Notions of Purity and the cohomology of Quiver-moduli

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Outline

1. Quick review of basic purity results in étale cohomology. Notions of weak and strong purity

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2. Application to GIT quotients: general result
3. Application to cohomology of quiver moduli: a case not overed by the last corollary
4. Notions of purity in the equivariant context:torsors

Results of Grothendieck and Deligne:I

X a smooth scheme of finite type over a finite field $\mathbb{F}_q$. Then the number of points of X rational over a finite extension $\mathbb{F}_{q^n}$ expressed by the Lefschetz trace formula

$$|X(\mathbb{F}_{q^n})| = \sum_{i \ge 0} (-1)^i Tr\big(Fr^n, H^i_c(\overline{X}, \mathbb{Q}_l)\big)$$

where $\overline{X} = X \underset{Spec\mathbb{F}_q}{\times} Spec\overline{\mathbb{F}_q}$ and Fr denotes the geometric Frobenius on $\overline{X}$. (Here $char(\mathbb{F}_q) = p$ and $l \neq p$.)

Results of Deligne:II

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- The zeta function of X defined in terms of $Q_i(t)$
- The $Q_i(t)$ have integral coefficients when X projective and smooth.
- Purity in the sense of Deligne: projective smooth varieties are pure. i.e. each eigen value on $H^i(\bar{X}, \bar{\mathbb{Q}}_l)$ has abs. value $q^{i/2}$

Weak Purity: for constant local systems

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- This implies there exists a positive integer N and polynomials $P_0(t), \cdots, P_{N-1}(t)$ in $\mathbb{Z}[t]$ so that $|X(\mathbb{F}_{q^n})| = P_r(q^n), n \equiv r \quad mod(N)$

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- Eigen value of Fr on H_c^i is q^{i-1}, i.e. q on H_c^2 and 1 on H_c^1.
- $|\mathbb{G}_m(\mathbb{F}_{q^n})| = q^n - 1$ (Exercise: check this is indeed correct!)

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$$\alpha^N = q^{(i-d)N}.$$
- Check that this agrees with Deligne's results: $|\alpha| = q^{i-d}$. Need to show $i - d \leq i/2$. Follows since $i \leq 2d$.

Weak Purity: homogeneous varieties

- Brion and Peyre proved that homogenous vars for linear algebraic groups, i.e. G/H, are weakly pure. (Readily follows from the case of tori. First reduce to the case both G and H are reductive, then to the case where H is replaced by $N_H(T_H)$ and then by T_H.)

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- Our results a sequel to their work.
- In any case, more information on the eigen-values translates into better information of the number of rational points. Our main interest on schemes provided with an action of a linear algebraic group: see how weak and strong purity behave with respect to taking GIT quotients and also under torsors. Then we apply these results to quiver moduli.

Strong Purity of the classifying spaces of linear algebraic groups: Example

For any linear algebraic group G, the l-adic cohomology $H^*_{et}(BG, \mathbb{Q}_l)$ is weakly pure and vanishes in odd degrees.

If G is split, then Fr acts on $H^{2n}_{et}(BG, \mathbb{Q}_l)$ via multiplication by q^n, for any integer $n \geq 0$.

The example: $H^*_{et}(BG, \mathbb{Q}_l)$

Can readily reduce to the case G is connected, then reductive, then Borel and finally a torus. In case G is split, the torus is a product of $\mathbb{G}_m$s.

$B\mathbb{G}_m = \mathbb{P}^\infty$, so its cohomology is strongly pure.

In general, G splits over some finite extension of the base field and therefore the statement on weak purity follows.

Application to GIT quotients: Theorem 1

X smooth provided with the action of a connected reductive group G and with an ample, G-linearized line bundle L, such that the following two conditions are satisfied:

- Every semi-stable point of X with respect to L is stable.

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- If G is split and X is strongly pure with respect to $\mathbb{Q}_l$, then so is $X//G$.

Explanation of the theorem: basic GIT notions

- Given G and L as above, $x \varepsilon X$ is *semi-stable* wrt (G, L), if there is some $f \varepsilon \Gamma(X, L^{\otimes r})^G$ with $f(x) \neq 0$ and X_f affine.

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- X^{ss} is open (though could be empty).

