Abstract. In this paper we discuss K"unneth decompositions for finite quotients of several classes of smooth projective varieties. We establish the strong K"unneth decomposition for finite quotients of projective smooth linear varieties and also Chow K"unneth decomposition for certain finite quotients of abelian varieties.

1. Introduction

Strong K"unneth decompositions are known to exist for linear varieties (cf. [8]), a class that includes projective spaces, flag schemes, toric schemes and spherical schemes (all over a field $k$). Previous work of the second author ([9], [10]) shows in particular that for varieties possessing a strong K"unneth decomposition, the cycle map to étale cohomology is an isomorphism. Chow-K"unneth decompositions, on the other hand, are conjectured (at least by optimists) to exist for all smooth projective varieties $X$; so far, they are known to exist (over $\mathbb{Q}$) for curves [13], surfaces [17], projective spaces [13], abelian varieties ([22], [3], [12]) certain classes of threefolds [4], [5], and other isolated examples. In each of these cases, though, one must construct the projectors explicitly and then check the various orthogonality conditions; this process depends heavily on particular properties of the variety under consideration. In this paper we extend the strong K"unneth decomposition and the Chow-K"unneth decomposition to varieties that are finite quotients of smooth projective varieties that admit the corresponding decomposition, working of course with rational coefficients throughout.

In the next section, we extend the strong K"unneth decomposition to finite quotients of projective smooth linear varieties. The main result is that if $X$ is a variety possessing a strong K"unneth decomposition and $f : X \rightarrow Y$ is a finite surjective map, then $Y$ possesses a strong K"unneth decomposition which we may describe explicitly in terms of that for $X$. The methods used are all elementary and require little more than the definitions and basic properties of Chow groups. As an application, we describe a strong K"unneth decomposition for symmetric products of projective spaces. We also discuss some formal consequences of such a strong K"unneth decomposition: we show that a strong K"unneth decomposition implies a Chow K"unneth decomposition and that the rational higher Chow groups are determined by the rational higher Chow groups of the base field and the rational (ordinary) Chow groups of the given variety.

In the following section we establish Chow K"unneth decomposition for finite quotients of abelian varieties for the action of a finite group so that the hypothesis in Theorem (3.2) are satisfied. The techniques are a modification of those of Deninger-Murre [3] and also [Be].

The second author thanks the Miami University, the IHES, the MPI and the NSA for support.
This gives Chow-Könneth decompositions for a number of interesting examples of nonabelian quotients of abelian varieties; examples of these may be found in [7], [15] some of which we briefly discuss. The existence of such a Chow-Könneth decomposition is guaranteed by the work of Kimura [11] on finite-dimensionality of motives; however, his results do not yield explicit formulas for the projectors, which is part of our goal.

We discuss, in an appendix, certain formal consequences of the existence of strong Künneth decompositions. This discussion is done in a somewhat more general setting so that it applies to other situations not considered in the body of the paper.

Acknowledgments. The authors would like to thank Claudio Pedrini and Vasudevan Srinivas for discussions which were essential in improving our understanding of the context of this paper.

1.1. Group Scheme Actions and Quotients. Following the treatment in [16], we review some definitions and results related to group scheme actions.

Definition 1.1. Let $G$ be a group scheme over a field $k$ with identity section $e : \text{Spec } k \to G$ and multiplication $m : G \times_k G \to G$.

An action of $G$ on a scheme $X$ is a morphism $\mu : G \times X \to X$ such that:

1. The composite

$$X \cong \text{Spec } k \times_k X \xrightarrow{e \times 1_X} G \times_k X \xrightarrow{\mu} X$$

is the identity map.

2. The diagram below commutes:

$$
\begin{array}{ccc}
G \times_k G \times_k X & \xrightarrow{m \times 1_X} & G \times_k X \\
\downarrow{1_G \times \mu} & & \downarrow{\mu} \\
G \times_k X & \xrightarrow{\mu} & X
\end{array}
$$

In the future, we will identify elements $g \in G$ with the morphism $\mu_g : X \to X$ defined by $\mu_g(x) = \mu(g, x)$ and (by abuse of notation) refer to this morphism simply as $g$.

If $G$ is a finite group scheme over $k$ acting on a quasi-projective variety $X$ (also over $k$), there exists a quasi-projective variety $Y$ together with a finite, surjective $G$-invariant morphism $f : X \to Y$ universal for $G$-invariant morphisms $X \to Z$. The scheme $Y$ is called the quotient of $X$ by $G$, and is typically denoted $Y = X/G$.

Definition 1.2. We say a variety $X$ is pseudo-smooth if it is the quotient of a smooth variety by the action of a finite group.

1.1.1. Notation and terminology: review of correspondences. In this section we define the category of rational correspondences and rational Chow motives for pseudo-smooth projective varieties.
Let $k$ be a field and $\mathcal{V}_k$ the category of schemes pseudo-smooth and projective over $k$. If $X, Y$ are objects of $\mathcal{V}_k$ and $X$ has pure dimension $d$, we define the group of degree $r$ correspondences from $X$ to $Y$ by $\text{Corr}^r(X, Y) = CH^{d+r}(X \times_k Y) \otimes \mathbb{Q}$, the group of codimension $d+r$ (rational) cycles on $X \times_k Y$ modulo rational equivalence. In general, let $X_1, \ldots, X_n$ be the irreducible components of $X$; we then define $\text{Corr}^r(X, Y) = \oplus_{i=1}^n \text{Corr}^r(X_i, Y)$. When $\alpha \in \text{Corr}^r(X, Y)$ and $\beta \in \text{Corr}^s(Y, Z)$, we define their composition $\beta \circ \alpha \in \text{Corr}^{r+s}(X, Z)$ by the formula

$$
\beta \circ \alpha = (p_{13})_*(p_{12}^* \alpha \circ p_{23}^* \beta);
$$

here $p_{ij}$ represents projection of $X \times_k Y \times_k Z$ on the $i$th and $j$th factors.

