RIEMANN-ROCH FOR ALGEBRAIC STACKS:III VIRTUAL STRUCTURE SHEAVES AND VIRTUAL FUNDAMENTAL CLASSES

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Abstract. In this paper we apply the Riemann-Roch and Lefschetz-Riemann-Roch theorems proved in our earlier papers to define virtual fundamental classes for the moduli stacks of stable curves in great generality and establish various formulae for them.

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The author was supported by the MSRI, the IHES, the MPI(Bonn) and a grant from the NSA.

1. **Introduction**

This is the last in a series of papers on the Riemann-Roch problem for algebraic stacks. The first part (see [J-4]) presented a solution to this problem in general for the natural transformation between the G-theory and topological G-theory of algebraic stacks. It also introduced a new site associated to algebraic stacks called the isovariant étale site using which we proved a descent theorem for the topological G-theory of algebraic stacks extending Thomason's basic results to algebraic stacks. Continuing along the same direction, we defined and studied Bredon style homology theories for algebraic stacks in [J-5]. We followed this up in [J-6], by establishing Riemann-Roch and Lefschetz-Riemann-Roch theorems as natural transformations between the G-theory of dg-stacks and these Bredon-style homology theories. These are only for algebraic stacks that admit coarse-moduli spaces which are quasi-projective schemes over a Noetherian excellent base scheme (for example, a field k). It is important to observe that these already include Artin stacks. One may recall that applications to virtual fundamental classes, dictated that we work out all these papers in the setting of dg-stacks.

In the present paper, we indeed establish various formulae for the virtual structure sheaves on dg-stacks associated to obstruction theories at the level of the G-theory of dg-stacks. Using Riemann-Roch and Lefschetz-Riemann-Roch theorems developed in the earlier papers, these provide pushforward and localization formulae for virtual fundamental classes. In fact we show that it is possible to derive most formulae for virtual fundamental classes (some not known before) , by first proving an appropriate formula at the level of virtual structure sheaves and then by applying Riemann-Roch to it. For example, we prove a general push-forward formula for virtual structure sheaves; then by applying Riemann-Roch to it we show it is possible to derive a general pushforward formula for virtual fundamental classes, special cases of which provide a proof of the conjecture of Cox, Katz and Lee as well as a strong form of the localization formula for virtual fundamental classes, both proven elsewhere by distinct and separate methods at the level of virtual fundamental classes. All of these seem to validate the idea, we believe due to Yuri Manin (and passed onto me by Bertrand Toen), that Riemann-Roch techniques could be used to derive most formulae for virtual fundamental classes, once the corresponding formulae for virtual structure sheaves are obtained. The latter seem more manageable and, as we show here, could be studied by standard techniques in G-theory, suitably modified to handle virtual objects.

We begin section 2 by defining first virtual structure sheaves and then virtual fundamental classes in great generality. This makes intrinsic use of the Riemann-Roch transformation. We show that our definition reduces to the more traditional cycle-theoretic definition (or definition in terms of homology classes) - see Theorem 2.4. The following is one of the main theorems proved in section 2.

Theorem 1.1. *(See Theorem 2.4 and Proposition 2.7). Let $\mathcal{S}$ denote a Deligne-Mumford stack provided with a perfect obstruction theory $E^\bullet$ in the sense of section 2.*

(i) Then the virtual fundamental class of $(\mathcal{S}, E^\bullet)$ is defined without any further assumptions on $\mathcal{S}$ or $E^\bullet$ except those assumed in 1.1.7 taking values both in Bredon-style homology theories as in [J-5] and also in homology theories defined on the smooth site of the stack $\mathcal{S}$.

(ii) Moreover, assume in addition to the above situation that the stack is an orbifold and that the complexes $\{\Gamma^h(r)|r\}$ are defined on the smooth site of all algebraic stacks. Let $\mathfrak{M}_0$ denote the open subscheme of the moduli space where the stabilizers are trivial. Assume further that Supp$([\mathcal{S}]^{virt}_{Br}) \cap \mathfrak{M}_0$ is non-empty, (where the support, Supp$([\mathcal{S}]^{virt}_{Br})$ is defined in Definition 2.5). Then the image of the class $[\mathcal{S}]^{virt}_{Br}$ in the smooth homology of the stack with respect to $\Gamma(*)$, agrees with the virtual fundamental classes defined cycle theoretically in the latter.

We begin the next section by reviewing basic definitions of obstructions theories and virtual structure sheaves. We discuss Gysin maps in the context of G-theory in section 3. This is done so that we obtain more convenient expressions for the virtual structure sheaves considered in section 2. Section 4 is devoted to a thorough study of pushforward for virtual structure sheaves and virtual fundamental classes for algebraic stacks. We begin section 4, by obtaining convenient expressions for the virtual structure sheaves: given a Deligne-Mumford stack $\mathcal{S}$ and an obstruction theory $E^\bullet$ for it, we obtain several expressions for $\mathcal{O}_S^{virt}$ as a class in $\pi_0(G(\mathcal{S}))$. One may assume one of these for the following discussion.

1.1. Next we consider pushforward for closed immersions of Deligne-Mumford stacks associated to compatible obstruction theories. The appropriate context for all of these is the following: assume that $u : \mathcal{T} \rightarrow \mathcal{S}$ and $v : \tilde{\mathcal{T}} \rightarrow \tilde{S}$ are closed immersions and that the square

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{u} & \mathcal{S} \\ \downarrow{\scriptstyle i_{\mathcal{T}}} & & \downarrow{\scriptstyle i} \\ \tilde{\mathcal{T}} & \xrightarrow{v} & \tilde{\mathcal{S}} \end{array}$$

is cartesian, with both $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ smooth Deligne-Mumford stacks and where the the vertical maps are *also closed immersions*. To handle the equivariant case, we may assume that all these stacks are provided with the action of a smooth group scheme G and the morphisms above are all G-equivariant. We will assume that (i) one is provided with a perfect obstruction theory $E^\bullet$ ($F^\bullet$) for $\mathcal{S} \to \tilde{\mathcal{S}}$ ($\mathcal{T} \to \tilde{\mathcal{T}}$, respectively) (ii) that $E^\bullet$ ($F^\bullet$) has a global resolution by a complex of vector bundles and (iii) that these are *weakly compatible* in the following sense: there is given a G-equivariant map $\phi : u^*(E^\bullet) \to F^\bullet$ of complexes so that there exists a distinguished triangle $K^\bullet \to u^*(E^\bullet) \to F^\bullet$ and $K^\bullet$ is of perfect amplitude contained in $[-1, 0]$. (For example, the two obstruction theories are weakly compatible if $E^\bullet$ and $F^\bullet$ may be replaced (upto G-equivariant quasi-isomorphism) by complexes of G-equivariant vector bundles and the given map $\phi : u^*(E^\bullet) \to F^\bullet$ is an *epimorphism* in each degree. It follows that, in this case, the kernel, $K^\bullet = ker(\phi)$ is a complex of vector bundles.)

We will also assume henceforth one of the following hypotheses:

• Let $\hat{\mathcal{S}} = Bl_{\mathcal{T}\times 0}(\mathcal{S} \times \mathbb{A}^1)$ = the blow-up of $\mathcal{S} \times \mathbb{A}^1$ along $\mathcal{T} \times 0$. We will assume that there exists a class (which we denote) $\lambda_{-1}(\hat{K}^0)$ in $\pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{\mathcal{S}}))$ so that for each $t\varepsilon\mathbb{A}^1$, $i_t^*(\lambda_{-1}(\hat{K}^0)\varepsilon\pi_0(G_{\mathcal{T}\times t}((\hat{\mathcal{S}})_t))$ identifies with the class of $\lambda_{-1}(K^0)$ in $\pi_0(G(\mathcal{T}))$. (The class $i_0^*(\lambda_{-1}(\hat{K}^0))$ will be denoted $\lambda_{-1}(K^0_{\mathcal{S}})$ henceforth.)

• We are in the equivariant case satisfying the hypotheses of Theorem 1.7.

Observe that in the latter case, we let $G(X, T)$ denote the T-equivariant G-theory of a stack X provided with the action of the torus T. ($K(X, T)$ will denote the corresponding T-equivariant K-theory and if X admits a closed immersion into $\tilde{X}$ onto which the T-action extends, $K_X(\tilde{X}, T)$ will denote the T-equivariant K-theory of $\tilde{X}$ with supports in X.) Now observe that one has the isomorphism $\pi_0(G(\mathcal{T} \times \mathbb{A}^1, T))_{(\mathfrak{p})} = \pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{\mathcal{S}}, T))_{(\mathfrak{p})} \cong \pi_0 G(\hat{\mathcal{S}}, T)_{(\mathfrak{p})}$ and hence the class $\lambda_{-1}(K^0)$ in the first group (i.e. in $\pi_0(G(\mathcal{T}, T))_{(\mathfrak{p})}$) lifts to a class in $\pi_0 G(\hat{\mathcal{S}}, T)_{(\mathfrak{p})}$ which we denote by $\lambda_{-1}(\hat{K}^0)$ and a class $\lambda_{-1}(K^0_{\mathcal{S}})\varepsilon\pi_0(G(\mathcal{S}, T))_{(\mathfrak{p})}$. Observe also that in either case one may identify $\lambda_{-1}(K^0_{\mathcal{S}})$ with a class in $\pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}, T))$ (or a localization of the latter in the equivariant case) so that tensor product with this class is well-defined and one may take its Chern-character (as a local Chern character). We let $\mathcal{O}^{virt}_{\mathcal{S}}$ ($\mathcal{O}^{virt}_{\mathcal{T}}$) denote the virtual structure sheaf associated to $\mathcal{S}$ ($\mathcal{T}$, respectively).

Definition 1.2. Assume that the complexes $\Gamma(*)$ and $\Gamma^h(*)$ extend to the smooth site of all algebraic stacks. We define the *virtual Todd class* of the obstruction theory $E^\bullet$ with values in $H^*_{smt}(\mathcal{S}, \Gamma(*))$ as $Td(E_0).Td(E_1)^{-1}$ where $E_i = (E^i)^\vee$. This will be denoted $Td(T\mathcal{S})^{virt}$. Then we will define the *virtual fundamental class* in $H^{smt}_*(\mathcal{S}, \Gamma(*))$, $[\mathcal{S}]^{virt}$, to be $\phi_*(\tau(\mathcal{O}^{virt}_{\mathcal{S}})).Td((T\mathcal{S})^{virt})^{-1}$. This will be denoted $[\mathcal{S}]^{virt}$.

Remark 1.3. The justification for the above definition is provided by Theorem 1.4 of [J-5]. It is also observed there that the Todd-classes considered above are invertible in the smooth cohomology of the stack.

Theorem 1.4. *(Push forward of virtual structure sheaves and virtual fundamental classes) Assume the above situation. Then* $\lambda_{-1}(\hat{K}^0)$ *defines a class in* $\pi_0(G_{\mathcal{T}}(\mathcal{S}))$ *and one obtains the formulae:*

(1.1.1) $$u_*(\mathcal{O}^{virt}_{\mathcal{T}}.\lambda_{-1}(K^{-1})) = \mathcal{O}^{virt}_{\mathcal{S}}.\lambda_{-1}(K^0_{\mathcal{S}})$$

in $\pi_0(G_{\mathcal{T}}(\mathcal{S}))$ *and*

(1.1.2) $$u_*(\tau(\mathcal{O}^{virt}_{\mathcal{T}}))ch(\lambda_{-1}(K^{-1})) = \tau(\mathcal{O}^{virt}_{\mathcal{S}}).ch(\lambda_{-1}(K^0_{\mathcal{S}})$$

in $H^{Br}_*(\mathcal{S}, \Gamma(*))$ *which is the Bredon-style homology defined in [J-5].*

Next assume that the complexes $\Gamma(*)$ *and* $\Gamma^h(*)$ *extend to the smooth site of all algebraic stacks. Then we obtain the formula*

(1.1.3) $$u_*(([\mathcal{T}]^{virt}).e(K_1)) = [\mathcal{S}]^{virt}.e((K^0_{\mathcal{S}})^\vee)$$

in $H^{smt}_*(\mathcal{S}, \Gamma(*))$. *Here* $K_1 = (K^{-1})^\vee$ *while* $e(V^\vee) = Td(V^\vee).Ch(\lambda_{-1}(V))$ *for a vector bundle* V *where* Ch *denotes the Chern character with values in smooth cohomology and* e *denotes the Euler class.*

Remarks 1.5. 1. Observe that the notion of compatibility of obstruction theories adopted above is indeed weaker than the usual notion of compatibility as in [BF] or [KKP]. Hence the adjective weak-compatibility is used in our situation. There seem to be obstruction theories that are weakly compatible and not compatible: for example, the obstruction theories as in the theorems below associated to the closed immersion of the fixed point stack for the action of a given torus on an algebraic stack.

2. The above theorem provides many useful formulae for virtual fundamental classes and virtual structure sheaves, some of which are considered next. For example, we answer the following strong form of the conjecture of Cox, Katz and Lee (see [CKL]).

1.1.4. Let X denote a smooth projective variety. Let $\beta \varepsilon CH_1(X)$ denote a class and let $\mathcal{M}_{0,n}(X,\beta)$ denote the moduli stack of n-pointed genus 0 stable maps to X of class β. Let V denote a vector bundle over X so that it is *convex.* i.e. $H^1(C, f^*(V)) = 0$ for all genus 0-stable maps $f : C \to X$. Let $e_{n+1} : \mathcal{M}_{0,n+1}(X,\beta) \to X$ send $(f, C, p_1, ..., p_{n+1})$ to $f(p_{n+1})$ and $\pi_{n+1} : \mathcal{M}_{0,n+1}(X,\beta) \to \mathcal{M}_{0,n}(X,\beta)$ denote the map forgetting the point p_{n+1}. Let $\mathcal{V}_{\beta,n} = \pi_{n+1*}e^*_{n+1}(V)$; this is a vector bundle on $\mathcal{M}_{0,n}(X,\beta)$ in view of the convexity of V. Let $i : Y \to X$ denote the inclusion of the zero locus of a regular section of V and for each $\gamma \varepsilon H_2(Y,\mathbb{Z})$ with $i_*(\gamma) = \beta$, let $i_\gamma : \mathcal{M}_{0,n}(Y,\gamma) \to \mathcal{M}_{0,n}(X,\beta)$ denote the induced closed immersion.

