Kunneth decomposition for quotient varieties

Terminology and notation

- k: a field of arbitrary characteristic

- All schemes projective over k

- A scheme Y is pseudo-smooth if it is of the form X/G, X smooth, G a finite group.

- The theory of correspondences extends to pseudo-smooth projective schemes with rational coefficients. (cf. Fulton). $CH^*_{\mathbb{Q}}(X) = CH^*(X) \otimes \mathbb{Q}$. $\circ$: composition of correspondences.

- Any Weil cohomology theory extends to pseudo-smooth schemes.

Definition: Chow Kunneth decomposition $d = dim(X)$. Then X has a Chow Kunneth decomposition if there exist $\pi_i \varepsilon CH^d_{\mathbb{Q}}(X \underset{k}{\times} X)$ so that $[\Delta_X] = \Sigma_{i=0}^{2d} \pi_i$, $\pi_i \circ \pi_j = 0$, $i \neq j$ and $\pi_i \circ \pi_i = \pi_i$. If cl denotes the cycle map into any Weil cohomology, $cl(\pi_i) =$ a Kunneth component of $cl([\Delta_X])$.

Theorem(Beauville, Denninger-Murre, Shermenev) Let A be an abelian variety of dimension d over a field k. Then there exists a Chow-Künneth decomposition for A:

$$\Delta_A = \sum_{i=0}^{2d} \pi_i. \quad \square$$

Our main result:

Theorem. Let A be an abelian variety of dimension d over a field k and G a finite group acting on A such that $g(0) \in A(k)$ is a torsion point for each $g \in G$. Let $f : A \longrightarrow A/G$ be the quotient map. Suppose $\Delta_A = \sum_{i=0}^{2d} \pi_i$ is a Chow-Künneth decomposition for A and let $\eta_i = \frac{1}{|G|}(f \times f)_* \pi_i$.

Then

$$\Delta_{A/G} = \sum_{i=0}^{2d} \eta_i$$

is a Chow-Künneth decomposition for A/G. $\square$

Examples

- Symmetric products of abelian varieties

- Smooth quotients of abelian varieties that are not abelian varieties (in pos. char) due to Igusa, Mehta and Srinivas

Remark.

The hypothesis $g(0)$ be a torsion point of A not always satisfied.

Key property:

$$f^* : CH^*_{\mathbf{Q}}(A/G) \longrightarrow CH^*_{\mathbf{Q}}(A)^G$$

is an isomorphism with inverse $\frac{1}{|G|} f_*$.

Descent Lemma. Suppose X is a pseudo-smooth projective variety of dimension d and G a finite group of automorphisms of X. Let $f : X \longrightarrow Y = X/G$ be the quotient map and suppose

$$\sum_{g,h \in G} (g \times h)^* \Delta_X = \sum_{i=0}^{2d} \rho_i$$

where $\rho_i \circ \rho_j = 0$ if $i \neq j$, $\rho_i \circ \rho_j = |G|^2 \rho_i$ if $i = j$ and the ρ_i are $G \times G$-invariant, i.e. for any $g, h \in G$, $(g \times h)^* \rho_i = \rho_i$.

Then

$$\Delta_Y = \sum_{i=0}^{2d} \frac{1}{|G|^3} (f \times f)_* \rho_i$$

is a Chow-Künneth decomposition for Y.

Proof(outline). We have

$(f \times f)_*(f \times f)^* = |G|^2$, $\sum_{g,h \in G}(g \times h)^* = (f \times f)^*(f \times f)_*$ and $(f \times f)_* \Delta_X = |G|\Delta_Y$,

and therefore:

$$|G|^2 (f \times f)_* \Delta_X = (f \times f)_* \sum_i \rho_i$$

Hence

$$\Delta_Y = \frac{1}{|G|^3} \sum_i (f \times f)_* \rho_i$$

Remains to show that $(f \times f)_* \rho_i$ are mutually orthogonal. Follows by similar argument. □

In Denninger-Murre, a crucial step is: for any integer n,

$$(\mathbf{n} \times 1)^* \ell = n\ell$$

where ℓ :the first Chern class of the Poincaré bundle.