One then constructs a new category $\mathcal{M}_k(\mathbb{Q})$, the category of (rational) Chow motives of pseudo-smooth projective varieties. The objects of $\mathcal{M}_k(\mathbb{Q})$ are pairs $(X, \pi)$, where $X$ is an object of $\mathcal{V}_k$ of dimension $d$ and $\pi \in \text{Corr}^0(X, X)$ is a projector; that is, an element satisfying $\pi \circ \pi = \pi$. For any two Chow motives $(X, \pi)$ and $(Y, \rho)$, one then defines

$$
\text{Hom}_{\mathcal{M}_k}((X, \pi), (Y, \rho)) = \bigoplus_j \rho \circ \text{Corr}^0(X, Y) \circ \pi.
$$

If $\Delta_X$ is the diagonal of $X \times_k X$ and $[\Delta_X]$ its class in $CH^*(X \times_k X) \otimes \mathbb{Q}$, a straightforward computation (cf. ??) shows that $\Delta_X$ is a projector, and furthermore that $\Delta_X \circ \alpha = \alpha = \alpha \circ \Delta_X$ for any pseudo-smooth projective scheme $Y$ and $\alpha \in \text{Corr}^r(X, Y)$. Thus, there is a functor $h : \mathcal{V}_k^{opp} \rightarrow \mathcal{M}_k(\mathbb{Q})$ defined on objects by $h(X) = (X, \Delta_X)$ and on morphisms by $h(X \xrightarrow{f} Y) = \Gamma_f$, where $\Gamma_f \in \text{Hom}_{\mathcal{M}_k}(h(Y), h(X))$ is the class of the graph of $f$. Furthermore, letting $\coprod$ denote disjoint union (of schemes), one may define the sum $\oplus$ and product $\otimes$ of motives thus:

$$(X, p) \oplus (Y, q) = (X \coprod Y, p \coprod q)$$

$$(X, p) \otimes (Y, q) = (X \times Y, p \times q)$$

We denote by $\mathbb{I}$ the “trivial” motive $h(\text{Spec } k)$, a neutral element for $\otimes$, and by $\mathbb{L}$ the “Lefschetz motive” $(\mathbb{P}_k^1, \mathbb{P}_k^1 \times_k \{x\})$; here $x \in \mathbb{P}_k^1$ is any rational point. Finally, if $\alpha \in \text{Corr}^*(X, Y)$ is any correspondence, we define its “transpose” $t\alpha = s^*(\alpha) \in \text{Corr}^*(Y, X)$, where $s : X \times_k Y \rightarrow Y \times_k X$ is the exchange of factors. For further discussion of motives, we refer the reader to [21]. Also see [Ful] Example (8.3.12) and Example (16.1.12) for discussion that shows one can in fact define a category of Chow motives for pseudo-smooth schemes as we have done. In fact it is possible to consider the above theory for all smooth Deligne-Mumford stacks over $k$; some of our results extend to this situation readily.

1.2. Abelian Varieties. In this section we establish notation and cite a rigidity property for abelian varieties necessary in the sequel. A comprehensive treatment of abelian varieties may be found in [16] or [14].

Let $k$ be a field and $A$ an abelian variety over $k$. Following [14], we denote by $m : A \times_k A \rightarrow A$ the morphism representing composition on (the group scheme) $A$ and use additive notation for
this (commutative) operation. For any \( a \in A(k) \), we denote by \( \tau_a : A \rightarrow A \) (translation by \( a \)) the map defined by \( \tau_a(x) = x + a \).

A morphism \( f : A \rightarrow B \) between abelian varieties is called a homomorphism if for every \( a, a' \in A \), \( f(a + a') = f(a) + f(a') \). When \( n \in \mathbb{Z} \) we define \( n : A \rightarrow A \) by \( n(a) = na \) and set \( A[n] = \text{Ker} (A \xrightarrow{n} A) \), the (group scheme of) \( n \)-torsion points on \( A \). For clarity of notation, we write \( \sigma \) instead of \( -1 \).

The following important result is a consequence of a general rigidity principle; see [14], Corollary 2.2 for details:

**Proposition 1.3.** Let \( h : A \rightarrow B \) be a morphism of abelian varieties. Then there exists a homomorphism \( h_0 : A \rightarrow B \) and an element \( a \in A(k) \) such that

\[
h = \tau_a \circ h_0.
\]

We remark that \( h_0 \) and \( a \) are in fact unique. Indeed, one must have \( a = -h(0) \); uniqueness of \( h_0 \) then follows immediately.

Let \( \hat{A} \) be the dual abelian variety; we will denote by \( \mathcal{L} \) the Poincaré bundle and \( \ell \) its class in \( \text{CH}^1_Q(A \times_k \hat{A}) \).

We conclude this section by recalling the definition of strong Künneth and Chow Künneth decompositions.

**Definition 1.4.** Let \( X \) be any variety of dimension \( d \) over a field \( k \).

We say that \( X \) possesses a strong Künneth decomposition if there exists \( n \geq 0 \) and elements \( a_{i,j}, b_{i,j} \in \text{CH}^i_Q(X) \) such that

\[
[\Delta_X] = \sum_i \sum_j a_{i,j} \times b_{d-i,j}
\]

Now suppose \( X \) is pseudo-smooth (over \( k \)). We say that \( X \) has a Chow-Künneth decomposition if there exist elements \( \pi_0, \ldots, \pi_{2d} \in \text{CH}^d_Q(X \times_k X) \) such that:

- \([\Delta_X] = \sum_{i=0}^{2d} \pi_i\)
- For every \( i \), \( \pi_i \circ \pi_i = \pi_i \) and for all \( j \neq i \), \( \pi_i \circ \pi_j = 0 \). (Thus, \( \pi_0, \ldots, \pi_{2d} \) form a system of mutually orthogonal projectors).
- Let \( H \) be a Weil cohomology theory \( H^* \) (cf. [?]) and, for any \( k \)-scheme \( Y \), let \( cl_Y : \text{CH}^*_Q(Y) \rightarrow H^*(Y) \) denote the cycle map. We require that \( cl_{X \times_k X}(\pi) = \Delta(i) \), where \( \Delta(i) \) is the codimension \( i \) Künneth component of the class of \( \Delta_X \) in \( H^*(X \times_k X) \).

Observe that if \( X \) is projective, \( X \) having a Chow-Künneth decomposition is equivalent to asserting that \( h(X) \cong \oplus_{i=0}^{2d} h^i(X) \) where \( h^i(X) \) is the motive \( (X, \pi_i) \).

2. The strong Künneth decomposition for finite quotients

Now suppose \( X \) is a pseudo-smooth, projective, equidimensional scheme over a field \( k \) and \( G \) a finite group of automorphisms of \( X \). As in [16], we may form the quotient variety \( Y = X/G \) and ask whether an explicit strong Künneth decomposition for \( X \) may be used to construct a strong Künneth decomposition for \( Y \). We answer this question in the affirmative below.
First we consider an elementary calculation showing that strong K"unneth decompositions are preserved under finite maps.

