

NOTIONS OF PURITY AND THE COHOMOLOGY OF QUIVER MODULI

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ABSTRACT. We explore several variations of the notion of *purity* for the action of Frobenius on schemes defined over finite fields. In particular, we study how these notions are preserved under certain natural operations like quotients for principal bundles and also geometric quotients for reductive group actions. We then apply these results to study the cohomology of quiver moduli. We prove that a natural stratification of the space of representations of a quiver with a fixed dimension vector is equivariantly perfect and from it deduce that each of the l -adic cohomology groups of the quiver moduli space is strongly pure.

0. Introduction

Consider a scheme X of finite type over a finite field \mathbb{F}_q . Then the number of points of X that are rational over a finite extension \mathbb{F}_{q^n} is expressed by the trace formula

$$(0.0.1) \quad |X(\mathbb{F}_{q^n})| = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(F^n, H_c^i(\bar{X}, \mathbb{Q}_l))$$

where $\bar{X} = X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \bar{\mathbb{F}}_q$ and F denotes the Frobenius morphism of \bar{X} . The results of Deligne (see [De74a, De77, De80]) show that every eigenvalue of F on $H_c^i(\bar{X}, \mathbb{Q}_l)$ has absolute value $q^{w/2}$ for some non-negative integer $w \leq i$.

In [BP10], the first named author and Peyre studied in detail the properties of the counting function $n \mapsto |X(\mathbb{F}_{q^n})|$, when X is a homogeneous variety under a linear algebraic group (all defined over \mathbb{F}_q). For this, they introduced the notion of a weakly pure variety X , by requesting that all eigenvalues of F in $H_c^*(\bar{X}, \mathbb{Q}_l)$ are of the form ζq^j , where ζ is a root of unity, and j a non-negative integer. This implies that the counting function of X is a periodic polynomial with integer coefficients, i.e., there exist a positive integer N and polynomials $P_0(t), \dots, P_{N-1}(t)$ in $\mathbb{Z}[t]$ such that $|X(\mathbb{F}_{q^n})| = P_r(q^n)$ whenever $n \equiv r \pmod{N}$. They also showed that homogeneous varieties under linear algebraic groups are weakly pure.

The present paper arose out of an attempt to study the notion of weak purity in more detail, and to see how it behaves with respect to torsors and geometric invariant theory quotients. While applying this notion to moduli spaces of quiver representations, it also became clear that they satisfy a stronger notion of purity, which in fact differs from the notion of a strongly pure variety, introduced in [BP10] as a technical device. Thus, we were led to define weak and strong purity in a more general setting, and to modify the notion of strong purity so that it applies to GIT quotients.

Here is an outline of the paper. In the first section, we introduce a notion of weak purity for equivariant local systems (generalizing that in [BP10] where only the constant local system is considered) and a closely related notion of strong purity. The basic definitions are in Definition 1.6. Then we study how these notions behave with respect to torsors and certain associated fibrations. The main result is the following.

Theorem 0.1. *(See Theorem 1.10.) Let $\pi : X \rightarrow Y$ denote a torsor under a linear algebraic group G , all defined over \mathbb{F}_q . Let \mathcal{C}_X denote a class of G -equivariant l -adic local systems on X , and \mathcal{C}_Y a class of G -equivariant l -adic local systems on Y , where Y is provided with the trivial G -action.*

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(i) Suppose $\mathcal{C}_X \supseteq \pi^*(\mathcal{C}_Y)$. Then if X is weakly pure with respect to \mathcal{C}_X so is Y with respect to \mathcal{C}_Y . In case G is split, and if X is strongly pure with respect to \mathcal{C}_X , then so is Y with respect to \mathcal{C}_Y .

(ii) Suppose G is also connected and $\mathcal{C}_X = \pi^*(\mathcal{C}_Y)$. Then if Y is weakly pure with respect to \mathcal{C}_Y , so is X with respect to \mathcal{C}_X . In case G is split and if Y is strongly pure with respect to \mathcal{C}_Y , so is X with respect to \mathcal{C}_X .

As explained just after Theorem 1.10, the above theorem applies with the following choice of the classes \mathcal{C}_X and \mathcal{C}_Y : let \mathcal{C}_X denote the class of G -equivariant l -adic local systems on X obtained as split summands of $\rho_{X*}(\underline{\mathbb{Q}}_l^{\oplus n})$ for some $n > 0$, where $\rho_X : X' \rightarrow X$ is a G -equivariant finite étale map and $\underline{\mathbb{Q}}_l^{\oplus n}$ is the constant local system of rank n on X' , and define \mathcal{C}_Y similarly.

In the second section, we first apply some of the above results to show that geometric invariant theory quotients of smooth varieties by connected reductive groups preserve the properties of weak and strong purity. Specifically, we obtain the following.

Theorem 0.2. (See Theorem 2.1.) *Consider a smooth variety X provided with the action of a connected reductive group G and with an ample, G -linearized line bundle L , such that the following two conditions are satisfied:*

- (i) *Every semi-stable point of X with respect to L is stable.*
- (ii) *X admits an equivariantly perfect stratification with open stratum the (semi)-stable locus.*

If X is weakly pure with respect to the constant local system \mathbb{Q}_l , then so is the geometric invariant theory quotient $X//G$.

If G is split and X is strongly pure with respect to \mathbb{Q}_l , then so is $X//G$.

Here we recall that a stratification by smooth, locally closed G -subvarieties is *equivariantly perfect*, if the associated long exact sequences in equivariant cohomology break up into short exact sequences (see e.g. [Kir84, p. 34]. When X is projective, such a stratification has been constructed by Kirwan via an analysis of semi-stability (see [Kir84, Theorem 13.5], and also the proof of Corollary 2.2 for details on equivariant perfection). We also obtain an extension of the above theorem with local systems in the place of \mathbb{Q}_l : this is in Theorem 2.5.

Next, we study in detail the quiver moduli spaces using these techniques. In particular, we show that the space of representations of a given quiver with a fixed dimension vector satisfies our assumption (ii) (since that space is affine, this assertion does not follow readily from Kirwan's theorem quoted above). Our main result in this setting is the following.

Theorem 0.3. (See Theorem 3.4.) *Let X denote the representation space of a given quiver with a given dimension vector. Then the stratification of X defined by using semi-stability with respect to a fixed character Θ is equivariantly perfect.*

Thus, the condition (ii) in Theorem 2.1 holds in this setting. Since the condition (i) holds for general values of Θ , it follows that the corresponding geometric invariant theory quotient (i.e. the quiver moduli space) is strongly pure with respect to \mathbb{Q}_l . i.e. We obtain the following result (see Corollary 3.6):

Corollary 0.4. *Assume in addition to the above situation that each semi-stable point is stable. Then the l -adic cohomology $H^*(M^{\Theta-s}(Q, \mathbf{d}), \mathbb{Q}_l)$ vanishes in all odd degrees, $H^*(M^{\Theta-s}(Q, \mathbf{d}), \mathbb{Q}_l)$ is strongly pure, and hence the number of \mathbb{F}_{q^n} -rational points of $M^{\Theta-s}(Q, \mathbf{d})$ is a polynomial function of q^n with integer coefficients.*

The last corollary recovers certain results of Reineke (see [Re03, Section 6] and [Re06, Theorem 6.2]) which are established by using the combinatorics of the Hall algebra associated to the quiver. Our proof is purely based on geometric invariant theory coupled with the theory of weak and strong purity developed in the first section of this paper.

It may be worth pointing out that several of the varieties that are weakly pure, for example, connected reductive groups, turn out to be mixed Tate. The weight filtration and the *slice-filtration* for such varieties are related in [HK06]. In view of this, we hope to explore the results of this paper in a motivic context in a sequel.

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1. Notions of purity

1.1. Equivariant local systems. Throughout the rest of the paper, we will only consider separated schemes of finite type which are defined over a finite field $\mathbb{F} = \mathbb{F}_q$ with q elements where q is a power of the characteristic p . Given such a scheme X , we will denote by $X_{\overline{\mathbb{F}}}$ (or \bar{X}) its base extension to the algebraic closure $\overline{\mathbb{F}}$ of \mathbb{F} , and by $F : \bar{X} \rightarrow \bar{X}$ the Frobenius morphism.

We will consider l -adic sheaves and local systems, where $l \neq p$ is a prime number. Recall the following from [BBD81, Chapitre 5]. Given a scheme X , an l -adic sheaf $\mathcal{L} = \{\mathcal{L}_\nu \mid \nu \in \mathbb{Z}_{\geq 1}\}$ on the étale site $X_{\text{ét}}$ will mean an inverse system of sheaves with each \mathcal{L}_ν a constructible sheaf of \mathbb{Z}/l^ν -modules. Such an l -adic sheaf defines by base extension an l -adic sheaf $\bar{\mathcal{L}}$ on \bar{X} provided with an isomorphism $F^*(\bar{\mathcal{L}}) \rightarrow \bar{\mathcal{L}}$. An l -adic local system on X is an l -adic *lisse* sheaf $\{\mathcal{L}_\nu | \nu\}$, that is, each \mathcal{L}_ν is locally constant on $X_{\text{ét}}$.

We now recall some basic properties of higher direct images of l -adic sheaves under fibrations. We say that a morphism of schemes $f : X \rightarrow Y$ is a *locally trivial fibration* if f is smooth, and there exists an étale covering $Y' \rightarrow Y$ such that the pull-back morphism $f' : X' := X \times_Y Y' \rightarrow Y'$ is isomorphic to the projection $Y' \times Z \rightarrow Y'$ for some scheme Z . Then Z is smooth, and all fibers of f at $\overline{\mathbb{F}}$ -rational points are isomorphic to $Z_{\overline{\mathbb{F}}}$.

The main importance of this notion comes from the following result, which is in fact a direct consequence of the definitions, the Künneth formula in étale cohomology, and [BBD81, 5.1.14].

Proposition 1.1. *Let $f : X \rightarrow Y$ denote a locally trivial fibration. Let $\nu > 0$ denote any fixed integer and let \mathcal{F} denote a constructible sheaf of \mathbb{Z}/l^ν -modules on $X_{\text{ét}}$ which is constant on some Galois covering of X .*

(i) *Then each $R^m f_*(\mathcal{F})$ is a constructible sheaf of \mathbb{Z}/l^ν -modules on $Y_{\text{ét}}$.*

(ii) *Let \bar{y} denote a fixed geometric point of Y and let $X_{\bar{y}} = X \times_Y \bar{y}$ denote the corresponding geometric fiber. Then $R^m f_*(\mathcal{F})_{\bar{y}} \simeq H^m(X_{\bar{y}}, \mathcal{F}|_{X_{\bar{y}}})$ for all y . In particular, if \mathcal{F} is a locally constant constructible sheaf of \mathbb{Z}/l^ν -modules on $X_{\text{ét}}$, then the sheaves $R^m f_*(\mathcal{F})$ for all $m \geq 0$ are locally constant constructible sheaves on $Y_{\text{ét}}$.*

(iii) *Moreover, if \mathcal{L} is an l -adic local system that is mixed and of weight $\geq w$, then each $R^m f_*(\mathcal{L})$ is also mixed and of weight $\geq m + w$.*

We now turn to equivariant local systems; for this, we first fix notations and conventions about algebraic groups and their actions. A smooth group scheme will be called an *algebraic group*; we will only consider *linear* algebraic groups in this paper. For such a group G , we denote by G^o its *neutral component*, that is, the connected component containing the identity element e_G . We recall that G^o is a closed normal subgroup of G , and that G/G^o is a finite group.

A scheme X provided with the action of an algebraic group G will be called a *G -scheme*. The action morphism $G \times X \rightarrow X$ will be denoted by $(g, x) \mapsto g \cdot x$.

