.5.5.*) $\mathbb{C}_{(s)} \underset{H^*(BG;\mathbb{C})}{\otimes} H^G_*(\overset{oo}{V}; \mathbb{C}) \cong \mathbb{C}_{(s)} \underset{H^*(BD(s);\mathbb{C})}{\otimes} H^{D(s)}_*(\overset{oo}{V}; \mathbb{C})$

$\mathbb{C}_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H^{Z_G(s)}_*(\overset{oo}{W}; \mathbb{C}) \cong \mathbb{C}_{(s)} \underset{H^*(BD(s);\mathbb{C})}{\otimes} H^{D(s)}_*(\overset{oo}{W}; \mathbb{C})$.

The first follows readily by tensoring both sides of (2.5.4.2) with $\mathbb{C}_{(s)}$ over $H^*(BD(s); \mathbb{C})$. Since $\overset{oo}{W} = (\overset{oo}{V})^s$, it follows from (2.5.4.1) that $H^{Z_G(s)}_*(\overset{oo}{W}; \mathbb{C})_{(s)} \cong H^{Z_G(s)}_*(\overset{oo}{V}; \mathbb{C})_{(s)}$ is a *projective module* over $H^*(BZ_G(s); \mathbb{C})_{(s)}$. Now (5.1.3) provides the isomorphism:

$$H^{D(s)}_*(\overset{oo}{W}; \mathbb{C})_{(s)} \cong H^*(BD(s); \mathbb{C})_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})_{(s)}}{\otimes} H^{Z_G(s)}_*(\overset{oo}{W}; \mathbb{C})_{(s)}$$

One may readily observe that these are isomorphisms of convolution algebras. Finally tensor both sides with $\mathbb{C}_{(s)}$ over $H^*(BD(s); \mathbb{C})_{(s)}$ to obtain the second isomorphism in (2.5.5.*). Now (2.5.3) applies to show that the induced map $id \otimes \widehat{res} : \mathbb{C}_{(s)} \underset{H^*(BD(s);\mathbb{C})}{\otimes} H^{D(s)}_*(\overset{oo}{V}; \mathbb{C}) \to \mathbb{C}_{(s)} \underset{H^*(BD(s);\mathbb{C})}{\otimes} H^{D(s)}_*(\overset{oo}{W}; \mathbb{C})$ is also an isomorphism. □

Let $u \in V^s$, let (s) denote also the corresponding maximal ideal in $H^*(BM^o(u, s); \mathbb{C})$ and let $\mathbb{C}_{(s)}$ denote the corresponding residue fields. Now we define functors:

(2.6.1) $\bar{\mathfrak{M}}^*_{u,s}, \bar{\mathfrak{M}}^!_{u,s} : D^{c, Z_G(s)}_b(W; \mathbb{C}) \to$(finitely generated left and right modules over $\mathbb{C}_{(s)} \underset{H^*(BZ_G(s);\mathbb{C})}{\otimes} H^{Z_G(s)}_*(\overset{oo}{W}; \mathbb{C})$)

by

$$\bar{\mathfrak{M}}^*_{u,s}(K) = \mathbb{C}_{(s)} \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \bar{\mathfrak{M}}^*_{u,s,+}(K) \text{ and}$$

$$\bar{\mathfrak{M}}^!_{u,s}(K) = \mathbb{C}_{(s)} \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \bar{\mathfrak{M}}^!_{u,s,+}(K)$$

(2.6.2) **Proposition**. Under the hypothesis that $H^G_*(\overset{oo}{V};\mathbb{C})$ is a projective module over $H^*(BG;\mathbb{C})$, the above functors take values in the category of finitely generated left and right modules over $\mathbb{C}_{(s)} \underset{H^*(BG;\mathbb{C})}{\otimes} H^G_*(\overset{oo}{V};\mathbb{C})$. Moreover, under the hypotheses (1.2.2) and (1.2.4), the compositions $\bar{\mathfrak{M}}^*_{u,s} \circ \kappa^*$ and $\bar{\mathfrak{M}}^!_{u,s} \circ R\kappa^!$ are exact functors when restricted to the abelian category $C^G(V)$.

Proof. This follows readily from (2.5.5) and (2.3). □

Remark. The hypothesis that $H^G_*(\overset{oo}{V};\mathbb{C})$ is a projective module over $H^*(BG;\mathbb{C})$ is satisfied in the cases (0.2.1) and (0.2.2) with $V = U = \mathcal{U}$. This follows from [K-L] or from [J-4] Theorem (5.5).

3. The general multiplicity formula

(3.0) Assume the following in addition to the hypotheses in (0.1): *$s \in G$ is a fixed semi-simple element*, V denotes a G-stable *open* subvariety of U and W denotes an *open* and $Z_G(s)$-stable subvariety of V^s. (One may now observe from (1.4) that the obvious restriction: $H^{Z_G(s)}_*(\overset{oo}{V^s};\mathbb{C}) \to H^{Z_G(s)}_*(\overset{oo}{W};\mathbb{C})$ is a map of convolution algebras.) We will assume throughout this section that the hypotheses (1.2.1), (1.2.5) as well as the hypothesis (1.2.6) on the existence of suitable transverse slices hold.

The main result of this section is a general multiplicity formula for the simple modules that turn up in the composition series associated $\mathfrak{M}^*_{u,s}(K)$ and $\mathfrak{M}^!_{u,s}(K)$, $K \in D^{c,Z_G(s)}_b(W;\mathbb{C})$ that will hold under the above hypotheses. Throughout the proof we will adopt the terminology of (5.3.0).

(3.1) Let $\mathcal{O}$ denote an orbit of $Z_G(s)$ in W and let $v \in \mathcal{O}$ denote a fixed element. We may now recall the basic terminology on equivariant sheaves from [J-1] (1.2) or [Fr] p.14. A sheaf $L = \{L_n|n\}$ of $\mathbb{C}$-vector spaces on a simplicial space $X.$ is locally constant if F_0 is locally constant on X_0 and each of the structure maps $\phi(\alpha) : \alpha^*(L_n) \to L_m$ is an isomorphism for any structure map $\alpha : X_m \to X_n$ of the simplicial space $X.$. Observe that if $X. = BG.=$ the classifying simplicial space for a topological group G, a locally constant sheaf L on $BG.$ is automatically G-equivariant and corresponds to a representation of $\pi_1(BG)$. These observations provide the equivalences:

(3.1.*) ($Z_G(s)$-equivariant local systems on $EZ_G(s) \underset{Z_G(s)}{\times} \mathcal{O}$)

$\simeq$ ($M(v,s)$-equivariant local systems on $BM(v,s)$) $\simeq$(local-systems on $BM(v,s)$)

$\simeq$ (representations of the finite group $\bar{M}(v,s)$)

(The first equivalence is provided by the functor that sends a $Z_G(s)$-equivariant local system $\mathcal{L}$ to its stalk at v.) For each irreducible representation σ of the finite group $\bar{M}(v,s)$, let $\mathcal{L}_\sigma$ denote the corresponding irreducible $Z_G(s)$-equivariant local system on $EZ_G(s) \underset{Z_G(s)}{\times} \mathcal{O}$ and let $IC^{Z_G(s)}(\mathcal{L}_\sigma)$ denote the $Z_G(s)$-equivariant intersection cohomology complex with the middle perversity.