The analogous strategy in our context:

Proposition There is an infinite subset $E \subset \mathbf{N}$ such that for all $n \in E$,

$$(\mathbf{n} \times 1)^*(g \times 1)^* \ell = n(g \times 1)^* \ell.$$

Proof. For each $g \in G$, $g = \tau_{a_g} \circ g_0$ where τ_{a_g} is translation by a_g, $a_g = -g(0)$, $g_0 =$ a homomorphism (of abelian varieties). Let m_g = the order of $a_g = -g(0)$. Next, let $m = \prod_{g \in G} m_g$, and

$$E = \{n \in \mathbf{N} : n \equiv 1 (\text{mod } m)\}$$

Note: if $n \in E$, m_g divides $n-1$ (for any g), so $na_g = a_g$.

For $n \in E$:

$(\mathbf{n} \times 1)^*(g \times 1)^*\ell = (\mathbf{n} \times 1)^*(g_0 \times 1)^*(\tau_{a_g} \times 1)^*\ell$

Since g_0 is a homomorphism, $\mathbf{n} \circ g_0 = g_0 \circ \mathbf{n}$; therefore the last expression equals

$$(g_0 \times 1)^*(\mathbf{n} \times 1)^*(\tau_{a_g} \times 1)^*\ell$$

Since $a_g = na_g$, this equals

$(g_0 \times 1)^*(\tau_{na_g} \times 1)^*(\mathbf{n} \times 1)^*\ell$

$= (g_0 \times 1)^*(\tau_{a_g} \times 1)^*(\mathbf{n} \times 1)^*\ell$

Since $(\mathbf{n} \times 1)^*\ell = n\ell$, the last term equals,

$$n(g_0 \times 1)^*(\tau_{a_g} \times 1)^*\ell = n(g \times 1)^*\ell \quad \square$$

The next step in the proof of our main Theorem is to construct the elements ρ_i appearing in Lemma; for each i, we simply set

$$\rho_i = \sum_{g,h \in G} (g \times h)^* \pi_i$$

where π_i are the Chow-Künneth components of Δ_A.

Clear from the formula that the ρ_i are $G \times G$-invariant and that $\sum_{i=0}^{2d} \rho_i = \sum_{g,h \in G} (g,h)^* \Delta_A$; so it remains to prove that they are mutually orthogonal. In preparation for this, we study the action of $(1 \times \mathbf{n})^*$ on ρ_i:

Proposition For $n \in E$, $(1 \times \mathbf{n})^* (g \times h)^* \pi_i = n^{2d-i} \pi_i$. Hence, $(1 \times \mathbf{n})^* \rho_i = n^{2d-i} \rho_i$.

Proof. Observe: $(1 \times \mathbf{n})^*(g \times h)^*\pi_i = (1 \times \mathbf{n})^*(g \times 1)^*(1 \times h)^*\pi_i = (g \times 1)^*(1 \times \mathbf{n})^*(1 \times h)^*\pi_i$, so it suffices to consider the case $g = 1$.

Next review the definition of the π_i: first, consider $A \times_k A$ as an abelian A-scheme via projection on the first factor; with respect to this structure, the dual abelian scheme is $A \times_k \widehat{A}$. Then the Fourier transform

$$F_{CH} : CH^*_{\mathbf{Q}}(A \times_k A) \longrightarrow CH^*_{\mathbf{Q}}(A \times_k \widehat{A})$$

is defined by $F_{CH}(\alpha) = p_{13*}({p_{12}}^*\alpha \cdot F)$, where

$$F = 1 \times \sum_{i=0}^{\infty} \frac{\ell^i}{i!} \in CH_{\mathbf{Q}}(A \times_k A \times_k \widehat{A})$$

p_{ij} represent projections from $A \times_k A \times_k \widehat{A}$ on the ith and jth factor. (Note:the sum defining F is actually finite.)