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- There exists a G-stable stratification of X by locally closed strata with the open stratum being X^{ss}. The other strata correspond to semistable points for the action of smaller subgroups of G.

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- The long exact sequence in equivariant cohomology:

$$\cdots \to H^*_G(X, \mathbb{Q}_l) \to H^*_G(X^{ss}, \mathbb{Q}_l) \to$$

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  breaks up into short exact sequences so that the last map is a surjection.
- If $X^s = X^{ss}$, $H^*_G(X^{ss}, \mathbb{Q}_l) \cong H^*(X//G, \mathbb{Q}_l)$ where

$$X//G = \text{ the GIT quotient }.$$

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- Not needed here, but often very useful: if Y projective and smooth, the above ss degenerates at E_2 (use Deligne's degeneration criterion using Hard-Lefschetz)
- One can work all these out in the setting of simplicial schemes (i.e. there are also other possibilities): my favorite reference for this simplicial framework is Eric's book from the early 1980s.

Proof of the theorem

In view of the above observations, it suffices to show $H^*_G(X, \mathbb{Q}_l)$ is weakly (strongly) pure if X is. Since the differentials of the ss are compatible with the action of Fr, it suffices to observe $E_2^{s,t} = H^s(BG, \mathbb{Q}_l) \otimes H^t(X, \mathbb{Q}_l)$ is weakly (strongly) pure if X is. This follows readily using the weak (strong) purity of $H^*(BG, \mathbb{Q}_l)$.

Application to GIT quotients: Corollary

Let G be a split connected reductive group and X a smooth projective G-variety equipped with an ample G-linearized line bundle such that every semi-stable point is stable. If X is strongly pure, then so is the GIT quotient $X//G$.

Proof. F. Kirwan has provided an equivariantly perfect stratification in this case so that the last theorem applies.

Some examples (where the corollary applies)

Corollary applies for instance to the case where X is a product of Grassmannians:

$$X = \prod_{i=1}^{m} \mathrm{Gr}(r_i, n), \quad L = \boxtimes_{i=1}^{m} \mathcal{O}_{\mathrm{Gr}(r_i,n)}(a_i)$$

where $\mathrm{Gr}(r,n)$ denotes the Grassmannian of r-dimensional linear subspaces of projective n-space, and $\mathcal{O}_{\mathrm{Gr}(r,n)}(a)$ denotes the a-th power of the line bundle associated with the Plücker embedding; here $G = \mathrm{PGL}(n+1)$ and $r_1, \ldots, r_m < n$, $a_1, \ldots, a_m$ are positive integers. Indeed, X is clearly strongly pure; moreover, $X^{\mathrm{ss}} = X^{\mathrm{s}}$ for general values of $a_1, \ldots, a_m$. The geometric quotient $X//G$ is called *the space of stable configurations*; examples include moduli spaces of m ordered points in $\mathbb{P}^n$.

Introduction to Quivers

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- $\mathbb{Z}\mathbf{I}$ = the free abelian group generated by I; the basis consisting of elements of I denoted by $\mathbf{I}$. An element $\mathbf{d} \in \mathbb{Z}\mathbf{I}$ written as $\mathbf{d} = \sum_{i \in I} d_i \, \mathbf{i}$.

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- $Q_0 = \{i, j\}$,
  $Q_1 = \{i \overset{n-arrows}{\to} j\}$.

Quiver representations

- A representation of a quiver $Q = (Q_0, Q_1)$ is $M = \big((M_i)_{i \in I}, (M_\alpha : M_i \to M_j)_{\alpha : i \to j}\big)$ each M_α being a k-vector space.

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- The *dimension vector* $\mathbf{dim}(M) \in \mathbb{N}\mathbf{I} = \mathbf{dim}(M) = \sum_{i \in I} \dim_{\mathbb{F}}(M_i)\,\mathbf{i}$. The *dimension* of M will be defined to be $\sum_{i \in I} \dim_{\mathbb{F}}(M_i)$ and denoted $\dim(M)$.

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- In the second example, one often lets V_i, $i = 1, \cdots n$ be all 1-dimensional.
- $\mathcal{H}om_{\mathbb{F}Q}(M, N)$ the $\mathbb{F}$-vector space of homomorphisms between two representations $M, N \in \mathrm{Mod}(\mathbb{F}Q)$.