Theorem 1.6. *(Conjecture of Cox, Katz and Lee: see [CKL] and also [CK] p. 386) Assuming the above situation*

$$\Sigma_{i_*(\gamma)=\beta} i_{\gamma*}(\mathcal{O}^{virt}_{\mathcal{M}_{0,n}(Y,\gamma)}) = \lambda_{-1}(\Gamma(\mathcal{V}_{\beta,n})).\mathcal{O}^{virt}_{\mathcal{M}_{0,n}(X,\beta)} \text{ in } \pi_0(G(\mathcal{M}_{0,n}(X,\beta), \mathcal{O}_{\mathcal{M}_{0,n}(X,\beta)})).$$

(Here $\Gamma(\mathcal{V}_{\beta,n})$ denotes the sheaf of sections of the vector bundle $\mathcal{V}_{\beta,n}$.) In particular, one obtains:

$$\Sigma_{i_*(\gamma)=\beta} i_{\gamma*}([\mathcal{M}_{0,n}(Y,\gamma)]^{virt}) = e(\Gamma(\mathcal{V}_{\beta,n})^\vee).[\mathcal{M}_{0,n}(X,\beta)]^{virt} \text{ in } H^{Br}_*(\mathcal{M}_{0,n}(X,\beta); \Gamma^h(*))$$

for any choice of homology theories $\Gamma^h(*)$ as above. Here $e(\Gamma(\mathcal{V}_{\beta,n})^\vee)$ denotes an Euler class, which is defined as the term of appropriate weight and degree in $ch(\lambda_{-1}(\Gamma(\mathcal{V}_{\beta,n})))$.

Assuming that the complexes $\Gamma(*)$ and $\Gamma^h(*)$ extend to the smooth site of all algebraic stacks, we also obtain:

$$\Sigma_{i_*(\gamma)=\beta} i_{\gamma*}([\mathcal{M}_{0,n}(Y,\gamma)]^{virt}) = e(\Gamma(\mathcal{V}_{\beta,n})^\vee).[\mathcal{M}_{0,n}(X,\beta)]^{virt} \text{ in } H^{smt}_*(\mathcal{M}_{0,n}(X,\beta); \Gamma^h(*)).$$

Here $e(\Gamma(\mathcal{V}_{\beta,n})^\vee)$ denotes the usual Euler class in smooth cohomology.

We conclude by considering localization formulae for virtual structure sheaves and virtual fundamental classes.

Theorem 1.7. *Assume in addition to the hypotheses in 1.1 that the base scheme is an algebraically closed field, the stacks $\mathcal{S}$ and $\tilde{S}$ are provided with actions by a torus T, T' is a given sub-torus with the associated prime ideal in $R(T)$ being $\mathfrak{p}$. Moreover, we require that $\mathcal{T} = \mathcal{S}^{T'}$ and $\tilde{T} = (\tilde{S})^{T'}$. (Here the fixed point stacks are defined in [J-5].) We let the obstruction theory $F^\bullet$ be defined as $u^*(E^\bullet)^{T'}$. Then the class $\lambda_{-1}(K^0)\varepsilon\pi_0(K(\mathcal{T}, T))$ lifts to a class $\lambda_{-1}(K^0_{\mathcal{S}})\varepsilon\pi_0(G(\mathcal{S}, T))_{(\mathfrak{p})}$ and one obtains the formula:*

$$u_*(\mathcal{O}^{virt}_{\mathcal{T}}.\lambda_{-1}(K^{-1})) = \mathcal{O}^{virt}_{\mathcal{S}}.\lambda_{-1}(K^0_{\mathcal{S}}) \tag{1.1.5}$$

in $\pi_0(G(\mathcal{S}))_{\mathfrak{p}}$. Assuming that the complexes $\Gamma(*)$ and $\Gamma^h(*)$ extend to the smooth site of all algebraic stacks, this implies the formula

$$u_*([\mathcal{T}]^{virt}.e(K_1)) = [\mathcal{S}]^{virt}.e((K^0_{\mathcal{S}})^\vee) \tag{1.1.6}$$

in smooth T-equivariant homology of $\mathcal{S}$ localized at the prime ideal $\mathfrak{p}$.

Theorem 1.8. *Assume the hypotheses of the last theorem. Then one has a Gysin map $u_* : \pi_0(K_{\mathcal{T}}(\tilde{T}, \mathcal{O}^{virt}_{\tilde{\mathcal{T}}}, T))_{(\mathfrak{p})} \to \pi_0(K_{\mathcal{S}}(\tilde{S}, \mathcal{O}^{virt}_{\tilde{\mathcal{T}}}, \mathcal{T}))_{(\mathfrak{p})}$ defined where the relative K-groups above are the Grothendieck groups of the category of perfect complexes of modules over dg-stacks defined as in the appendix. This has the property that*

$$u_*(\mathcal{O}^{virt}_{\tilde{\mathcal{T}}} \underset{\mathcal{O}_{\tilde{\mathcal{T}}}}{\otimes} \lambda_{-1}(K^{-1})) = \mathcal{O}^{virt}_{\tilde{\mathcal{S}}} \underset{\mathcal{O}_{\tilde{\mathcal{S}}}}{\otimes} \lambda_{-1}(K^0_{\mathcal{S}})$$

where $K^0_{\mathcal{S}}$ is viewed as a class in $\pi_0(K_{\mathcal{S}}(\tilde{S}, T))_{(\mathfrak{p})}$. Consequently one obtains

$$u^*u_*(\mathcal{F}) = \mathcal{F} \otimes \lambda_{-1}(K^0) \otimes \lambda_{-1}(K^{-1})^{-1},\ \mathcal{F}\varepsilon\pi_0(K(\mathcal{T}, \mathcal{O}^{virt}_{\mathcal{T}}, T)).$$

Assuming the complexes $\Gamma(*)$ and $\Gamma^h(*)$ extend to the smooth site of all algebraic stacks, a pull-back is defined on smooth homology (under our hypothesis) and we obtain:

$$u_*([\mathcal{T}]^{virt}.e(K_1).e(K_0)^{-1}) = [\mathcal{S}]^{virt}$$

in $H^T_*(\mathcal{T}, \Gamma(*))_{(\mathfrak{p})}$. Here $H^T_*(\mathcal{S}, \Gamma(*))$ denotes the homology of the stack $[\mathcal{T}/T]$ computed on the smooth site with respect to the complex $\Gamma(*)$ and $\mathfrak{p}$ is the prime ideal in $R(T)$ corresponding to the sub-torus T'. Moreover, $K_0 = (K^0)^\vee$, $K_1 = (K^{-1})^\vee$ and $e(K_i)$ is the corresponding Euler class in in $H^*_T(\mathcal{T}, \Gamma(*))_{(\mathfrak{p})}$.

Remark 1.9. If we let the *Euler class of the virtual normal bundle* be defined by $e(K_1)^{-1}.e(K_0)$ we recover the main result in [GP] proven there by other means. Observe that the use of *dg*-stacks and Riemann-Roch simplifies the proof considerably. Moreover the formula in (1.1.6) seems to be not known before.

Acknowledgments. We would like to thank Dan Edidin, Bertrand Toen and Angelo Vistoli on several discussions over the years on algebraic stacks. As one can see a key role is played by the push-forward formula in Proposition 3.2 originally proved by Vistoli in the context of intersection theory on algebraic stacks: see [Vi-1]. The relevance of dg-stacks and the possibility of defining pushforward and other formulae for the virtual fundamental classes using Riemann-Roch theorems on stacks, became clear to the author at the MSRI program on algebraic stacks in 2001 and especially during many conversations with Bertrand Toen while they were both supported by the MSRI.

After this paper was written up, we learned from David Cox that an alternate solution of the conjecture of Cox, Katz and Lee appears in the recent paper [KKP]. However, as one can see, there are several important differences in the proofs. The most important of course is that we prove an analogue of this formula for virtual structure sheaves first as a corollary to our more general push-forward formulae in Theorem 4.9 and Theorem 1.4, making use of standard methods from K-theory and deformation to the normal cone. The conjectured formula of virtual fundamental classes then follows by applying our Riemann-Roch to the formula at the level of virtual structure sheaves. Another difference that seems worth mentioning is that our formula holds in all possible homology theories defined with respect to the complexes $\Gamma^h(*)$ satisfying the basic hypotheses in [J-5] section 3.

1.1.7. **Basic frame work.** We will adopt the terminology and conventions from [J-6] throughout the paper. For the sake of completeness we will recall these here. Let S denote an excellent Noetherian separated scheme which will serve as the base scheme. All objects we consider will be *locally finitely presented over S, and locally Noetherian.* In particular, all objects we consider are locally quasi-compact. However, our results are valid, for the most part only for objects that are finitely presented over the base scheme S or for disjoint unions of such objects. Since we consider mostly dg-stacks, G-theory and K-theory will always mean the theory associated to the dg-stack as in the appendix. i.e. If $\mathcal{S}$ is an algebraic stack provided with a dg-structure sheaf $\mathcal{A}$ and an action by a smooth group scheme G, we will let $\mathbf{G}(\mathcal{S}, \mathcal{A}, G)$ ($\mathbf{K}(\mathcal{S}, \mathcal{A}, G)$, respectively) denote the G-theory spectrum (the K-theory spectrum, respectively) of the category of coherent G-equivariant $\mathcal{A}$-modules on $\mathcal{S}$, (perfect G-equivariant $\mathcal{A}$-modules, respectively) as defined in Definition 5.2.

We will adopt the following conventions regarding moduli spaces. A *coarse moduli-space* for an algebraic stack $\mathcal{S}$ will be a *proper map* $p : \mathcal{S} \rightarrow \mathfrak{M}_{\mathcal{S}}$ (with $\mathfrak{M}_{\mathcal{S}}$ an algebraic space) which is a uniform categorical quotient and a uniform geometric quotient in the sense of [KM] 1.1 Theorem. Moreover, for purposes of Riemann-Roch, we will assume that p always has *finite cohomological dimension.* (Observe that this hypothesis is satisfied if the order of the residual gerbes are prime to the residue characteristics, for example in characteristic 0 for all Deligne-Mumford stacks. Observe also that the notion of coarse moduli space above may be a bit different from the notion adopted in [Vi-1].) It is shown in [KM] that if the stack $\mathcal{S}$ is Deligne-Mumford, of finite type over k and the obvious map $I_{\mathcal{S}} \rightarrow \mathcal{S}$ is finite, then a coarse moduli space exists with all of the above properties, except the map p may not be proper (i.e. finite). However, if $\mathcal{S}$ is also separated over k, then the map p will also be proper (i.e. finite). To see this observe (see [Vi-1]) that one may find an étale covering $\mathfrak{M}' \rightarrow \mathfrak{M}_{\mathcal{S}}$ so that the induced map $p' : \mathcal{S} \underset{\mathfrak{M}_{\mathcal{S}}}{\times} \mathfrak{M}' \rightarrow \mathfrak{M}'$ is finite. (In fact one may assume that the stack $\mathcal{S} \underset{\mathfrak{M}_{\mathcal{S}}}{\times} \mathfrak{M}'$ is the quotient stack associated a finite group action.) Therefore, in this case p itself is finite and a coarse moduli space in our sense exists.

Convention 1.10. Henceforth a stack will mean a DG-stack. DG-stacks whose associated underlying stack is of Deligne-Mumford type will be referred to as Deligne-Mumford DG-stacks. We will assume that all coarse-moduli spaces that we consider are quasi-projective schemes. In the presence of an action by a smooth affine group scheme, we will assume these are G-quasi-projective in the sense that they admit G-equivariant locally closed immersion into a projective space on which the group G acts linearly. Given a presheaf of spectra P, $P_{\mathbb{Q}}$ will denote its localization at $\mathbb{Q}$. (Observe that then $\pi_*(P_{\mathbb{Q}}) = \pi_*(P) \otimes \mathbb{Q}$.)

2. Virtual structure sheaves and virtual fundamental classes: definitions and basic properties

2.0.8. *Virtual structure sheaves and virtual fundamental classes.* Presently we will define virtual structure sheaves and virtual fundamental classes associated to perfect obstruction theories: *our approach using the Riemann-Roch makes it possible to define virtual fundamental classes even when global resolutions of coherent sheaves by vector bundles do not exist.*

Throughout this discussion we will fix a base object B which will be in general any *smooth* Artin stack of finite type over the given base scheme. (The base scheme may be assumed to be a field or a general Noetherian excellent scheme of finite type over a field.) Let $b = dim(B)$. All objects and morphisms we consider in this section will be over B and therefore we will often omit the adjective *relative*. We begin by recalling briefly the definition of the *intrinsic normal cone* from [BF] section 3 or [CK] pp. 178-179. *Convention: in what follows we will ignore the fact the base is a smooth stack and not a field. Since this stack is smooth, all this does is to necessitate modifying the dimensions by adding b to them.*

First we proceed to define *virtual structure sheaves* associated to perfect obstruction theories, following [BF]. Let $\mathcal{S}$ denote a Deligne-Mumford stack with $u : U \rightarrow \mathcal{S}$ an atlas and let $i : U \rightarrow M$ denote a closed immersion into a smooth scheme. Let $C_{U/M}$ ($N_{U/M}$) denote the normal cone (normal bundle, respectively) associated to the closed immersion i. (Recall that if $\mathcal{I}$ denotes the sheaf of ideals associated to the closed immersion i, $C_{U/M} = Spec \underset{n}{\oplus} \mathcal{I}^n/\mathcal{I}^{n+1}$ and $N_{U/M} = SpecSym(\mathcal{I}/\mathcal{I}^2)$. Now $[C_{U/M}/i^*(T_M)]$ ($[N_{U/M}/i^*(T_M)]$) denotes the *intrinsic normal cone* denoted $\mathcal{C}_\mathcal{S}$ (the *intrinsic abelian normal cone* denoted $\mathcal{N}_\mathcal{S}$, respectively). In case the algebraic stack $\mathcal{S}$ is provided with the action of a smooth group scheme G, we will assume that this action lifts to an action on the intrinsic normal cone and the intrinsic abelian normal cone. This hypothesis is satisfied, for example, if the stack $\mathcal{S}$ admits a closed immersion into a smooth Deligne-Mumford stack onto which the action of G extends making the above closed immersion G-equivariant.

Let $E^\bullet$ denote a complex of $\mathcal{O}_\mathcal{S}$-modules so that it is trivial in positive degrees and whose cohomology sheaves in degrees 0 and -1 are coherent. Let $L^\bullet_\mathcal{S}$ denote the *cotangent complex* of the stack $\mathcal{S}$ over the base B. A morphism $\phi : E^\bullet \rightarrow L^\bullet_\mathcal{S}$ in the derived category of complexes of $\mathcal{O}_\mathcal{S}$-modules is called an *obstruction theory* if ϕ induces an isomorphism (surjection) on taking the cohomology sheaves in degree 0 (in degree -1, respectively). In case $\mathcal{S}$ is provided with the action of a smooth group scheme G, we will assume that $E^\bullet$ is a complex of G-equivariant sheaves of $\mathcal{O}_\mathcal{S}$-modules and that the homomorphism ϕ is G-equivariant. (Observe that, in this case, the cotangent complex $L^\bullet_\mathcal{S}$ is automatically a complex of G-equivariant $\mathcal{O}_\mathcal{S}$-modules.) We call the obstruction theory $E^\bullet$ *perfect* if $E^\bullet$ is of perfect amplitude contained in $[-1, 0]$ (i.e. locally on the étale site of the stack, it is quasi-isomorphic to a complex of vector bundles concentrated in degrees 0 and -1). In this case, one may define the *virtual dimension* of $\mathcal{S}$ with respect to the obstruction theory $E^\bullet$ as $rank(E^0) - rank(E^{-1}) + b$. Moreover, in this case, we let $\mathcal{E}_\mathcal{S} = h^1/h^0(E^\bullet) = [\mathcal{E}_1/\mathcal{E}_0]$ where $\mathcal{E}_i = SpecSym(E^{-i})$. We will denote this by $C(E^{-i})$.