Given a G -scheme X , a *G -equivariant l -adic local system* on X will denote an l -adic sheaf

$$\{\mathcal{L}(m) | m \geq 0\} = \{\mathcal{L}_\nu(m) | \nu \in \mathbb{Z}_{\geq 1}, m \geq 0\}$$

on the simplicial scheme $EG \times_G X$ such that the following two conditions are satisfied:

(i) Each $\mathcal{L}_\nu(m)$ is a locally constant sheaf of \mathbb{Z}/l^ν -modules on the étale site

$$(EG \times_G X)(m)_{\text{et}} = (G^m \times X)_{\text{et}}.$$

(ii) For any $\nu \geq 0$ and for any structure map $\alpha : [m] \rightarrow [n]$ in Δ , the induced map $\alpha^*(\mathcal{L}_\nu(n)) \rightarrow \mathcal{L}_\nu(m)$ is an isomorphism.

We say that \mathcal{L} is mixed and of weight w (mixed and of weight $\geq w$), if the l -adic sheaf $\{\mathcal{L}_\nu(m)|\nu\}$ on $(EG \times_G X)_m$ is mixed and of weight w (mixed and of weight $\geq w$, respectively) for each $m \geq 0$. (Since G is smooth, it suffices to verify this condition just for $m = 0$.) Given an l -adic local system $\mathcal{L} = \{\mathcal{L}_\nu|_\nu\}$ as above, we let $\tilde{\mathcal{L}}$ denote the corresponding l -adic local system on \bar{X} obtained by base extension.

Remark 1.2. Proposition 1.1 (iii) justifies the condition on the weights put into the definition of the class of local systems. In this paper we only consider the derived direct image functors and not the derived direct image functors with proper supports. The latter functors send complexes of l -adic sheaves that are mixed and of weight $\leq w$ to complexes of l -adic sheaves that are mixed and of weight $\leq w$. Therefore, to consider these functors or l -adic cohomology with proper supports, one needs to consider classes of local systems that are mixed and of weight $\leq w$ for some positive integer w .

We now relate equivariant local systems for the actions of a connected algebraic group G and of various subgroups. Recall that G contains a Borel subgroup B (defined over \mathbb{F}) which in turn contains a maximal torus T (also defined over \mathbb{F}). Moreover, the pairs (B, T) as above are all conjugate under the group $G(\mathbb{F})$.

Proposition 1.3. *Let X denote a G -scheme, where G is a connected algebraic group. Fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. Then the restriction functors induce equivalences of categories*

$$\begin{aligned} & (G\text{-equivariant } l\text{-adic local systems on } X) \simeq (B\text{-equivariant } l\text{-adic local systems on } X) \\ & \simeq (T\text{-equivariant } l\text{-adic local systems on } X), \end{aligned}$$

where all local systems are considered on the corresponding schemes defined over $\bar{\mathbb{F}}$.

Proof. Since G is assumed to be connected, each connected component of X is stable by G , so that we may assume X is also connected. We will show below that there exists a fibration sequence of étale topological types in the sense of [Fr83, Chapter 10]:

$$(1.1.1) \quad \begin{aligned} (G/B)_{\text{et}} & \rightarrow (EB \times_B X)_{\text{et}} \xrightarrow{\alpha} (EG \times_G X)_{\text{et}}, \\ (B/T)_{\text{et}} & \rightarrow (ET \times_T X)_{\text{et}} \xrightarrow{\beta} (EB \times_B X)_{\text{et}} \end{aligned}$$

where the subscript *et* denotes the étale topological types. Recall this means the maps α and β are maps of an inverse system of pointed simplicial sets and that the homotopy fiber of α (resp. β) is weakly equivalent in the sense of [Fr83, Definition 6.1] to $(G/B)_{\text{et}}$ (resp. $(B/T)_{\text{et}}$). Assuming this, one obtains the exact sequences of pro-groups:

$$\pi_1((G/B)_{\text{et}}) \rightarrow \pi_1((EB \times_B X)_{\text{et}}) \rightarrow \pi_1((EG \times_G X)_{\text{et}}) \rightarrow 1 \text{ and}$$

$$\pi_1((B/T)_{\text{et}}) \rightarrow \pi_1((ET \times_T X)_{\text{et}}) \rightarrow \pi_1((EB \times_B X)_{\text{et}}) \rightarrow 1.$$

Since the flag variety G/B is the same for the reductive quotient $G/R_u(G)$, we may assume G is reductive when considering G/B . Therefore, G/B lifts to characteristic 0 so that one may see $\pi_1((G/B)_{\text{et}})^\wedge = 0$ where the completion is away from p . Next observe that B/T is an affine space. Therefore, its étale topological type completed away from p is trivial: this follows from the observation that the étale cohomology of affine spaces are trivial with respect to locally constant torsion sheaves, with torsion prime to the characteristic. (See [Mi80, Chapter Vi, Corollary 4.20].) Applying completion away from p to the above exact sequences and observing that such a completion is right-exact

(see for example, [BK72, Chapter 3, 8.2]) and that $\pi_1((G/B)_{\text{et}})^\wedge$ and $\pi_1((B/T)_{\text{et}})^\wedge$ are trivial provides the isomorphisms:

$$\begin{aligned}\pi_1((EB \times_B X)_{\text{et}})^\wedge &\simeq \pi_1((EG \times_G X)_{\text{et}})^\wedge, \\ \pi_1((ET \times_T X)_{\text{et}})^\wedge &\simeq \pi_1((EB \times_B X)_{\text{et}})^\wedge.\end{aligned}$$

Since l -adic equivariant local systems correspond to continuous l -adic representations of the above completed fundamental groups (see, for example, [Jo93, Appendix (A.3.3)]), the statements in the proposition follow.

Now we proceed to show that the first fibration sequences of étale topological types as in (1.1.1) exists assuming that one has a weak-equivalence: $(EB \times_B X)_{\text{et}} \simeq (EG \times_G G \times_B X)_{\text{et}}$. For this we observe first that the inclusion $X \rightarrow G \times_B X$ given by sending $x \mapsto (e, x)$ composed with the map $\pi : G \times_B X \rightarrow X$ given by $(g, x) \mapsto gx$ is the identity. Therefore, one may readily show that the homotopy fiber of the map $\pi_{\text{et}} : (G \times_B X)_{\text{et}} \rightarrow X_{\text{et}}$ is given by $(G/B)_{\text{et}}$. Next we observe that

$$(EG \times_G G \times_B X)_n = G^n \times G \times_B X \text{ and that } (EG \times_G X)_n = G^n \times X.$$

Let $p : EG \times_G G \times_B X \rightarrow EG \times_G X$ be the map induced by π . Then the homotopy fiber of the map $(p_n)_{\text{et}}$ also identifies with $(G/B)_{\text{et}}$ for all $n \geq 0$. (This follows from a Künneth formula for the étale topological types which may be readily deduced from [Fr83, Theorem 10.7].) Next observe (see [Wa78, Lemma 5.2]) that given a diagram of bi-simplicial sets $F_{\bullet, \bullet} \rightarrow E_{\bullet, \bullet} \rightarrow B_{\bullet, \bullet}$ so that for each fixed n , $F_{\bullet, n} \rightarrow E_{\bullet, n} \rightarrow B_{\bullet, n}$ is a fibration sequence up to homotopy and $B_{\bullet, n}$ is connected for each n , then the diagram $\Delta F_{\bullet, \bullet} \rightarrow \Delta E_{\bullet, \bullet} \rightarrow \Delta B_{\bullet, \bullet}$ is also a fibration sequence up to homotopy. Since each $(EG \times_G X)_n = G^n \times X$ is connected, so is $\pi(U_{\bullet, n})$ where $U_{\bullet, \bullet}$ is an étale hypercovering of $EG \times_G X$ and π denotes the functor of connected components. Therefore, one may apply this with $B_{\bullet, \bullet}$ denoting the bi-simplicial sets forming $(EG \times_G X)_{\text{et}}$ and $E_{\bullet, \bullet}$ denoting the bi-simplicial sets forming $(EG \times_G G \times_B X)_{\text{et}}$. It follows that one obtains the first fibration sequence of étale topological types appearing in (1.1.1) assuming that one has a weak-equivalence $(EB \times_B X)_{\text{et}} \simeq (EG \times_G G \times_B X)_{\text{et}}$. Next we proceed to sketch the existence of such a weak-equivalence.

For this one needs the intermediary simplicial scheme $E(G \times B) \times_{(G \times B)} (G \times X)$ where $G \times B$ acts on $G \times X$ by $(g_1, b_1) \circ (g, x) = (g_1 g b_1^{-1}, b_1 x)$. This simplicial scheme maps to both $EB \times_B X$ and $EG \times_G G \times_B X$. The simplicial geometric fibers of the map to $EB \times_B X$ identify with EG and the simplicial geometric fibers of the map to $EG \times_G G \times_B X$ identify with EB . Moreover an argument as in the last paragraph will show that the homotopy fibers of the corresponding maps of the étale topological types are $(EG)_{\text{et}}$ and $(EB)_{\text{et}}$, respectively: clearly these are contractible so that one obtains a weak-equivalence between $(EB \times_B X)_{\text{et}}$, $(E(G \times B) \times_{G \times B} (G \times X))_{\text{et}}$ and $(EG \times_G G \times_B X)_{\text{et}}$. This completes the proof of the existence of the first fibration sequence of étale topological types appearing in (1.1.1). A similar argument with G (resp. B) replaced by B (resp. T) proves the second fibration sequence of étale topological types appearing in (1.1.1) and completes the proof of the Proposition. \square

1.2. Torsors. Recall that a *torsor under an algebraic group* G (also called a *principal G -bundle*) consists of a G -scheme X together with a faithfully flat, G -invariant morphism of schemes $\pi : X \rightarrow Y$ such that the morphism

$$G \times X \longrightarrow X \times_Y X, \quad (g, x) \longmapsto (g \cdot x, x)$$

is an isomorphism. Then $Y = X/G$ as topological spaces, and π is the quotient map. Moreover, π is a locally trivial fibration with fiber G (as follows from [Ra70, Lemme XIV 1.4], since G is assumed to be linear).

Lemma 1.4. *Let $\pi : X \rightarrow Y$ denote a torsor under an algebraic group G . Then π^* induces an equivalence of categories:*

$$(l\text{-adic local systems on } Y) \simeq (G\text{-equivariant } l\text{-adic local systems on } X).$$

Proof. The conditions for a sheaf to be G -equivariant correspond to descent data for the map π . Therefore, the conclusion follows by descent theory. \square

We will need the following result, which is probably known but for which we could not locate any reference.

Lemma 1.5. *Let $\pi : X \rightarrow Y$ denote a torsor under an algebraic group G , and $H \subset G$ a closed subgroup. Then π factors uniquely as*

$$X \xrightarrow{\varphi} Z \xrightarrow{\psi} Y = X/G$$

where φ is an H -torsor, and ψ is a locally trivial fibration with fiber G/H .

If H is a normal subgroup of G , then the quotient algebraic group G/H acts on Z , and ψ is a G/H -torsor.

In particular, there is a unique factorization

$$X \xrightarrow{\varphi} Z = X/G^o \xrightarrow{\psi} Y = X/G$$

where φ is a G^o -torsor and ψ is a Galois cover with group G/G^o .

Proof. For the uniqueness of the factorization, note that $Z = X/H$ as topological spaces, and the structure sheaf \mathcal{O}_Z is the subsheaf of H -invariants in $\varphi_*\mathcal{O}_X$.