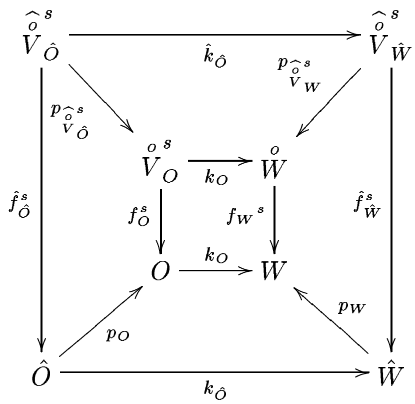
(3.2) *Let O denote a closed orbit of $Z_G(s)$ on W fixed throughout the rest of this section.* Let $u \in O$ denote a fixed point and let S_u denote the *transverse slice* to O in W at u as in (1.2.6).

(3.3) **Proposition**. Assume the above situation. Let $\hat{O} = M^o(u,s)_{red}\backslash Z_G(s)$ and $\hat{W} = M^o(u,s)_{red}\backslash(Z_G(s) \times S_u)$. Now there exists a $Z_G(s)$-equivariant retraction $r_{\hat{O}} : \hat{W} \to \hat{O}$.

Proof. Let $r : S_u \to Spec\ \mathrm{C} \to u$ denote the composition of the obvious projection of S_u to Spec $\mathbb{C}$ followed by the obvious map sending Spec $\mathbb{C}$ to the point u. Since $M^o(u,s)_{red}$ acts trivially on u, this map is clearly equivariant for the action of $M^o(u,s)_{red}$. Now r defines a $Z_G(s)$-equivariant map $\hat{W} = M^o(u,s)_{red}\backslash(Z_G(s) \times S_u) \to \hat{O} =$

$M^o(u,s)_{red}\backslash Z_G(s)$ which we denote by $r_{\hat{O}}$. If $k_{\hat{O}} : \hat{O} \to \hat{W}$ is the obvious map induced by the inclusion $u \to S_u$, it follows readily that $r_{\hat{O}} \circ k_{\hat{O}} =$ the identity of $\hat{O}$. $\square$

Let $M^o(u,s)_{red}$ denote a maximal reductive subgroup of $M^o(u,s)$ = the connected component containing the identity element in $M(u,s)$ (which is the stabilizer of u in $Z_G(s)$) that leaves S_u stable under its action. Let $f^s : (\overset{o}{V})^s \to V^s$ denote the obvious map induced by f. Now we obtain the commutative diagram where the outer and central squares are *cartesian*:

(3.4.*) 

Here $\hat{O} = (M^o(u,s)_{red})\backslash Z_G(s) \cong (M^o(u,s)_{red})\backslash(Z_G(s) \times u)$, $\hat{W} = (M^o(u,s)_{red})\backslash(Z_G(s) \times S_u)$, $\overset{\widehat{o}}{V}{}^s_{\hat{O}} = \hat{O} \underset{O}{\times} \overset{o}{V}{}^s_O \cong M^o(u,s)_{red}\backslash(Z_G(s) \times f^{-1}(u)^s)$, $\overset{\widehat{o}}{V}{}^s_{\hat{W}} = M^o(u,s)_{red}\backslash(Z_G(s) \times (f^s)^{-1}(S_u))$ and $\overset{o}{V}{}^s_O = f^{-1}(O)^s$.

The maps $p_W : \hat{W} \to W$ and $p_O : \hat{O} \to O = M(u,s)\backslash(Z_G(s))$ are the obvious maps. The remaining maps are also the obvious ones: for example, the map $\hat{f}^s_{\hat{W}}$ sends the class of $(g,x) \in M^o(u,s)_{red}\backslash(Z_G(s) \times (f^s)^{-1}(S_u))$ to the class of $(g, f(x))$ in $M^o(u,s)_{red}\backslash(Z_G(s) \times S_u)$. *Observe that the only $Z_G(s)$-orbits $\mathcal{O}$ in W are the ones whose closures contain the given orbit O. Since p_W is smooth, $\hat{W}$ is stratified by the inverse images of these orbits which we will denote by $\hat{\mathcal{O}}$.*

Remark. The above constructions are made necessary due to the fact that the transverse slice S_u will not in general be stable under the action of the group $M^o(u,s)$, but stable only under the action of a maximal reductive subgroup $M^o(u,s)_{red}$.

One may apply the functor $EZ_G(s) \underset{Z_G(s)}{\times}$ to the above diagram to obtain a commutative diagram over $BZ_G(s)$, where the induced maps will be denoted by the same symbols. Let $K \in D_b^{c,Z_G(s)}(W;\mathbb{C})$. Now we may first of all observe the isomorphism:

(3.5) $\bar{\mathfrak{M}}^!_{u,s}(K) \cong \mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{Z_G(s)}(\overset{\widehat{o}}{V}{}^s_{\hat{O}};\ R\hat{f}^{s!}_{\hat{O}} Rp^!_O Rk^!_O(K))$.

To see this, first one observes:

(3.5.1) $\mathbb{H}^*_{Z_G(s)}(\overset{\widehat{o}}{V}{}^s_{\hat{O}};\ R\hat{f}^{s!}_{\hat{O}} Rp^!_O Rk^!_O(K)) \cong \mathbb{H}^*_{M^o(u,s)_{red}}(f^{-1}(u)^s;\ \hat{i}^{s*}_u(R\hat{f}^{s!}_{\hat{O}} Rp^!_O Rk^!_O(K)))$,

where $\hat{i}^s_u : f^{-1}(u)^s \to \overset{\widehat{o}}{V}{}^s_{\hat{O}}$ is the obvious closed immersion.

(This follows from [J-1] (2.P.3)' by taking $X = f^{-1}(u)^s$, $\bar{H} = Z_G(s)$ and $H = M^o(u,s)_{red}$.) Let $\bar{i}^s_u : f^{-1}(u)^s \to \overset{o}{V}{}^s_O$ denote the obvious map as before. Now (5.3.4) shows that the last term on the right in (3.5.1) is isomorphic to:

(3.5.2) $\mathbb{H}^*_{M^o(u,s)_{red}}(f^{-1}(u)^s; R\hat{i}^{s!}_u(R\hat{f}^{s'}_{\hat{O}}{}^! Rp^!_O Rk^!_O(K)))$

Since $H^*(BM^o(u,s);\ \mathbb{C})) \cong H^*(BM^o(u,s)_{red};\ \mathbb{C})$ and since $p_O \circ \hat{f}^s_{\hat{O}} \circ \hat{i}^s_u = f^s_O \circ \bar{i}^s_u$, the Kunneth spectral sequence in (5.1.1) with $G = M^o(u,s)$ and $H = M^o(u,s)_{red}$ shows that the last term is isomorphic to $\mathbb{H}^*_{M^o(u,s)}(f^{-1}(u)^s; R\bar{i}^{s!}_u Rf^{s!}_O Rk^!_O(K)) = \bar{\mathfrak{M}}^!_{u,s,+}(K)$. Now the definition of $\bar{\mathfrak{M}}^!_{u,s}(K)$ as in (2.6.1) provides the identification in (3.5).