Now the morphism ϕ defines a closed immersion $\phi^\vee : \mathcal{N}_\mathcal{S} \rightarrow \mathcal{E}_\mathcal{S}$. Composing with the closed immersion $\mathcal{C}_\mathcal{S} \rightarrow \mathcal{N}_\mathcal{S}$ one observes that $\mathcal{C}_\mathcal{S}$ is a closed cone substack of $\mathcal{E}_\mathcal{S}$. Let the corresponding closed immersion be denoted $i_{\mathcal{C}_\mathcal{S}}$. We let $C(E^\bullet)$ be defined by the cartesian square:

$$\begin{array}{ccc} C(E^\bullet) & \xrightarrow{i_{C(E^\bullet)}} & \mathcal{E}_1 \\ \downarrow & & \downarrow \\ \mathcal{C}_\mathcal{S} & \xrightarrow{i_{\mathcal{C}_\mathcal{S}}} & \mathcal{E}_\mathcal{S} \end{array} \tag{2.0.9}$$

In view of our hypotheses, $C(E^\bullet)$ has an induced action by the smooth group scheme G in the G-equivariant situation.

Definition 2.1. (Virtual structure sheaf) Let $\mathcal{E}_1 = C(E^{-1})$ and let $\mathcal{O}_{\mathcal{E}_1} : \mathcal{S} \rightarrow \mathcal{E}_1$ denote the vertex of the cone stack $\mathcal{E}_1$. We let $\mathcal{O}^{virt}_\mathcal{S} = L0^*_{\mathcal{E}_1}(\mathcal{O}_{C(E^\bullet)}) = \mathcal{O}_\mathcal{S} \underset{0^{-1}_{\mathcal{E}_1}(\mathcal{O}_{\mathcal{E}_1})}{\overset{L}{\otimes}} 0^{-1}_{\mathcal{E}_1}(\mathcal{O}_{C(E^\bullet)})$ and call it the *virtual structure sheaf* of the stack $\mathcal{S}$. (Observe that in the G-equivariant case this defines a complex of G-equivariant $\mathcal{O}_\mathcal{S}$-modules.)

One may now observe that $(\mathcal{S}, \mathcal{O}^{virt}_\mathcal{S})$ is a DG-stack in the sense of the appendix as follows. Recall that the sheaf $\mathcal{O}_{C(E^\bullet)}$ is defined by a coherent sheaf of ideals in $\mathcal{O}_{\mathcal{E}_1}$; locally on the étale site of the stack $\mathcal{S}$, one may find a resolution of $\mathcal{O}_{C(E^\bullet)}$ by a complex of the form $\mathcal{O}_{\mathcal{E}_1} \leftarrow P^{-1} \leftarrow \cdots P^{-n} \leftarrow \cdots$ with each P^{-i} a locally free coherent sheaf on $\mathcal{E}_1$. Therefore, on applying $L0_{\mathcal{S}}^*$ to the above complex (where $0_{\mathcal{S}}$ is the zero section $\mathcal{S} \rightarrow \mathcal{E}_S$), one gets a complex of locally free coherent $\mathcal{O}_{\mathcal{S}}$-modules, again locally on the étale site. Therefore the cohomology sheaves of $\mathcal{O}_{\mathcal{S}}^{virt}$ are all coherent $\mathcal{O}_{\mathcal{S}}$-modules. Proposition 2.2 below shows that $\mathcal{H}^i(\mathcal{O}_{\mathcal{S}}^{virt}) = 0$ for $i << 0$. Making use of the hypothesis that the stack is Noetherian, one may now replace $\mathcal{O}_{\mathcal{S}}^{virt}$ upto quasi-isomorphism by a bounded complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules. Therefore the hypotheses in the Definition 5.1 are satisfied. We will denote $(\mathcal{S}, \mathcal{O}_{\mathcal{S}}^{virt})$ for simplicity by $\mathcal{S}^{virt}$.

Often in the literature, one uses a Gysin map $0_{\mathcal{E}_1}^!$ in the place of $L0_{\mathcal{E}_1}^*$. Therefore, we next proceed to define such *Gysin maps at the level of G-theory of algebraic stacks* and show that one could use it in the place of $L0_{\mathcal{E}_1}^*$. (Further properties of Gysin maps are discussed in the next section.)

Consider a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ \downarrow g & & \downarrow f \\ Y' & \xrightarrow{y} & Y \end{array} \tag{2.0.10}$$

of Deligne-Mumford stacks where y is a *regular closed immersion* of algebraic stacks. We may assume all the stacks are provided with the action of a smooth group scheme G and that all the maps above are G-equivariant. We will assume that these are all non-dg stacks, or stacks in the usual sense. We will now define the refined *Gysin-map* (or often what will be simply called the Gysin map)

$$y^! : G(X, G) \rightarrow G(X', G) \tag{2.0.11}$$

Since y is assumed to be a regular immersion, it follows that if $\mathcal{O}_{Y'}$ is the structure sheaf of Y', $y_*(\mathcal{O}'_Y) \varepsilon \pi_0(K_{Y'}(Y, G))$. (Recall $K_{Y'}(Y, G)$ is the Waldhausen K-theory spectrum of perfect complexes on Y with supports in Y'.) Now pull-back of this class by f defines the class $f^*(y_*(\mathcal{O}_{Y'})) \varepsilon \pi_0(K_{X'}(X, G))$. Next observe the natural pairing $\circ : \pi_* K_{X'}(X, G) \otimes \pi_* G(X, G) \rightarrow \pi_* G_{X'}(X, G) \overset{\simeq}{\rightarrow} \pi_* G(X', G)$. Therefore, we define for any $F \varepsilon \pi_* G(X, G)$, $y^!(F) =$ the class of $F \circ f^*(y_*(\mathcal{O}_{Y'}))$ in $\pi_* G(X', \mathcal{O}_{X'})$. (In case f and g are the identity maps, one may verify, that $y^!(F)$ identifies with $y^*(F)$) as classes in $\pi_* G(Y', G)$.) *In case g is a closed immersion, we will often identify $y^!(F)$ with $g_*(y^!(F))$, i.e. the image of the class $y^!(F)$ defined above in $\pi_* G(Y', G)$.*

In the above case, one may define a refined Gysin map

$$y^! : D_-(Mod(X, G)) \rightarrow D_{-,X'}(Mod(X, G)) \tag{2.0.12}$$

where $Mod(X, G)$ ($Mod(X', G)$) denotes the category of G-equivariant coherent $\mathcal{O}_X$ ($\mathcal{O}_{X'}$) modules. $D_-(Mod(X, G))$ ($D_{-,X'}(Mod(X, G))$) will denote the derived category of complexes in $Mod(X, G)$ that are bounded above (complexes in $Mod(X, G)$ that are bounded above and whose cohomology sheaves have support in X', respectively). We let $y^!(M) = M \underset{\mathcal{O}_X}{\overset{L}{\otimes}} Lf^*(y_*(\mathcal{O}_{Y'}))$.

Proposition 2.2. *Assume the situation in (2.0.9). Then $\mathcal{O}_{\mathcal{S}}^{virt} \simeq 0_{\mathcal{E}_1}^!(\mathcal{O}_{C(E^\bullet)})$ as $\mathcal{O}_{\mathcal{S}}$-modules with the natural $\mathcal{O}_{\mathcal{S}}$-module structure on $\mathcal{O}_{\mathcal{S}}^{virt}$. In particular, both $\mathcal{O}_{\mathcal{S}}^{virt}$ and $0_{\mathcal{E}_1}^!(\mathcal{O}_{C(E^\bullet)})$ define the same class in $\pi_0(G(\mathcal{S}, \mathcal{O}_{\mathcal{S}}))$.*

Proof. Observe that $0_{\mathcal{E}_1,*}(L0_{\mathcal{E}_1}^*(\mathcal{O}_{C(E^\bullet)})) = 0_{\mathcal{E}_1,*}(\mathcal{O}_{\mathcal{S}}) \underset{\mathcal{O}_{\mathcal{E}_1}}{\otimes} P^\bullet$ while $0_{\mathcal{E}_1}^!(\mathcal{O}_{C(E^\bullet)}) = Q^\bullet \underset{\mathcal{O}_{\mathcal{E}_1}}{\otimes} \mathcal{O}_{C(E^\bullet)}$ where $P^\bullet \rightarrow \mathcal{O}_{C(E^\bullet)}$ and $Q^\bullet \rightarrow 0_{\mathcal{E}_1,*}(\mathcal{O}_{\mathcal{S}})$ are resolutions by complexes of locally free coherent $\mathcal{O}_{\mathcal{E}_1}$-modules. Since $\mathcal{O}_{\mathcal{E}_1}$ is a sheaf of commutative rings, it is clear that the two complexes $0_{\mathcal{E}_1,*}(L0_{\mathcal{E}_1}^*(\mathcal{O}_{C(E^\bullet)}))$ and $0_{\mathcal{E}_1}^!(\mathcal{O}_{C(E^\bullet)})$ are quasi-isomorphic as $\mathcal{O}_{\mathcal{E}_1}$-modules. The $\mathcal{O}_{\mathcal{S}}$-module structures on the two complexes $0_{\mathcal{E}_1,*}(L0_{\mathcal{E}_1}^*(\mathcal{O}_{C(E^\bullet)}))$ and $0_{\mathcal{E}_1}^!(\mathcal{O}_{C(E^\bullet)})$ are obtained now by using the map $0_{\mathcal{E}_1,*}(\mathcal{O}_{\mathcal{S}}) \rightarrow \mathcal{O}_{\mathcal{E}_1}$. However, the $\mathcal{O}_{\mathcal{E}_1}$-module structure on $0_{\mathcal{E}_1,*}(L0_{\mathcal{E}_1}^*(\mathcal{O}_{C(E^\bullet)}))$ is induced from the obvious $0_{\mathcal{E}_1,*}(\mathcal{O}_{\mathcal{S}})$-module structure using the map $\mathcal{O}_{\mathcal{E}_1} \rightarrow 0_{\mathcal{E}_1,*}(\mathcal{O}_{\mathcal{S}})$. Since the composition of the two maps $0_{\mathcal{E}_1,*}(\mathcal{O}_{\mathcal{S}}) \rightarrow \mathcal{O}_{\mathcal{E}_1} \rightarrow 0_{\mathcal{E}_1,*}(\mathcal{O}_{\mathcal{S}})$ is the identity, it follows that the quasi-isomorphism $0_{\mathcal{E}_1,*}(L0_{\mathcal{E}_1}^*(\mathcal{O}_{C(E^\bullet)})) \simeq 0_{\mathcal{E}_1}^!(\mathcal{O}_{C(E^\bullet)})$ is one of $0_{\mathcal{E}_1,*}(\mathcal{O}_{\mathcal{S}})$-modules where the $\mathcal{O}_{\mathcal{S}}$-module structure on $L0_{\mathcal{E}_1}^*(\mathcal{O}_{C(E^\bullet)})$ is the obvious one. $\square$

We let $G(\mathcal{S}^{virt}, G)$ denote the G-equivariant G-theory of the DG-stack $\mathcal{S}^{virt}$. Let $\mathbb{H}^*_{Br}(\mathcal{S}^{virt}, G; \Gamma^h(*))$ denote a Bredon-style G-equivariant homology theory associated to the DG-stack $\mathcal{S}^{virt}$. Let $\tau = \tau^G_{\mathcal{S}^{virt}} : \pi_*(G(\mathcal{S}^{virt}, G)) \to H^*_{Br}(\mathcal{S}^{virt}, G; \Gamma^h(*))$ denote the Riemann-Roch transformation considered in [J-6] section 2.

2.0.13. It will be important to use the relationship between the Riemann-Roch transformation and the local Chern character to be able to define the *virtual fundamental classes.* We will do this presently. Let $\mathcal{S}$ denote a Deligne-Mumford stack with coarse moduli space $\mathfrak{M}$ and let $p : \mathcal{S} \to \mathfrak{M}$ denote the obvious map. Let $\bar{F}$ denote a perfect complex on $\mathfrak{M}$ and let $F = p^*(\bar{F})$ denote its inverse image on $\mathcal{S}$. Then $\tau(F)$ corresponds to the map that sends a perfect complex $\mathcal{E}$ on the stack $\mathcal{S}$ to $\tau_{\mathfrak{M}}(p_*(p^*(\bar{F}) \otimes \mathcal{E}) = \tau_{\mathfrak{M}}(\bar{F} \otimes p_*(\mathcal{E})) = \tau_{\mathfrak{M}}(\bar{F}).ch^{\tilde{\mathfrak{M}}|\mathfrak{M}}(i_*p_*(\mathcal{E}))$, where $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ is a closed immersion of $\mathfrak{M}$ into a smooth scheme, and $ch^{\tilde{\mathfrak{M}}|\mathfrak{M}}$ is the local Chern character. In particular, if $\mathcal{E}$ has supports in a closed algebraic sub-stack $\mathcal{S}_0$ of $\mathcal{S}$ with pure codimension c, $i_*p_*(\mathcal{E})$ has supports in a closed sub-scheme of $\mathfrak{M}$ of pure codimension c. Therefore, in this case, (in view of our cohomological semi-purity hypothesis - see [J-5] section 3)) $ch^{\tilde{\mathfrak{M}}|\mathfrak{M}}(i_*p_*(\mathcal{E}))(j)$ is trivial in $H^i_{\mathfrak{M}}(\tilde{\mathfrak{M}}; \Gamma(j))$ for $j < c$. If, in addition, $ch^{\tilde{\mathfrak{M}}|\mathfrak{M}}(i_*p_*(\mathcal{E}))(c) \neq 0$ as well, it follows that, in this case the *non-trivial term of highest weight* in $\tau(F)$ is in $d - c$, where d= the weight of the non trivial term in $\tau_{\mathfrak{M}}(\bar{F})$ of highest weight. Moreover if the non-trivial term in $\tau_{\mathfrak{M}}(\bar{F})$ of highest weight is in weight d and degree $2d$, the non-trivial term of highest weight in $\tau(F)$ is in weight $d - c$ and degree $2d - 2c$.

Definition 2.3. Let d denote the virtual dimension of the stack $\mathcal{S}$ with respect to the given obstruction theory. We define *the virtual fundamental class* of the stack $\mathcal{S}$ in Bredon homology to be $\tau(\mathcal{O}_{\mathcal{S}^{virt}})_{2d}(d)$, i.e. the part of $\tau(\mathcal{O}_{\mathcal{S}^{virt}})$ in degree $2d$ and weight d. This will be denoted $[\mathcal{S}]^{virt}_{Br}$. If $p : \mathcal{S} \to \mathfrak{M}$ denotes the obvious map from the stack to its moduli space, we will also let $[\mathcal{S}]^{virt}_{Br}$ denote $p_*(\tau(\mathcal{O}_{\mathcal{S}^{virt}})_{2d}(d)) \varepsilon H^{Br}_{2d}(\mathfrak{M}, \Gamma(d))$.