For the existence, we may assume that Y is affine: indeed, if we have factorizations $X_i \rightarrow Z_i \rightarrow Y_i$ where (Y_i) is a covering of Y by open affine subschemes, then they may be glued to a global factorization, by uniqueness. Now X is affine, since so is the morphism π . Thus, the product $X \times G/H$ is quasi-projective. Moreover, the projection $p_2 : X \times G/H \rightarrow X$ admits a relatively ample G -linearized invertible sheaf (indeed, by a classical theorem of Chevalley, the homogeneous space G/H is isomorphic to a G -orbit in the projectivization $\mathbb{P}(V)$, where V is a finite-dimensional rational representation of G . So the pull-back of the invertible sheaf $\mathcal{O}_{\mathbb{P}(V)}(1)$ yields an ample, G -linearized invertible sheaf on G/H). By [MFK94, Proposition 7.1], it follows that the quotient morphism $\varpi : X \times G/H \rightarrow X \times_G G/H$ exists and is a G -torsor, where $X \times_G G/H$ is a quasi-projective scheme. Moreover, the G -equivariant morphism p_2 descends to a morphism $\psi : X \times_G G/H \rightarrow Y$ such that the square

$$\begin{array}{ccc} X \times G/H & \xrightarrow{p_2} & X \\ \varpi \downarrow & & \pi \downarrow \\ X \times_G G/H & \xrightarrow{\psi} & Y \end{array}$$

is cartesian. We define φ as the composite map

$$X \times H/H \xrightarrow{i} X \times G/H \xrightarrow{\varpi} X \times_G G/H,$$

where i denotes the natural inclusion. Since i yields a section of p_2 , we have $\pi = \psi \circ \varphi$.

To show that φ is an H -torsor, we may perform an étale base change on Y , and hence assume that $X \simeq G \times Y$ as torsors over Y . Then $Z \simeq G/H \times Y$ over Y , and the statement is obvious. The remaining assertions are proved along similar lines. \square

1.3. Weak and strong purity. Given a G -scheme X as above, let \mathcal{C} denote a class (or collection) of G -equivariant l -adic local systems $\mathcal{L} = \{\mathcal{L}_\nu | \nu\}$ on X such that the following hypotheses hold:

- (i) Each local system \mathcal{L} is mixed and of weight $\geq w$ for some non-negative integer w .
- (ii) The class \mathcal{C} is closed under extensions.
- (iii) \mathcal{C} contains the local system \mathbb{Q}_l .

Definition 1.6. (i) A finite-dimensional \mathbb{Q}_l -vector space V provided with an endomorphism F will be called *strongly pure*, if each eigenvalue α of F (in the algebraic closure $\bar{\mathbb{Q}}_l$) satisfies $\alpha = q^j$ for some integer $j = j(\alpha) \geq 0$.

We will say that the pair (V, F) is *weakly pure* if each eigenvalue α of F satisfies $\alpha = \zeta q^j$ for some root of unity $\zeta = \zeta(\alpha)$ and some integer $j = j(\alpha) \geq 0$. Equivalently, $\alpha^n = q^{jn}$ for some positive integer n .

(ii) Given a G -scheme X and a class of G -equivariant local systems \mathcal{C} on X satisfying the above hypotheses, we will say that X is *weakly (strongly) pure* with respect to the class \mathcal{C} , if the cohomology space $H_{\text{et}}^*(\bar{X}, \bar{\mathcal{L}})$, provided with the action of the Frobenius F , is weakly (strongly, respectively) pure for each $\mathcal{L} \in \mathcal{C}$.

We will say X is *weakly (strongly) pure with respect to \mathbb{Q}_l* if the above hypotheses hold for the action of the trivial group and for the class generated by the constant l -adic local system \mathbb{Q}_l .

Remarks 1.7. (i) The notion of weak purity generalizes that of [BP10, Definition 3.2], which is in fact equivalent to the above definition of weak purity for the constant local system. However, the notion of strong purity of [loc. cit.] differs from the above notion (again, for the constant local system): rather than requiring that all eigenvalues of F acting on $H_{\text{et}}^*(\bar{X}, \mathbb{Q}_l)$ are integer powers of q , it requires that each eigenvalue α of F acting on $H_{\text{et}}^i(\bar{X}, \mathbb{Q}_l)$ satisfies $\alpha^n = q^{in/2}$ for some positive integer n .

(ii) The notions of weak and strong purity are stable by extension of the base field \mathbb{F}_q to \mathbb{F}_{q^n} (which replaces the Frobenius endomorphism F by F^n , and its eigenvalues by their n -th powers).

Specifically, if X is strongly pure over \mathbb{F}_q , then so are the base extensions $X_{\mathbb{F}_{q^n}}$ for all $n \geq 1$. Also, X is weakly pure if and only if $X_{\mathbb{F}_{q^n}}$ is weakly pure for some n ; then $X_{\mathbb{F}_{q^n}}$ is weakly pure for all n . In particular, weak purity is independent of the base field \mathbb{F} .

(iii) For this reason, *we will often write X for $X_{\bar{\mathbb{F}}}$ or \bar{X} , when there is no cause of confusion as to what is intended. Also, since we only work with étale cohomology, the subscript *et* will always be omitted. As an example, we will denote $H_{\text{et}}^*(\bar{X}, \bar{\mathcal{L}})$ by $H^*(X, \mathcal{L})$.*

(iv) If X is smooth and weakly (strongly) pure with respect to \mathbb{Q}_l , then so is $H_c^*(X, \mathbb{Q}_l)$ by Poincaré duality. In view of the trace formula (0.0.1), it follows that the counting function $n \mapsto |X(\mathbb{F}_{q^n})|$ is a periodic polynomial with integer coefficients (a polynomial with integer coefficients, respectively).

(v) Recall that a scheme X is said to be *pure*, if all the complex conjugates of eigenvalues of F on $H^i(\bar{X}, \bar{\mathbb{Q}}_l)$ have absolute value $q^{i/2}$: see [De74a, De80]. It may be important to observe that a scheme X may be weakly pure without being pure or strongly pure in the above sense: for example, split tori. Clearly, for a scheme X that is pure, a necessary condition for strong purity is that all the odd l -adic cohomology vanishes. Typical examples of schemes that are strongly pure are those projective smooth varieties that are stratified by affine spaces.

We now record some basic properties of these notions:

Lemma 1.8. (i) *Let $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ denote a short exact sequence of finite-dimensional \mathbb{Q}_l -vector spaces provided with compatible endomorphisms $F_{V'}, F_V, F_{V''}$. Then (V, F_V) is weakly pure (strongly pure) if and only if $(V', F_{V'})$ and $(V'', F_{V''})$ are weakly pure (strongly pure, respectively).*

(ii) *Given a finite-dimensional \mathbb{Q}_l -vector space V provided with an endomorphism F , there exists a canonical decomposition $V = V_{wp} \oplus V_{wi}$ into F -stable subspaces, where V_{wp} is the maximal subspace of V which is weakly pure. A similar decomposition holds with weakly pure replaced by strongly pure.*

(iii) *Given pairs (V, F_V) and (W, F_W) that are weakly pure (strongly pure), the pair $(V \otimes W, F_V \otimes F_W)$ is also weakly pure (strongly pure, respectively).*

Proof. For (i), just note that the set of eigenvalues of F_V in V is the union of the corresponding sets for $(V', F_{V'})$ and $(V'', F_{V''})$.

For (ii), consider the minimal polynomial P_F of the endomorphism F , that is, the polynomial $P(t) \in \mathbb{Q}_l[t]$ of smallest degree such that $P(F) = 0$ and that P has leading term 1. Recall that the eigenvalues of F are exactly the roots of P_F . We have a unique factorization $P_F(t) = P_{wp}(t)P_{wi}(t)$ in $\mathbb{Q}_l[t]$, where $P_{wp}(t)$ denotes the largest factor of $P_F(t)$ with roots the weakly pure eigenvalues. Since the set of these eigenvalues is stable under the action of the Galois group of $\overline{\mathbb{Q}_l}$ over \mathbb{Q}_l , we see that both $P_{wp}(t)$ and $P_{wi}(t)$ have coefficients in \mathbb{Q}_l . Also, since $P_{wp}(t)$ and $P_{wi}(t)$ are coprime, we have

$$V = \text{Ker}(P_{wp}(F)) \oplus \text{Ker}(P_{wi}(F))$$

which yields the desired decomposition in the weakly pure case. The strongly pure case is handled by the same argument.

Finally, (iii) follows from the fact that the eigenvalues of $F_V \otimes F_W$ are exactly the products $\alpha\beta$, where α is an eigenvalue of F_V , and β of F_W . \square

Remark 1.9. In particular, weak and strong purity are stable by negative Tate twists in the following sense. Given an integer $n \geq 0$, we let $\mathbb{Q}_l(-n)$ denote the \mathbb{Q}_l -vector space of dimension 1 provided with the action of the Frobenius F by multiplication with q^n . Let V denote a \mathbb{Q}_l -vector space provided with an endomorphism F_V . Then $(V \otimes \mathbb{Q}_l, F_V \otimes F)$ is weakly pure (strongly pure) if (V, F) is weakly pure (strongly pure, respectively).

We now come to the main result of this section:

Theorem 1.10. *Let $\pi : X \rightarrow Y$ denote a torsor for the action of an algebraic group G . Let \mathcal{C}_Y denote a class of G -equivariant l -adic local systems on Y , and \mathcal{C}_X a class of G -equivariant l -adic local systems on X .*

(i) *Suppose $\mathcal{C}_X \supseteq \pi^*(\mathcal{C}_Y)$. Then if X is weakly pure with respect to \mathcal{C}_X so is Y with respect to \mathcal{C}_Y . In case G is split, and if X is strongly pure with respect to \mathcal{C}_X , then so is Y with respect to \mathcal{C}_Y .*

(ii) *Suppose G is also connected and $\mathcal{C}_X = \pi^*(\mathcal{C}_Y)$. Then if Y is weakly pure with respect to \mathcal{C}_Y , so is X with respect to \mathcal{C}_X . In case G is split and if Y is strongly pure with respect to \mathcal{C}_Y , so is X with respect to \mathcal{C}_X .*

It may be important to point out that the second statement in the above theorem applies to the class \mathcal{C}_X of G -equivariant l -adic local systems on X obtained as split summands of $\rho_{X*}(\underline{\mathbb{Q}_l}^{\oplus n})$ for some $n > 0$, where $\rho_X : X' \rightarrow X$ is a G -equivariant finite étale map and $\underline{\mathbb{Q}_l}^{\oplus n}$ is the constant local system of rank n on X' , and to the similarly defined class \mathcal{C}_Y . Indeed, taking $X' = X \times_Y Y'$, the proper base change theorem applied to the map ρ_Y shows that $\pi^*(\rho_{Y*}(\underline{\mathbb{Q}_l}^{\oplus n})) = \rho_{X*}(\pi'^*(\underline{\mathbb{Q}_l}^{\oplus n}))$, where $\pi' : X' \rightarrow Y'$ is the map induced by π . Therefore, \mathcal{C}_X contains $\pi^*(\mathcal{C}_Y)$ in this case.

The proof of Theorem 1.10 consists in decomposing π into a sequence of torsors under particular groups, and of fibrations with particular fibers, and in examining each step separately. Specifically, choose a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$, and denote by $R_u(G)$ the unipotent radical of G . Then $R_u(G) \subset B \subset G^0$. Moreover, $G' := G^0/R_u(G)$ is a connected reductive group with Borel subgroup $B' := B/R_u(G)$ and maximal torus $T' \subseteq B'$, isomorphic to T via the quotient map $G \rightarrow G'$. Now π factors as the composition of the following maps:

(i) $\pi_1 : X \rightarrow X' := X/R_u(G)$, a torsor under $R_u(G)$ and hence a locally trivial fibration with fiber an affine space,

(ii) $\pi_2 : X' \rightarrow X'/T'$, a torsor under $T' \simeq T$,

(iii) $\pi_3 : X'/T' \rightarrow X'/B' \simeq X/B$, a locally trivial fibration with fiber $B'/T' \simeq R_u(B')$, an affine space again,

(iv) $\pi_4 : X'/B' \rightarrow X'/G' \simeq X/G^o$, a locally trivial fibration with fiber $G'/B' \simeq G^o/B$, the flag variety of G' ,

(v) $\pi_5 : X/G^o \rightarrow X/G$, a Galois cover with group G/G^o .