Let $r_{\hat{O}} : \hat{W} \to \hat{O}$ be the $Z_G(s)$-equivariant retraction in (3.3). Now $r_{\hat{O}} \circ k_{\hat{O}}$ = the identity. Therefore, $k_O \circ p_O \circ \hat{f}^s_{\hat{O}} = k_O \circ p_O \circ r_{\hat{O}} \circ k_{\hat{O}} \circ \hat{f}^s_{\hat{O}} = k_O \circ p_O \circ r_{\hat{O}} \circ \hat{f}^s_{\hat{W}} \circ \hat{k}_{\hat{O}}$. *From now onwards, we will denote the composition* $k_O \circ p_O \circ r_{\hat{O}}$ *by* ϕ_O. The above arguments show that one obtains the identification:

$$(3.6)\ \bar{\mathfrak{M}}^!_{u,s}(K) = \mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{Z_G(s)}(\overset{\widehat{o}}{V}_{\hat{O}};\ R\hat{k}^!_{\hat{O}} R\hat{f}^{s!}_{\hat{W}} R\phi^!_O(K))$$

Next we apply the decomposition theorem in equivariant intersection cohomology (see [J-3] (5.3.12) for example) to the *proper* map $f^s_W : \overset{o}{W} \to W$. Recall that W is stratified in such a manner that the $Z_G(s)$-orbits form the strata. Let $\underline{\mathbb{C}}$ denote the obvious constant sheaf on $\overset{o}{W}$. Now $\underline{\mathbb{C}}[dim\ \ \overset{o}{W}]$ is a perverse sheaf on $\overset{o}{W}$. Since $f^s|$ (the inverse image of each stratum in W) is a locally trivial fibration, one observes that $Rf^s_{W*}(\underline{\mathbb{C}}[dim\ \ \overset{o}{W}])$ have locally constant cohomology sheaves on each stratum and therefore:

$$(3.7.1)\ Rf^s_{W*}(\underline{\mathbb{C}}[dim\ \ \overset{o}{W}]) = \sum_{\mathcal{O}}\sum_{\sigma}\sum_{c_{\mathcal{O}_\sigma}} V_{\mathcal{O}_\sigma}(c_{\mathcal{O}_\sigma}).IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma})[c_{\mathcal{O}_\sigma}]$$

where the outer most sum varies over all strata $\mathcal{O}$ in W.

For each $\mathcal{O}$ = an orbit appearing in the sum on the right hand side, let $v \in \mathcal{O}$ denote a fixed point. Now the second sum varies over all irreducible representations σ of $M(v,s)$ (i.e. over all $Z_G(s)$-equivariant local systems $\mathcal{L}_{\mathcal{O}_\sigma}$ of $\mathbb{C}$-vector spaces on the simplicial space $EZ_G(s) \underset{Z_G(s)}{\times} \mathcal{O}$.) Here $IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma})$ denotes the equivariant intersection cohomology complex with the middle perversity (see [Bryl] or [J-0]) and obtained by starting with the local system $\mathcal{L}_{\mathcal{O}_\sigma}$ on $\mathcal{O}$.) The inner-most sum varies over all shifts $c_{\mathcal{O}_\sigma}$ with which the complex $IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma})$ appears on the right-hand-side while each $V_{\mathcal{O}_\sigma}(c_{\mathcal{O}_\sigma})$ is a finite dimensional $\mathbb{C}$-vector space.

Fix an orbit $\mathcal{O}$ and an irreducible local system $\mathcal{L}_{\mathcal{O}_\sigma}$ on it; let $v \in \mathcal{O}$ be a fixed element. As in [C-G] (7.6) one may now show that the sum $\oplus V_{\mathcal{O}_\sigma}(c_{\mathcal{O}_\sigma})$, where the sum varies over all $c_{\mathcal{O}_\sigma}$, with $v \in \mathcal{O}$ fixed and for a fixed irreducible representation σ of the finite group $\bar{M}(v,s)$ may be identified with the simple module $L_{\mathcal{O},\sigma}$ over the algebra $\mathbb{C}_{(s)} \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^{Z_G(s)}_*(\overset{oo}{W};\mathbb{C})$. (To see this merely observe that the derived functor

$Rf^s_{W*} : D^{c,Z_G(s)}_b(\overset{o}{W};\ \mathbb{C}) \to D^{c,Z_G(s)}_b(W;\ \mathbb{C})$ is a collection of derived functors

$\{Rf^s_{W,n_*} : D^c_b((EZ_G(s) \underset{Z_G(s)}{\times} \overset{o\ s}{V}_W)_n;\ \mathbb{C}) \to D^c_b((EZ_G(s) \underset{Z_G(s)}{\times} W)_n;\ \mathbb{C}) | n \geq 0\}$

and that $Rf^s_{W,0_*} = Rf^s_{W*} : D^c_b(\overset{o\ s}{V}_W;\ \mathbb{C}) \to D^c_b(W;\ \mathbb{C})$. The analysis in [C-G](7.6) applies to $Rf^s_{W,0_*}$.)

(3.7.2) *It is shown in ([C-G](7.6.8)) that* $\{L_{\mathcal{O},\sigma} | \mathcal{O}, \sigma\}$ *forms a complete collection of all the simple modules, some of them possibly zero.* (Note: as remarked in [C-G] (7.6) whether or not a particular $L_{\mathcal{O},\sigma}$ is non-zero is rather a delicate question depending on the particular situation.)

For our purposes it is more convenient to consider $Rf^s_{W*}(\underline{\mathbb{C}})$. This is obtained from (3.7.1) by applying the shift $[-dim\ \ \overset{o}{W}]$ to each term on the right hand side (as well as on the left-hand side). We will continue to denote the resulting decomposition as:

$$(3.7.3)\ Rf^s_{W*}(\underline{\mathbb{C}}) = \sum_{\mathcal{O}}\sum_{\sigma}\sum_{c_{\mathcal{O}_\sigma}} V_{\mathcal{O}_\sigma}(c_{\mathcal{O}_\sigma}).IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma})[c_{\mathcal{O}_\sigma} - dim\ \ \overset{o}{W}]$$

Recall that the map $p_W : (M^o(u,s)_{red})\backslash(Z_G(s)\times S_u) \to W$ is smooth; therefore one may identify $p_W^*(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))$ with $IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma})$, where $\mathcal{L}_{\hat{\mathcal{O}}_\sigma}$ is the pull-back of $\mathcal{L}_\sigma$ to $\hat{\mathcal{O}}$. Now proper-base-change shows that on applying p_W^* to (3.7.3) one obtains:

(3.8) $R\hat{f}^s_{\hat{W}*}(\underline{\mathbb{C}}) = \sum_{\mathcal{O}}\sum_{\sigma}\sum_{c_{\mathcal{O}_\sigma}} V_{\mathcal{O}_\sigma}(c_{\mathcal{O}_\sigma}).IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma})[c_{\mathcal{O}_\sigma} - dim \quad \overset{o}{W}]$

where the outer sum varies over all strata $\hat{\mathcal{O}}$ in $\hat{W}$ and the second sum varies over all $Z_G(s)$-equivariant local systems $\mathcal{L}_{\mathcal{O}_\sigma}$ on the simplicial space $EZ_G(s) \underset{Z_G(s)}{\times} \mathcal{O}$.