Space of quiver representations

Fix a quiver Q and a dimension vector $\mathbf{d} = \sum_i d_i\, \mathbf{i}$, and consider the affine space

$$X = R(Q, \mathbf{d}) := \bigoplus_{\alpha: i \to j} \mathcal{H}om_{\mathbb{F}}(\mathbb{F}^{d_i}, \mathbb{F}^{d_j}).$$

Its points $M = (M_\alpha | \alpha)$ parametrize representations of Q with dimension vector $\mathbf{d}$. The connected reductive algebraic group

$$G(Q, \mathbf{d}) := \prod_{i \in I} \mathrm{GL}(d_i)$$

acts on $R(Q, \mathbf{d})$ via base change:

$$\big((g_i) \cdot (M_\alpha)\big)_\alpha = (g_j M_\alpha g_i^{-1})_{\alpha: i \to j}.$$

Space of quiver representations

By definition, the orbits $G(Q,\mathbf{d})\cdot M$ in $R(Q,\mathbf{d})$ correspond bijectively to the isomorphism classes $[M]$ of $\mathbb{F}$-representations of Q of dimension vector $\mathbf{d}$.

The subgroup of G consisting of tuples $(t\,\mathrm{id}_{d_i})_{i\in I}$, $t\in\mathbb{G}_m$, is a central one-dimensional torus and acts trivially on X.

Therefore we let $G := PG(Q,\mathbf{d})$ and $X := R(Q,\mathbf{d})$.

A choice of character Θ

The only characters of GL_n are powers of the determinant map; therefore, the only characters of the group $PG(Q, \mathbf{d})$ are of the form

$$(g_i)_i \longmapsto \prod_{i \in I} \det(g_i)^{m_i},$$

for a tuple $(m_i)_{i \in I}$ such that $\sum_{i \in I} m_i d_i = 0$ to guarantee well-definedness on $PG(Q, \mathbf{d})$.

Thus, one may choose a linear function $\Theta : \mathbb{Z}\mathbf{I} \to \mathbb{Z}$ and associate to it a character

$$\chi_\Theta((g_i)_i) := \prod_{i \in I} \det(g_i)^{\Theta(\mathbf{d}) - \dim(\mathbf{d}) \cdot \Theta(\mathbf{i})}$$

of $PG(Q, \mathbf{d})$. For convenience, we will call Θ itself a *character*.

Moduli space of quivers with fixed dimension vector and character:I

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- This bundle trivial on forgetting the G-action. A global section of L_χ^n, for any $n \geq 1$, given by $f \otimes_{\mathbb{F}} t$, $f \in \mathbb{F}[X]$, $t \in \mathbb{F}[\mathbb{A}^1]$. G acts on such pairs (f, t) by $g.(f, t) = (f \circ g, \chi(g)^{-n} t)$ with $f \circ g$ defined by $(f \circ g)(x) = f(gx)$.

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- Therefore, such a global section will be G-invariant precisely when f is χ-*semi-invariant with weight* n, i.e.

$$f(gx) = \chi^n(g) f(x) \text{ for all } g \in G \text{ and all } x \in X.$$

Moduli space of quivers with fixed dimension vector and character:II

One may invoke the usual definitions of GIT to define the semi-stable points and stable points.

A point $x \in R(Q, \mathbf{d})$ semi-stable (stable) precisely when there exists a G-invariant global section of some positive power of the above line bundle that does not vanish at x (when, in addition, the orbit of x is closed in the semi-stable locus, and the stabilizer at x is finite).

Moduli space of quivers:III

The corresponding varieties of Θ-semi-stable and stable points with respect to the line bundle L_χ will be denoted by

$$R(Q,\mathbf{d})^{\mathrm{ss}} = R(Q,\mathbf{d})^{\Theta-\mathrm{ss}} = R(Q,\mathbf{d})^{\Theta-\mathrm{ss}}$$

and

$$R(Q,\mathbf{d})^{\mathrm{s}} = R(Q,\mathbf{d})^{\Theta-\mathrm{s}} = R(Q,\mathbf{d})^{\Theta-\mathrm{s}}.$$

These are open subvarieties of X, possibly empty. The corresponding quotient varieties will be denoted as follows:

$M^{\Theta-\mathrm{s}}(Q,\mathbf{d}) = R(Q,\mathbf{d})^{\Theta-\mathrm{s}}/G$ and

$M^{\Theta-\mathrm{ss}}(Q,\mathbf{d}) = R(Q,\mathbf{d})^{\Theta-\mathrm{ss}}//G = X//G.$

Criterion for semi-stability (well-known)

A representation $M \in R(Q, \mathbf{d})$ is Θ-semi-stable if and only if $\mu(N) \leq \mu(M)$ for all non-zero sub-representations N of M. The representation M is Θ-stable if and only if $\mu(N) < \mu(M)$ for all non-zero proper sub-representations N of M.

Slope $\mu(M) = \frac{\Theta(M)}{dim(M)}$.

Semistability for the second example

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- $dim(M) = m\mathbf{i}_0 + \mathbf{i}_1 + \cdots \mathbf{i}_n$. We let $\Theta = -\mathbf{i}_0^*$.
- Now a semi-stable(stable)representation M corresponds to a tuple $(v_1, \cdots, v_n)$ of nonzero vectors in V_{i_0} so that the number of vectors $\{v_k\}$ in U is $\leq$ ($<$, respectively) $\frac{n}{m} dim(U)$ for each non-zero subspace $U \subseteq V_{i_0}$.

Stratification of X using semi-stability

If a representation M is unstable, one can define a finite increasing filtration by sub-representations (the so-called *Harder-Narasimhan filtration*),

$$\{0\} = M^0 \subset M^1 \subset M^2 \subset \cdots \subset M^{n-1} \subset M^n = M,$$

such that:

- each M^i/M^{i-1} is semi-stable, and

Let $\mathbf{d}_i$ = the dimension vector of the representation M^i/M^{i-1}. Varying $i = 1, \cdots, n$, we obtain a sequence $(\mathbf{d}^1, \cdots, \mathbf{d}^n)$ of dimension vectors. The sequence of slopes of the sub-quotients given by

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- $\mu(M^i/M^{i-1}) > \mu(M^j/M^{j-1})$ for all $j > i$.

Let $\mathbf{d}_i$ = the dimension vector of the representation M^i/M^{i-1}. Varying $i = 1, \cdots, n$, we obtain a sequence $(\mathbf{d}^1, \cdots, \mathbf{d}^n)$ of dimension vectors. The sequence of slopes of the sub-quotients given by

Stratification of X using semi-stability:II

$$\big(\mu(M^1/M^0), \cdots, \mu(M^n/M^{n-1})\big)$$

together with the above sequence of dimension vectors- called *the instability type* of the given unstable representation M. Pairs of such sequences

$$\beta := \big((\mu^1, \cdots, \mu^n), (\mathbf{d}^1, \cdots, \mathbf{d}^n)\big),$$

with $\mu^1 > \mu^2 > \cdots > \mu^n$ and the $\mathbf{d}^i$ are dimension vectors so that $\sum_i \mathbf{d}^i = \mathbf{dim}(M)$ and $\mu^i = \mu(\mathbf{d}^i)$ will be used to index the strata of a natural stratification of the representation space $R(Q, \mathbf{d})$.

Theorem 2: Equivariant perfection of the above stratification

The above stratification is equivariantly perfect. If in addition, every semi-stable point is stable, then $M^{\Theta-\mathrm{s}}(Q, \mathbf{d})$ is weakly-pure.

Remarks (i) Suppose $\mathbf{d}$ is co-prime for Θ, i.e. $\mu(\mathbf{e}) \neq \mu(\mathbf{d})$ for all $0 \neq \mathbf{e} < \mathbf{d}$. (For a generic choice of Θ, this is equivalent to g.c.d$\{d_i | i \in I\} = 1$.) In this case, every semi-stable point is stable.

(ii) F. Kirwan had developed a theory in the 1980s to prove similar results, but her results valid only when X is projective. In our case X is affine, so the proof is non-trivial.

Corollary

Assume that each semi-stable point is stable. Then the l-adic cohomology $H^*\big(M^{\Theta-\mathrm{s}}(Q,\mathbf{d}),\mathbb{Q}_l\big)$ vanishes in all odd degrees. Moreover, Fr acts on each $H^{2n}\big(M^{\Theta-\mathrm{s}}(Q,\mathbf{d}),\mathbb{Q}_l\big)$ via multiplication by q^n.