As in Proposition 1.3, the étale fundamental groups of the schemes $(X'/T')_{\bar{\mathbb{F}}}$, $(X'/B')_{\bar{\mathbb{F}}}$ and $(X'/G')_{\bar{\mathbb{F}}}$ are all isomorphic, showing that the l -adic local systems on the above schemes correspond bijectively. Now let $\mathcal{C}_{X'/T'}$ ($\mathcal{C}_{X'/B'}$, $\mathcal{C}_{X'/G'}$) denote classes of l -adic local systems on these three schemes that are in bijective correspondence after base-extension to $\bar{\mathbb{F}}$ under the pull-back maps.

Proposition 1.11. *With the above notation and assumptions, X'/T' is weakly (strongly) pure with respect to $\mathcal{C}_{X'/T'}$ if and only if so is X'/B' with respect to $\mathcal{C}_{X'/B'}$, if and only if so is X'/G' with respect to $\mathcal{C}_{X'/G'}$.*

Proof. Since π_3 is a fibration in affine spaces, the first equivalence is clear.

We will next assume that X'/B' is weakly pure (strongly pure) with respect to $\mathcal{C}_{X'/B'}$, and show that X'/G' is weakly pure (strongly pure, respectively) with respect to $\mathcal{C}_{X'/G'}$. Let $\mathcal{L}' \in \mathcal{C}_{X'/G'}$ and put $\mathcal{L} := \pi_4^*(\mathcal{L}')$. The Leray spectral sequence

$$E_2^{s,t} = H^s(X'/G', R^t \pi_{4*}(\mathcal{L})) \Rightarrow H^{s+t}(X'/B', \mathcal{L})$$

and the isomorphism

$$R^t \pi_{4*}(\mathcal{L}) = R^t \pi_{4*}(\pi_4^*(\mathcal{L}')) \simeq \mathcal{L}' \otimes R^t \pi_{4*}(\mathbb{Q}_l)$$

yield a spectral sequence

$$(1.3.1) \quad E_2^{s,t} = H^s(X'/G', \mathcal{L}' \otimes R^t \pi_{4*}(\mathbb{Q}_l)) \Rightarrow H^{s+t}(X'/B', \pi_4^*(\mathcal{L}')).$$

By Deligne's degeneracy criterion (see [De68, Proposition 2.4 and (2.6.3)]), that spectral sequence degenerates at E_2 . In particular, $H^*(X'/G', \mathcal{L}') = E_2^{s,0} = E_\infty^{s,0}$ injects into $H^s(X'/B', \mathcal{L})$, which yields our assertion.

For the converse, let $\mathcal{L} \in \mathcal{C}_{X'/B'}$. Without loss of generality, we may assume that $\mathcal{L} = \pi_4^*(\mathcal{L}')$ for some $\mathcal{L}' \in \mathcal{C}_{X'/G'}$. Then we still have the spectral sequence (1.3.1) that degenerates at E_2 . Thus, the desired assertion will follow if we show that the l -adic sheaf $R^t \pi_{4*}(\mathbb{Q}_l)$ is isomorphic to the constant sheaf $H^t(G'/B'_{\bar{\mathbb{F}}}, \mathbb{Q}_l) = \bigoplus_i \mathbb{Q}_l(-t_i)$, which is a finite sum with each $t_i \geq 0$. For this we will use the Leray-Hirsch theorem (see [Hu94, Chapter 17, 1.1 Theorem]) adapted to the present framework as follows.

First the l -adic cycle map on the flag variety G'/B' is shown to be an isomorphism in [Jo01, Theorem 5.2]. The Chow ring of G'/B' is generated over \mathbb{Q}_l by the Chern classes of the equivariant line bundles associated with a basis $\{\chi_i | i = 1, \dots, n\}$ of the character group of B' , as shown in [Gr58, Corollaire 4]. We will denote these classes by $\{f_i | i = 1, \dots, n\}$. Each χ_i also defines a line bundle over X'/B' via the B' -torsor $X' \rightarrow X'/B'$; this defines Chern classes $\{e_i | i = 1, \dots, n\}$ in the l -adic cohomology of X'/B' , which lift the above classes f_i under pull-back to a fiber G'/B' . In this setting, the Leray-Hirsch theorem is the statement that the map $\alpha \otimes f_i \mapsto \pi_4^*(\alpha)e_i$ defines an isomorphism

$$H^*(X'/G', \mathbb{Q}_l) \otimes H^*(G'/B', \mathbb{Q}_l) \cong H^*(X'/B', \mathbb{Q}_l).$$

In case the torsor is locally trivial over a Zariski open covering, one may prove the Leray-Hirsch theorem using a Mayer-Vietoris sequence with respect to the covering over which the torsor is trivial. (See [Hu94, Chapter 17, 1.1 Theorem] for a proof in the setting of singular cohomology, but one can see readily that the proof works also in l -adic cohomology.) Since we are considering torsors that are trivial with respect to the étale topology, this argument needs to be modified as follows. First, let $U \rightarrow X'/G'$ denote an étale covering over which the given torsor $\mathcal{E} = X'/B'$ (with fibers G'/B') is trivial. Let $U_\bullet = \text{cosk}_0^{X'/G'}(U)$ denote the hypercovering of X'/G' defined by U . If \mathcal{E}_{U_\bullet} denotes the pull-back of the torsor \mathcal{E} to U_\bullet , then \mathcal{E}_{U_\bullet} trivializes. Therefore, the Künneth formula, along with the hypotheses provides the isomorphism: $H^*(\mathcal{E}_{U_\bullet}, \mathbb{Q}_l) \simeq H^*(U_\bullet, \mathbb{Q}_l) \otimes H^*(G'/B', \mathbb{Q}_l)$. However, since U_\bullet is a hypercovering of X'/G' and \mathcal{E}_{U_\bullet} is a hypercovering of X'/B' , one obtains the identification $H^*(U_\bullet, \mathbb{Q}_l) \simeq H^*(X'/G', \mathbb{Q}_l)$ and $H^*(\mathcal{E}_{U_\bullet}, \mathbb{Q}_l) \simeq H^*(X'/B', \mathbb{Q}_l)$, see [Fr83, Proposition 3.7].

We now show that the sheaves $R^t\pi_{4*}(\mathbb{Q}_l)$ are of the form as stated. For this we fix a geometric point \bar{x} on X'/G' and consider the geometric fiber of π_4 at \bar{x} ; this fiber identifies with G'/B' . Now we consider the commutative diagram:

$$\begin{array}{ccc} G'/B' & \xrightarrow{i} & X'/B' \\ \downarrow & & \downarrow \pi_4 \\ \bar{x} & \longrightarrow & X'/G' \end{array}$$

This induces a map of the corresponding Leray spectral sequences. Now one obtains the commutative diagram where the $E_r^{s,t}$ -terms of the spectral sequence for the trivial fibration $G'/B' \rightarrow \bar{x}$ are marked as $E_r^{s,t}(2)$:

$$\begin{array}{ccccccc} H^t(X'/B', \mathbb{Q}_l) \twoheadrightarrow & E_\infty^{0,t} & \cong E_{t+1}^{0,t} & \simeq \cdots \simeq & E_2^{0,t} = H^0(X'/G', R^t\pi_{4*}(\mathbb{Q}_l)) & & \\ \downarrow i^* & & & & \downarrow & & \\ H^t(G'/B', \mathbb{Q}_l) \twoheadrightarrow & E_\infty^{0,t}(2) & \cong E_{t+1}^{0,t}(2) & \simeq \cdots \simeq & E_2^{0,t}(2) = H^t(G'/B', \mathbb{Q}_l) & & \\ & & & & \downarrow id & & \\ & & & & R^t\pi_{4*}(\mathbb{Q}_l)_{\bar{x}} \cong H^t(G'/B', \mathbb{Q}_l) & & \end{array}$$

The isomorphisms on the top row follow from the degeneration of the spectral sequence at the E_2 -terms as observed above. Since the spectral sequence for the map $G'/B' \rightarrow \bar{x}$ clearly degenerates with $E_2^{s,t} = 0$ for all $s \neq 0$, the maps in the bottom row (including the first) are all isomorphisms. The naturality of the Leray spectral sequences then shows that the composition of the maps in the top-row and the right column identifies with the restriction map $i^* : H^t(X'/B', \mathbb{Q}_l) \rightarrow H^t(G'/B', \mathbb{Q}_l)$. Now the map $H^0(X'/G', \mathbb{Q}_l) \otimes H^t(G'/B', \mathbb{Q}_l) \rightarrow H^t(X'/B', \mathbb{Q}_l)$ sending the classes f_i to e_i composed with the maps in the top row and the right column of the above diagram is an isomorphism. Since the geometric point \bar{x} was chosen arbitrarily, this shows that the corresponding map of the constant sheaf $H^t(G'/B', \mathbb{Q}_l)$ to the sheaf $R^t\pi_{4*}(\mathbb{Q}_l)$ is an isomorphism. Since the l -adic cohomology of G'/B' is isomorphic to the Chow ring with coefficients in \mathbb{Q}_l via the cycle map, this yields the decompositions $H^t(G'/B', \mathbb{Q}_l) \cong \oplus_i \mathbb{Q}_l(-t_i)$. This completes the proof of the theorem. \square

Remark 1.12. With the above notation and assumptions, the G -equivariant l -adic local systems on X correspond bijectively to the $G/R_u(G)$ -equivariant local systems on X' , by Lemma 1.4. Moreover, if \mathcal{C}_X ($\mathcal{C}_{X'}$) denote classes of G -equivariant local systems on X (of $G/R_u(G)$ -equivariant local systems on X' , respectively) that are in bijective correspondence (after base extension to $\bar{\mathbb{F}}$) under pull-back, then X is weakly pure (strongly pure) with respect to \mathcal{C}_X if and only if X' is weakly pure (strongly pure, respectively) with respect to $\mathcal{C}_{X'}$.

In other words, weak and strong purity are preserved under the above maps π_1 , π_3 and π_4 . The case of the map π_5 is handled by the following.

Proposition 1.13. *Let $\pi : X \rightarrow Y$ denote a torsor under an algebraic group G , and $\psi : Z = X/G^o \rightarrow X/G = Y$ the associated Galois cover with group G/G^o . Let \mathcal{C}_Y (\mathcal{C}_Z) denote a class of G -equivariant l -adic local systems on Y (Z , respectively).*

Suppose ψ_ sends \mathcal{C}_Z to \mathcal{C}_Y . If Y is weakly pure (strongly pure) with respect to \mathcal{C}_Y , then so is Z with respect to \mathcal{C}_Z .*

Suppose ψ^ sends \mathcal{C}_Y to \mathcal{C}_Z . If Z is weakly pure (strongly pure) with respect to \mathcal{C}_Z , then so is Y with respect to \mathcal{C}_Y .*

Proof. Once again one considers the fibration $G/G^o \rightarrow Z \xrightarrow{\psi} Y$. Since G acts trivially on Y , observe that $EG \times_Y G \simeq BG \times Y$, so that G -equivariant l -adic local systems on Y correspond to l -adic local

systems on Y_{et} . (Strictly speaking one needs to consider the tensor product $V \otimes_{\mathbb{Q}_l} \mathcal{L}$, where \mathcal{L} is an l -adic local system on Y_{et} , and V is an l -adic representation of the finite group G/G^o . But $V \otimes_{\mathbb{Q}_l} \mathcal{L}$ is also an l -adic local system on Y_{et} .)