Recall that $\overset{\widehat{o}}{V}{}^s_{\hat{W}}$ is smooth; therefore the dualizing complex (for the category of complexes of sheaves of $\mathbb{C}$-vector spaces) on $\overset{o}{V}{}^s_{\hat{W}}$ may be identified with $\underline{\mathbb{C}}[2n]$ where $n =$ the dimension of $\overset{\widehat{o}}{V}{}^s_{\hat{W}}$. Therefore we apply the projection formula and (5.2.4) to (3.8) to conclude:

(3.9) $R\hat{f}^s_{\hat{W}*}(R\hat{f}^{s!}_{\hat{W}}R\phi^!_O(K)) \simeq R\Delta^![R\hat{f}^s_{\hat{W}*}(\underline{\mathbb{C}}[2n]) \boxtimes R\phi^!_O(K)]$

$\cong \sum_{\mathcal{O}}\sum_{\sigma}\sum_{c_{\mathcal{O}_\sigma}} V_{\hat{\mathcal{O}}_\sigma}(c_{\mathcal{O}_\sigma}).R\Delta^![(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma})[c_{\mathcal{O}_\sigma} - dim \quad \overset{o}{W}]) \boxtimes R\phi^!_O(K)[2n]]$

$\cong \sum_{\mathcal{O}}\sum_{\sigma}\sum_{c_{\mathcal{O}_\sigma}} V_{\hat{\mathcal{O}}_\sigma}(c_{\mathcal{O}_\sigma}).(R\Delta^![(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma})) \boxtimes R\phi^!_O(K)])[c_{\mathcal{O}_\sigma} - dim \quad \overset{o}{W} + 2n]$

To compute the multiplicities of the simple-modules in $\bar{\mathfrak{M}}^!_{u,s}(K)$ one now considers:

(3.10) $\bar{\mathfrak{M}}^!_{u,s}(K) = \mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{Z_G(s)}(\overset{\widetilde{o}}{V}{}^s_{\hat{O}};\ R\hat{k}^!_{\hat{O}}R\hat{f}^{s!}_{\hat{W}}R\phi^!_O(K))$

$\cong \mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{Z_G(s)}(\hat{O};\ R\hat{f}^s_{\hat{O}*}R\hat{k}^!_{\hat{O}}R\hat{f}^{s!}_{\hat{W}}R\phi^!_O(K))$

$\cong \mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{Z_G(s)}(\hat{O};\ Rk^!_{\hat{O}}R\hat{f}^s_{\hat{W}*}R\hat{f}^{s!}_{\hat{W}}R\phi^!_O(K))$

$\cong \sum_{\mathcal{O}}\sum_{\sigma}\sum_{c_{\mathcal{O}_\sigma}} V_{\mathcal{O}_\sigma}(c_{\mathcal{O}_\sigma}).(\mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{Z_G(s)}(\hat{O};\ Rk^!_{\hat{O}}R\Delta^!(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma}) \boxtimes R\phi^!_O(K)[c_{\mathcal{O}_\sigma} - dim \quad \overset{o}{W} + 2n])))$

Let $\mathcal{O}$ denote a fixed $Z_G(s)$-orbit on W, let $v \in \mathcal{O}$ and let σ denote an irreducible representation of the finite group $\bar{M}(v,s)$. As remarked in (3.7.1), the simple modules forming the composition series of $\bar{\mathfrak{M}}^!_{u,s}(K)$ are given by $\{L_{\mathcal{O},\sigma} = \sum_{c_{\mathcal{O}_\sigma}} V_{\mathcal{O}_\sigma}(c_{\mathcal{O}_\sigma})\}$. Now *the multiplicity of the simple module* $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^!_{u,s}(K)$ is clearly given by

(3.10)' $dim(\mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{Z_G(s)}(\hat{O};\ Rk^!_{\hat{O}}R\Delta^!(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma}) \boxtimes R\phi^!_O(K))))$

(3.11) **Theorem** (*The general multiplicity formula*). Assume the above situation. (i) Let $K \in D_b^{c,Z_G(s)}(W;\mathbb{C})$.

The multiplicity of the simple module $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^!_{u,s}(K)$ is given by

$dim(\mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{M^o(u,s)}(u;\ Ri^{s!}_u(Rk^!_O(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri^{s!}_u Rk^!_O(K)))$

where $i^s_u : u \to O$ is the obvious immersion of u in O.

(ii) Similarly the multiplicity of the simple module $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^*_{u,s}(K)$ is given by

$dim(\mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{M^o(u,s)}(u;\ i^{s*}_u(k^*_O(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes i^{s*}_u k^*_O(K)))$

Let $V \subseteq U$ be a G-stable *open* subvariety so that $W \subseteq V^s$ as a $Z_G(s)$-stable *open* subvariety. Let $\kappa : W \to V$ denote the obvious locally closed immersion. Let $P \,\epsilon\, \mathcal{C}^G(V)$. *Assume also that the hypotheses (1.2.2) and (1.2.4) also hold.* If $K = R\kappa^!(P) \,\epsilon\, D_b^{c,Z_G(s)}(W;\,\mathbb{C})$, the multiplicity of the simple module $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^!_{u,s}(K)$ is given by

(iii) $\sum_i dim H^i(Ri_u^{s!}(Rk_O^!(IC(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri_u^{s!}Rk_O^!(K))$

If $K = \kappa^*(P) \,\epsilon\, D_b^{c,Z_G(s)}(W;\,\mathbb{C})$, the multiplicity of the simple module $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^*_{u,s}(K)$ is given by

(iv) $\sum_i dim H^i(i_u^{s*}(k_O^*(IC(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes i_u^{s*}k_O^*(K))$

Proof. Since the proofs of (i) and (ii) are quite similar we will only prove (i). We identify the term in (3.11)(i) with the term in (3.10)' as follows:

$$Rk^!_{\hat{O}}R\Delta^!(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma}) \boxtimes R\phi^!_O(K)) \simeq Rk^!_{\hat{O}}R\Delta^!(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma}) \boxtimes Rr^!_{\hat{O}}Rp^!_ORk^!_O(K))$$

$$\simeq R\Delta^!(Rk^!_{\hat{O}}(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma})) \boxtimes Rk^!_{\hat{O}}Rr^!_{\hat{O}}Rp^!_ORk^!_O(K))$$

$$\simeq R\Delta^!(Rk^!_{\hat{O}}(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma})) \boxtimes Rp^!_ORk^!_O(K))$$