**Proposition 2.1.** Let $X$ and $Y$ be pseudo-smooth proper varieties and $f : X \rightarrow Y$ a finite surjective map. If $X$ has a strong K"unneth decomposition, then $Y$ also has a strong K"unneth decomposition.

**Proof.**

Let $d = \dim X$, $m = \deg f$. The hypothesis that $X$ has a strong K"unneth decomposition allows us to write

$$[\Delta_X] = \sum_i \sum_j a_{i,j} \times b_{d-i,j}$$

where as before $a_{i,j}, b_{i,j} \in CH^*_Q(X)$. Furthermore, $(f \times f)_*[\Delta_X] = m[\Delta_Y]$, so it suffices to prove that $(f \times f)_*(a_{i,j} \times b_{d-i,j}) = f_*a_{i,j} \times f_*b_{d-i,j}$. This is accomplished by the next lemma.

**Lemma 2.2.** Let $f : X \rightarrow Y$ be a morphism of pseudo-smooth varieties

1. If $f$ is proper, then for all $\alpha, \beta \in CH^*_Q(X)$, $f_*(\alpha \times \beta) = f_*(\alpha) \times f_*(\beta)$.
2. For all $\gamma, \delta \in CH^*_Q(Y)$, $f^*(\gamma \times \delta) = f^*(\gamma) \times f^*(\delta)$.

**Proof.**

We will prove the first statement, the second being similar. To establish some notation, let $p_1$ ($p_2$) denote the projection of $X \times_k Y$ onto the first (second) factor; we similarly define $q_i$ as the appropriate projections of $Y \times_k Y$ and $r_i$ as the appropriate projections of $X \times_k Y$.

Then

$$f_*(\alpha \times \beta) = (f \times 1_Y)_*(p_1^*\alpha \cdot p_2^*\beta)$$

$$= (f \times 1_Y)_*(p_1^*\alpha \cdot (1_X \times f)_*p_2^*\beta)$$

$$= (f \times 1_Y)_*(1_X \times f)_*((1_X \times f)_*r_1^*\alpha \cdot p_2^*\beta)$$

$$= (f \times 1_Y)_*(r_1^*\alpha \cdot (1_X \times f)_*p_2^*\beta)$$

$$= (f \times 1_Y)_*(r_1^*\alpha \cdot r_2^*f_*\beta)$$

$$= (f \times 1_Y)_*(r_1^*\alpha \cdot (f \times 1_Y)^*q_2^*f_*\beta)$$

$$= (f \times 1_Y)_*r_1^*\alpha \cdot q_2^*f_*\beta$$

$$= q_1^*f_*\alpha \cdot q_2^*f_*\beta = f_*\alpha \times f_*\beta$$

We note the following as a special case:

**Corollary 2.3.** Let $X$ be a pseudo-smooth quasi-projective variety, $G$ a finite group of automorphisms of $X$. If $X$ possesses a strong K"unneth decomposition, so does $Y = X/G$.

The utility of the previous statements becomes evident from the following easy result:

**Proposition 2.4.** Let $X$ be a pseudo-smooth projective variety possessing a strong K"unneth decomposition. Then $X$ has a Chow-K"unneth decomposition.
Proof. [\Delta_X] = \sum_{r=0}^{d} \sum_{s \neq r} \pi_s \circ \pi_r = \sum_{r=0}^{d} \sum_{s \neq r} \pi_s \circ \pi_r = \Delta_X \circ \pi_r = \pi_r

Lemma 2.5. With notation as above, suppose \(a_r \in CH^r(X), b_{d-r} \in CH^{d-r}(X), a_s \in CH^s(X), b_s \in CH^{d-s}(X),\) and set \(\gamma_r = a_r \times b_{d-r}, \gamma_s = a_s \times b_{d-s}.\) If \(r \neq s,\) then \(\gamma_s \circ \gamma_r = 0.\)

Proof.

\[
\gamma_s \circ \gamma_r = p_{13}^{23} (p_{12}^{13} \gamma_r \cdot p_{23}^{13} \gamma_s) = \sum_j p_{13}^{23} (p_{12}^{13} (p_1^{12} a_r \cdot p_2^{12} b_{d-r}) \cdot p_{23}^{13} (p_2^{23} a_s \cdot p_3^{23} b_{d-s})) = p_{13}^{13} a_r \cdot p_3^{13} b_{d-s} \cdot p_{13}^{13} b_{d-s} = p_{13}^{13} (a_s \cdot b_{d-r})
\]

Note that \(\sigma_s (a_s \cdot b_{d-r}) \in CH^{s-r}(X),\) so if \(r \neq s,\) then \(\sigma_s (a_s \cdot b_{d-r}) = 0,\) and hence \(\gamma_s \circ \gamma_r = 0.\) This concludes the proof of Lemma 2.5.

As an application, we compute the strong Künneth decomposition for the \(n\)th symmetric product of projective space \(P_k^n.\) Let \(\ell \in CH^1_Q(P_k^n)\) be the class of a generic hyperplane in \(P_k^n.\) It is well-known (cf. [13], p. 455) that \(P_k^n\) has a strong Künneth decomposition:

\[
\Delta P_k^n = \sum_{i=0}^{m} \ell^i \times \ell^{m-i}
\]

Let \(X = (P_k^n)^n.\) By the Künneth formula for motives, we have

\[
\Delta_X = \sum_{0 \leq i_1, \ldots, i_n \leq m} f_{i_1, \ldots, i_n}
\]

where \(f_{i_1, \ldots, i_n} = \ell^{i_1} \times \ldots \times \ell^{i_n} \times \ell^{m-i_1} \times \ldots \ell^{m-i_n} \in CH_Q^m(X \times_k X).\)