Suppose that Y is weakly pure with respect to \mathcal{C}_Y , and ψ_* sends \mathcal{C}_Z to \mathcal{C}_Y . Let $\mathcal{L} \in \mathcal{C}_Z$. Then $\psi_*(\mathcal{L})$ is a G -equivariant l -adic local system on Y , i.e. a local system on Y . Moreover, if \mathcal{L} is mixed of weight $\geq w$, then so is $\pi_*(\mathcal{L})$. The hypothesis shows that $H^*(Y, \psi_*(\mathcal{L}))$ is weakly pure. Recall that one has a spectral sequence $E_2^{s,t} = H^s(Y, R^t \psi_*(\mathcal{L})) \Rightarrow H^{s+t}(Z, \mathcal{L})$. This spectral sequence clearly degenerates since $E_2^{s,t} = 0$ for $t > 0$. Therefore, the $E_2^{s,t}$ terms identify with the $E_\infty^{s,t}$ -terms and it follows that the abutment $H^*(Z, \mathcal{L})$ is weakly pure.

Next suppose that Z is weakly pure with respect to \mathcal{C}_Z , that ψ^* sends \mathcal{C}_Y to \mathcal{C}_Z and that $\mathcal{L} \in \mathcal{C}_Y$. Then $\psi^*(\mathcal{L})$ is a G -equivariant l -adic local system on Z . This is mixed of weight $\geq w$ if \mathcal{L} is mixed of weight $\geq w$. Therefore, by the hypothesis, $H^*(Z, \pi^*(\mathcal{L}))$ is weakly pure. Now $H^*(Y, \mathcal{L})$ is a summand of $H^*(Y, \psi_*(\psi^*(\mathcal{L}))) = H^*(Z, \psi^*(\mathcal{L}))$ so that it is also weakly pure. This completes the proof of the assertions about weak purity; those about strong purity are obtained along the same lines. \square

Finally, the case of the map π_2 is handled by an induction on the dimension of the torus T' (possibly after a finite extension of fields so that it becomes split) together with the following:

Proposition 1.14. *Let $\pi : X \rightarrow Y$ denote a \mathbb{G}_m -torsor. Let \mathcal{C}_X (\mathcal{C}_Y) denote a class of \mathbb{G}_m -equivariant local systems on X (Y equipped with the trivial \mathbb{G}_m -action, respectively) such that π^* sends \mathcal{C}_Y to \mathcal{C}_X . Then the following hold: (i) if X is weakly pure (strongly pure) with respect to the class \mathcal{C}_X , then so is Y with respect to the class \mathcal{C}_Y . (ii) If in addition, $\mathcal{C}_X = \pi^*(\mathcal{C}_Y)$, then the converse to (i) also holds.*

Proof. Assume that X is weakly pure with respect to \mathcal{C}_X , and let $\mathcal{L}' \in \mathcal{C}_Y$. Then $\mathcal{L} := \pi^*(\mathcal{L}')$ belongs to the class \mathcal{C}_X and satisfies $R^i \pi_*(\mathcal{L}) \simeq \mathcal{L}' \otimes R^i \pi_*(\mathbb{Q}_l)$ for all $i \geq 0$. Also, since \mathbb{G}_m is affine and of dimension 1, Proposition 1.1 yields that $R^i \pi_*(\mathbb{Q}_l) = 0$ for $i > 1$. Moreover, we have $R^i \pi_*(\mathbb{Q}_l) \simeq \mathbb{Q}_l$ for $i = 0$ and we will presently show that $R^1 \pi_*(\mathbb{Q}_l) = \mathbb{Q}_l(-1)$ so that $R^i \pi_*(\mathcal{L}) \simeq \mathcal{L}'$ for $i = 0$, $R^1 \pi_*(\mathcal{L}) \simeq \mathcal{L}'(-1)$ and $R^i \pi_*(\mathcal{L}) = 0$ for $i > 1$. The identification $\pi_*(\mathbb{Q}_l) \simeq \mathbb{Q}_l$ is straightforward; to obtain the identification $R^1 \pi_*(\mathbb{Q}_l) \simeq \mathbb{Q}_l(-1)$, one may proceed as follows. Since $R^2 \pi_!(\mathbb{Q}_l) \simeq \mathbb{Q}_l(-1)$ by the trace map, it suffices to show using Poincaré duality that $R^1 \pi_!(\mathbb{Q}_l) \simeq \mathbb{Q}_l$. Let $\pi_1 : L \rightarrow Y$ denote the line bundle associated with the \mathbb{G}_m -torsor $\pi : X \rightarrow Y$. Then we have an open immersion $j : X \rightarrow L$ with complement the zero section L_0 , isomorphic to Y . Denoting that isomorphism by $\pi_2 : L_0 \rightarrow Y$, one obtains a distinguished triangle $R\pi_!(\mathbb{Q}_l) \rightarrow R\pi_{1!}(\mathbb{Q}_l) \rightarrow R\pi_{2!}(\mathbb{Q}_l) \simeq \mathbb{Q}_l[0] \rightarrow R\pi_!(\mathbb{Q}_l)[1]$. Computing the cohomology sheaves, one obtains the isomorphism $\mathbb{Q}_l \simeq R^1 \pi_!(\mathbb{Q}_l)$ since $R^1 \pi_{1!}(\mathbb{Q}_l) = 0$.

Now the Leray spectral sequence

$$E_2^{s,t} = H^s(Y, R^t \pi_*(\mathcal{L})) \Rightarrow H^{s+t}(X, \mathcal{L})$$

provides us with the long exact sequence (see, for example, [CE56, Chapter XV, Theorem 5.11]):

$$(1.3.2) \quad \dots \rightarrow E_2^{n,0} \rightarrow H^n(X, \mathcal{L}) \rightarrow E_2^{n-1,1} \xrightarrow{d^2} E_2^{n+1,0} \rightarrow H^{n+1}(X, \mathcal{L}) \rightarrow E_2^{n,1} \xrightarrow{d^2} E_2^{n+2,0} \rightarrow \dots$$

It is important to observe that all the maps in the above long exact sequence are compatible with the action of the Frobenius. In view of the isomorphisms

$$E_2^{n,0} = H^n(Y, \mathcal{L}' \otimes \pi_*(\mathbb{Q}_l)) \simeq H^n(Y, \mathcal{L}'),$$

$$E_2^{n,1} = H^n(Y, \mathcal{L}' \otimes R^1 \pi_*(\mathbb{Q}_l)) \simeq H^n(Y, \mathcal{L}'(-1)) \simeq H^n(Y, \mathcal{L}')(-1)$$

for all $n \geq 0$, this yields exact sequences (of maps compatible with the Frobenius)

$$(1.3.3) \quad H^n(X, \mathcal{L}) \xrightarrow{\alpha} H^{n-1}(Y, \mathcal{L}')(-1) \xrightarrow{\beta} H^{n+1}(Y, \mathcal{L}') \xrightarrow{\gamma} H^{n+1}(X, \mathcal{L})$$

for all $n \geq 1$, and an isomorphism

$$H^0(Y, \mathcal{L}') \simeq H^0(X, \mathcal{L}),$$

all compatible with the Frobenius. In particular, $H^0(Y, \mathcal{L}')$ is weakly pure. We will now argue by ascending induction on n and assume that we have already shown the weak purity of $H^i(Y, \mathcal{L}')$ for all $i \leq n$. Then in the long exact sequence (1.3.3), all the terms except for $H^{n+1}(Y, \mathcal{L}')$ are weakly pure. Now one may break up that sequence into the short exact sequence:

$$0 \rightarrow \text{Im}(\beta) \rightarrow H^{n+1}(Y, \mathcal{L}') \rightarrow \text{Im}(\gamma) \rightarrow 0.$$

Since $\text{Im}(\gamma)$ is a sub-vector space of $H^{n+1}(X, \mathcal{L})$, it is weakly pure by Lemma 1.8 (i). Moreover,

$$\text{Im}(\beta) \simeq H^{n-1}(Y, \mathcal{L}')(-1)/\text{Ker}(\beta)$$

and hence $\text{Im}(\beta)$ is weakly pure by the induction assumption and Lemma 1.8 (i) again. This completes the inductive argument, and shows that Y is weakly pure with respect to \mathcal{C}_Y .

By the same argument, if X is strongly pure with respect to \mathcal{C}_X , then so is Y with respect to \mathcal{C}_Y .

For the converse direction, one argues similarly by using the exact sequence

$$H^{n-2}(Y, \mathcal{L}')(-1) \xrightarrow{\beta} H^n(Y, \mathcal{L}') \xrightarrow{\gamma} H^n(X, \mathcal{L}) \xrightarrow{\alpha} H^{n-1}(Y, \mathcal{L}')(-1) \xrightarrow{\beta} H^{n+1}(Y, \mathcal{L}')$$

□

To conclude this section, we recall a result of Deligne (see [De74b, 9.1.4]) that fits very well into our framework, and will be a key ingredient of the proof of Theorem 2.1.

Theorem 1.15. *For any algebraic group G , the l -adic cohomology $H^*(BG, \mathbb{Q}_l)$ is weakly pure and vanishes in odd degrees.*

If G is split, then F acts on $H^{2n}(BG, \mathbb{Q}_l)$ via multiplication by q^n , for any integer $n \geq 0$.

Proof. We follow the argument sketched in [loc. cit.]. We first assume that G is split, and show the vanishing of cohomology in odd degrees and the second assertion. Using the fibration $G/G^o \rightarrow BG^o \rightarrow BG$, one reduces as above to the case where G is connected. Likewise, using the fibration $G/B \rightarrow BB \rightarrow BG$, and arguing as in the proof of Proposition 1.11 one may further assume that G is solvable. Then $G = R_u(G)T$ and the fibration $R_u(G) \rightarrow BT \rightarrow BG$ yields a reduction to the case where $G = T$. Since G is split, then T is a direct product of copies of \mathbb{G}_m . Now an induction on the dimension, together with the isomorphism $B(T \times T') \simeq BT \times BT'$ (for two tori T and T') reduce to the case where $T = \mathbb{G}_m$. In this case, it is well-known that $\mathbb{P}^\infty = \lim_{n \rightarrow \infty} \mathbb{P}^n$ is also a model for $B\mathbb{G}_m$, i.e. the l -adic cohomology of the latter is approximated in any given finite range by the l -adic cohomology of a sufficiently large projective space. In view of the structure of $H^*(\mathbb{P}^n, \mathbb{Q}_l)$, this yields the last assertion and the vanishing in odd degrees.

If G is arbitrary, then it splits over some finite extension \mathbb{F}_{q^n} , and therefore the arguments above together with Remark 1.7(ii) yield the assertion on weak purity. □

2. Preservation of the notions of purity by GIT quotients

In this section, we show that the properties of weak purity and strong purity are preserved by taking certain geometric invariant theory quotients.

We consider a connected reductive group G and a smooth G -variety X equipped with an ample, G -linearized line bundle L . We denote by X^{ss} the open G -stable subset of X consisting of semi-stable points with respect to L , in the sense of [MFK94, Definition 1.7], and by

$$\pi : X^{\text{ss}} \longrightarrow X^{\text{ss}}//G$$

the categorical quotient; we will also use the notation $X//G$ for the GIT quotient variety $X^{\text{ss}}//G$. The complement $X \setminus X^{\text{ss}}$ is the unstable locus of X . The subset of (properly) stable points will be denoted by X^{s} ; recall from [loc.cit.] that X^{s} consists of those points of X^{ss} having a finite stabilizer and a closed orbit in X^{ss} . Also, X^{s} is an open G -stable subset of X , and π restricts to a geometric quotient $X^{\text{s}} \rightarrow X^{\text{s}}/G$.

Theorem 2.1. *Let X be a smooth G -variety equipped with an ample G -linearized line bundle such that the two following conditions are satisfied:*

- (a) *Every semi-stable point of X is stable.*
- (b) *X admits an equivariantly perfect stratification with open stratum $X^{\text{ss}} = X^{\text{s}}$.*

If X is weakly pure with respect to the constant local system \mathbb{Q}_l , then so is the GIT quotient $X//G$.