The last quasi-isomorphism follows from the observation that $r_{\hat{O}} \circ k_{\hat{O}} =$ the identity. It follows that

(3.11.1) $\mathbb{H}^*_{Z_G(s)}(\hat{O};\; Rk^!_{\hat{O}}R\Delta^!(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma}) \boxtimes R\phi^!_O(K)))$

$\simeq \mathbb{H}^*_{M^\circ(u,s)_{red}}(u;\; R\psi_u^{s!}R\Delta^!(Rk^!_{\hat{O}}(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma})) \boxtimes Rp^!_ORk^!_O(K)))$

as $H^*(BZ_G(s);\mathbb{C})$-modules where $\psi_u^s : u \to \hat{O}$ denotes the obvious closed immersion. Now one obtains the identifications:

$$R\psi_u^{s!}R\Delta^!(Rk^!_{\hat{O}}(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma})) \boxtimes Rp^!_ORk^!_O(K))$$

$$\simeq R\Delta^![R\psi_u^{s!}(Rk^!_{\hat{O}}(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma}))) \boxtimes R\psi_u^{s!}Rp^!_ORk^!_O(K)]$$

Since the last Δ is the map $u \to u \times u$, it is an isomorphism. Therefore one may identify the last term with

$$R\psi_u^{s!}(Rk^!_{\hat{O}}(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma}))) \otimes R\psi_u^{s!}Rp^!_ORk^!_O(K)$$

Observe that $p_O \circ \psi_u^s = i_u^s$, where $i_u^s : u \to O$ denotes the obvious closed immersion. (See the diagram (3.4.*).) Therefore one may identify the above term with:

$$R\psi_u^{s!}(Rk^!_{\hat{O}}(IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma}))) \otimes Ri_u^{s!}Rk^!_O(K)$$

Recall $IC^{Z_G(s)}(\mathcal{L}_{\hat{\mathcal{O}}_\sigma}) \simeq p_W^*(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma})) \simeq Rp_W^!(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))$ (modulo an even dimensional shift). Therefore one may identify the last term with $Ri_u^{s!}(Rk^!_O(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri_u^{s!}Rk^!_O(K)$

One may now identify the last term in (3.11.1) with

(3.11.2) $\mathbb{H}^*_{M^\circ(u,s)_{red}}(u;\; Ri_u^{s!}(Rk^!_O(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri_u^{s!}Rk^!_O(K))$

$\mathbb{H}^*_{M^\circ(u,s)}(u;\; Ri_u^{s!}(Rk^!_O(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri_u^{s!}Rk^!_O(K))$

This completes the proof of (3.11)(i).

Next we will prove (iii). Let $K = R\kappa^!(P)$, for some $P \,\epsilon\, C^G(V)$. First consider the term $\mathbb{H}^*_{Z_G(s)}(\hat{O};\, Rk^!_{\hat{O}}R\hat{f}^s_{\hat{W}*}R\hat{f}^{s!}_{\hat{W}}R\phi^!_O(K))$ in (3.10). Clearly this is isomorphic to

$$\mathbb{H}^*_{Z_G(s)}(\hat{O}; R\hat{f}^s_{\hat{O}*}Rk^!_{\hat{O}}R\hat{f}^{s!}_{\hat{W}}R\phi^!_O(K)) \cong \mathbb{H}^*_{Z_G(s)}(\hat{O}; R\hat{f}^s_{\hat{O}*}R\hat{f}^{s!}_{\hat{O}}Rk^!_{\hat{O}}R\phi^!_O(K))$$

$$\cong \mathbb{H}^*_{Z_G(s)}(\widehat{\overset{o}{V}}_{\hat{O}};\; R\hat{f}^{s!}_{\hat{O}}Rp^!_ORk^!_O(K)).$$

The arguments right after (3.5) show that this is isomorphic to

$$\mathbb{H}^*_{M^o(u,s)_{red}}(f^{-1}(u)^s;\ R\hat{i}^{s!}_u(Rf^{s!}_{\hat{O}}Rp^!_O Rk^!_O(K))) \cong \mathbb{H}^*_{M^o(u,s)}(f^{-1}(u)^s;\ R\bar{i}^{s!}_u(Rf^{s!}_O Rk^!_O(K)))$$

By (2.3)(ii) this is a finitely generated *projective* module over $H^*(BM^o(u,s);\ \mathbb{C})$. By (3.9) each of the terms

$$\mathbb{H}^*_{Z_G(s)}(\hat{O};\ Rk^!_{\hat{O}}R\Delta^!(IC^{Z_G(s)}(\mathcal{L}_{\hat{O}_\sigma}) \boxtimes R\phi^!_O(K)))$$

$$\cong \mathbb{H}^*_{M^o(u,s)}(u;\ Ri^{s!}_u(Rk^!_O(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri^{s!}_u Rk^!_O(K))$$

that appear in (3.11.2) is a split summand of the above $H^*(BM^o(u,s);\ \mathbb{C})$-module and hence also *projective*. In fact, in the spectral sequence

$$E_2^{s,t} = H^s(BM^o(u,s); \mathcal{H}^t(Ri^{s^!}_u(Rk^!_O(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri^{s^!}_u Rk^!_O(K)))$$

$$\Rightarrow \mathbb{H}^{s+t}_{M^o(u,s)}(u; Ri^{s^!}_u(Rk^!_O(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri^{s^!}_u Rk^!_O(K))$$

$E_2^{s,t} = 0$ if s or t is *odd* so that the spectral sequence degenerates and one obtains the isomorphism as in (2.1)(ii). Therefore one may now tensor with $\mathbb{C}_s$ to obtain the isomorphism:

$$\mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} \mathbb{H}^*_{Z_G(s)}(\hat{O};\ Rk^!_{\hat{O}}R\Delta^!(IC^{Z_G(s)}(\mathcal{L}_{\hat{O}_\sigma}) \boxtimes R\phi^!_O(K)))$$

$$\cong \mathbb{H}^*(u;\ Ri^{s!}_u(Rk^!_O(IC^{Z_G(s)}(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri^{s!}_u Rk^!_O(K)) \cong H^*(Ri^{s!}_u(Rk^!_O(IC(\mathcal{L}_{\mathcal{O}_\sigma}))) \otimes Ri^{s!}_u Rk^!_O(K))$$

It follows that, under the hypotheses of (3.11)(iii) the multiplicity formula in (3.11)(i) reduces to the one in (3.11)(iii); this completes the proof of (3.11)(iii). The proof of (3.11)(iv) is quite similar to that of (3.11)(iii) and is therefore skipped. □

(3.12) To see that the multiplicity formulae in [C-G] (7.6.12) are special cases of the above formula one may argue as follows. Assume that W is chosen so that $k_O : O \to W$ is *a closed immersion* and $u \in O$. Assume in addition the following:

(i) $\mathcal{C}$ is a G-orbit in V so that $O \subseteq \mathcal{C}^s$

(ii) $V \subseteq U$ is a G-stable open subvariety so that $W \subseteq V^s$.