Now consider the action of the symmetric group on \(n\) letters (denoted \(S_n\)) on \(X = (P_k^n)^n\) by interchanging of factors. Let \(Y = X/S_n\) and \(q: X \rightarrow Y\) the quotient map. Note also that for any \(\sigma \in S_n,\) \((q \times q)*f_{i_1, \ldots, i_n} = (q \times q)*f_{\sigma(i_1), \ldots, \sigma(i_n)}.\)
Applying \((q \times q)_*\) to the strong K"unneth decomposition for \(\Delta_X\) given above, and noting that \(\deg q = n!\), we obtain

\[
(n!)\Delta_Y = \sum_{0 \leq i_1, \ldots, i_n \leq m} (q \times q)_* f_{i_1, \ldots, i_n}
\]

\[
= \sum_{0 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq m} \sum_{\sigma \in S_n} (q \times q)_* f_{\sigma(i_1), \ldots, \sigma(i_n)}
\]

\[
= \sum_{0 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq m} n! (q \times q)_* f_{i_1, \ldots, i_n}
\]

Now let \(\bar{\ell}^i = q_*(\ell^i)\). Then

\[
(2.0.3) \quad \Delta_Y = \sum_{0 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq m} (q \times q)_* f_{i_1, \ldots, i_n}
\]

\[
= \sum_{0 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq m} \bar{\ell}^{i_1} \times \ldots \times \bar{\ell}^{i_n} \times \bar{\ell}^{m-i_1} \times \ldots \times \bar{\ell}^{m-i_n}
\]

giving a strong K"unneth decomposition for \(Y\).

**Corollary 2.6.** Let \(Y\) denote the \(n\)-th symmetric product of \(\mathbb{P}^m_k\). Then

\[
CH^*(Y, Q, r) \cong CH^*(Y, Q, 0) \otimes CH^*(\text{Spec } k, Q, r)
\]

where \(CH^*(Z, Q, r) = \pi_r(z^*(Z, .) \otimes Q)\) and \(z^*(Z, .)\) denotes the higher cycle complex of the scheme \(Z\).

**Proof.** This follows readily from the above strong K"unneth decomposition for the class \(\Delta_Y\) and Theorem 4.1.

### 3. Chow-K"unneth Decomposition for Quotients of Abelian Varieties

Our goal in this section is to exhibit an explicit Chow-K"unneth decomposition for the quotient of an abelian variety \(A\) by the action of a finite group \(G\), assuming only that \(g(0)\) is a torsion point for each \(g \in G\). As before, the quotient \(A/G\) may be singular. We rely on the following result, originally due to Shermenev [22], but later proved in a somewhat more functorial setting by Deninger and Murre ([3], Theorem 3.1); in this latter source the result is proved more generally for abelian schemes over a smooth quasi-projective base:

**Theorem 3.1.** Let \(A\) be an abelian variety of dimension \(d\) over a field \(k\). Then there exists a Chow-K"unneth decomposition for \(A\):

\[
\Delta_A = \sum_{i=0}^{2d} \pi_i
\]

Since we need to make explicit use of the projectors \(\pi_i\), we will presently review their construction. First, consider \(A \times_k A\) as an abelian \(A\)-scheme via projection on the first factor; with respect to this structure, the dual abelian scheme is \(A \times_k \hat{A}\). Consider then the Fourier transform (cf. [3], 2.12, [12], 1.3):
defined by \( F_{CH}(\alpha) = p_{13*}(p_{12*}\alpha \cdot F) \), where

\[
F = \sum_{i=0}^{\infty} \frac{1 \times \ell_i}{i!} \in CH(Q(A \times_k A) \times \hat{A})
\]

and the various \( p_{ij} \) represent projections from \( A \times_k A \times_k \hat{A} \) on the \( i \)th and \( j \)th factor. Note that the sum defining \( F \) is actually finite.

Dualizing this construction, we may define \( \hat{F}_{CH} : CH_{Q}(A \times_k \hat{A}) \rightarrow CH_{Q}(A \times_k A) \) by

\[
\hat{F}_{CH}(\gamma) = q_{13*}(q_{12*}\gamma \cdot \hat{F})
\]

by \( \hat{F}_{CH}(\gamma) = q_{13*}(q_{12*}\gamma \cdot \hat{F}) \), where

\[
\hat{F} = \sum_{i=0}^{\infty} \frac{1 \times \ell_i}{i!} \in CH^Q(A \times_k \hat{A} \times_k A)
\]

and \( q_{ij} \) represent the various projections from \( A \times_k \hat{A} \times_k A \). By switching the last two factors and changing notation appropriately, we see that in fact

\[
\hat{F}_{CH}(\gamma) = p_{12*}(p_{13*}\gamma \cdot F).
\]

An argument involving the theorem of the square (cf. [3], Cor. 2.22, also [1], Prop. 3) then shows that \( \hat{F}_{CH}(F_{CH}(\alpha)) = (-1)^d\sigma^*\alpha \) for all \( \alpha \in CH^*(A \times_k A) \), and similarly for the other composition.

Observe that \( [\Delta_A] \in CH^d(A \times_k A) \), and write \( F_{CH}([\Delta_A]) = \sum_{i=0}^{2d} \beta_i \), where \( \beta_i \in CH^i_{Q}(A \times_k \hat{A}) \). It is a fact ([3], p. 214-216) that \((1 \times n)^*\beta_i = n^i\beta_i \). Now define

\[
(3.0.4) \quad \pi_i = (-1)^d\sigma^*\hat{F}_{CH}(\beta_i)
\]

The main result to be proved is:

**Theorem 3.2.** Let \( A \) be an abelian variety of dimension \( d \) over a field \( k \) and \( G \) a finite group acting on \( A \) such that \( g(0) \in A(k) \) is a torsion point for each \( g \in G \). Let \( f : A \rightarrow A/G \) be the quotient map. Suppose \( \Delta_A = \sum_{i=0}^{2d} \pi_i \) is a Chow-Künneth decomposition for \( A \) and let

\[
\eta_i = \frac{1}{|G|}(f \times f)_*\pi_i.
\]

Then

\[
\Delta_{A/G} = \sum_{i=0}^{2d} \eta_i
\]

is a Chow-Künneth decomposition for \( A/G \).
We remark that the morphisms $g : A \to A$ associated to the action of $G$ on $A$ are automorphisms of $A$ as a scheme but not necessarily automorphisms of $A$ as a group scheme. (i.e. we do not require that $g(0) = 0$). It is because of this semantic ambiguity that we talk of quotients of $A$ "by the action of $G"' rather than simply speaking of quotients by a finite group of automorphisms. Furthermore, the hypothesis that $g(0)$ be a torsion point of $A$ is not always satisfied. For example, one could choose any non-torsion point $a \in A(k)$; then the automorphism $x \mapsto x + a$ defines an action of $\mathbb{Z}/2\mathbb{Z}$ on $A$. Nevertheless, if $k$ is an algebraic extension of a finite field, this hypothesis is always satisfied.