The term of highest weight i and degree $2i$ in $\tau(\mathcal{O}^{virt}_{\mathcal{S}})$ that is non-trivial will be called the *the leading term of* $\tau(\mathcal{O}^{virt}_{\mathcal{S}})$. We proceed to show that, when the the stack $\mathcal{S}$ is flat over its moduli space (for example, if the stack is a gerbe), the leading term of $\tau(\mathcal{O}^{virt}_{\mathcal{S}})$ is of weight d and degree $2d$, where d is the virtual dimension defined by the obstruction theory. For this, first observe from [J-6] Definition 2.13, that the Riemann-Roch transformation localizes on $\tilde{\mathfrak{M}}_{et}$ and hence on $\mathfrak{M}_{et}$. Clearly the virtual structure sheaf localizes on $\mathcal{S}_{et}$. Therefore, it suffices to prove this for the stack $\mathcal{S}_1$ which is defined as the pull-back $\mathcal{S} \underset{\mathfrak{M}}{\times} \mathfrak{M}_1$ where $\mathfrak{M}_1 \to \mathfrak{M}$ is a finite étale map. Therefore, we reduce to the situation where the stack $\mathcal{S}$ is the quotient stack associated to a finite étale constant group-scheme action, i.e. $\mathcal{S} = [X/G]$ and the map $\bar{\pi} : X \to X/G = \mathfrak{M}$ is finite surjective.

Next one may pull-back all objects defined over $\mathfrak{M}$ to those defined over X. i.e. We let $\bar{\mathcal{E}}_i = \bar{\pi}^*(\mathcal{E}_i)$, $\bar{C}(E^\bullet) = \bar{\pi}^*(C(E^\bullet))$. Now $\bar{C}(E^\bullet)$ is a closed algebraic sub-stack of $\bar{\mathcal{E}}_1$ and hence defines a class $[\bar{C}(E^\bullet)] \varepsilon H^{Br}_*(\bar{\mathcal{E}}_1, \Gamma(*)) \cong \mathbb{H}^{et}_*(\bar{\mathcal{E}}_1, \Gamma(*))$.

Recall from [J-5] Theorem 1.2 that there is defined a map $\phi : H^{Br}_*(\mathcal{S}, \Gamma(*)) \to H^{smt}_*(\mathcal{S}, \Gamma(*))$ (compatible with push-forwards by proper representable morphisms) under the assumption that the complexes $\{\Gamma(i)|i\}$ and $\{\Gamma^h(i)|i\}$ extend to the big smooth site of all algebraic stacks.

Theorem 2.4. *Assume in addition to the above situation that the base stack B is a scheme (i.e. the obstruction theory is absolute) and that the stack $\mathcal{S}$ is flat over its coarse moduli space $\mathfrak{M}$.*

(i) Then $\tau(\mathcal{O}_{\mathcal{S}^{virt}})(i)_{2i} = 0$ for all $i > d$, where d = the virtual dimension. Moreover if $[\bar{C}(E^\bullet)] \varepsilon H^{Br}_*(\bar{\mathcal{E}}_1, \Gamma(*)) \cong \mathbb{H}^{et}_*(\bar{\mathcal{E}}_1, \Gamma(*))$ is nonzero, the leading term of $\tau(\mathcal{O}_{\mathcal{S}^{virt}})$ is in weight = d and degree $2d$, where d = the virtual dimension.

(ii) Let $[\mathcal{S}]^{virt} = [C(E^\bullet)] \bullet Ch(pr^*(\lambda_{-1}(E^{-1}))$ where $[C(E^\bullet)]_{Br}$ = the leading term of $\tau(\mathcal{O}_{C(E^\bullet)})$ with values in Bredon homology, $[C(E^\bullet)]$ = its image in smooth homology and $Ch(pr^*(\lambda_{-1}(E^{-1}))$ is the Chern character with values in smooth cohomology, where $pr : \mathcal{E}_1 \to \mathcal{S}$ is the obvious projection. Moreover $\bullet$ denotes the obvious pairing between smooth homology and cohomology. Then $[\mathcal{S}]^{virt}$ = the term of $\phi_*(\tau(\mathcal{O}^{virt}_{\mathcal{S}}))$ in smooth homology with highest weight and degree 2∗the weight.

Proof. For simplicity we will only consider the non-equivariant case: our arguments readily extend to the equivariant case. Let $\mathfrak{M}$ denote the coarse moduli space of the stack $\mathcal{S}$.

Step 1. Next we show that it is possible to reduce to the case where $\mathfrak{M}$ has been replaced by X, i.e. after pulling back all objects defined over $\mathcal{S}$ to those defined over $\mathcal{S}' = \mathcal{S} \underset{\mathfrak{M}}{\times} X$ by the obvious map $\pi : \mathcal{S}' \to \mathcal{S}$. We will let $\mathcal{S}'$ with the obvious induced dg-structure-sheaf i.e. $\pi^*(\mathcal{O}^{virt}_{\mathcal{S}})$, though this is not defined by an obstruction theory. To see that this in fact defines a class in $\pi_0 G(\mathcal{S}')$ we may argue as follows. First one applies the pull-back $\mathcal{S}' \underset{\mathcal{S}}{\times} -$ to the square in (2.0.9). Observe that $\mathcal{S}' \underset{\mathcal{S}}{\times} \mathcal{E}_1 = C(\pi^*(E^{-1}))$. Now $C(\pi^*(E^\bullet)) = \mathcal{S}' \underset{\mathcal{S}}{\times} C(E^\bullet)$. Therefore

$$\pi^*(\mathcal{O}_{\mathcal{S}}^{virt}) = \mathcal{O}_{C(\pi^*(E^\bullet))}.\Lambda_{-1}(\pi^*(pr^{-1}E^{-1})) \tag{2.0.14}$$

and hence $\pi^*(\mathcal{O}_{\mathcal{S}}^{virt})$ defines a class in $\pi_0(G(\mathcal{S}'))$. Next observe that, by the projection formula, one obtains:

$$\pi_*(\pi^*(\mathcal{O}_{\mathcal{S}}^{virt})) = \pi_*(\mathcal{O}_{\mathcal{S}'}) \otimes \mathcal{O}_{\mathcal{S}}^{virt} \tag{2.0.15}$$

The projection $\pi : \mathcal{S}' \to \mathcal{S}$ factors as the composition of the two maps $\mathcal{S}' = \mathcal{S} \underset{X/G}{\times} X \overset{\pi_1}{\to} \mathcal{S} \underset{X/G}{\times} \mathcal{S} \overset{\pi_2}{\to} \mathcal{S} \underset{X/G}{\times} X/G \cong \mathcal{S}$. The first map π_1 is finite étale since it is obtained by base-change from the map $X \to [X/G]$. Let n denote the degree of this map. Then $\pi_{1*}(\mathcal{O}_{\mathcal{S}'}) = \pi_{1*}\pi_1^*(\mathcal{O}_{\mathcal{S} \underset{X/G}{\times} \mathcal{S}}) = n\mathcal{O}_{\mathcal{S} \underset{X/G}{\times} \mathcal{S}}$.

Observe next that for the finite map $\bar{\pi}_2 : \mathcal{S} = [X/G] \to X/G$, $\bar{\pi}_{2*}(\mathcal{O}_{\mathcal{S}}) = \mathcal{O}_{X/G}$. The map $\pi_2 : \mathcal{S} \underset{X/G}{\times} \mathcal{S} \to \mathcal{S}$ is induced by *flat base-change* from the map $\bar{\pi}_2$. Therefore $\pi_{2*}(\mathcal{O}_{\mathcal{S} \underset{X/G}{\times} \mathcal{S}}) = \mathcal{O}_{\mathcal{S}}$. It follows, therefore, that

$$\pi_*(\mathcal{O}_{\mathcal{S}'}) = n\mathcal{O}_{\mathcal{S}} \quad and \quad hence$$

$$\pi_*(\pi^*(\mathcal{O}_{\mathcal{S}}^{virt})) = n\mathcal{O}_{\mathcal{S}}^{virt} \tag{2.0.16}$$

Let $p : \mathcal{S} \to \mathfrak{M}$ and $p' : \mathcal{S}' \to X$ denote the obvious maps and let $\mathfrak{M} \to \tilde{\mathfrak{M}}$ and $\mathfrak{M}' \to \tilde{\mathfrak{M}}'$ denote closed immersions into smooth schemes. Recall that, by the Riemann-Roch theorem proved in section 3, $p_*(\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}}^{virt})) = \tau_{\mathfrak{M}}(p_*(\mathcal{O}_{\mathcal{S}}^{virt}))$ and $p'_*\tau_{\mathcal{S}'}(\pi^*(\mathcal{O}_{\mathcal{S}}^{virt})) = \tau_X(p'_*(\pi^*(\mathcal{O}_{\mathcal{S}}^{virt})))$, where $\tau_{\mathfrak{M}}$ (τ_X) denotes the Riemann-Roch transformation into the homology of the moduli space $\mathfrak{M}$ (X, respectively). Now $\bar{\pi}_*p'_*\tau_{\mathcal{S}'}(\pi^*(\mathcal{O}_{\mathcal{S}}^{virt})) = \bar{\pi}_*\tau_X(p'_*(\pi^*(\mathcal{O}_{\mathcal{S}}^{virt}))) = \tau_{\mathfrak{M}}(\bar{\pi}_*p'_*(\pi^*(\mathcal{O}_{\mathcal{S}}^{virt}))) = \tau_{\mathfrak{M}}(p_*\pi_*(\pi^*(\mathcal{O}_{\mathcal{S}}^{virt})))$, where $\bar{\pi} : X \to \mathfrak{M}$ is the induced map. This agrees with $n\tau_{\mathfrak{M}}(p_*(\mathcal{O}_{\mathcal{S}}^{virt})) = np_*(\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}}^{virt}))$ by (2.0.16). Therefore, the non-trivial term of highest weight occurs in the same weight in both of the classes $\bar{\pi}_*p'_*\tau_{\mathcal{S}'}(\pi^*(\mathcal{O}_{\mathcal{S}}^{virt}))$ and $p_*(\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}}^{virt}))$.

Step 2. From now onwards, we will let $\mathfrak{M}$ itself denote X, $\mathcal{S}$ denote $\mathcal{S}'$ and $\mathcal{O}_{\mathcal{S}}^{virt}$ denote $\pi^*(\mathcal{O}_{\mathcal{S}}^{virt})$ defined above. Observe that now the vector bundles $\mathcal{E}_1$ and $\mathcal{E}_0$ (the cone $C(E^\bullet)$) descend to vector bundles $\bar{\mathcal{E}}_1$ and $\bar{\mathcal{E}}_0$ (a cone $C(\bar{E}^\bullet)$ defined on the coarse moduli space $\mathfrak{M}$ with $\bar{\mathcal{E}}_i$ being a coarse-moduli space for $\mathcal{E}_i$). (Observe that, the $\mathcal{E}_i$ no longer define an obstruction theory, which is fine for the proof: all we require is that the virtual structure sheaf $\pi^*(\mathcal{O}_{\mathcal{S}}^{virt})$ (i.e. what we denote $\mathcal{O}_{\mathcal{S}}^{virt}$ henceforth) be defined as in (2.0.14).

Next consider the class $[\mathcal{O}_{C(E^\bullet)}].pr^*(\Lambda_{-1}(E^{-1}))$ in $\pi_0(G_{\mathcal{S}}(\mathcal{E}_1)) \cong \pi_0(G(\mathcal{S}))$ where $pr : \mathcal{E}_1 \to \mathcal{S}$ is the obvious projection. Observe that $\Lambda_{-1}(E^{-1})$ is the Koszul complex associated to the zero section imbedding $0 : \mathcal{S} \to \mathcal{E}_1$. The hypothesis that the bundles $\mathcal{E}_1$ and $\mathcal{E}_0$ descend to vector bundles $\bar{\mathcal{E}}_1$ and $\bar{\mathcal{E}}_0$ on the moduli space $\mathfrak{M}$ shows that there exist finite flat maps $p_i : \mathcal{E}_i \to \bar{\mathcal{E}}_i$, $i = 0$, $i = 1$. Now $Ch(\Lambda_{-1}(E^{-1}))$ in smooth cohomology is the pull-back of the class $Ch(\Lambda_{-1}(\bar{E}^{-1}))$ from Bredon cohomology of $\bar{\mathcal{E}}_1$ which identifies with its étale cohomology. Observe also that the class $[C(E^\bullet)]$ is the pull-back of the class $[C(\bar{E}^\bullet)] \varepsilon H_*(\bar{\mathcal{E}}_1; \Gamma(*))$. (Moreover the map to the smooth hypercohomology of $\mathcal{E}_1$ from the hypercohomology of $\bar{\mathcal{E}}_1$ is compatible with the obvious pairings: see [J-5] Theorem 1.2.) Therefore, it suffices to prove (ii) with $Ch(\overline{pr}^*\Lambda_{-1}(\bar{E}^{-1}))$ ($[C(\bar{E}^\bullet)]$) in the place of $Ch(pr^*\Lambda_{-1}(E^{-1}))$ ($[C(E^\bullet)]$, respectively), where $\overline{pr} : \bar{\mathcal{E}}_1 \to \mathfrak{M}$ is the obvious projection.

Step 3. When we view $\mathcal{O}_{\mathcal{S}}^{virt}$ as a class in $\pi_0(G_{\mathcal{S}}(\mathcal{E}_1))$, we may apply p_{1*} to it to get a class in $\pi_0(G(\bar{\mathcal{E}}_1)) \cong \pi_0(G_{\bar{\mathcal{E}}_1}(\tilde{\mathcal{E}}_1))$ and then apply the Riemann-Roch transformation $\tau_{\bar{\mathcal{E}}_1}$: this defines a class in $H_*^{Br}(\bar{\mathcal{E}}_1; \Gamma(*))$. (Here $\bar{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_1$ is the closed immersion into a smooth scheme.) We proceed to determine this class.