If G is split and X is strongly pure with respect to \mathbb{Q}_l , then so is $X//G$.

Proof. By Remark 1.7 (ii), it suffices to show the final assertion. We begin by obtaining an isomorphism

$$(2.0.1) \quad H^*(X//G, \mathbb{Q}_l) \simeq H^*(EG \times_G X^{\text{ss}}, \mathbb{Q}_l).$$

For this, consider the morphism $EG \times_G X^{\text{ss}} \rightarrow X^{\text{ss}}/G = X//G$, with fiber at the image of the geometric point $\bar{x} \in \bar{X}^{\text{ss}}$ being $EG \times_G G/G_{\bar{x}} = EG/G_{\bar{x}} = BG_{\bar{x}}$. By [Jo93, (A.3.1)], this yields a Leray spectral sequence

$$E_2^{s,t} = H^s(X//G, R^t \pi_*(\mathbb{Q}_l)) \Rightarrow H^{s+t}(EG \times_G X^{\text{ss}}, \mathbb{Q}_l)$$

together with the identification of the stalks $R^t(\pi_*(\mathbb{Q}_l))_{\bar{x}} \cong H^t(BG_{\bar{x}}, \mathbb{Q}_l)$. Since the stabilizers $G_{\bar{x}}$ are all assumed to be finite groups, it follows that $E_2^{s,t} = 0$ for $t > 0$ and $E_2^{s,0} \cong H^s(X//G, \mathbb{Q}_l)$; this implies the isomorphism (2.0.1).

Also, one obtains that the pull-back map

$$H^*(EG \times_G X, \mathbb{Q}_l) \longrightarrow H^*(EG \times_G X^{\text{ss}}, \mathbb{Q}_l)$$

is a surjection by arguing as in [Kir84, p. 98]. Together with the isomorphism (2.0.1) again, it follows that it suffices to show that $H^*(EG \times_G X, \mathbb{Q}_l)$ is strongly pure. For this, we consider the projection $\pi : EG \times_G X \rightarrow BG$ and the associated Leray spectral sequence

$$E_2^{s,t} = H^s(BG, R^t \pi_*(\mathbb{Q}_l)) \Rightarrow H^{s+t}(EG \times_G X, \mathbb{Q}_l).$$

Since G is assumed to be connected, $\pi_1(BG_{\text{et}}) = 0$ and hence the l -adic local system $R^t \pi_*(\mathbb{Q}_l)$ is constant. (See [Jo93, (A.8) Theorem] and also [Jo02, (4.2) Theorem].) Therefore, the $E_2^{s,t}$ -term takes on the form $H^s(BG, \mathbb{Q}_l) \otimes H^t(X, \mathbb{Q}_l)$. Moreover, since X is strongly pure, Theorem 1.15 and Lemma 1.8(iii) show that the $E_2^{s,t}$ terms above are strongly pure, and hence so are the E_∞ terms as well as the abutment. This completes the proof of the theorem. \square

Corollary 2.2. *Let G be a split connected reductive group and X a smooth projective G -variety equipped with an ample G -linearized line bundle such that every semi-stable point is stable. If X is strongly pure, then so is the GIT quotient $X//G$.*

Proof. Let $(S_\beta)_{\beta \in \mathbf{B}}$ denote the stratification of X constructed by Kirwan (see [Kir84, Theorem 13.5]). Each S_β is a smooth, locally closed G -stable subvariety of X , and the open stratum S_0 equals X^{ss} . Moreover,

$$S_\beta = G \times_{P_\beta} Y_\beta^{\text{ss}}$$

for some parabolic subgroup P_β of G and some smooth, closed P_β -stable subvariety $Y_\beta^{\text{ss}} \subset S_\beta$. Finally, there exists a fibration $p_\beta : Y_\beta^{\text{ss}} \rightarrow Z_\beta^{\text{ss}}$ with fibers affine spaces, where Z_β^{ss} denotes the subset of semi-stable points of a smooth, closed subvariety $Z_\beta \subset X$, stable under a Levi subgroup L_β of P_β ; also, Z_β is a union of connected components of the fixed point locus X^{T_β} , where T_β is a subtorus of G with centralizer L_β .

By the arguments of [Kir84, Part I], this stratification is equivariantly perfect. Specifically, by a criterion of Atiyah and Bott (see [AB83, 1.4]), it suffices to show that the equivariant Euler class of the normal bundle N_β to S_β in X is not a zero divisor in $H_G^*(S_\beta, \mathbb{Q}_l)$. But

$$H_G^*(S_\beta, \mathbb{Q}_l) \cong H_{P_\beta}^*(Y_\beta^{\text{ss}}, \mathbb{Q}_l) \cong H_{L_\beta}^*(Y_\beta^{\text{ss}}, \mathbb{Q}_l) \cong H_{L_\beta}^*(Z_\beta^{\text{ss}}, \mathbb{Q}_l)$$

and this identifies the equivariant Euler class of N_β with that of the restriction $N_\beta|_{Z_\beta^{\text{ss}}}$. But that restriction is a quotient of the normal bundle N'_β to Z_β^{ss} in X , and the action of T_β on each fiber of N'_β has no non-zero fixed vector. By the lemma below, it follows that the equivariant Euler class of N'_β is not a zero divisor in $H_{L_\beta}^*(Z_\beta^{\text{ss}}, \mathbb{Q}_l)$; thus, the same holds for the equivariant Euler class of N_β . \square

Lemma 2.3. *Let L be an algebraic group, Z a L -variety, and N a L -linearized vector bundle on Z . Assume that a subtorus T of L acts trivially on Z and fixes no non-zero point in each fiber of N . Then the equivariant Euler class of N is not a zero divisor in $H_L^*(Z, \mathbb{Q}_l)$.*

Proof. We adapt the argument of [AB83, 13.4]. Choose a maximal torus T_L of L containing L . Then the natural map $H_L^*(Z, \mathbb{Q}_l) \rightarrow H_{T_L}^*(Z, \mathbb{Q}_l)$ is injective; thus, we may replace L with T_L , and assume that L is a torus. Now $L \cong T \times T'$ for some subtorus T' of L . Therefore, $H_L^*(Z, \mathbb{Q}_l) \cong H^*(BT, \mathbb{Q}_l) \otimes H_{T'}^*(Z, \mathbb{Q}_l)$, since T fixes Z pointwise. Moreover, N decomposes as a direct sum of L -linearized vector bundles N_χ on which T acts via a non-zero character χ . Thus, we may further assume that $N = N_\chi$. Then the equivariant Euler class of N satisfies $c_d^L(N) = \prod_{i=1}^d (\chi + \alpha_i)$, where d denotes the rank of N , and α_i its T' -equivariant Chern roots. This is a non-zero divisor in $H^*(BT, \mathbb{Q}_l) \otimes H_{T'}^*(Z, \mathbb{Q}_l)$ since $\chi \neq 0$. \square

Corollary 2.2 applies for instance to the case where X is a product of Grassmannians:

$$X = \prod_{i=1}^m \text{Gr}(r_i, n), \quad L = \boxtimes_{i=1}^m \mathcal{O}_{\text{Gr}(r_i, n)}(a_i)$$

where $\text{Gr}(r, n)$ denotes the Grassmannian of r -dimensional linear subspaces of projective n -space, and $\mathcal{O}_{\text{Gr}(r, n)}(a)$ denotes the a -th power of the line bundle associated with the Plücker embedding; here $G = \text{PGL}(n+1)$ and $r_1, \dots, r_m < n$, a_1, \dots, a_m are positive integers. Indeed, X is clearly strongly pure; moreover, $X^{\text{ss}} = X^s$ for general values of a_1, \dots, a_m (see [Do03, Section 11.1]). The geometric quotient $X//G$ is called the space of stable configurations; examples include moduli spaces of m ordered points in \mathbb{P}^n .

Presently we will provide an extension of Corollary 2.2 to local systems. We need a general result on lifting group actions under finite étale covers.

Proposition 2.4. *Let G be a connected algebraic group, X a complete G -variety, and $f : X' \rightarrow X$ a finite étale cover. Then there exist a finite étale cover of connected algebraic groups $\pi : G' \rightarrow G$ and an action of G' on X' which lifts the given action of G on X . If G is reductive, then so is G' .*

Proof. Let $\alpha : G \times X \rightarrow X$ denote the action map and let $F : Z' \rightarrow Z = G \times X$ denote the pull-back of f by α . Then F is again a finite étale cover. Since X is complete, [SGA1, Exposé X, Corollaire 1.9] shows that the coverings $F_g : X'_g \rightarrow X$ are all isomorphic, where $g \in G(k)$, X is identified with the subscheme $\{g\} \times X$ of $G \times X$, and F_g denotes the pull-back of F to $\{g\} \times X$.

Therefore, for each $g \in G(k)$, we have an isomorphism $X' \rightarrow X'_g$ of schemes over X . On the other hand, since α is the action map, then X'_g is the pull-back of X' by the automorphism g of X . Thus, we obtain an isomorphism $X'_g \rightarrow X'$ lifting $g : X \rightarrow X$. Composing with the isomorphism $X' \rightarrow X'_g$ yields an automorphism of X' which lifts g . Therefore, every closed point of G lifts to an automorphism of X' .

Now recall that the functor of automorphisms of X is represented by a group scheme $\text{Aut}(X)$, locally of finite type. Also, we have a group scheme $\text{Aut}(X', X)$ of pairs of compatible automorphisms of X' and X , equipped with a homomorphism

$$f_* : \text{Aut}(X', X) \longrightarrow \text{Aut}(X).$$

The kernel of f_* is the group scheme of relative automorphisms $\text{Aut}(X'/X)$; this is a finite reduced group scheme, since its Lie algebra (the derivations of $\mathcal{O}_{X'}$ over \mathcal{O}_X) is trivial. On the other hand, we just showed that the image of f_* contains the image of G in $\text{Aut}(X)$. We may now take the pull-back of $G \rightarrow \text{Aut}(X)$ by f_* to obtain a finite étale cover $\pi : G' \rightarrow G$ such that G' acts on X' by lifting the G -action on X . In case G' is not connected, we replace it by its neutral component to obtain the desired cover.

To complete the proof, note that the unipotent radical $R_u(G')$ is a finite cover of $R_u(G)$. If G is reductive, then $R_u(G')$ is a finite scheme, and hence a point. In other words, G' is reductive. \square

Theorem 2.5. *With the assumptions of Corollary 2.2, let $\mathcal{L} = \{\mathcal{L}_\nu | \nu\}$ denote a G -equivariant l -adic local system on X which satisfies the following two assumptions:*

(c) *There exists some finite étale cover X' of X on which \mathcal{L} is the constant G -equivariant local system (i.e. each \mathcal{L}_ν is a G -equivariant constant sheaf).*

(d) *The restriction of \mathcal{L} to X^{ss} is the pull-back of an l -adic local system \mathcal{M} on $X//G$.*

If X is weakly pure (strongly pure) with respect to \mathcal{L} , then so is $X//G$ with respect to \mathcal{M} .

Proof. Let $f : X' \rightarrow X$ denote the given finite étale cover, L the given ample G -linearized line bundle on X , and $\pi : G' \rightarrow G$ the cover obtained in Proposition 2.4. Then $L' := f^*(L)$ is an ample line bundle on X' , and we may assume (replacing G' with a further covering) that L' is G' -linearized. It follows that the strata for the Kirwan stratification of X' (with respect to G' and L') are exactly the pull-backs of the strata of X . For this stratification, the long exact sequence in G' -equivariant cohomology with respect to the constant local system breaks up into short exact sequences.