Our method of proof is based on that of [3], Theorem 3.1; however, there are further technicalities which complicate it somewhat. The content of the proof is, of course, to show that the elements $\frac{1}{|G|} (f \times f)_* \pi_i, 0 \leq i \leq 2d$, are mutually orthogonal projectors. Unfortunately, $A/G$ is in general not an abelian variety, so we cannot exploit any special properties of this variety. However, the map $f^*$ establishes an isomorphism ([6], Example 1.7.6):

$$CH^*_Q(A/G) \to CH^*_Q(A)^G$$

with inverse $\frac{1}{|G|} f_*$. Thus, we will work in the group $CH^*_Q(A)^G$, constructing mutually orthogonal $G \times G$-invariant elements which may be descended to elements of $CH^*_Q(A/G)$ by the following device:

**Lemma 3.3.** Suppose $X$ is a pseudo-smooth projective variety of dimension $d$ and $G$ a group of automorphisms of $X$. Let $f : X \to Y = X/G$ be the quotient map and suppose

$$\sum_{g,h \in G} (g \times h)^* \Delta_X = \sum_{i=0}^{2d} \rho_i$$

where $\rho_i \circ \rho_j = 0$ if $i \neq j$ and the $\rho_i$ are $G \times G$-invariant, i.e. for any $g, h \in G$, $(g \times h)^* \rho_i = \rho_i$.

Then

$$\Delta_Y = \sum_{i=0}^{2d} \frac{1}{|G|^2} (f \times f)_* \rho_i$$

is a Chow-K"unneth decomposition for $Y$.

**Proof.**

We have

$$\sum_{g,h \in G} (g \times h)^* = (f \times f)^* (f \times f)_* \text{ and } (f \times f)_* \Delta_X = |G| \Delta_Y,$$

and therefore:

$$|G|^2 (f \times f)_* \Delta_X = (f \times f)_* \sum_i \rho_i$$
Hence
\[
\Delta_y = \frac{1}{|G|^3} \sum_i (f \times f)_* \rho_i
\]

It remains to show that \((f \times f)_* \rho_i\) are mutually orthogonal. As in Proposition 2.4, we add subscripts and superscripts to \(p\) (respectively, \(q\)) to denote the various projections between products of \(X\) (respectively, \(Y\)), and for convenience of notation set \(r = (f \times f \times f) : X \times_k X \times_k X \to Y \times_k Y \times_k Y\). Now,

\[
(3.0.5) \quad (f \times f)_* \rho_i \circ (f \times f)_* \rho_j = q_{13}^{123} (q_{12}^{123} (f \times f)_* \rho_i \cdot q_{23}^{123} (f \times f)_* \rho_j)
\]

\[
= \frac{1}{|G|^3} q_{13}^{123} (r_* r^* q_{12}^{123} (f \times f)_* \rho_i \cdot q_{23}^{123} (f \times f)_* \rho_j)
\]

\[
= \frac{1}{|G|^3} q_{13}^{123} (r_* p_{12}^{123} (f \times f)_* \rho_i \cdot q_{23}^{123} (f \times f)_* \rho_j)
\]

The first and last equality above are obvious, whereas the second one follows from the observation \(r_* r^* = \text{multiplication by } |G|^3\). Because the \(\rho_i\) are \(G \times G\)-invariant, the last term

\[
= \frac{1}{|G|^3} q_{13}^{123} r_* (p_{12}^{123} \rho_i \cdot r^* q_{23}^{123} (f \times f)_* \rho_j)
\]

\[
= \frac{1}{|G|} p_{13}^{123} r_* (p_{12}^{123} \rho_i \cdot p_{23}^{123} (f \times f)_* \rho_j)
\]

\[
= |G| p_{13}^{123} (p_{12}^{123} \rho_i \cdot p_{23}^{123} \rho_j)
\]

\[
= |G| (\rho_i \circ \rho_j) = 0
\]

In [3], the crucial step in the proof of the Chow-K"unneth decomposition for abelian varieties is the following computation, which may be proved using the seesaw theorem ([14], Corollary 5.2):

**Proposition 3.4.** ([3], 2.15)

For any integer \(n\),

\[
(1 \times n)^* \ell = n \ell
\]

The analogous strategy in our context would seem to be to study the action of \((1 \times n)^*\) on \((g \times h)^* \ell\); however, there is a priori no action of \(G\) on \(\hat{A}\). (If \(G\) acts on \(\hat{A}\) “by isogenies”; that is, if all of the maps \(g : A \to A\) are in fact homomorphisms of \(A\), then duality gives a natural action of \(G\) on \(\hat{A}\), but we are not assuming this). Instead, we rely on the fact ([14], p.119) that the Poincaré bundle on \(\hat{A} \times_k A\) is the transpose of the Poincaré bundle on \(A \times_k \hat{A}\). Hence:

\[
(3.0.6) \quad (n \times 1)^* \ell = t(1 \times n)^* t \ell = t(n \ell) = n \ell
\]

and we prove the following:

**Proposition 3.5.** There is an infinite subset \(E \subset \mathbb{N}\) such that for all \(n \in E\),

\[
(n \times 1)^*(g \times 1)^* \ell = n(g \times 1)^* \ell.
\]
Proof. For each $g \in G$, write $g = a_g \circ g_0$ as in Proposition 1.3. Let $m_g$ be the order of $a_g = -g(0)$; this is guaranteed to be finite by our hypothesis. Next, let $m = \prod_{g \in G} m_g$, and

$$E = \{n \in \mathbb{N} : n \equiv 1 \pmod{m}\}$$

Note that if $n \in E$, $m_g$ divides $n - 1$ (for any $g$), so $na_g = a_g$.