In particular, $H^*\big(M^{\Theta-\mathrm{s}}(Q,\mathbf{d}),\mathbb{Q}_l\big)$ is strongly pure, and hence the number of $\mathbb{F}_{q^n}$-rational points of $M^{\Theta-\mathrm{s}}(Q,\mathbf{d})$ is a polynomial function of q^n with integer coefficients.

Such a result, could be established prior to our proof, only in a round-about way using Hall algebras.

The G-equivariant setting

Given a G-scheme X as above, $\mathcal{C}$ denotes a class of G-equivariant l-adic local systems $\mathcal{L} = \{\mathcal{L}_\nu|\nu\}$ on X such that the following hypotheses hold:

- Each local system $\mathcal{L}$ is mixed and of weight $\geq w$ for some non-negative integer w

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- The class $\mathcal{C}$ is closed under extensions
- $\mathcal{C}$ contains the local system $\mathbb{Q}_l$

Weak and Strong purity

- A finite-dimensional $\mathbb{Q}_l$-vector space V provided with an endomorphism F is *strongly pure*, if each eigenvalue α of F (in the algebraic closure $\bar{\mathbb{Q}}_l$) satisfies $\alpha = q^j$ for some integer $j = j(\alpha) \geq 0$.

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- The pair (V, F) is *weakly pure* if each eigenvalue α of F satisfies $\alpha = \zeta q^j$ for some root of unity $\zeta = \zeta(\alpha)$ and some integer $j = j(\alpha) \geq 0$. Equivalently, $\alpha^n = q^{jn}$ for some positive integer n.

Weak and strong purity in the G-equivariant case

- Given a G-scheme X and a class of G-equivariant local systems $\mathcal{C}$ on X as above, X is *weakly (strongly) pure* with respect to the class $\mathcal{C}$, if the cohomology space $H^*_{\text{et}}(\bar{X}, \bar{\mathcal{L}})$, provided with the action of the Frobenius Fr, is weakly (strongly, respectively) pure for each $\mathcal{L} \in \mathcal{C}$.

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- We will say X is *weakly (strongly) pure with respect to* $\mathbb{Q}_l$ if the above hypotheses hold for the action of the trivial group and for the class generated by the constant l-adic local system $\mathbb{Q}_l$.

Theorem 3: Notions of purity and torsors

$\pi : X \to Y$ denote a torsor under a linear algebraic group G, all defined over $\mathbb{F}_q$.

$\mathcal{C}_X$ ($\mathcal{C}_Y$)a class of G-equivariant l-adic local systems on X (on Y, respectively) , with Y provided with the trivial G-action.

Suppose $\mathcal{C}_X \supseteq \pi^*(\mathcal{C}_Y)$. Then (i) if X is weakly pure with respect to $\mathcal{C}_X$ so is Y with respect to $\mathcal{C}_Y$. In case G is split, and if X is strongly pure with respect to $\mathcal{C}_X$, then so is Y with respect to $\mathcal{C}_Y$.

Theorem 3: contd

(ii) Suppose G is also connected and $\mathcal{C}_X = \pi^*(\mathcal{C}_Y)$. Then if Y is weakly pure with respect to $\mathcal{C}_Y$, so is X with respect to $\mathcal{C}_X$. In case G is split and if Y is strongly pure with respect to $\mathcal{C}_Y$, so is X with respect to $\mathcal{C}_X$.

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$\pi_3 : X'/T' \rightarrow X'/B'$, $\pi_4 : X'/B' \rightarrow X'/G' \cong X/G^o$ and
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$\pi_5 : X/G^o \rightarrow X/G$.
- Fibers of π_1 and π_3 are acyclic in l-adic étale cohomology, the fibers of π_4 are G'/B' (simply-connected) and one uses Galois descent for π_5.

Outline of Proof:II

- This reduces the proof to proving the assertions for π_2. After possibly extending the field, one reduces to the case the torus is one dimensional. This is rather tricky and uses a rather elaborate Leray spectral sequence argument.

See http://www.math.ohio-state.edu:/~joshua/pub.html