Dualize this construction, to define

$$\widehat{F}_{CH} : CH^*_{\mathbf{Q}}(A \times_k \widehat{A}) \longrightarrow CH^*_{\mathbf{Q}}(A \times_k A)$$

by $\widehat{F}_{CH}(\gamma) = q_{13*}(q^*_{12}\gamma \cdot \widehat{F})$, where

$$\widehat{F} = 1 \times \sum_{i=0}^{\infty} \frac{t_{\ell}{}^{i}}{i!} \in CH^*_{\mathbf{Q}}(A \times_k \widehat{A} \times_k A)$$

and q_{ij} represent the various projections from $A \times_k \widehat{A} \times_k A$. Switching the last two factors,

$$\widehat{F}_{CH}(\gamma) = p_{12*}(p^*_{13}\gamma \cdot F).$$

Theorem of the square then shows $\widehat{F}_{CH}(F_{CH}(\alpha)) = (-1)^d\sigma^*\alpha$ for all $\alpha \in CH^*(A \times_k A)$, and similarly for the other composition.

Observe that $[\Delta_A] \in CH^d(A \times_k A)$, and write $F_{CH}([\Delta_A]) = \sum_{i=0}^{2d} \beta_i$, where $\beta_i \in CH^i_{\mathbf{Q}}(A \times_k \widehat{A})$.

$$\pi_i = (-1)^d \sigma^* \widehat{F}_{CH}(\beta_i)$$

Now:

$$(1 \times \mathbf{n})^*(1 \times h)^* \pi_i = (-1)^d \sigma^* (1 \times \mathbf{n})^*(1 \times h)^* \widehat{F}_{CH}(\beta_i)$$

From the definition of $\widehat{F}_{CH}$ this identifies with:

$$(-1)^d \sigma^* (1 \times \mathbf{n})^*(1 \times h)^* p_{12*}(p_{13}{}^* \beta_i \cdot (1 \times \sum_{i=0}^{\infty} \frac{\ell^\mu}{\mu!}))$$

However, $deg(\widehat{F}_{CH}(\beta_i) = d$, all terms except with $\mu = 2d - i$ vanish. Using base-change one identifies the latter with:

$$(-1)^d \sigma^* p_{12*}(p_{13}^* \beta_i \cdot (1 \times \tfrac{1}{(2d-i)!}((\mathbf{n} \times 1)^*(h \times 1)^* \ell^{2d-i})))$$

At this point, our earlier observations imply that for an infinite set of ns, this identifies with:

$$n^{2d-i}(-1)^d\sigma^* p_{12*}(p_{13}^*\beta_i\cdot(1\times\frac{1}{(2d-i)!}((h\times 1)^*\ell^{2d-i})))$$

Reversing the arguments, one identifies this with $n^{2d-i}(1\times h)^*\pi_i$. This concludes the proof of the last proposition. $\square$

To prove orthogonality of the ρ_i, we need a version of Liebermann's trick; first the following simple lemma:

Lemma For every $g, h \in G$, $\rho_j\circ(g\times h)^*\Delta_A = \rho_j$.

Proposition(Liebermann's trick) For every i, j, $i \neq j$, $\rho_i \circ \rho_j = 0$.

Outline of proof. Suppose $n \in E$. By our earlier result,

$$n^{2d-j}\rho_j = (1 \times \mathbf{n})^* \rho_j$$

$$= (1 \times \mathbf{n})^* (\rho_j \circ \Delta_A)$$

By the last Lemma, the last term equals

$$\frac{1}{|G|^2}(1 \times \mathbf{n})^* (\rho_j \circ \textstyle\sum_{g,h} (g \times h)^* \Delta_A)$$

$$= \frac{1}{|G|^2}(1 \times \mathbf{n})^* (\rho_j \circ \textstyle\sum_{i=0}^{2d} \rho_i)$$

One shows this is equal to: $\frac{1}{|G|^2} \sum_{i=0}^{2d} n^{2d-i} (\rho_j \circ \rho_i)$

Hence

$$n^{2d-j}((\rho_j \circ \rho_j) - |G|^2\rho_j) + \sum_{i \neq j} n^{2d-i}(\rho_i \circ \rho_j) = 0$$

for all $n \in E$. Since E is infinite, this forces $\rho_i \circ \rho_j = 0$ for all $i \neq j$, and also $\rho_j \circ \rho_j = |G|^2\rho_j$.
$\square$

Remarks

• One may show readily that the cycle map is compatible with Kunneth decomposition.