Now, if $n \in E$, we have

$$(3.0.7) \quad (n \times 1)^* (g \times 1)^* \ell = (n \times 1)^* (g_0 \times 1)^* (\tau_{a_g} \times 1)^* \ell$$

Since $g_0$ is a homomorphism, $n \circ g_0 = g_0 \circ n$; therefore the last terms

$$(g_0 \times 1)^* (n \times 1)^* (\tau_{a_g} \times 1)^* \ell$$

Since $a_g = na_g$, this equals

$$(g_0 \times 1)^* (n \times 1)^* (\tau_{na_g} \times 1)^* \ell = (g_0 \times 1)^* (\tau_{a_g} \times 1)^* (n \times 1)^* \ell$$

By (3.0.6) the last term equals,

$$n(g_0 \times 1)^* (\tau_{a_g} \times 1)^* \ell = n(g \times 1)^* \ell$$

The next step in the proof of Theorem 3.2 is to construct the elements $\rho_i$ appearing in Lemma 3.3; for each $i$, we simply set

$$\rho_i = \sum_{g,h \in G} (g \times h)^* \pi_i$$

where $\pi_i$ are the Chow-Künneth components of $\Delta_A$ from Theorem 3.1. It is clear from the formula that the $\rho_i$ are $G \times G$-invariant and that $\sum_{i=0}^{2d} \rho_i = \sum_{g,h \in G} (g, h)^* \Delta_A$; so it remains to prove that they are mutually orthogonal. In preparation for this, we study the action of $(1 \times n)^*$ on $\rho_i$:

**Proposition 3.6.** For $n \in E$, $(1 \times n)^* (g \times h)^* \pi_i = n^i \pi_i$. Hence, $(1 \times n)^* \rho_i = n^i \rho_i$.

**Proof.**

Observe that $(1 \times n)^* (g \times h)^* \pi_i = (1 \times n)^* (g \times 1)^* (1 \times h)^* \pi_i = (g \times 1)^* (1 \times n)^* (1 \times h)^* \pi_i$, so it suffices to consider the case $g = 1$. 

We recall the construction of $\pi_i$ from (3.0.4):

$$n^i(-1)^g\sigma^*p_{12*}(p_{13}\beta_i \cdot \sum_{i=0}^{\infty} \frac{(1 \times \ell^i)}{i!}) = n^i(-1)^g\sigma^*p_{12*}(p_{13}\beta_i \cdot (1 \times h \times 1)^* \sum_{i=0}^{\infty} \frac{(1 \times \ell^i)}{i!})$$

By Proposition 3.5 the last term is given by

$$n^i(-1)^g\sigma^*p_{12*}(p_{13}\beta_i \cdot \sum_{i=0}^{\infty} \frac{1}{i!} (1 \times (h \times 1)^* \ell^i)) = n^i(-1)^g\sigma^*p_{12*}(p_{13}\beta_i \cdot (1 \times h \times 1)^* \sum_{i=0}^{\infty} \frac{(1 \times \ell^i)}{i!})$$

To prove orthogonality of the $\rho_i$, we need a version of Liebermann’s trick (cf. [3], Proof of Theorem 3.1); first we prove the following simple lemma:

**Lemma 3.7.** For every $g, h \in G$, $\rho_j \circ (g \times h)^* \Delta_A = \rho_j$.

**Proof.**

Certainly the lemma is true if $g = h = 1$. In the general case,

$$\rho_j \circ (g \times h)^* \Delta_A = p_{13*}(p_{12}(g \times h)^* \Delta_A \cdot p_{23}\rho_j) = p_{13*}((g \times h \times 1)^*p_{12}\Delta_A \cdot p_{23}\rho_j)$$

$$= p_{13*}((g \times h \times 1)^*(p_{12}\Delta_A \cdot (g^{-1} \times h^{-1} \times 1)^*p_{23}\rho_j))$$

$$= (g \times 1)^*p_{13*}(p_{12}\Delta_A \cdot p_{23}(h^{-1} \times 1)^*\rho_j)$$

Since $\rho_j$ is $G \times G$-invariant the last term equals

$$(g \times 1)^*p_{13*}(p_{12}\Delta_A \cdot p_{23}\rho_j) = (g \times 1)^*(\rho_j \circ \Delta_A) = (g \times 1)^*(\rho_j) = \rho_j$$
Proposition 3.8. (Liebmann’s trick) For every $i, j$, $i \neq j$, $\rho_i \circ \rho_j = 0$.

Proof.
Suppose $n \in E$. By Proposition 3.6,

$$n^j \rho_j = (1 \times n)^* \rho_j$$

$$= (1 \times n)^*(\rho_j \circ \Delta_A)$$

By Lemma 3.7, the last term equals

$$\frac{1}{|G|^2}(1 \times n)^*(\rho_j \circ \sum_{g,h} (g \times h)^* \Delta_A) = \frac{1}{|G|^2}(1 \times n)^*(\rho_j \circ \sum_{i=0}^{2g} \rho_i)$$

$$= \frac{1}{|G|^2} \sum_{i=0}^{2g} (1 \times n)^* p_{13*} (p_{12}^* \rho_j \cdot p_{23}^* \rho_i)$$

$$= \frac{1}{|G|^2} \sum_{i=0}^{2g} p_{13*} (1 \times 1 \times n)^* (p_{12}^* \rho_j \cdot p_{23}^* \rho_i)$$

$$= \frac{1}{|G|^2} \sum_{i=0}^{2g} p_{13*} (p_{12}^* \rho_j \cdot p_{23}^* (1 \times n)^* \rho_i)$$

$$= \frac{1}{|G|^2} \sum_{i=0}^{2g} n^i (\rho_j \circ \rho_i)$$

Hence

$$n^j((\rho_j \circ \rho_j) - |G|^2 \rho_j) + \sum_{i \neq j} n^i (\rho_i \circ \rho_j) = 0$$

for all $n \in E$. Since $E$ is infinite, this forces $\rho_i \circ \rho_j = 0$ for all $i \neq j$, and also $\rho_j \circ \rho_j = |G|^2 \rho_j$.

This final step in the proof of Theorem 3.2 is to show that the images of the $\eta_i$ under the cycle map $cl_{A/G \times_k A/G} : CH^*(A/G \times_k A/G) \rightarrow H^*(X/G \times_k X/G)$ to any Weil cohomology theory are in fact the Künneth components of the class of the diagonal. This follows easily from the analogous fact for the variety $A$ and commutativity of the following diagram:

$$\begin{align*}
CH^*_{Q}(A \times_k A) &\xrightarrow{cl_{A\times_k A}} H^*(A \times_k A) \\
CH^*_{Q}(A/G \times_k A/G) &\xrightarrow{cl_{A/G \times_k A/G}} H^*(A/G \times_k A/G)
\end{align*}$$

\(\downarrow (f \times_k f)_* \quad \downarrow (f \times_k f)_*\)

This concludes the proof of Theorem 3.2.
Among the formulas proved by Künemann is the so-called Poincaré duality for abelian varieties ([12], Theorem 3.1.1 (iii)); in our notation, this reads $\pi_{2d-i} = i^! \pi_i$ for each $i$. This fact immediately implies the analogue for quotients:

**Corollary 3.9.** (Poincaré duality for quotients) The Chow-Künneth decomposition for $A/G$ of Theorem 3.2 satisfies Poincaré duality; that is, for any $i$, $\eta_{2d-i} = i^! \eta_i$.