Also, the composition of the maps $\mathcal{L} \rightarrow f_* f^* \mathcal{L} \rightarrow \mathcal{L}$ of sheaves on $EG \times_G X$ is an isomorphism, where the first map is given by adjunction, and the second one is the trace map. In view of assumption (c), it follows that \mathcal{L} is a direct factor of $f_*(\mathbb{Q}_l^{\oplus n})$. Now consider the long exact sequence in G -equivariant cohomology with respect to the local system \mathcal{L} provided by the stratification of X : the terms in this sequence are summands in the corresponding long exact sequence in the G' -equivariant cohomology of X' . Therefore, this breaks up into short exact sequences, and hence the restriction map

$$(2.0.2) \quad H_G^*(X, \mathcal{L}) \longrightarrow H_G^*(X^{\text{ss}}, \mathcal{L})$$

is surjective.

We now adapt the proof of Theorem 2.1. Recall the spectral sequence in equivariant cohomology: $E_2^{s,t} = H^s(BG, R^t \pi_* (\mathcal{L})) \Rightarrow H_G^{s+t}(X, \mathcal{L})$ where $\pi : EG \times_G X \rightarrow BG$ denotes the projection. Since G is connected, the local system $R^t \pi_* (\mathcal{L})$ is constant on BG , so that $E_2^{s,t} = H^s(BG, \mathbb{Q}_l) \otimes H^t(X, \mathcal{L})$. The hypothesis of weak purity (strong purity) of X and Lemma 1.8 (iii) show that the E_2 -terms are weakly pure (strongly pure, respectively) and hence so are the abutments $H_G^{s+t}(X, \mathcal{L})$. By the surjectivity of the map in (2.0.2), $H_G^*(X^{\text{ss}}, \mathcal{L})$ is also weakly pure (strongly pure, respectively). But $H_G^*(X^{\text{ss}}, \mathcal{L}) \cong H^*(X//G, \mathcal{M})$ in view of assumption (d). This completes the proof of the theorem. \square

Remarks 2.6. (i) The above assumption (d) is satisfied whenever the quotient morphism $\pi : X^{\text{ss}} \rightarrow X^{\text{ss}}//G$ is a G -torsor. Equivalently, the stabilizer of any semi-stable point is trivial as a subgroup scheme.

(ii) If the unstable locus $X \setminus X^{\text{ss}}$ has codimension ≥ 2 , then the fundamental groups of X and X^{ss} are naturally isomorphic, by [SGA1, Exposé X, Corollaire 3.3]. Thus, the local systems on X and X^{ss} are in bijective correspondence via restriction, and the same holds for the G -equivariant local systems. In particular, given any local system \mathcal{M} on $X//G$, the pull-back $\pi^* \mathcal{M}$ extends to a unique G -equivariant local system on X .

If in addition X satisfies the assumptions of Corollary 2.2, then we see that $X//G$ is weakly (strongly) pure with respect to all local systems associated with finite étale covers, if so is X .

3. The l -adic cohomology of quiver-moduli

In this section, we begin by recalling some basic facts about quiver representations and the associated moduli spaces. While the material we discuss is well-known, it seems to be scattered in the literature (see e.g. [Kin94, Re08]): we summarize the relevant details from the point of view of GIT.

A *quiver* Q is a finite directed graph, possibly with oriented cycles. That is, Q is given by a finite set of vertices I (often also denoted Q_0) and a finite set of arrows Q_1 . The arrows will be denoted by $\alpha : i \rightarrow j$. We will denote by $\mathbb{Z}\mathbf{I}$ the free abelian group generated by I ; the basis consisting of elements of I will be denoted by \mathbf{I} . An element $\mathbf{d} \in \mathbb{Z}\mathbf{I}$ will be written as $\mathbf{d} = \sum_{i \in I} d_i \mathbf{i}$.

Let $\text{Mod}(\mathbb{F}Q)$ denote the abelian category of finite-dimensional representations of Q over the finite field \mathbb{F} (or, equivalently, finite-dimensional representations of the path algebra $\mathbb{F}Q$). Its objects are thus given by tuples

$$(3.0.1) \quad M = ((M_i)_{i \in I}, (M_\alpha : M_i \rightarrow M_j)_{\alpha: i \rightarrow j})$$

of finite-dimensional \mathbb{F} -vector spaces and \mathbb{F} -linear maps between them.

The *dimension vector* $\mathbf{dim}(M) \in \mathbb{N}\mathbf{I}$ is defined as $\mathbf{dim}(M) = \sum_{i \in I} \dim_{\mathbb{F}}(M_i) \mathbf{i}$. The *dimension* of M will be defined to be $\sum_{i \in I} \dim_{\mathbb{F}}(M_i)$, i.e. the sum of the dimensions of the \mathbb{F} -vector spaces M_i . This will be denoted $\dim(M)$.

We denote by $\text{Hom}_{\mathbb{F}Q}(M, N)$ the \mathbb{F} -vector space of homomorphisms between two representations $M, N \in \text{Mod}(\mathbb{F}Q)$.

We will fix a quiver Q and a dimension vector $\mathbf{d} = \sum_i d_i \mathbf{i}$, and consider the affine space

$$X = R(Q, \mathbf{d}) := \bigoplus_{\alpha: i \rightarrow j} \text{Hom}_{\mathbb{F}}(\mathbb{F}^{d_i}, \mathbb{F}^{d_j}).$$

Its points $M = (M_\alpha)_\alpha$ obviously parametrize representations of Q with dimension vector \mathbf{d} . (Strictly speaking only the \mathbb{F} -rational points of X define such representations; in general, a point of X over a field extension k of \mathbb{F} will define only a representation of Q over k with dimension vector \mathbf{d} . We will however, ignore this issue for the most part.)

The connected reductive algebraic group

$$G(Q, \mathbf{d}) := \prod_{i \in I} \text{GL}(d_i)$$

acts on $R(Q, \mathbf{d})$ via base change:

$$((g_i) \cdot (M_\alpha))_\alpha = (g_j M_\alpha g_i^{-1})_{\alpha: i \rightarrow j}.$$

By definition, the orbits $G(Q, \mathbf{d}) \cdot M$ in $R(Q, \mathbf{d})$ correspond bijectively to the isomorphism classes $[M]$ of \mathbb{F} -representations of Q of dimension vector \mathbf{d} . We will set for simplicity $G := G(Q, \mathbf{d})$ and $X := R(Q, \mathbf{d})$. For any \mathbb{F} -rational point M of X , the stabilizer $G_M = \text{Aut}_{\mathbb{F}Q}(M)$ is smooth and connected, since it is open in the affine space $\text{End}_{\mathbb{F}Q}(M)$. Also, note that the subgroup of G consisting of tuples $(t \text{id}_{d_i})_{i \in I}$, $t \in \mathbb{G}_m$, is a central one-dimensional torus and acts trivially on X ; moreover, the quotient $PG(Q, \mathbf{d})$ by that subgroup acts faithfully. So one may replace G henceforth by $PG(Q, \mathbf{d})$.

We next proceed to consider certain geometric quotients associated to the above action. For this, it is important to choose a character of G , that is, a morphism of algebraic groups $\chi : G \rightarrow \mathbb{G}_m$. If \mathcal{O}_X denotes the trivial line bundle on X , we will linearize it by using the character χ^{-1} : the resulting G -linearized line bundle on X will be denoted L_χ . Since this bundle is trivial on forgetting the G -action, a global section of L_χ^n , for any $n \geq 1$, corresponds to $f \otimes_{\mathbb{F}} t$, where $f \in \mathbb{F}[X]$, $t \in \mathbb{F}[\mathbb{A}^1]$, and $f \otimes_{\mathbb{F}} t$ denotes their tensor product over \mathbb{F} . Now G acts on such pairs (f, t) by $g \cdot (f, t) = (f \circ g, \chi(g)^{-n} t)$ where $f \circ g$ denotes the regular function defined by $(f \circ g)(x) = f(gx)$. Therefore, such a global section will be G -invariant precisely when f is χ -semi-invariant with weight n , i.e.

$$f(gx) = \chi^n(g) f(x) \text{ for all } g \in G \text{ and all } x \in X.$$

We will denote the set of all such global sections by $\mathbb{F}[X]^{G, \chi^n}$. Therefore, the corresponding geometric quotient (see e.g. [Do03, Section 8.1]) will be defined by

$$X//G = \text{Proj} \left(\bigoplus_{n \geq 0} \mathbb{F}[X]^{G, \chi^n} \right).$$

Next observe that the only characters of GL_n are powers of the determinant map; therefore, the only characters of the group $PG(Q, \mathbf{d})$ are of the form

$$(g_i)_i \mapsto \prod_{i \in I} \det(g_i)^{m_i},$$

for a tuple $(m_i)_{i \in I}$ such that $\sum_{i \in I} m_i d_i = 0$ to guarantee well-definedness on $PG(Q, \mathbf{d})$.

Thus, one may choose a linear function $\Theta : \mathbb{Z}^I \rightarrow \mathbb{Z}$ and associate to it a character

$$\chi_\Theta((g_i)_i) := \prod_{i \in I} \det(g_i)^{\Theta(\mathbf{d}) - \dim(\mathbf{d}) \cdot \Theta(\mathbf{i})}$$

of $PG(Q, \mathbf{d})$. For convenience, we will call Θ itself a *character*. (This adjustment of Θ by a suitable multiple of the function $\dim : (d_i) \mapsto \sum_i d_i$ has the advantage that a fixed Θ can be used to formulate stability for arbitrary dimension vectors, and not only those with $\Theta(\mathbf{d}) = 0$. However, this notation is a bit different from the one adopted in [Kin94].)

Associated to each character Θ , we define the *slope* μ . This is the function defined by $\mu(\mathbf{d}) = \frac{\Theta(\mathbf{d})}{\dim(\mathbf{d})}$. With this framework, one may invoke the usual definitions of geometric invariant theory to define the semi-stable points and stable points. Observe that now a point $x \in R(Q, \mathbf{d})$ will be semi-stable (stable) precisely when there exists a G -invariant global section of some positive power of the above line bundle that does not vanish at x (when, in addition, the orbit of x is closed in the semi-stable locus, and the stabilizer at x is finite). Since all stabilizers are smooth and connected, the latter condition is equivalent to the stabilizer being trivial.

The corresponding varieties of Θ -semi-stable and stable points with respect to the line bundle L_χ will be denoted by

$$R(Q, \mathbf{d})^{\text{ss}} = R(Q, \mathbf{d})^{\Theta-\text{ss}} = R(Q, \mathbf{d})^{\Theta-\text{ss}}$$

and

$$R(Q, \mathbf{d})^{\text{s}} = R(Q, \mathbf{d})^{\Theta-\text{s}} = R(Q, \mathbf{d})^{\Theta-\text{s}}.$$

These are open subvarieties of X , possibly empty. The corresponding quotient varieties will be denoted as follows:

$$M^{\Theta-\text{s}}(Q, \mathbf{d}) = R(Q, \mathbf{d})^{\Theta-\text{s}}/G \text{ and } M^{\Theta-\text{ss}}(Q, \mathbf{d}) = R(Q, \mathbf{d})^{\Theta-\text{ss}}//G = X//G.$$

Observe that the variety $M^{\Theta-\text{s}}(Q, \mathbf{d})$ parametrizes isomorphism classes of Θ -stable representations of Q with dimension vector \mathbf{d} .

An orbit $G \cdot M = PG(Q, \mathbf{d}) \cdot M$ is closed in X if and only if the corresponding representation M is semi-simple, by [Ar69]. The quotient variety $X//G$ (for the trivial line bundle with the trivial linearization) therefore parametrizes isomorphism classes of semi-simple representations of Q of dimension vector \mathbf{d} . It will be denoted by $M^{\text{ssimp}}(Q, \mathbf{d})$ and called the moduli space of semi-simple representations.

We now may state some results taken from [Kin94, Sections 3 and 4].