We consider:

$$\begin{aligned} p_{1*}(\tau_{\mathcal{E}_1}(\mathcal{O}_{\mathcal{S}^{virt}})) &= \tau_{\bar{\mathcal{E}}_1}(p_{1*}(\mathcal{O}_{C(E^\bullet)}.pr^*(\Lambda_{-1}(E^{-1})))) \\ &= \tau_{\bar{\mathcal{E}}_1}(p_{1*}(p_1^*(\mathcal{O}_{C(\bar{E}^\bullet)}.\overline{pr}^*(\Lambda_{-1}(\bar{E}^{-1}))))) \\ &= \tau_{\bar{\mathcal{E}}_1}(p_{1*}\mathcal{O}_{C(E^\bullet)}).Ch(\overline{pr}^*(\Lambda_{-1}(\bar{E}^{-1}))) \\ &= p_{1*}(\tau_{\mathcal{E}_1}(\mathcal{O}_{C(E^\bullet)})).Ch(\overline{pr}^*(\Lambda_{-1}(\bar{E}^{-1}))) \end{aligned} \tag{2.0.17}$$

where Ch denotes the local Chern character on $\tilde{\mathcal{E}}_1$ with supports in $\bar{\mathcal{E}}_1$ and $\bar{pr} : \bar{\mathcal{E}}_1 \to \mathfrak{M}$ is the obvious projection. (The identifications $p_{1*}(\tau_{\mathcal{E}_1}(\mathcal{O}_{\mathcal{S}^{virt}})) = \tau_{\bar{\mathcal{E}}_1}(p_{1*}(\mathcal{O}_{C(E^\bullet)}.pr^*(\Lambda_{-1}(E^{-1}))))$ and $\tau_{\bar{\mathcal{E}}_1}(p_{1*}\mathcal{O}_{C(E^\bullet)}) = p_{1*}(\tau_{\mathcal{E}_1}(\mathcal{O}_{C(E^\bullet)}))$ follow readily from the definition of the Riemann-Roch transformation. See the basic example in [J-6] Example

3.8. Observe also that $\tau_{\bar{\mathcal{E}}_1}$ is the usual Riemann-Roch transformation associated to the scheme $\bar{\mathcal{E}}_1$ (imbedded in $\tilde{\mathcal{E}}_1$) and that $C(\bar{E}^\bullet)$ is a sub-scheme of the cone $\bar{\mathcal{E}}_1$. $\tau_{\mathcal{E}_1}$ is the Riemann-Roch transformation defined in [J-6] section 2 at the level of stacks.)

Since $\overline{pr}^*(\Lambda_{-1}(\bar{E}^{-1}))$ has supports contained in $\mathfrak{M}$, it is clear $Ch(\overline{pr}^*(\Lambda_{-1}(\bar{E}^{-1})))(i) = 0$ for all $i < c$ where $c =$ the codimension of $\mathfrak{M}$ in $\bar{\mathcal{E}}_1$ (= the codimension of $\mathcal{S}$ in $\mathcal{E}_1$.) It follows therefore that the nonzero term of highest weight in $p_{1*}(\tau_{\mathcal{E}_1}(\mathcal{O}_{\mathcal{S}^{virt}})$ occurs in weight $n-c$ and degree $2n-2c$ where $n=$ the dimension of $C(E^\bullet)$. (See 2.0.13.) From our discussion below (see 4.0.21) one may identify the dimension of $C(\bar{E}^\bullet)$ with $rank(E_0)+b$, where $b =$ the dimension of the base stack B.

Therefore, now it suffices to identify $p_{1*}\tilde{\tau}_{\mathcal{E}_1}(\mathcal{O}_{\mathcal{S}^{virt}})$ with $p_*\tau(\mathcal{O}_{\mathcal{S}^{virt}})$. Observe that in the latter $\mathcal{O}_{\mathcal{S}}^{virt}$ is viewed as a class in $\pi_0(G(\mathcal{S}))$ whereas in the former $\mathcal{O}_{\mathcal{S}}^{virt}$ is viewed as a class in $\pi_0(G_{\mathcal{S}}(\mathcal{E}_1))$ (i.e. actually as $0_*\mathcal{O}_{\mathcal{S}}^{virt}$, where $0 : \mathcal{S} \to \mathcal{E}_1$ is the zero section). Moreover, for the latter we apply $\tau_{\mathfrak{M}}$ to $p_*(\mathcal{O}_{\mathcal{S}}^{virt})$, whereas for the former we apply $\tau_{\mathcal{E}_1}$ to $p_{1*}(\mathcal{O}_{\mathcal{S}}^{virt})$. Therefore, a straightforward application of the usual Riemann-Roch theorem for the closed immersions $\bar{0} : \mathfrak{M} \to \bar{\mathcal{E}}_1$ and $0 : \mathcal{S} \to \mathcal{E}_1$ shows that one may in fact identify $p_{1*}\tilde{\tau}_{\mathcal{E}_1}(\mathcal{O}_{\mathcal{S}^{virt}})$ and $p_*\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}^{virt}})$. This completes the proof of the first assertion in the proposition. The second assertion also follows readily from the above arguments: see for example, the formulae in 2.0.17 above. □

Definition 2.5. We define the *support* of any class $\alpha\varepsilon H_n^{Br}(\mathcal{S}, \Gamma_{\mathcal{S}}^h(*))_{\mathbb{Q}})$ as follows. Clearly one has an obvious map $K(\quad)_{\mathfrak{M}} \to p_*(\mathbf{K}(\quad)_{\mathcal{S}})$ of presheaves on $\mathfrak{M}_{et}$; this provides a map $\bar{p}_* : H_n^{Br}(\mathcal{S}, \Gamma_{\mathcal{S}}^h(*))_{\mathbb{Q}}) \to H_n(\mathfrak{M}; Sp(\Gamma^h(*)))_{\mathbb{Q}})$ compatible with localization on the Zariski (or even étale) site of $\mathfrak{M}$. We define the support of the class α as $\{m\varepsilon\mathfrak{M} | 0 \neq \pi_*(\alpha)_m \varepsilon H^{-n}(\quad, Sp(\Gamma^h(*))))\tilde{}_m\}$.

Example 2.6. Take the complex $\Gamma(r) = \mathcal{Z}^r(X, .) =$ the higher cycle complex. Now the support of any class $\alpha\varepsilon H_{2n}^{Br}(\mathcal{S}, \Gamma(n))$ is the closed subscheme of $\mathfrak{M}$ defined by the union of the irreducible closed subvarieties of $\mathfrak{M}$ appearing in the algebraic cycle $p_*([\mathcal{S}]^{virt})$.

Proposition 2.7. *Assume in addition to the above situation that the stack is an orbifold and that the complexes $\{\Gamma^h(r)|r\}$ are defined on the smooth site of all algebraic stacks. Let $\mathfrak{M}_0$ denote the open subscheme of the moduli space where the stabilizers are trivial. Assume further that $Supp([\mathcal{S}]_{Br}^{virt}) \cap \mathfrak{M}_0$ is non-empty. Then the image of the class $[\mathcal{S}]_{Br}^{virt}$ in the smooth homology of the stack with respect to $\Gamma(*)$, agrees with the virtual fundamental classes defined cycle theoretically in the latter (i.e. modulo classes that are supported on strictly smaller dimensional sub-spaces of the moduli space).*

Proof. Observe that the cycle theoretic definition of the virtual fundamental class in smooth homology is

$$[C(E^\bullet)].Ch(\lambda_{-1}(E^{-1}))$$

where $[C(E^\bullet)]$ $(Ch(\lambda_{-1}(E^{-1})))$ is the class of $C(E^\bullet)$ in smooth homology (is the Chern character of $\lambda_{-1}(E^{-1})$ in smooth cohomology. On the other hand, the image of $[\mathcal{S}]_{Br}^{virt}$ in smooth homology is given by $p_{smt}^*(p_*([\mathcal{S}]_{Br}^{virt}.ch(\lambda_{-1}(E^{-1}))))$: see [J-5] Theorem 1.2. Here $p : \mathcal{S} \to \mathfrak{M}$ is the obvious map, $p_*([\mathcal{S}]_{Br}^{virt}.ch(\lambda_{-1}(E^{-1})))$ is the corresponding image in $\mathbb{H}_{et}^*(\mathfrak{M}, \Gamma^h(*))_{\mathbb{Q}}$, $p_{smt}^* : \mathbb{H}_{et}^*(\mathfrak{M}, \Gamma^h(*))_{\mathbb{Q}} \cong \mathbb{H}_{smt}^*(\mathfrak{M}, \Gamma^h(*)) \to \mathbb{H}_{et}^*(\mathcal{S}, \Gamma^h(*))_{\mathbb{Q}}$ is the corresponding pull-back to the smooth homology of the stack and ch denotes the Chern character with values in Bredon cohomology. Since the map $p_0 : \mathcal{S}_0 = \mathcal{S} \underset{\mathfrak{M}}{\times} \mathfrak{M}_0 \to \mathfrak{M}_0$ is the identity, one can see that we have equality of $[C(E^\bullet_{|\mathcal{S}_0})].Ch(\lambda_{-1}(E^{-1}_{|\mathcal{S}_0}))$ and $p_{smt}^*(p_*([\mathcal{S}_0]_{Br}^{virt}.ch(\lambda_{-1}(E^{-1}_{|\mathcal{S}_0}))))$. Moreover the difference between these and the corresponding classes for the stack $\mathcal{S}$ is given by classes supported on strictly lower dimensional sub-spaces of $Supp([\mathcal{S}]^{virt})$. □

Remark 2.8. Observe that we are able to define the virtual fundamental class without the hypothesis that one can replace the obstruction theory $E^\bullet$ (upto quasi-isomorphism) by a complex of G-equivariant vector bundles.

3. Gysin maps in G-theory

In this section we explore basic properties of Gysin maps at the level of G-theory with the goal of applying these in the next section. This roughly parallels the treatment in [F] where such Gysin maps are defined at the level of algebraic cycles.

Proposition 3.1. *Assume the situation in (2.0.10). Let $\alpha\varepsilon\pi_0(G(X))$, $\beta\varepsilon\pi_0(G(X'))$ so that $y^!(\alpha) = \beta$. Then $x_*(\beta) = \alpha \otimes Lf^*(y_*(\mathcal{O}_{Y'}))$ in $\pi_0(G_{X'}(X))$ and in $\pi_0(G(X))$.*

Proof. Observe that the map $x_* : \pi_0G(X') \to \pi_0G_{X'}(X)$ is an isomorphism with its inverse given by the devissage theorem in G-theory. The hypotheses imply that under this inverse isomorphism the class $\alpha \otimes Lf^*(y_*(\mathcal{O}_{Y'}))$ maps to the class β. Therefore, $x_*(\beta) = \alpha \otimes Lf^*(y_*(\mathcal{O}_{Y'}))$. □

Proposition 3.2. *Consider the commutative square*

$$\begin{array}{ccccc} N\underset{Y}{\times}C & \longrightarrow & C' & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ N\underset{Y}{\times}X & \longrightarrow & X' & \xrightarrow{x} & X \\ \downarrow & & \downarrow g & & \downarrow f \\ N & \xrightarrow{\rho} & Y' & \xrightarrow{y} & Y \end{array}$$

where the following hold: the bottom right square is as in (2.0.10) with f a local immersion with Y smooth, $C = C_X(Y)$ = the cone associated to this immersion and the rest of the diagram is defined so that all the squares are cartesian. Then

$$y^!(\mathcal{O}_C) = \mathcal{O}_{\mathcal{C}_{X'}(Y')} \text{ in } \pi_0(G(N\underset{Y}{\times}C)) \cong \pi_0(G(C'))$$

Proof. This is a rather well known result; the corresponding results for algebraic cycles appears in [Vi-1] and may be proved along similar lines by reducing to the case when Y' and X are divisors in Y. The key observation is that we define $y^! : G(C) \to G(C')$ by taking for the map f in (2.0.10) not the map f above but instead the composition of the two maps forming the right-most column. We skip the details. □

Proposition 3.3. *Assume the square*

$$\begin{array}{ccc} X' & \xrightarrow{i'} & X \\ \downarrow g & & \downarrow f \\ Y' & \xrightarrow{i} & Y \end{array}$$

is cartesian and that one is given maps $\pi : Y \to Y'$ and $s : Y' \to X'$ so that $g \circ s = id_{Y'}$ and $\pi \circ i = id_{Y'}$. Assume also, that both the maps i and s are regular closed immersions. Then the composite map

$$\pi_0(G(X, \mathcal{O}_X)) \xrightarrow{i^!} \pi_0(G_{X'}(X, \mathcal{O}_X)) \xrightarrow{\simeq} \pi_0(G(X', \mathcal{O}_{X'})) \xrightarrow{s^!} \pi_0(G(Y', \mathcal{O}_{Y'}))$$

is also equal to the map induced by the map $M \mapsto \alpha \underset{\mathcal{O}_X}{\otimes} (f^*\pi^*(\lambda_{-1}(N_{Y'/Y}) \underset{\mathcal{O}_{Y'}}{\otimes} (\lambda_{-1}(N_{Y'/X'})))),\ M \varepsilon Coh(X, G) = Coh([X/G])$. Here $N_{Y'/Y}$ $(N_{Y'/X'})$ is the conormal sheaf associated to the closed immersion $Y' \to Y$ $(Y' \to X'$, respectively).

Proof. If M denotes a coherent $\mathcal{O}_X$-module, it follows from the definition that

$$i^!(M) = [M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})]$$

which is the class of $M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y}) \varepsilon \pi_0(G(X'))$.

Similarly, $g^*(\lambda_{-1}(N_{Y'/X'}))$ is a resolution of $s_*(\mathcal{O}_{Y'})$. It follows therefore, that for M a coherent $\mathcal{O}_X$-module,

$$s^!i^!(M) = [M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})] \underset{\mathcal{O}_{X'}}{\otimes} g^*\lambda_{-1}(N_{Y'/X'})).$$

Observe, in view of our hypothesis that $N_{Y'/X'}$ is a locally free $\mathcal{O}_{Y'}$-module. Moreover $\pi^*(\lambda_{-1}(N_{Y'/Y}))$ is a resolution of $i_*(\mathcal{O}_{Y'})$. Therefore, each term of $F^\bullet = f^*\pi^*\lambda_{-1}(N_{Y'/X'})$ is a locally free $\mathcal{O}_X$-module. Moreover the commutativity of the square in the proposition shows that $i'^*(F^\bullet) = g^*(\lambda_{-1}(N_{Y'/X'}))$. Therefore

$$s^!i^!(M) = [M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})] \underset{\mathcal{O}_{X'}}{\otimes} i'^*(F^\bullet).$$

Since $M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})$ has supports contained in X' and each term of the complex $F^\bullet$ is a locally free $\mathcal{O}_X$-module, one obtains the identification

$$[M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})] \underset{\mathcal{O}_{X'}}{\otimes} i'^*(F^\bullet) = [M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y}) \underset{\mathcal{O}_X}{\otimes} F^\bullet]$$

as classes in $\pi_0(G_{Y'}(Y)) \cong \pi_0(G(Y'))$. □

Remark 3.4. The above proposition enables us to obtain a convenient reformulation of the virtual structure sheaves as in Definition 4.1.