3.1. **Examples.** 1. **Symmetric products of abelian varieties** Let $X$ denote an abelian variety. We let $X^n/\Sigma_n$ denote the $n$-fold symmetric power of $X$. Observe that for every $\sigma \in \Sigma_n$, $\sigma(0, \cdots, 0) = (0, \cdots, 0)$. Therefore the hypotheses of Theorem (3.2) are satisfied irrespective of the base field $k$. Therefore, we obtain a Chow Kunneth decomposition for $X^n/\Sigma_n$. (Observe that the action of $\Sigma_n$ is not free so that the quotient $X^n/\Sigma_n$ is only pseudo-smooth and not smooth.)

2. **Example of Mehta and Srinivas** (See [15].) Let $X$ be an elliptic curve (or more generally any abelian variety) over $k$, with $\text{char}(k) \neq 2$. Let $t$ denote a point of order 2 on $X$. Let the group $\mathbb{Z}/2\mathbb{Z}$ act on $X \times X$ by : $(x, y) \mapsto (x + t, -y)$. Let $Y$ denote the quotient variety. Now one may see easily that the action is free so that $Y$ is smooth. Nevertheless, in positive characteristics, $Y$ need not be an abelian variety as is shown in [15]. Theorem (3.2) provides a Chow Kunneth decomposition for $Y$. 
4. Appendix: The strong relative K"unneth decomposition of cohomology

In this appendix, we consider, with a view to applications elsewhere, a relative form of the strong K"unneth decomposition in arbitrary cohomology theories satisfying certain mild conditions. It should be remarked that in this section, it suffices to assume the cohomology theory is, at least in principle, part of a twisted duality theory in the sense of Bloch-Ogus. (See [2]...) It should be added that the arguments are modifications of the ones in [20] where he considered the case of K-theory.

Accordingly we will denote $H^*(X, \Gamma(r))$ by $H^*(X, (r))$. Throughout this section we will make the following additional hypotheses on our cohomology theories. (Observe these hypotheses are not identical to the ones in [2], but are implied by them.)

(H.1): for every flat map $f : X \to Y$, there is an induced map $f^* : H^*(Y, (r)) \to H^*(X, (r))$ and this is natural in $f$.

(H.2): for every proper smooth map $f : X \to Y$ of relative dimension $d$, there is a push-forward $f_* : H^i(X; j) \to H^{i-2d}(Y; j-d)$ so that if $g : Y \to Z$ is another proper smooth map of relative dimension $d'$, one obtains $g_* \circ f_* = (g \circ f)_*$. In this case the obvious projection formula $f_*(x \circ f^*(y)) = f_*(x) \circ y, x \in H^r(X, (r)), y \in H^s(Y, (r))$ holds.

(H.3): for each smooth scheme $X$ and closed smooth sub-scheme $Y$ of pure codimension $c$, there exists a canonical class $[Y] \in H^{2c}(X; c)$. Moreover the last class lifts to a canonical class $[Y] \in H^{2c}_{\text{Y}}(X; c)$. (The latter has the obvious meaning in the setting Bloch-Ogus twisted duality theories. In case the cohomology theory is defined as hyper-cohomology with respect to a complex, we let $\mathbb{H}(X; c) = \text{the canonical homotopy fiber of the obvious map } \mathbb{H}(X; c) \to \mathbb{H}(X - Y; c)$; now $H^{2c}_{\text{Y}}(X, c) = H^{2c}(\mathbb{H}(X; \Gamma(c)))$.) The cycle classes are required to pull-back under flat pull-back and push-forward under proper push-forwards.

(H.4): if $X$ is a smooth scheme, there exists the structure of a graded commutative ring on $H^*(X, (..)) = \bigoplus_{r,s} H^r(X; s)$. i.e. $\circ : H^r(X; s) \otimes H^s(X; s') \to H^{r+s'}(X; s + s')$. In addition to this, there exists an external product $H^r(X; s) \otimes H^s(X; s') \to H^{r+s'}(X \times X; s + s')$ so that the internal product is obtained from the latter by pull-back with the diagonal.

For the purposes of this section it is also convenient to consider only cohomology theories that are singly graded or non-weighted. Given a bigraded cohomology theory $H^*(X; (r))$, we will re-index it as follows: we let

$$h^r(X; 2r - s) = H^s(X; (r)) \quad \text{and} \quad h^*(X; n) = \bigoplus_r h^r(X; n)$$

We view $\{h^*(X; n)|n\}$ as a singly graded cohomology theory. Observe that if $f : X \to Y$ is a proper smooth map of relative dimension $d$, the induced map $f_*$ sends $h^*(X; n)$ to $h^*(Y; n)$. Similarly if $f : X \to Y$ is a flat map, the induced map $f^*$ sends $h^*(Y; n)$ to $h^*(X; n)$.
Theorem 4.1. Let \( f : X \to Y \) denote a proper smooth map of smooth schemes of relative dimension \( d \) and let \([\Delta] \in H^{2d}(X \times X; d)\) denote the class of the diagonal. Assume that \([\Delta] = \Sigma_{i,j}a_{i,j} \times b_{d-i,j}, \) with each \( a_{i,j} \in H^{2i}(X; i), \) \( b_{d-i,j} \in H^{2d-2i}(X; d-i)\). Then for every fixed integer \( n \) one obtains the isomorphism:

\[
(4.0.2) \quad h^*(X; n) \cong h^*(X; 0) \otimes_{h^*(Y; 0)} h^*(Y; n)
\]

Proof. We will first prove that the classes \( \{a_{i,j}\} \) generate \( h^*(X; n) \) as a module over \( h^*(Y; .) \) i.e. the obvious map from the right hand side to the left hand side of 4.0.2 (which we will denote by \( \rho \)) is surjective.