(iii) One also assumes that $k_{\mathcal{C}} : \mathcal{C} \to V$ is a closed immersion so that one has the cartesian square:

$$\begin{array}{ccc} O & \longrightarrow & \mathcal{C} \\ {\scriptstyle k_O}\downarrow & & \downarrow{\scriptstyle k_{\mathcal{C}}} \\ W & \xrightarrow{\kappa} & V \end{array}$$

(One may let V be obtained from U by removing all G-orbits different from $\mathcal{C}$ and which are contained in $\bar{\mathcal{C}}$. Now $\mathcal{C}$ will be closed in V. Now $\mathcal{C}^s$ breaks up into the union of finitely many $Z_G(s)$-orbits each of which is closed. Let W be obtained from V^s by removing all these orbits in $\mathcal{C}^s$ except O.) Now $P = Rk_{\mathcal{C}*}(\underline{\mathbb{C}}[dim\ \ \mathcal{C}]) = k_{\mathcal{C}!}(\underline{\mathbb{C}}[dim\ \mathcal{C}])$ belongs to $C^G(V)$ and if $K = R\kappa^!(P)$ or if $K = \kappa^*(P)$, one may readily identify K with $Rk_{O*}(\underline{\mathbb{C}}) = k_{O!}(\underline{\mathbb{C}})$ (modulo an even-dimensional shift). Therefore $Rk^!_O(K) \cong k^*_O(K) \simeq \underline{\mathbb{C}}$ modulo an even-dimensional shift. (It follows that $Ri^{s^!}_u Rk^!_O(K) \simeq i^{s^*}_u k^*_O(K) \simeq \underline{\mathbb{C}}$ (modulo an even dimensional shift). Now one may readily observe

$$\bar{\mathfrak{M}}^*_{u,s}(K) \cong \mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} H^*_{M^o(u,s)}(f^{-1}(u)^s;\ \mathbb{C}) \cong H^*(f^{-1}(u)^s;\ \mathbb{C}) \text{ and}$$

$$\bar{\mathfrak{M}}^!_{u,s}(K) \cong \mathbb{C}_s \underset{H^*(BM^o(u,s);\mathbb{C})}{\otimes} H_*^{M^o(u,s)}(f^{-1}(u)^s;\ \mathbb{C}) \cong H_*(f^{-1}(u)^s;\ \mathbb{C}).$$

Let $\mathcal{O}$ denote a $Z_G(s)$-orbit on W whose closure contains O, $v \in \mathcal{O}$ and σ an irreducible representation of $\bar{M}(v,s)$. Now (3.11)(iii) and (iv) show that the multiplicty formula for the multiplicity of $L_{\mathcal{O},\sigma}$ in $\bar{\mathfrak{M}}^*_{u,s}(K)$ ($\bar{\mathfrak{M}}^!_{u,s}(K)$) is given by the sum

$$\sum_i dimH^i(i^{s*}_u k^*_O(IC(\mathcal{L}_{\mathcal{O}_\sigma})))$$

$$(\sum_i dimH^i(Ri^{s!}_u Rk^!_O(IC(\mathcal{L}_{\mathcal{O}_\sigma}))), respectively).$$

This shows that the multiplicity formula in [C-G] (7.6.12) is in fact a special case of the general multiplicity formula in (3.10).

4. Examples.

(4.1) *Modules over affine and graded Hecke alegbras*

To obtain this we assume the situation in (0.2.2) with $U = \mathcal{U}$. Now $\mathbf{H}_{gr} = H_*^G(\overset{oo}{U}; \mathbb{C})$ where $G = \mathbf{G} \times \mathbb{C}^*$ and $\mathbf{H}_{gr}$ is the graded Hecke-algebra associated to $\mathbf{G}$. The simple modules over the affine Hecke-algebra coincide with the simple modules over the graded Hecke algebra - see ([C-G]). Moreover the natural map from the affine Hecke-algebra to the graded Hecke algebra preserves the convolution operation. (See [C-G] or [J-4] for example.) A semi-simple element in the group G corresponds to a pair (s, q) with s a semi-simple element in $\mathbf{G}$ and $q \epsilon \mathbb{C}^*$. Therefore the functors $\bar{\mathfrak{M}}^*_{u,(s,q)}$ and $\bar{\mathfrak{M}}^!_{u,(s,q)}$ produce modules over the affine Hecke algebra. The hypotheses (1.2.1) - (1.2.6) are easily seen to be satisfied in this case. The multiplicity formula in Theorem (3.11) provides the multiplicity of the simple modules in the composition series of these modules.

(4.2) *Modules over affine quantum universal enveloping algebras of type* A_n

To obtain this we assume the situation of (0.2.4). Now the variety N is a union of conjugacy classes of unipotent elements and therefore one may readily verify the hypotheses in (1.2.1), (1.2.5) and (1.2.2). Moreover (1.2.4) and (1.2.6) are satisfied as well: the transverse slice at any point of N may be obtained by intersecting the transverse slice on $\mathcal{N}$ with N.)

5. Additional results on equivariant derived categories

Kunneth-spectral sequences.

(5.1.1) Assume that G is a *connected* linear algebraic groups acting on the variety X and that H is a closed subgroup of G. Let $L \in D_b^{c,G}(X; \mathbb{C})$; now one obtains a Kunneth-spectral sequence:

$$E^2_{p,q} = Tor_{p,q}^{H^*(BG;\mathbb{C})}(H^*(BH; \mathbb{Q}), \mathbb{H}^*_G(X; L)) \Rightarrow \mathbb{H}^{p+q}_H(X; i^*(L))$$

where $i : EH \underset{H}{\times} X \rightarrow EG \underset{G}{\times} X$ is the obvious closed immersion. (This is an Eilenberg-Moore-type spectral sequence obtained from the pull-back square

$$\begin{array}{ccc} EH \underset{H}{\times} X & \longrightarrow & BH \\ \downarrow & & \downarrow \\ EG \underset{G}{\times} X & \longrightarrow & BG \end{array}$$

of simplicial spaces. See [K-M] Theorem (7.3), Part V where similar spectral sequences are established in a general setting.) It follows that, if $\mathbb{H}^*_G(X; L)$ is a projective module over $H^*(BG; \mathbb{C})$, we obtain the isomorphism:

$$\mathbb{H}^*_H(X; i^*(L)) \simeq H^*(BH; \mathbb{C}) \underset{H^*(BG;\mathbb{C})}{\otimes} \mathbb{H}^*_G(X; L)$$