Later on in this section, we will need the following alternate definition of the refined Gysin maps defined using deformation to the normal cone. We begin by defining the specialization map to the normal cone at the level of G-theory. If $X' \to X$ is a closed immersion of Deligne-Mumford stacks, one performs the blow-up of $X \times \mathbb{P}^1$ along $X \times \{\infty\}$; let this be denoted M and let $\tilde{X}$ be the blow-up of X along X'. Let M^o denote the complement of $\tilde{X}$ in M. Now $j : X \times \mathbb{A}^1$ imbeds as an open sub-stack of M^o with complement $C = C_{X'}X$ = the normal cone to X' in X. Therefore one obtains the localization sequence: $G(C) \overset{i_*}{\to} G(M^o) \overset{j^*}{\to} G(X \times \mathbb{A}^1)$ where $i : C \to M^o$ is the obvious closed immersion. Since C is a divisor in M^o, it follows that i is a regular closed immersion of codimension 1 and therefore that one has a pull-back $i^* : G(M^o) \to G(C)$. Moreover the composition $i^* \circ i_* : G(C) \to G(C)$ is null-homotopic, since the normal bundle to the immersion i is trivial. Therefore, the map $i^* : G(M^o) \to G(C)$ factors through j^*. The induced map $G(X \times \mathbb{A}^1) \to G(C)$ will be denoted sp'. We define the *specialization map* $sp : G(X) \to G(C)$ as the composition $sp' \circ pr_1^*$, where $pr_1 : X \times \mathbb{A}^1 \to X$ is the obvious projection.

Given a diagram as in (2.0.10), one may first replace it with the diagram:

$$\begin{array}{ccc} X' & \xrightarrow{x_0} & C_{X'}X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_0} \\ Y' & \xrightarrow{y_0} & N_{Y'}Y \end{array}$$

One has a refined Gysin map ${y_0}^! : G(C_{X'}X) \to G(X')$. We may pre-compose this with the specialization map $sp : G(X) \to G(C_{X'}X)$ to define the *alternate refined Gysin map* $y^!_{alt} : G(X) \to G(X')$.

Proposition 3.5. $y^! = y^!_{alt} : \pi_0(G(X)) \to \pi_0(G(X'))$

Proof. First observe by the localization sequence that the restriction $j^* : \pi_0(G(M^o)) \to \pi_0(G(X \times \mathbb{A}^1)$ is surjective. (See [Qu] section 5, Theorem 5: observe that this is stated for abelian categories and therefore applies to algebraic (non-dg) stacks as well.) Therefore the specialization map on the Grothendieck groups is simply defined by starting with a class α in $\pi_0(G(X))$, pulling it back to $\pi_0(G(X \times \mathbb{A}^1))$ by pr_1^*, lifting this to a class in $\pi_0(G(M^o))$ and then applying i^*. Therefore, the specialization map at the level of Grothendieck groups is compatible with pairings in the following sense: assume the situation of (2.0.10). Now the specializations $sp : G_{Y'}(Y) \to G_{Y'}(N_{Y'}Y)$ and $sp : G(X) \to G(C_{X'}X)$ are compatible in the sense the following square commutes:

$$\begin{array}{ccc} \pi_0(G(X)) & \xrightarrow{y^!} & \pi_0 G(X') \\ \downarrow{\scriptstyle sp} & & \downarrow{\scriptstyle id} \\ \pi_0(G(C_{X'}X)) & \xrightarrow{y_0^!} & \pi_0 G(X') \end{array}$$

For this observe that both the Gysin maps $y^!$ and $y_0^!$ are defined by pairing with the Koszul-Thom class of Y' in Y. Therefore, it suffices to show that the Koszul-Thom class of Y' in Y specializes to the Koszul-Thom class of Y' in $N_{Y'}Y$. We skip this verification to the reader. □

4. **Pushforward and localization formulae for virtual structure sheaves and virtual fundamental classes**

Next we proceed to establish a push-forward formula for the virtual fundamental classes. Using Lefschetz-Riemann-Roch, it suffices to establish a push-forward formula for the virtual structure sheaves instead. For this, we will first find another more convenient alternate definition of the virtual structure sheaf. We will assume henceforth that the given stack $\mathcal{S}$ admits a *G-equivariant closed immersion* into a smooth Deligne-Mumford stack

$\tilde{\mathcal{S}}$ onto which the G-action extends. Assuming this closed immersion is denoted i and is defined locally by the sheaf of ideals $\mathcal{I}$, the cotangent complex of $\mathcal{S}$ truncated outside the interval $[-1, 0]$ can be identified with the complex:

(4.0.18) $$\tau_{\geq -1} L^{\bullet}\mathcal{S} : \mathcal{I}/\mathcal{I}^2 \rightarrow i^*(\Omega_{\tilde{\mathcal{S}}})$$

4.0.19. *Basic pushforward hypothesis. We will also assume henceforth that the obstruction theory is given by a strict map of complexes* $E^{\bullet} \rightarrow \tau_{\geq -1} L^{\bullet}_{\mathcal{S}}$ *and that* E^i, $i = -1, 0$ *are vector bundles.* As observed in [GP], the hypotheses that every coherent sheaf on the stack is a quotient of a vector bundle, implies one may make the above assumption without further loss of generality.

One may show that our hypothesis that $E^{\bullet}$ is an obstruction theory associated to the immersion i (in the above sense) implies that the sequence of sheaves $E^{-1} \rightarrow E^0 \oplus \mathcal{I}/\mathcal{I}^2 \xrightarrow{\gamma} \Omega_{\tilde{\mathcal{S}}|\mathcal{S}} \rightarrow 0$ is *exact*. Then one obtains the associated exact sequence of abelian cones:

(4.0.20) $$0 \rightarrow T\tilde{S}_{|\mathcal{S}} \rightarrow C(\mathcal{I}/\mathcal{I}^2) \underset{\mathcal{S}}{\times} \mathcal{E}_0 \rightarrow C(Q) \rightarrow 0$$

where $C(Q)$ is the cone associated to $Q = ker(\gamma)$ and $\mathcal{E}_0 = C(E_0^{\vee})$. Since Q is a quotient of E^{-1}, $C(Q)$ imbeds in $\mathcal{E}_1$. The normal cone to $\mathcal{S}$ in $\tilde{\mathcal{S}}$, $C_{\mathcal{S}}(\tilde{\mathcal{S}})$ is a closed substack of $C(\mathcal{I}/\mathcal{I}^2)$. Observe that $C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0$ is a $T\tilde{\mathcal{S}}_{|\mathcal{S}}$-cone.

If Q' denotes the kernel of $C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0 \rightarrow E^0 \oplus \mathcal{I}/\mathcal{I}^2 \xrightarrow{\gamma} \Omega_{\tilde{\mathcal{S}}|\mathcal{S}}$, we obtain the short exact sequence $0 \rightarrow Q' \rightarrow E^0 \oplus C_{\mathcal{S}}(\tilde{\mathcal{S}}) \rightarrow \Omega_{\tilde{\mathcal{S}}|\mathcal{S}} \rightarrow 0$ and therefore the exact sequence $0 \rightarrow T\tilde{S}_{|\mathcal{S}} \rightarrow C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0 \rightarrow C(Q') \rightarrow 0$.

Observe that $C(Q') = C(E^{\bullet})$ in the terminology used earlier. Viewing the above as an exact sequence of objects over $\mathcal{S}$, one may compute the dimension of $C(Q')$ as follows:

(4.0.21) $$dim(C(E^{\bullet})) = dim(C(Q')) = rank(E^0) + b$$

Moreover, $\mathcal{O}^{virt}_{\mathcal{S}} = 0^!_{\mathcal{E}_1}(\mathcal{O}_{C(Q')})$. Alternatively, one has the cartesian square

(4.0.22) $$\begin{array}{ccc} T\tilde{\mathcal{S}}_{|\mathcal{S}} & \longrightarrow & C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0 \\ \big\downarrow p & & \big\downarrow f \\ \mathcal{S} & \xrightarrow{0_{\mathcal{E}_1}} & \mathcal{E}_1 \end{array}$$

Here $p : T\tilde{\mathcal{S}}_{|\mathcal{S}} \rightarrow \mathcal{S}$ is the obvious projection and f is the map induced by the map $E^{-1} \rightarrow E^0 \oplus \mathcal{I}/\mathcal{I}^2$; let $s_{T\tilde{\mathcal{S}}_{|\mathcal{S}}} : \mathcal{S} \rightarrow T\tilde{\mathcal{S}}_{|\mathcal{S}}$ denote the obvious zero-section. Now one obtains a quasi-isomorphism:

(4.0.23) $$\mathcal{O}^{virt}_{\mathcal{S}} \simeq s^!_{T\tilde{\mathcal{S}}_{|\mathcal{S}}} 0^!_{\mathcal{E}_1}(\mathcal{O}_{C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0})$$

(This follows from the observation: $s^!_{T\tilde{\mathcal{S}}_{|\mathcal{S}}} 0^!_{\mathcal{E}_1}(\mathcal{O}_{C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0}) \simeq s^*_{T\tilde{\mathcal{S}}_{|\mathcal{S}}} 0^!_{\mathcal{E}_1}(\mathcal{O}_{C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0}) \simeq 0^!_{\mathcal{E}_1}(s^*_{T\tilde{\mathcal{S}}_{|\mathcal{S}}} \mathcal{O}_{C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0}) = 0^!_{\mathcal{E}_1}(\mathcal{O}_{C(Q')})$ as classes in $\pi_0(G(\mathcal{S}))$. The last but one $\simeq$ follows from the observation that locally, $C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0$ is a product of $C(Q')$ and $T\tilde{\mathcal{S}}_{|\mathcal{S}}$.)

Proposition 3.3 shows that as classes in $\pi_0 G(\mathcal{S})$, one has the identification:

$[\mathcal{O}^{virt}_{\mathcal{S}}] = [f^* \pi^*_{\mathcal{E}}(\lambda_{-1}(E^{-1}) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}}))]$.

Here $\pi_{\mathcal{E}} : \mathcal{E}_1 \rightarrow \mathcal{S}$ is the obvious projection. Observe that the right-hand-side is only a complex of quasi-coherent sheaves on $\mathcal{S}$: nevertheless it is a complex of *coherent sheaves* on the stack $C\mathcal{E}_0$ with supports in the closed sub-stack $\mathcal{S}$.

Definition 4.1. Henceforth we will let

(4.0.24) $$\mathcal{O}^{virt}_{\mathcal{S}} = f^* \pi^*_{\mathcal{E}}(\lambda_{-1}(E^{-1}) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}}))$$

viewed as a complex of sheaves on the stack $C\mathcal{E}_0$. Proposition 7.8 in [J-6] shows that if I is the sheaf of ideals defining $\mathcal{S}$ in $C\mathcal{E}_0$, then, $\Sigma_i \mathcal{RH}om_{\mathcal{O}_{C\mathcal{E}_0}}(I^{i-1}/I^i, f^*\pi_{\mathcal{E}}^*(\lambda_{-1}(E^{-1}) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}})))[-i]$ is a complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules and that as classes in $\pi_0 G(\mathcal{S})$ this identifies with the class $[\mathcal{O}_{\mathcal{S}}^{virt}]$. (In fact, the sum on the right is a finite sum.) Therefore, it is often convenient to use the following variant of the virtual structure sheaf:

(4.0.25)
$$\bar{\mathcal{O}}_{\mathcal{S}}^{virt} = \Sigma_i \mathcal{RH}om_{\mathcal{O}_{C\mathcal{E}_0}}(I^{i-1}/I^i, f^*\pi_{\mathcal{E}}^*(\lambda_{-1}(E^{-1}) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}})))[-i]$$

4.0.26. Next assume that $i_0 : \mathcal{T} \to \tilde{\mathcal{T}}$ and $i : \mathcal{S} \to \tilde{\mathcal{S}}$ are *closed immersions* and that the square

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{u} & \mathcal{S} \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i} \\ \tilde{\mathcal{T}} & \xrightarrow{v} & \tilde{\mathcal{S}} \end{array}$$

is cartesian, with both $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ smooth Deligne-Mumford stacks and where the maps u and v are closed immersions.

4.0.27. *Weak compatibility of obstruction theories.* We will assume that one is provided with a perfect obstruction theory $E^\bullet$ ($F^\bullet$) for $\mathcal{S} \to \tilde{\mathcal{S}}$ ($\mathcal{T} \to \tilde{\mathcal{T}}$, respectively) satisfying the hypotheses as in 4.0.19 and that these are *weakly compatible* in the following sense: there is given a G-equivariant map $\phi : u^*(E^\bullet) \to F^\bullet$ of complexes so that there exists a distinguished triangle $K^\bullet \to Lu^*(E^\bullet) \to F^\bullet$ and $K^\bullet$ is of perfect amplitude contained in $[-1, 0]$. For example, the two obstruction theories are compatible if one has G-equivariant resolutions of coherent sheaves by vector bundles, $E^\bullet$ and $F^\bullet$ may be replaced by complexes of vector bundles and the given map $\phi : Lu^*(E^\bullet) \to F^\bullet$ is an *epimorphism*. It follows that, in this case, the kernel, $K^\bullet = ker(\phi)$ is a complex of vector bundles.