Let \( p_i : X \times X \to X \) denote the projection to the \( i \)-th factor. For each \( x \in h^*(X; n) \) we will first observe the equality:

\[
(4.0.3) \quad x = p_{1*}(\Delta \circ p_{2*}(x))
\]

(To see this observe that \([\Delta] = \Delta_*(1), 1 \in H^*(X; \Gamma(\cdot)). \) Therefore, \( \Delta \circ p_{2*}(x) = \Delta(\Delta^* p_2^*(x)) \) and hence \( p_{1*}(\Delta \circ p_{2*}(x)) = p_{1*} \Delta_* (\Delta^* p_2^*(x)) = (p_1 \circ \Delta)_*((p_2 \circ \Delta)^*(x)) = x. \)

Now we substitute \([\Delta] = \Sigma_{i,j} p_i^*(a_{i,j}) \circ p_2^*(b_{d-i,j})\) into the above formula to obtain:

\[
(4.0.1) \quad x = p_{1*}(\Sigma_{i,j} p_i^*(a_{i,j}) \circ p_2^*(b_{d-i,j} \circ p_2^*(x)))
\]

\[
= p_{1*}(\Sigma_{i,j} p_i^*(a_{i,j}) \circ p_2^*(b_{d-i,j} \circ x))
\]

\[
= \Sigma_{i,j} a_{i,j} \circ p_{1*} p_2^*(b_{d-i,j} \circ x)
\]

\[
= \Sigma_{i,j} a_{i,j} \circ f^*(f_*(b_{d-i,j} \circ x))
\]

This proves the assertion that the classes \( \{a_{i,j}\} \) generate \( h^*(X; .) \) i.e. the map \( \rho \) is surjective.

The rest of the proof is to show that the map \( \rho \) is injective. The key is the following diagram:

\[
\begin{array}{ccc}
 h^*(X; n) & \xrightarrow{\rho} & h^*(X, 0) \otimes_{h^*(Y, 0)} h^*(Y; n) \\
 \downarrow{\alpha} & & \downarrow{\mu}
\end{array}
\]

\[
H_{h^*(Y, 0)}(h^*(X, 0), h^*(Y; n))
\]

where the map \( \alpha (\mu(x), x \in h^*(X, 0)) \) is defined by \( \alpha(x \otimes y) = \mu(x' \circ x \circ y) \) (the map \( x' \mapsto f'_*(x' \circ x) \), respectively). The commutativity of the above diagram is an immediate consequence of the projection formula: observe \( \rho(x \otimes y) = x \circ f^*(y) \). Therefore, to show the map \( \rho \) is injective, it suffices to show the map \( \alpha \) is injective. For this we define a map \( \beta \) to be a splitting for \( \alpha \) as follows: if \( \phi \in H_{h^*(Y, 0)}(h^*(X, 0), h^*(Y; n)) \), we let \( \beta(\phi) = \Sigma_{i,j} a_{i,j} \otimes (\phi(b_{d-i,j})) \). Observe that \( \beta(\alpha(x \otimes y)) = \beta(\text{the map } x' \mapsto f'_*(x' \circ x) \otimes y) = (\Sigma_{i,j} a_{i,j} \otimes f_*(b_{d-i,j} \circ x)) \otimes y. \)

Now observe that \( f_*(b_{d-i,j} \circ x) \in h^*(Y; 0) \) so that we may write the last term as \( = (\Sigma_{i,j} a_{i,j} \circ f^* f_*(b_{d-i,j} \circ x)) \otimes y. \) By (4.0.1), the last term = \( x \otimes y. \) This proves that \( \alpha \) is injective and hence that so is \( \rho. \)

We end this section by considering some explicit examples of what the last theorem implies for various cohomology theories.
Examples 4.2. 1. \textit{K-theory}. In this case the theorem takes on the form:
\begin{equation}
K^i(X) \cong K^0(X) \otimes_{K^0(Y)} K^i(Y)
\end{equation}
See [20] for a proof of this. The last part of the proof of the above theorem is clearly an adaptation of this argument.

2. The theorem takes on the following simple form in the case of any of the following bigraded cohomology theories: absolute, motivic or Deligne cohomology.
\begin{equation}
H^n(X; t) = \bigoplus_a H^{2a}(X; a) \otimes \left( \bigoplus_a H^{n-2a}(Y; t-a) \right)
\end{equation}

Remarks 4.3. 1. Taking \( n = 0 \), the argument in the last part of the proof of the theorem shows that one obtains a non-degenerate pairing \( < , > : h^*(X, 0) \otimes h^*(Y, 0) \to h^*(Y, 0) \) by \( \alpha \otimes \beta \mapsto f_*(\alpha \otimes \beta) \).

2. As an example of the usefulness of the last theorem or the formula in 2, we consider the following result. Let \( H \) denote motivic cohomology. Recall that the Beilinson-Soulé vanishing conjecture for the motivic cohomology of a scheme is the following statement: \( H^s(X, r; \mathbb{Q}) = 0 \) if \( s < 0 \) or if \( (s = 0 \text{ and } r \neq 0) \) while \( H^0(X, 0; \mathbb{Q}) = \mathbb{Q} \). Now we leave it as an easy exercise to prove from 2 the following proposition.

\textbf{Proposition 4.4.} If \( X \to Y \) satisfies the hypotheses of Theorem 4.1 and \( X \) and \( Y \) are both connected, then the Beilinson-Soulé vanishing conjecture holds for \( X \) if it holds for \( Y \).

\textbf{Corollary 4.5.} The following class of varieties over a number field \( k \) satisfy the Beilinson-Soulé vanishing conjecture.

\begin{itemize}
\item All toric varieties over \( k \)
\item All spherical varieties over \( k \). (A variety \( X \) is spherical if there exists a reductive group \( G \) defined over \( k \) acting on \( X \) so that there exists a Borel subgroup scheme \( S \) having a dense orbit. The orbits are products of tori and affine spaces (over \( k \)).)
\item Any variety over \( k \) on which a connected solvable group scheme defined over \( k \) acts with finitely many orbits. (For example projective spaces and flag varieties over \( k \)).
\item Any variety over \( k \) that has a stratification into strata each of which is the product of a torus with an affine space.
\end{itemize}

\textbf{References}


Department of Mathematics, Miami University, Oxford, Ohio, 45056, USA.

E-mail address: reza@calico.mth.muohio.edu

Department of Mathematics, Ohio State University, Columbus, Ohio, 43210, USA

E-mail address: joshua@math.ohio-state.edu