• The Kunneth components for pseudo-smooth schemes satisfy Poincaré duality, i.e. $\eta_{2d-i} = {}^t\eta_i$

Definition: Strong Kunneth decomposition

X any scheme of pure dimension d. X possesses a *strong Künneth decomposition* if there exist elements $a_{i,j}, b_{i,j} \in CH^i_{\mathbf{Q}}(X)$ such that

$$[\Delta_X] = \sum_i \sum_j a_{i,j} \times b_{d-i,j}$$

□

Exercise: Strong Kunneth decomposition implies a Chow Kunneth decomposition

Proposition Let X and Y be pseudo-smooth proper varieties and $f : X \longrightarrow Y$ a finite surjective map. If X has a strong Künneth decomposition, then Y also has a strong Künneth decomposition.

Corollary Let X be a pseudo-smooth quasi-projective variety, G a finite group of automorphisms of X. If X possesses a strong Künneth decomposition, so does $Y = X/G$.

Example (Symmetric Products of projective spaces)

Let $\ell \in CH^1_{\mathbf{Q}}(\mathbf{P}^m_k)$ be the class of a generic hyperplane in $\mathbf{P}^m_k$. $\mathbf{P}^m_k$ has a strong Künneth decomposition:

$\Delta_{\mathbf{P}^m_k} = \sum_{i=0}^m \ell^i \times \ell^{m-i}$

Let $X = (\mathbf{P}^m_k)^n$. By the Künneth formula:

$\Delta_X = \sum_{0 \leq i_1, \ldots, i_n \leq m} f_{i_1, \ldots, i_n}$

where $f_{i_1,\ldots,i_n} = \ell^{i_1} \times \ldots \times \ell^{i_n} \times \ell^{m-i_1} \times \ldots \ell^{m-i_n} \in CH^{mn}_{\mathbf{Q}}(X \times_k X)$.

Let $Y = X/S_n$ and $q : X \longrightarrow Y$ the quotient map.

Applying $(q \times q)_*$ to the strong Künneth decomposition for Δ_X given above, and noting that deg $q = n!$:

$$(n!)\Delta_Y = \sum_{0 \le i_1, \ldots, i_n \le m} (q \times q)_* f_{i_1,\ldots,i_n}$$
$$= \sum_{0 \le i_1 \le i_2 \le \ldots \le i_n \le m} \sum_{\sigma \in S_n} (q \times q)_* f_{\sigma(i_1),\ldots,\sigma(i_n)}$$
$$= \sum_{0 \le i_1 \le i_2 \le \ldots \le i_n \le m} n!(q \times q)_* f_{i_1,\ldots,i_n}$$

Now let $\bar{\ell}^i = q_*(\ell^i)$. Then

$$\Delta_Y = \sum_{0 \le i_1 \le i_2 \le \ldots \le i_n \le m} (q \times q)_* f_{i_1,\ldots,i_n}$$
$$= \sum_{0 \le i_1 \le i_2 \le \ldots \le i_n \le m} \bar{\ell}^{i_1} \times \ldots \times \bar{\ell}^{i_n} \times \bar{\ell}^{m-i_1} \times \ldots \times \bar{\ell}^{m-}$$

giving a strong Künneth decomposition for Y.

Corollary $CH^*(Y, Q, r)$

$\cong CH^*(Y, Q, 0) \otimes CH^*(Spec \quad k, Q, r)$

where $CH^*(Z,Q,r) = \pi_r(z^*(Z,.)\otimes\mathbb{Q})$ and $z^*(Z,.)$ denotes the higher cycle complex of the scheme Z.

Proof This follows readily from the above strong Künneth decomposition for the class Δ_Y and a Theorem on the higher Chow groups of linear schemes. $\square$

See:

http://www.math.ohio-state.edu/~ joshua/pub.html or

http://www.math.ias.edu/~ joshua