Lemma 4.2. *$E^\bullet$ and $F^\bullet$ are weakly compatible if and only if there exists a distinguished triangle $K'^\bullet \to E'^\bullet \to L\pi^*(F^\bullet) \to K'^\bullet[1]$ of complexes of $\mathcal{O}_{C_{\mathcal{T}}(\mathcal{S})}$-modules so that (i) $K'^\bullet$ and $E'^\bullet$ are complexes of perfect amplitude contained in $[-1, 0]$ and (ii) $L0^*(E'^\bullet) = Lu^*(E^\bullet)$. Here $\pi : C_{\mathcal{T}}(\mathcal{S}) \to \mathcal{T}$ is the obvious projection while $0 : \mathcal{T} \to C_{\mathcal{T}}(\mathcal{S})$ is the obvious closed immersion of the vertex of the cone.*

Proof. Assume that one is given a distinguished triangle $K'^\bullet \to E'^\bullet \to L\pi^*(F^\bullet) \to K'^\bullet[1]$ satisfying the above hypotheses. Taking $K^\bullet = L0^*(K'^\bullet)$ provides a distinguished triangle $K^\bullet \to Lu^*(E^\bullet) \to F^\bullet \to K^\bullet[1]$ showing the weak compatibility of the obstruction theories. Conversely given a distinguished triangle, $K^\bullet \to Lu^*(E^\bullet) \to F^\bullet \to K^\bullet[1]$ with $K^\bullet$ a complex of perfect amplitude contained in $[-1, 0]$, one may take $E'^\bullet = L\pi^*(Lu^*(E^\bullet))$ and $K'^\bullet = L\pi^*(K^\bullet)$. □

4.0.28. *The deformed virtual structure sheaf.* Let $C_{\mathcal{T}}(\mathcal{E}_0)$ denote the normal cone associated to the composite closed immersion $\mathcal{T} \to \mathcal{S} \to \mathcal{E}_0$. Now $C_{\mathcal{T}}(\mathcal{S})$ is a closed subscheme of $C_{\mathcal{T}}(\mathcal{E}_0)$: moreover the obvious projection $\mathcal{E}_0 \to \mathcal{S}$ induces a splitting to the above map so that $C_{\mathcal{T}}(\mathcal{S})$ is a factor of the cone $C_{\mathcal{T}}(\mathcal{E}_0)$. Moreover $C_{\mathcal{T}}(\mathcal{E}_{0|\mathcal{T}})$ is also a sub-scheme of $C_{\mathcal{T}}(\mathcal{E}_0)$. Now a local computation will show that the obvious map $C_{\mathcal{T}}(\mathcal{S}) \underset{\mathcal{T}}{\times} C_{\mathcal{T}}(\mathcal{E}_{0|\mathcal{T}}) \to C_{\mathcal{T}}(\mathcal{E}_0)$ is an isomorphism. In addition, one readily obtains the isomorphism $C_{\mathcal{T}}(\mathcal{E}_{0|\mathcal{T}}) \cong C_{\mathcal{F}_0}(\mathcal{E}_{0|\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0$. Therefore, one obtains the isomorphism

(4.0.29)
$$C_{\mathcal{T}}(\mathcal{E}_0) \cong C_{\mathcal{T}}(\mathcal{S}) \underset{\mathcal{T}}{\times} C_{\mathcal{F}_0}(\mathcal{E}_{0|\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0$$

We consider the commutative diagram:

(4.0.30)
$$\begin{array}{ccccc} C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0) & \xrightarrow{\pi_0} & C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 & \xrightarrow{\beta} & C_{\mathcal{T}}(\tilde{\mathcal{T}}) \\ \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \alpha} \\ C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) & \xrightarrow{\phi_1} & C_{\mathcal{T}}(\mathcal{S}) & \xrightarrow{\pi} & \mathcal{T} \end{array}$$

where π_0 is the obvious projection induced by the projections $C_{\mathcal{T}}(\mathcal{S}) \to \mathcal{T}$, $C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \to C_{\mathcal{T}}(\tilde{\mathcal{T}})$ and $C_{\mathcal{T}}(\mathcal{E}_0) \to \mathcal{F}_0$. Moreover π_1 denotes the projection to the first factor. We let the composition of maps $\alpha \circ \beta$ by f_0 and let $C = C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))$. Let $0 : C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 \to C \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0)$ denote the obvious closed immersion. (Observe that this is a section to the map π_0.) Then we provide the following definition of the deformed virtual structure sheaf:

Definition 4.3. $\mathcal{O}_C^{virt} = \pi_0^* f_0^*(\lambda_{-1}(E_{|\mathcal{T}}^{-1}) \otimes \lambda_{-1}(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}})) = \pi_1^* \phi_1^* \pi^*(\lambda_{-1}(E_{|\mathcal{T}}^{-1}) \otimes \lambda_{-1}(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}))$ and call this the *deformed virtual structure sheaf.*

Remark 4.4. Having replaced $\mathcal{S}$ by the cone $C_{\mathcal{T}}(\mathcal{S})$ and the virtual structure sheaf $\mathcal{O}_{\mathcal{S}}^{virt}$ by its deformation, $\mathcal{O}_{C_{\mathcal{T}}(\mathcal{S})}$, we have greater flexibility: the main advantage is the presence of the morphism $\pi : C_{\mathcal{T}}(\mathcal{S}) \to \mathcal{T}$ so that $\pi \circ 0 = id_{\mathcal{T}}$, where $0 : \mathcal{T} \to C_{\mathcal{T}}(\mathcal{S})$ is the obvious zero-section imbedding. See 4.0.46 below for more details on this deformation. Throughout the following theorem we will let $C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0)$ $(C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0)$ be denoted by $C\mathcal{E}_0$ $(C\mathcal{F}_0$, respectively).

Theorem 4.5. *Assume the above situation. Now one obtains the formula*

$$\pi_0^* f_0^*(\lambda_{-1}(K^{-1})) \otimes 0_*(\mathcal{O}_{\mathcal{T}}^{virt}) = 0_*(f_0^*(\lambda_{-1}(K^{-1})) \underset{\mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0}}{\otimes} \mathcal{O}_{\mathcal{T}}^{virt}) = \lambda_{-1}(\pi_0^* f_0^* K^0) \otimes \mathcal{O}_C^{virt}$$

in $\pi_0(G_{\mathcal{T}}(C\pi^*(\mathcal{E}_{0|\mathcal{T}})))$. *In case* $\mathcal{S}$, $\mathcal{T}$ *are provided with a compatible action by a smooth group scheme* G *and the obstruction theories are* G*-equivariant, the last formula holds in* $\pi_0(G_{\mathcal{T}}(C(\pi^*(\mathcal{E}_{0|\mathcal{T}})), G))$. *(Here* $\otimes$ *denotes the tensor product over* $\mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0)}$.*)*

Proof. Let $\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}$ $(\Omega_{\tilde{\mathcal{S}}_0|\mathcal{S}_0})$ denote the restriction of $\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})}$ $(\Omega_{\tilde{\mathcal{T}}}$, respectively) to $\mathcal{T}$. Let g and π_F be defined by the following obvious diagram:

(4.0.31)
$$\begin{array}{ccccc} C_{\mathcal{T}}\mathcal{S} & & & & C\mathcal{E}_0 \\ \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \pi_0} \\ \mathcal{T} & \underset{\pi_{\mathcal{F}}}{\longleftarrow} & \mathcal{F}_1 & \underset{g}{\longleftarrow} & C\mathcal{F}_0 \end{array}$$

Step 1. By definition, the right-hand-side identifies with

(4.0.32)
$$\lambda_{-1}(\pi_0^* f_0^* \Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) \underset{\mathcal{O}_{C\mathcal{E}_0}}{\otimes} \lambda_{-1}(\pi_0^* f_0^*(E_{|\mathcal{T}}^{-1})) \underset{\mathcal{O}_{C\mathcal{E}_0}}{\otimes} \lambda_{-1}(\pi_0^* f_0^* K^0)$$

Definition 4.1 applied to the cartesian square

(4.0.33)
$$\begin{array}{ccc} T\tilde{\mathcal{T}}_{|\mathcal{T}} & \longrightarrow & C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathcal{T} & \underset{0_{\mathcal{F}_1}}{\longrightarrow} & \mathcal{F}_1 \end{array}$$

shows that the left-hand-side identifies with

(4.0.34)
$$0_*(\lambda_{-1}(g^* \pi_{\mathcal{F}}^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \underset{\mathcal{O}_{C\mathcal{F}_0}}{\otimes} \lambda_{-1}(g^* \pi_{\mathcal{F}}^*(F^{-1}))) \underset{\mathcal{O}_{C\mathcal{E}_0}}{\otimes} \lambda_{-1}(\pi_0^* f_0^* K^{-1}))$$

Henceforth $\otimes$ *will denote* $\underset{\mathcal{O}_{C\mathcal{E}_0}}{\otimes}$.

Step 2. Next, one considers the obvious immersion $0 : C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0$ in $C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0)$ (which we denoted 0). This factors as

$$C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 \overset{\alpha}{\to} C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} \pi^*(\mathcal{F}_0) \overset{v}{\to} C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0).$$

The first observation here is that, in this situation, one obtains the identifications:

(4.0.35)
$$C_{\mathcal{T}}(\mathcal{E}_0) = C_{\mathcal{T}}(\mathcal{S}) \underset{\mathcal{T}}{\times} C_{\mathcal{F}_0}(\mathcal{E}_{0|\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0, \quad \pi^*(\mathcal{F}_0) = C_{\mathcal{T}}(\mathcal{S}) \underset{\mathcal{T}}{\times} \mathcal{F}_0$$

4.0.36. Therefore, the map α identifies with the map $C_{\mathcal{T}}(\tilde{\mathcal{T}})\underset{\mathcal{T}}{\times}\mathcal{F}_0 \overset{i\times id}{\to} C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{\mathcal{T}}{\times}\mathcal{F}_0$ where $i : C_{\mathcal{T}}(\tilde{\mathcal{T}}) \to C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))$ is the obvious immersion.

Next we apply the Proposition 3.2 to the bottom square of the following diagram (i.e. the bottom square corresponds to the bottom right square in Proposition 3.2):

(4.0.37)
$$\begin{array}{ccc} C_{\mathcal{T}}(\tilde{\mathcal{T}}) & \overset{x}{\longrightarrow} & C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \\ \downarrow & & \downarrow \phi_1 \\ \mathcal{T} & \longrightarrow & C_{\mathcal{T}}(\mathcal{S}) \\ \downarrow & & \downarrow \phi_0 \\ \tilde{\mathcal{T}} & \overset{y}{\longrightarrow} & N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}) \end{array}$$

Therefore, it follows first that $y^!(\mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))}) = \mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}})}$ and then by invoking Proposition 3.1 that

$$x_*(\mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}})}) = \mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))} \otimes \lambda_{-1}(\phi^*\tilde{\pi}^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})))$$

in $\pi_0(G(C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})), \mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))}))$.

Here $\phi = \phi_0 \circ \phi_1$ ($\tilde{\pi}$) is the map forming the right vertical column in the above square (is the projection $N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}) \to \tilde{\mathcal{T}}$, respectively). Moreover, it is clear that $\lambda_{-1}(\phi^*\tilde{\pi}^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))) = \lambda_{-1}(\phi_1^*\pi^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))_{\mathcal{T}})$. It follows from the observation about the map α in 4.0.36 above that

$$\alpha_*(\mathcal{O}_{C\mathcal{F}_0}) = \mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{C_{\mathcal{T}}(\tilde{\mathcal{S}})}{\times}\pi^*(\mathcal{F})} \otimes \lambda_{-1}(\phi^*\tilde{\pi}^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))) \tag{4.0.38}$$

in $\pi_0(G_{C\mathcal{F}_0}(C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{C_{\mathcal{T}}(\tilde{\mathcal{S}})}{\times}\pi^*(\mathcal{F}), \mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{C_{\mathcal{T}}(\tilde{\mathcal{S}})}{\times}\pi^*(\mathcal{F}_{0|\mathcal{T}})}))$.

The short exact sequence $0 \to K'^0 \to E'^0 \to \pi^*(F^0) \to 0$ shows on taking the symmetric algebras associated to E'^0 and $\pi^*(F^0)$ that the kernel of the obvious surjection $Sym(E'^0) \to Sym(\pi^*(F^0))$ is the ideal $K'^0 \otimes Sym(E'^0)$. One may identify $(K'^0 \otimes Sym(E'^0)/(K'^0 \otimes Sym(E'^0))^2$ with $(K'^0 \otimes \pi^*(Sym(F^0)))/(K'^0 \otimes \pi^*(Sym(F^0)))^2 = K'^0/(K'^0)^2 \otimes \pi^*(Sym(F^0))$. Clearly one obtains a natural map of the last term to $(K'^0 \otimes Sym(E'^0)/(K'^0 \otimes Sym(E'^0))^2$; by working locally one may show this is an isomorphism. Therefore one gets the formula:

$$v_*(\mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))}\underset{C_{\mathcal{T}}(\tilde{\mathcal{S}})}{\times}\pi^*(\mathcal{F}_0)) = \mathcal{O}_{C\mathcal{E}_0} \otimes \lambda_{-1}(\pi_0^* f_0^* K^0) \tag{4.0.39}$$

$$= \mathcal{O}_{C\mathcal{E}_0} \otimes \lambda_{-1}(\pi_1^*\phi_1^*\pi^*(K^0)) \tag{4.0.40}$$

in the Grothendieck group $\pi_0(G(C\mathcal{E}_0, \mathcal{O}_{C\mathcal{E}_0}))$. (Recall $K^{\bullet\prime} = \pi^*(K^\bullet)$. Therefore the commutative diagram in (4.0.30) shows that $\pi_0^* f_0^*(\lambda_{-1}(K^0)) = \pi_1^*(\phi_1^*(\pi^*(\lambda_{-1}(K^0))))$.) Combining these provides the identification

$$\begin{aligned} 0_*(\mathcal{O}_{C\mathcal{F}_0}) &= \mathcal{O}_{C\mathcal{E}_0} \otimes \lambda_{-1}(\pi_1^*\phi_1^*\pi^* N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \otimes \lambda_{-1}(\pi_0^* f_0^* K^0) \\ &= \mathcal{O}_{C\mathcal{E}_0} \otimes \lambda_{-1}(\pi_0^* f_0^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})_{|\mathcal{T}})) \otimes \lambda_{-1}(\pi_0^* f_0^* K^0) \end{aligned} \tag{4.0.41}$$

in $\pi_0 G_{C\mathcal{F}_0}(C\mathcal{E}_0, \mathcal{O}_{C\mathcal{E}_0})$.

Step 3. Here we show that

$$\begin{aligned} \lambda_{-1}(\pi_0^* f_0^* \Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) &= \lambda_{-1}(\pi_0^* g^* \pi_{\mathcal{F}}^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* g^* \pi_{\mathcal{F}}^* N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \\ &= \lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^* N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})_{|\mathcal{T}}) \end{aligned} \tag{4.0.42}$$

Observe that the normal cone to the immersion $\tilde{\mathcal{T}}$ in $C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})$ identifies with the normal bundle $N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})$. We begin with the the *split short exact sequence* $0 \to N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}) \to \tilde{0}^*(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})}) \to \Omega_{\tilde{\mathcal{T}}} \to 0$. Here $\tilde{0} : \tilde{\mathcal{T}} \to C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})$ is the obvious map. Let $\tilde{\pi} : C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}) \rightarrow \tilde{\mathcal{T}}$ denote the obvious projection. We apply the pull-back by $\tilde{\pi}^*$ and restriction to $C_{\mathcal{S}_0}(\mathcal{S})$ (= restriction to $\mathcal{T}$ and pull-back by π^*) to obtain:

$$\lambda_{-1}(\pi^*\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) = \lambda_{-1}(\pi^*(\Omega_{\tilde{\mathcal{T}}|\mathcal{T}})) \otimes \lambda_{-1}(\pi^*N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})_{|\mathcal{T}}) \tag{4.0.43}$$

of perfect complexes. Recall from the commutative diagram (4.0.30) that $\pi \circ \phi_1 \circ \pi_1 = f_0 \circ \pi_0$. Therefore, the pull-back of this by $\pi_1^* \circ \phi_1^*$ then provides the identification

$$\lambda_{-1}(\pi_0^* f_0^* \Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) = \lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^* N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})_{|\mathcal{T}})$$

Finally observe that the map $\pi_{\mathcal{F}} \circ g : C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 \rightarrow \mathcal{T}$ also identifies with the map f_0 defined in (4.0.30). See also (4.0.31). This provides the identification in (4.0.42).