(5.1.2) In case G is not connected, for the purposes of this paper, it suffices to observe that there is a similar spectral sequence when L is in fact the constant sheaf $\underline{\mathbb{C}}$. (See for example [Hs] p. 38.) Given this, we obtain the following extension:

(5.1.3) **Proposition**. Let X denote a G-quasi-projective variety so that $H^G_*(X; \mathbb{C})$ is a *projective module* over $H^*(BG; \mathbb{C})$. If H is any closed subgroup of G, one obtains the isomorphism:

$$H^H_*(X; \mathbb{C}) \cong H^*(BH; \mathbb{C}) \underset{H^*(BG;\mathbb{C})}{\otimes} H^G_*(X; \mathbb{C}).$$

Similarly if (s) denotes any maximal ideal in $H^*(BG; \mathbb{C})$ and $H^G_*(X; \mathbb{C})_{(s)}$ is a projective module over $H^*(BG : \mathbb{C})_{(s)}$, then one obtains the isomorphism $H^H_*(X; \mathbb{C})_{(s)} \cong H^*(BH; \mathbb{C})_{(s)} \underset{H^*(BG;\mathbb{C})_{(s)}}{\otimes} H^*(BG; \mathbb{C})_{(s)}$.

Proof. Given an algebraic group K, a K-variety Z and $L \in D_b^{c,K}(Z; \mathbb{C})$, we define the *$K$-equivariant hypercohomology spectrum* of X to be the complex $R\Gamma(EK \underset{K}{\times} X; L)$. We will denote this by $\mathbb{H}(EK \underset{K}{\times} X; L)$. This is a differential graded module over the differential graded algebra $R\Gamma(BK; \mathbb{C})$; the latter will be denoted $\mathbb{H}(BK)$. The cohomology groups of $\mathbb{H}(BK)$ ($\mathbb{H}(EK \underset{K}{\times} X; L)$) will be denoted $H^*(BK)$ ($\mathbb{H}^*(EK \underset{K}{\times} X; L)$, respectively).

Since X is G-quasi-projective, one obtains a G-equivariant closed immersion of X into a smooth G-quasi-projective variety $\tilde{X}$. We will denote the maps $EH \underset{H}{\times} X \rightarrow EH \underset{H}{\times} \tilde{X}$ and $EG \underset{G}{\times} X \rightarrow EG \underset{G}{\times} \tilde{X}$ by i. Let $U = \tilde{X} - X$. Now one obtains the commutative diagram whose rows are distinguished triangles:

$$\begin{array}{ccccc} \mathbb{H}(BH) \overset{L}{\underset{H(BG)}{\otimes}} \mathbb{H}(EG \underset{G}{\times} X; Ri^!\underline{\mathbb{C}}) & \longrightarrow & \mathbb{H}(BH) \overset{L}{\underset{\mathbb{H}(BG)}{\otimes}} \mathbb{H}(EG \underset{G}{\times} \tilde{X}) & \longrightarrow & \mathbb{H}(BH) \overset{L}{\underset{\mathbb{H}(BG)}{\otimes}} \mathbb{H}(EG \underset{G}{\times} U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}(EH \underset{H}{\times} X; Ri^!\underline{\mathbb{C}}) & \longrightarrow & \mathbb{H}(EH \underset{H}{\times} \tilde{X}; \mathbb{C}) & \longrightarrow & \mathbb{H}(EH \underset{H}{\times} U; \mathbb{C}) \end{array}$$

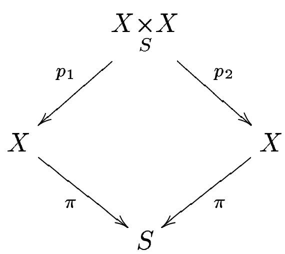
The last two vertical maps are quasi-isomorphisms; therefore so is the first. Now the spectral sequence in [K-M] Theorem (7.3) provides a spectral sequence with $E^2_{p,q} = Tor^{H^*(BG;\mathbb{C})}(H^*(BH; \mathbb{C}), H^*(EG \underset{G}{\times} X; Ri^!\underline{\mathbb{C}}))$ converging to the cohomology of the complex $\mathbb{H}(BH) \overset{L}{\underset{\mathbb{H}(BG)}{\otimes}} \mathbb{H}(EG \underset{G}{\times} X; Ri^!\underline{\mathbb{C}})$. This degenerates since $H^G_*(X; \mathbb{C})$ is a projective module over $H^*(BG; \mathbb{C})$. (Observe that $\tilde{X}$ is smooth and that therefore one may identify

$\mathbb{H}^*(EG\underset{G}{\times}X; Ri^!\underline{\mathbb{C}})$ $(\mathbb{H}^*(EH\underset{H}{\times}X; Ri^!\underline{\mathbb{C}}))$ with $H_*^G(X;\mathbb{C})$ $(H_*^H(X;\mathbb{C})$, respectively).) The proof of the second statement follows by localizing the above spectral sequence at the ideal (s). $\square$

(5.2) *Relations in the equivariant derived category of complexes of finite tor dimension and in the equivariant derived category of perfect complexes*

Let G denote a complex linear algebraic group acting on a scheme of finite type X over $\mathbb{C}$. Let R denote a *regular* Nöetherian ring or a *regular* graded Nöetherian ring. (For example $R = \mathbb{Z}$, $R = \mathbb{Q}$, or $R = \mathbb{C}$.) All modules we consider will be *left* modules over R. Now $D^G_{ctf}(X;\, R)$ will denote the full subcategory of the equivariant derived category $D_b^{c,G}(X;\, R)$ consisting of complexes of finite tor-dimension. In case $R = \mathbb{Q}$, or $R = \mathbb{C}$ one may observe that $D^G_{ctf}(X;\, R) = D_b^{c,G}(X;\, R)$. We will also let $D^G_{perf}(X;\, R)$ denote the full subcategory of $D_b^{c,G}(X;\, R)$ of complexes with locally constant cohomology sheaves.

We summarise the various relations valid in $D^G_{ctf}(X;\, R)$. (See [SGA] 5, Exposé III and [SGA] 6, Exposé I for more details.) Assume $\pi : X \to S$ is a map of schemes of finite type over *Spec* $\mathbb{C}$. Now consider the cartesian square:



Let $K,\, M \,\epsilon\, D^G_{ctf}(X;\, R)$. Now we let $K \boxtimes M = p_1^*(K) \otimes p_2^*(M)$ and $\mathcal{RH}om_S(K,\, M) = \mathcal{RH}om(p_1^*(K),\, Rp_2^!(M))$ and we let $\mathcal{RH}om(K,\, M)$ denote the internal hom (between K and M) in the derived category $D^G(X;\, R)$. Now

(5.2.1) $\mathcal{RH}om_S(K,\, M) \simeq D(K)\overset{L}{\underset{R}{\boxtimes}}M$

(5.2.2) $\mathcal{RH}om(K,\, M) \simeq D(K\overset{L}{\underset{R}{\otimes}}D(M)) \simeq R\Delta^!(\mathcal{RH}om_S(K,\, M)) \simeq R\Delta^!(D(K)\overset{L}{\underset{R}{\boxtimes}}M)$

(5.2.3) Assume in addition that $\pi : X \to S$ is smooth and S is smooth so that the diagonal immersion $\Delta : X \to X\underset{S}{\times}X$ is a regular immersion. If K and M belong to $D^G_{perf}(X;\, R)$, one also obtains $D(K)\overset{L}{\underset{R}{\otimes}}M \simeq \mathcal{RH}om(K,\, M^\cdot)$ modulo an even dimensional shift. (This follows readily from (5.2.2) since $R\Delta^!(D(K)\overset{L}{\underset{R}{\boxtimes}}M^\cdot) \simeq \Delta^*[-2.dimX](D(K)\overset{L}{\underset{R}{\boxtimes}}M^\cdot)$.)