Proposition 3.1. *With the above notation and assumptions, the following assertions hold:*

- *The variety $M^{\text{ssimp}}(Q, \mathbf{d})$ is affine.*
- *There is a natural projective morphism $M^{\Theta-\text{ss}}(Q, \mathbf{d}) \rightarrow M^{\text{ssimp}}(Q, \mathbf{d})$. In particular, if the quiver has no oriented cycles, then every G -invariant regular functions on $R(Q, \mathbf{d})$ is constant: therefore, in this case $M^{\Theta-\text{ss}}(Q, \mathbf{d})$ is a projective variety.*
- *The quotient map $R(Q, \mathbf{d})^{\Theta-\text{s}} \rightarrow M^{\Theta-\text{s}}(Q, \mathbf{d})$ is a $PG(Q, \mathbf{d})$ -torsor.*

The following characterization of Θ -(semi-)stable points in $R(Q, \mathbf{d})$ is also given in [Kin94]:

Theorem 3.2. *A representation $M \in R(Q, \mathbf{d})$ is Θ -semi-stable if and only if $\mu(N) \leq \mu(M)$ for all non-zero sub-representations N of M . The representation M is Θ -stable if and only if $\mu(N) < \mu(M)$ for all non-zero proper sub-representations N of M .*

Next we discuss a procedure for determining *the instability type* of an unstable quiver representation, based on the above theorem. One starts with a given representation M of Q . Assume that it is *unstable*. In view of the above proposition, it follows that there is some sub-representation N of M for which $\mu(N) > \mu(M)$. We let

$$\mathcal{U} := \{N \mid N \text{ sub-representation of } M, \mu(N) > \mu(M)\}.$$

Let M^1 denote a representation in \mathcal{U} such that $\mu(M^1) \geq \mu(N)$ for all $N \in \mathcal{U}$ and such that M^1 is maximal among all such sub-representations of M . Since $\mu(N) \leq \mu(M^1)$ for all sub-representations of M^1 , it is clear that M^1 is semi-stable.

Next we consider the quotient representation M/M^1 and apply the above procedure to M replaced with M/M^1 . In case M/M^1 is not semi-stable and nonzero, one then obtains a sub-representation \bar{M}^2 of M/M^1 such that:

- (i) $\mu(\bar{M}^2) \geq \mu(\bar{N})$ for any sub-representation \bar{N} of M/M^1 , and
- (ii) \bar{M}^2 is maximal among sub-representations \bar{N} of M/M^1 such that $\mu(\bar{N}) > \mu(M/M^1)$.

Let M^2 be the sub-representation of M obtained as the inverse image of \bar{M}^2 under the quotient map $M \rightarrow M/M^1$. Then the choice of M^1 shows that $\mu(M^2) < \mu(M^1)$, unless $M^1 = M$ or M/M^1 is semi-stable. Now consider the short exact sequence

$$0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^2/M^1 \longrightarrow 0$$

of representations. [Re08, Lemma 4.1] shows that, since $\mu(M^2) < \mu(M^1)$, it follows that $\mu(M^2) > \mu(M^2/M^1)$. Combining this with $\mu(M^1) > \mu(M^2)$ shows that $\mu(M^1/M^0) = \mu(M^1) > \mu(M^2/M^1)$. Clearly both M^1/M^0 and M^2/M^1 are semi-stable.

One may now *repeat the above procedure* to define a finite increasing filtration by sub-representations (the so-called *Harder-Narasimhan filtration*),

$$(3.0.2) \quad \{0\} = M^0 \subset M^1 \subset M^2 \subset \dots \subset M^{n-1} \subset M^n = M,$$

such that:

- (i) each M^i/M^{i-1} is semi-stable, and
- (ii) $\mu(M^i/M^{i-1}) > \mu(M^j/M^{j-1})$ for all $j > i$.

Let \mathbf{d}_i denote the dimension vector of the representation M^i/M^{i-1} . Varying $i = 1, \dots, n$, we obtain a sequence $(\mathbf{d}^1, \dots, \mathbf{d}^n)$ of dimension vectors. The sequence of slopes of the sub-quotients given by

$$(\mu(M^1/M^0), \dots, \mu(M^n/M^{n-1}))$$

together with the above sequence of dimension vectors will be called *the instability type* of the given unstable representation M . Pairs of such sequences

$$\beta := ((\mu^1, \dots, \mu^n), (\mathbf{d}^1, \dots, \mathbf{d}^n)),$$

where $\mu^1 > \mu^2 > \dots > \mu^n$ is a sequence of rational numbers and the \mathbf{d}^i are dimension vectors with the properties that $\sum_i \mathbf{d}^i = \mathbf{dim}(M)$ and $\mu^i = \mu(\mathbf{d}^i)$ will be used to index the strata of a natural stratification of the representation space $R(Q, \mathbf{d})$. (One may observe that there is a slight redundancy in the above data: since we assume the character Θ is fixed once and for all, the choice of the dimension vectors \mathbf{d}^i determines the slopes μ^i by the formula $\mu^i = \Theta(\mathbf{d}^i)/\dim(\mathbf{d}^i)$. Nevertheless, we keep the present notation for its clarity.)

Observe that the closed orbits of G in $R(Q, \mathbf{d})^{\Theta\text{-ss}}$ correspond to the Θ -polystable representations, that is, the direct sums of stable representations of the same slope. They can also be viewed as the semi-simple objects in the abelian subcategory of semi-stable representations of fixed slope μ .

By [Re03, Proposition 3.7], the closure relations between the strata may be described as follows. For $\beta = ((\mu^1, \dots, \mu^n), (\mathbf{d}^1, \dots, \mathbf{d}^n))$ we construct the polygon with vertices, the integral points in the plane defined by $(\sum_{i=1}^k \dim(\mathbf{d}^i), \sum_{i=1}^k \Theta(\mathbf{d}^i))$, $k = 1, \dots, n$. Then the closure of the stratum indexed by β is contained in the union of the strata indexed by $\gamma = ((\nu^1, \dots, \nu^m), (\mathbf{e}^1, \dots, \mathbf{e}^m))$, when the polygon corresponding to γ lies on or above the polygon corresponding to β .

Next we proceed to describe in detail the stratum associated to an index β as above, adopting the setting of GIT as in [Kir84]. Accordingly the stratum corresponding to β , to be denoted S_β , will be given as follows.

- (1) S_β is the set of those $M \in X$ admitting a filtration (3.0.2) such that

$$((\mu(M^1/M^0), \dots, \mu(M^n/M^{n-1})), (\mathbf{dim}(M^1/M^0), \dots, \mathbf{dim}(M^n/M^{n-1}))) = \beta.$$

- (2) A prescribed filtration (3.0.2) by sub-representations satisfying the above condition, induces a filtration on each space M_i and therefore a parabolic subgroup of $\mathrm{GL}(d_i)$. The product of these parabolic subgroups is a parabolic subgroup P_β of G .
- (3) We choose a 1-parameter subgroup $\lambda_\beta(t)$ of G such that the associated parabolic subgroup is exactly P_β ; in other words,

$$P_\beta = \{g \in \mathrm{PGL}(Q, \mathbf{d}) \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t^{-1}) \text{ exists in } \mathrm{PGL}(Q, \mathbf{d})\}.$$

- (4) Y_β is the set of those representations $M \in R(Q, \mathbf{d})$ provided with a chosen filtration by subrepresentations M^i satisfying the condition that $\mu(M^i/M^{i-1}) > \mu(M^j/M^{j-1})$ for all $j > i$. (i.e. we do not require the quotients M^i/M^{i-1} to be semi-stable.) Y_β^{ss} will denote the subset of those representations M so that the successive quotients M^i/M^{i-1} of the chosen filtration are also semi-stable.
- (5) Z_β is the set of associated graded modules, $\mathrm{gr}(M) = \bigoplus_{i=1}^n M^i/M^{i-1}$, where $M \in Y_\beta$. Similarly Z_β^{ss} is the set of associated graded modules, $\mathrm{gr}(M)$, where $M \in Y_\beta^{\mathrm{ss}}$.

Proposition 3.3. (i) $Z_\beta = \{\lim_{t \rightarrow 0} \lambda_\beta(t) \cdot M \mid M \in Y_\beta\}$. Similarly, $Z_\beta^{\mathrm{ss}} = \{\lim_{t \rightarrow 0} \lambda_\beta(t) \cdot M \mid M \in Y_\beta^{\mathrm{ss}}\}$.

(ii) Z_β^{ss} is open in the fixed point locus for the action of $\lambda_\beta(t)$ on X .

Proof. (i) is standard, see [Kin94, Section 3].

(ii) Clearly, the fixed point locus is exactly the set of associated graded modules. Moreover, the semi-stability of each M^i/M^{i-1} is an open condition (it amounts to semi-stability with respect to the centralizer L_β of λ_β , a Levi subgroup of P_β , and to the restriction of Θ to L_β). \square

Theorem 3.4. *The stratification of X defined above is equivariantly perfect.*

Proof. Since X and all strata are clearly smooth, it suffices to show that the equivariant Euler classes of the normal bundles to the strata S_β are non-zero divisors. But this follows from Proposition 3.3 by arguing as in the proof of Corollary 2.2. Specifically, since Z_β^{ss} is open in the fixed locus X^{λ_β} , it is a locally closed smooth subvariety of X , and λ_β has no non-zero fixed point in the normal spaces to Z_β^{ss} in X . By Lemma 2.3, it follows that the Euler class of the normal bundle to Z_β^{ss} in X is not a zero divisor. \square

Remark 3.5. It is important to observe that this result does not follow readily from the theory discussed in [Kir84], since one of the key assumptions there is that the variety X be *projective*. The projectivity was needed there, however, only to make sure that the limits as considered in the definition of Z_β exist. Here we prove the existence of these limits directly.

Corollary 3.6. *Assume in addition to the above situation that each semi-stable point is stable. Then the l -adic cohomology $H^*(M^{\Theta-s}(Q, \mathbf{d}), \mathbb{Q}_l)$ vanishes in all odd degrees. Moreover, F acts on each $H^{2n}(M^{\Theta-s}(Q, \mathbf{d}), \mathbb{Q}_l)$ via multiplication by q^n .*

In particular, $H^(M^{\Theta-s}(Q, \mathbf{d}), \mathbb{Q}_l)$ is strongly pure, and hence the number of \mathbb{F}_{q^n} -rational points of $M^{\Theta-s}(Q, \mathbf{d})$ is a polynomial function of q^n with integer coefficients.*

Proof. We adapt the argument of the proof of Theorem 2.1. By Theorem 3.4, the pull-back map

$$H^*(EG \times_G X, \mathbb{Q}_l) \longrightarrow H^*(EG \times_G X^{\text{ss}}, \mathbb{Q}_l)$$

is surjective. The right-hand side is isomorphic to $H^*(M^{\Theta-s}(Q, \mathbf{d}), \mathbb{Q}_l)$ by our assumption and Proposition 3.1. On the other hand, since X is an affine space, the left-hand side is isomorphic to $H^*(BG, \mathbb{Q}_l)$. Since G is split, the assertions follow from Theorem 1.15. \square

Remarks 3.7. (i) Suppose \mathbf{d} is co-prime for Θ , i.e. $\mu(\mathbf{e}) \neq \mu(\mathbf{d})$ for all $0 \neq \mathbf{e} < \mathbf{d}$. (For a generic choice of Θ , this is equivalent to $\text{g.c.d}\{d_i | i \in I\} = 1$.) In this case, every semi-stable point is stable.

(ii) The above corollary also recovers certain results of Reineke (see [Re03, Section 6] and [Re06, Theorem 6.2]) which are established by using the combinatorics of the Hall algebra associated to the quiver. Our proof is purely based on geometric invariant theory coupled with the theory of weak and strong purity developed in the first section of this paper.

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