(5.2.4) Let $f : Z \to X$ denote a *proper* map of schemes of finite type over $\mathbb{C}$ provided with a G-action so that it is G-equivariant. Assume that f induces functors:

$Rf_* : D^G_{ctf}(Z;\, R) \to D^G_{ctf}(X;\, R)$ and $Rf^! : D^G_{ctf}(X;\, R) \to D^G_{ctf}(Z;\, R)$.

Now $Rf_*(Rf^!K) \simeq R\Delta^!(Rf_*\mathbb{D}_Z\overset{L}{\underset{R}{\boxtimes}}K)$ where $\mathbb{D}_Z$ is the dualizing complex for the category $D^G_{ctf}(Z;\, R)$. This may be obtained as follows. $R\Delta^!(Rf_*\mathbb{D}_Z\overset{L}{\underset{R}{\boxtimes}}K) \simeq \mathcal{RH}om(Rf_*(\underline{R}),\, K)$ by (5.2.2). The latter $\simeq Rf_*(\mathcal{RH}om(\underline{R},\, Rf^!K)) \simeq Rf_*(Rf^!K)$. $\square$

(5.3.0) Let $\bar{H}$ denote a complex linear algebraic group and let H denote a closed subgroup. Assume that we are provided with an action of H on a variety X which we assume is also H-quasi-projective. (We may assume that $\bar{H}$ and H are *not necessarily connected*). Let H act on $\bar{H} \times X$ by $h.(\bar{h},\, x) = (\bar{h}.h^{-1},\, hx)$, $h \,\epsilon\, H$, $\bar{h} \,\epsilon\, \bar{H}$ and

$x \in X$. Then a geometric quotient $H\backslash(\bar{H} \times X)$ exists for this action and the map $s : \bar{H} \times X \to H\backslash(\bar{H} \times X)$ is smooth with fibers isomorphic to H. Now $\bar{H}$ acts on $\bar{H} \times X$ by translation on the first factor; this induces an $\bar{H}$-action on $H\backslash(\bar{H} \times X)$ as well. One verifies that the map s is equivariant for these actions of $\bar{H}$.

Let $r : \bar{H} \times X \to X$ denote the projection to the second factor. Next let $\bar{H} \times H$ act on $\bar{H} \times X$ by $(\bar{h}_1, h_1).(\bar{h}, x) = (\bar{h}_1.\bar{h}.h_1^{-1}, h_1 x)$, $\bar{h}_1$, $\bar{h} \in \bar{H}$, $h_1 \in H$ and $x \in X$. We observe that the maps r and s are such that we obtain the commutative squares:

(5.3.1)
$$\begin{array}{ccc} (\bar{H} \times H) \times (\bar{H} \times X) & \longrightarrow & \bar{H} \times X \\ {\scriptstyle pr_1 \times s}\downarrow & & {\scriptstyle s}\downarrow \\ \bar{H} \times (H\backslash(\bar{H} \times X)) & \longrightarrow & H\backslash(\bar{H} \times X) \end{array} \qquad \begin{array}{ccc} (\bar{H} \times H) \times (\bar{H} \times X) & \longrightarrow & \bar{H} \times X \\ {\scriptstyle pr_2 \times r}\downarrow & & {\scriptstyle r}\downarrow \\ H \times X & \longrightarrow & X \end{array}$$

It follows that r and s induce maps $\bar{r} : E(\bar{H} \times H) \underset{\bar{H}\times H}{\times} (\bar{H} \times X) \to EH \underset{H}{\times} X$ and $\bar{s} : E(\bar{H} \times H) \underset{\bar{H}\times H}{\times} (\bar{H} \times X) \to E\bar{H} \underset{\bar{H}}{\times} (H\backslash(\bar{H} \times X))$.

Let $\Delta : H \to \bar{H} \times H$ denote the diagonal and let $j : X \to \bar{H} \times X$ denote the map $x \to (e, x)$ where e is the identity element of G. We now observe that the square

(5.3.2)
$$\begin{array}{ccc} H \times X & \longrightarrow & X \\ {\scriptstyle \Delta \times j}\downarrow & & {\scriptstyle j}\downarrow \\ (\bar{H} \times H) \times (\bar{H} \times X) & \longrightarrow & \bar{H} \times X \end{array}$$

commutes. It follows that j and Δ induce a map $\bar{j} : EH \underset{H}{\times} X \to E(\bar{H} \times H) \underset{\bar{H}\times H}{\times} (\bar{H} \times X)$; one checks readily that $\bar{r} \circ \bar{j} =$ the identity. We denote $\bar{s} \circ \bar{j}$ by $\bar{i}$.

Under the above assumptions we had proved in [J-1] (A.1) a thereom that shows the functors

(5.3.3) $D_b^{c,H}(X; \mathbb{C}) \xrightarrow{\bar{r}^*} D_b^{c,\bar{H}\times H}(\bar{H} \times X; \mathbb{C})$ and $D_b^{c,\bar{H}}(H\backslash(\bar{H} \times X); \mathbb{C}) \xrightarrow{\bar{s}^*} D_b^{c,\bar{H}\times H}(\bar{H} \times X; \mathbb{C})$

are equivalences of categories. Hence so are the functors $\bar{j}^*$ and $\bar{i}^*$.

(5.3.4) **Proposition**. Assume the above situation. Let c denote the codimension of X in $H\backslash(\bar{H} \times X)$ by the closed immersion $\bar{i}$. Now there exists a natural isomorphism of functors: $R\bar{i}^!$ and $\bar{i}^*[-2c]$.

Proof. Let d denote the dimension of H. Recall $\bar{i} = \bar{s} \circ \bar{j}$. Since s is a smooth map with fibers isomorphic to H, it follows that $R\bar{s}^!$ is naturally isomorphic to $\bar{s}^*[2d]$. Now j is a closed immersion with the codimension of X in $\bar{H} \times X$ given by $c + d$. Therefore it suffices to show that there is a natural isomorphism of functors: $R\bar{j}^! \simeq \bar{j}^*[-2c - 2d]$. This follows readily since we already know that $\bar{r}^*$ is an equivalence. □

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