Step 4. Next, using the observation that $0^*\pi_0^* = id$, the projection formula and the diagram (4.0.31), one may identify the term in (4.0.34) with

$$\begin{aligned}\lambda_{-1}(\pi_0^* g^* \pi_{\mathcal{F}}^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* g^* \pi_{\mathcal{F}}^* F^{-1}) \otimes \lambda_{-1}(\pi_0^* f_0^* K^{-1}) \otimes 0_*(\mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0}) \\ = \lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^* F^{-1}) \otimes \lambda_{-1}(\pi_0^* f_0^* K^{-1}) \otimes 0_*(\mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0})\end{aligned} \tag{4.0.44}$$

Now we consider the term in (4.0.32). In view of (4.0.42), clearly this may be written as

$$\lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^* N_{\tilde{\mathcal{T}}|\mathcal{S}}(\tilde{\mathcal{S}})_{|\mathcal{T}})) \otimes \lambda_{-1}(\pi_0^* f_0^* (E_{|\mathcal{T}}^{-1})) \otimes \lambda_{-1}(\pi_0^* f_0^* K^0) \tag{4.0.45}$$

Therefore, a comparison of the terms in (4.0.44) with that in (4.0.45) (making use of (4.0.41)) shows that the left-hand-side (right-hand-side) of the equation we wish to establish in the theorem is obtained by tensoring the left-hand-side (right-hand-side, respectively) of (4.0.41) by $\lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^* (F_{-1})) \otimes \lambda_{-1}(\pi_0^* f_0^* K^{-1})$. (Recall the short exact sequence $0 \rightarrow K^{-1} \rightarrow E_{|\mathcal{T}}^{-1} \rightarrow F^{-1} \rightarrow 0$, shows $\lambda_{-1}(E_{|\mathcal{T}}^{-1}) = \lambda_{-1}(K^{-1}) \otimes \lambda_{-1}(F^{-1})$.)

So far the arguments show that the required formula holds in the Grothendieck group of sheaves of modules over $\mathcal{O}_{C_{\mathcal{T}}(\mathcal{E}_0)}$ with supports contained in $C\mathcal{F}_0$. However, it is clear (see Definition 4.1 and Proposition 3.3) that the term $\mathcal{O}_{\mathcal{T}}^{virt}$ has supports in $\mathcal{T}$. Therefore, we obtain the required formula in the Grothendieck group with supports contained in $\mathcal{T}$. This completes the proof of the theorem. □

4.0.46. *Deformation to the normal cone.* We will presently define a deformation of the virtual structure sheaf making use of the deformation to the normal cone. This will produce the deformed virtual structure sheaf considered above. We begin with

$$\hat{D} = C_{Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)}(Bl_{\tilde{\mathcal{T}}\times 0}(\tilde{\mathcal{S}} \times \mathbb{A}^1)) \underset{Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)}{\times} Bl_{\mathcal{T}\times 0}(\mathcal{E}_0 \times \mathbb{A}^1) \tag{4.0.47}$$

We begin with the composite map $\hat{D} \rightarrow C_{\mathcal{S}\times\mathbb{A}^1}(\tilde{\mathcal{S}} \times \mathbb{A}^1) \underset{\mathcal{S}\times\mathbb{A}^1}{\times} \mathcal{E}_0 \times \mathbb{A}^1 \cong (C_{\mathcal{S}}(\tilde{\mathcal{S}}) \times \mathbb{A}^1) \underset{\mathcal{S}\times\mathbb{A}^1}{\times} \mathcal{E}_0 \times \mathbb{A}^1 \rightarrow \mathcal{E}_1 \times \mathbb{A}^1$ where the last map is defined by the given obstruction theory on $\mathcal{S}$. The composition of this map with the obvious projection to $\mathcal{S} \times \mathbb{A}^1$ factors also as the projection of $\hat{D}$ to the factor $C_{Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)}(Bl_{\tilde{\mathcal{T}}\times 0}(\tilde{\mathcal{S}} \times \mathbb{A}^1))$ followed by the projection to the vertex of the cone given by $Bl_{\mathcal{T}\times 0}(\mathcal{S} \times \mathbb{A}^1)$ and the projection of the latter to $\mathcal{S} \times \mathbb{A}^1$. We will denote the composite map $\hat{D} \rightarrow C_{Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)}(Bl_{\tilde{\mathcal{T}}\times 0}(\tilde{\mathcal{S}} \times \mathbb{A}^1)) \rightarrow Bl_{\mathcal{T}\times 0}(\mathcal{S} \times \mathbb{A}^1)$ by $\hat{\pi}\hat{f}$: observe that this map is a map between schemes flat over $\mathbb{A}^1$. $\hat{\pi}\hat{f}_{t=1}$ identifies with the map $\pi \circ f : C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{S}$ as in (4.0.22) and $\hat{\pi}\hat{f}_{t=0}$ identifies with the map $\phi_1 \circ \pi_1$ as in (4.0.30).

Let $\hat{p} : Bl_{\mathcal{T}\times 0}(\mathcal{E}_1 \times \mathbb{A}^1) \rightarrow Bl_{\mathcal{T}\times 0}(\mathcal{S} \times \mathbb{A}^1)$ denote the obvious projection. Observe that the obvious maps $Bl_{\mathcal{T}\times 0}(\mathcal{S} \times \mathbb{A}^1)$ and $Bl_{\mathcal{T}\times 0}(\mathcal{S} \times \mathbb{A}^1)$ to $\mathbb{A}^1$ are flat. Therefore, to show $\hat{p}$ is smooth, it suffices to show that for each fiber of $\hat{p}$ over each point t of $\mathbb{A}^1$: see [AK] Chapter VII, Corollary (1.9). This assertion is clear. Let $0_{Bl} : Bl_{\mathcal{T}\times 0}(\mathcal{S} \times \mathbb{A}^1) \rightarrow Bl_{\mathcal{T}\times 0}(\mathcal{E}_1 \times \mathbb{A}^1)$ denote the map induced by the obvious zero-section $\mathcal{S} \times \mathbb{A}^1 \rightarrow \mathcal{E}_1 \times \mathbb{A}^1$. Since this is a section to $\hat{\pi}$, it follows readily that 0_{Bl} is a regular immersion locally. (See, for example, [F], (B.7.3).) Let $\hat{E}^1$ denote the conormal sheaf associated to the regular immersion 0_{Bl}. Let $\hat{\mathcal{S}} = Bl_{\mathcal{T}\times 0}(\mathcal{S} \times \mathbb{A}^1)$ and $\hat{\tilde{\mathcal{S}}} = Bl_{\tilde{\mathcal{T}}\times 0}(\tilde{\mathcal{S}} \times \mathbb{A}^1)$. Observe that the obvious map $\hat{\tilde{\mathcal{S}}} \rightarrow \mathbb{A}^1$ is *flat.*

We let

$$\mathcal{O}^{virt}_{\hat{\mathcal{S}}} = (\hat{\pi}\hat{f})^*(\lambda_{-1}(\Omega_{\hat{\hat{\mathcal{S}}}/\mathbb{A}^1|\hat{\mathcal{S}}} \otimes \lambda_{-1}(\hat{E}^{-1})) \tag{4.0.48}$$

This is a complex of coherent sheaves on $\hat{D}$, and for each $t\varepsilon\mathbb{A}^1$, is a *perfect complex* on $\hat{D}_t$. (Observe that when $t=1$, the corresponding complex is just the virtual structure sheaf $\mathcal{O}^{virt}_{\mathcal{S}}$ as in (4.0.24). When $t=0$, the corresponding complex is the deformed virtual structure sheaf as in Definition 4.3.) We let $\bar{\mathcal{O}}^{virt}_{\hat{\mathcal{S}}}$ denote the corresponding complex of coherent sheaves on $\hat{\mathcal{S}}$ defined as in (4.0.25).

Recall that one has the isomorphisms $K_{\mathcal{T}}(\tilde{T}) \simeq G(T)$ and $K_{\mathcal{S}}(\tilde{\mathcal{S}}) \simeq G(\mathcal{S})$. Therefore, one has a restriction map $G(\mathcal{S}) \rightarrow G(\mathcal{T})$. Next we will also need to consider the equivariant case where a torus acts on the algebraic stacks $\mathcal{S}$ and $\tilde{\mathcal{S}}$. In this case we will assume the following :

- the base scheme is an algebraically closed field so that the results on the fixed point stacks as in [J-4] section 6 apply,
- $\mathcal{T} = \mathcal{S}^{T'}$ and $\tilde{\mathcal{T}} = \tilde{\mathcal{S}}^{T'}$ for a fixed sub-torus T' of T and
- $\mathfrak{p} \subseteq R(T)$ is the prime ideal corresponding to T'.

4.0.49. *Basic pushforward hypothesis:II.* We will assume henceforth that the vector bundle K^0 satisfies one of the following hypotheses:

- there exists a class (which we denote) $\lambda_{-1}(\hat{K}^0)$ in $\pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{\mathcal{S}}))$ so that for each $t\varepsilon\mathbb{A}^1$, $i_t^*(\lambda_{-1}(\hat{K}^0))\varepsilon\pi_0(G_{\mathcal{T}\times t}((\hat{\mathcal{S}})_t))$ identifies with the class of $\lambda_{-1}(K^0)$ in $\pi_0(G(\mathcal{T}))$ *or*
- we are in the equivariant case.

Observe that in the latter case, one has the isomorphism

$$(\hat{\mathcal{S}})^{T'} = \mathcal{T} \times \mathbb{A}^1 \tag{4.0.49}$$

To see this it suffices to observe that there are no fixed vectors in the normal cone $C_{\mathcal{T}}(\mathcal{S}) \subseteq C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})$. Since the fixed point stack $\mathcal{T} = \mathcal{S}^{T'}$ ($\tilde{T} = (\tilde{\mathcal{S}})^{T'}$) is defined as a closed sub-stack of $\mathcal{S}$ ($\tilde{\mathcal{S}}$, respectively) (see [J-4] section 6), one may reduce this assertion to the case of schemes where it is well-known. (See, for example, the proof of Proposition 6.8 in [J-4].) Therefore: $\pi_0(G(\mathcal{T}\times\mathbb{A}^1), T)_{(\mathfrak{p})} = \pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{\mathcal{S}}); T)_{(\mathfrak{p})} \cong \pi_0 G(\hat{\mathcal{S}}; T)_{(\mathfrak{p})}$ and hence the class $\lambda_{-1}(K^0)$ in the first group lifts to a class in $\pi_0 G(\hat{\mathcal{S}}; T)_{(\mathfrak{p})}$. Observe also that in either case one may identify $\lambda_{-1}(\hat{K}^0)$ with a class in $\pi_0(K_{\hat{\mathcal{S}}}(\hat{\tilde{\mathcal{S}}}; T))$ (or a localization of the latter in the equivariant case) so that tensor product with this class is well-defined. A similar argument applies to show that the tensor product with the class $\lambda_{-1}(K^{-1})$ is well defined.

Definition 4.6. Observe that the class $i_1^*(\lambda_{-1}(\hat{K}^0))\varepsilon\pi_0(G_{\mathcal{T}\times 1}((\hat{\mathcal{S}})_1))$ ($\varepsilon\pi_0(G_{\mathcal{T}\times 1}((\hat{\mathcal{S}})_1), T)$ in the equivariant case) maps to a class in $\pi_0(G(\mathcal{S})) \cong \pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}))$ (in $\pi_0(G(\mathcal{S}, T)) \cong \pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}, T))$, respectively). (Recall $(\hat{\mathcal{S}})_{t=1} = \mathcal{S}$.) We will denote this class by $\lambda_{-1}(K^0_{\mathcal{S}})$.

Examples 4.7. There are various situations where the hypothesis (4.0.49) is satisfied. The simplest is where the stacks $\mathcal{T}$ and $\mathcal{S}$ are smooth so that the above K-groups identify with the corresponding homotopy groups of G-theory. In this case the required hypothesis is satisfied, by taking the obstruction theories to be $\Omega_{\mathcal{S}}[0]$ and $\Omega_{\mathcal{T}}[0]$. Observe that now K^0 identifies with the conormal sheaf. Using deformation to the normal cone, one may define a class as required.

An alternate situation is the following. Assume that there exists a vector bundle $\mathcal{K}^0_{\mathcal{S}}$ on $\mathcal{S}$ and a section s of $\mathcal{K}^0_{\mathcal{S}}$ so that $\mathcal{T}$ is defined as the sub-stack where s vanishes. Let $K^0_{\mathcal{S}} = \Gamma(\mathcal{K}^0_{\mathcal{S}})$ = the sheaf of sections of $\mathcal{K}^0_{\mathcal{S}}$. Then $\lambda_{-1}(\mathcal{K}^0_{\mathcal{S}})$ is a perfect complex of $\mathcal{O}_{\mathcal{S}}$-modules which is a resolution of $u_*(\mathcal{O}_{\mathcal{T}})$. Let $\hat{\mathcal{K}}^0_{\mathcal{S}} = Bl_{\mathcal{T}\times 0}(\mathcal{K}^0_{\mathcal{S}} \times \mathbb{A}^1)$: this is a vector bundle on $\hat{S}$. *Observe that* $(\hat{\mathcal{K}}^0_{\mathcal{S}})_{t=t_0|\mathcal{T}} \cong K^0$ *where* $t_0\varepsilon\mathbb{A}^1$ *is any closed point and* K^0 *is defined as in 4.0.27.* Therefore, the class of $\lambda_{-1}(\hat{K}^0_{\mathcal{S}})\varepsilon\pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{S}))$ satisfies the hypotheses in (4.0.49).

Henceforth we will denote $C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0$ by $D_{\mathcal{T}}$ and the corresponding closed immersion $D_{\mathcal{T}} \rightarrow D = C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0$ by w.

Proposition 4.8. *(Preliminary pushforward formula) Assume the above hypotheses. Now one obtains the formulae* *i)* $w_*(\mathcal{O}_{\mathcal{T}}^{virt} \otimes g^* \pi_{\mathcal{F}}^* \lambda_{-1}(K^{-1})) = \mathcal{O}_{\mathcal{S}}^{virt} \otimes f^* \pi_{\mathcal{E}}^*(\lambda_{-1}(K_{\mathcal{S}}^0))$ *in* $\pi_0(G_{\mathcal{T}}(D)) \cong \pi_0(G(\mathcal{T}, \mathcal{O}_{\mathcal{T}}))$ *and hence in* $\pi_0(G_{\mathcal{S}}(D) \cong \pi_0(G(\mathcal{S}))$. *In the equivariant case, the corresponding formula holds in the above Grothendieck groups localized at the prime ideal* $\mathfrak{p}$.

ii) $w_*(\tau(\mathcal{O}_{\mathcal{T}}^{virt}) \otimes ch(g^* \pi_{\mathcal{F}}^* \lambda_{-1}(K^{-1}))) = \tau(\mathcal{O}_{\mathcal{S}}^{virt}) \otimes ch(f^* \pi_{\mathcal{E}}^*(\lambda_{-1}(K_{\mathcal{S}}^0))$ *in* $H_*^{Br}(\mathcal{S}, \Gamma(*))$.

(Here $\pi_{\mathcal{E}} : \mathcal{E}_1 \to \mathcal{S}$ *is the obvious projection and* $f : D \to \mathcal{E}_1$ *is the map considered in Definition 4.0.22.)* τ *is the Riemann-Roch transformation defined with values in Bredon-style homology and ch is the Chern-character with values in Bredon-style cohomology: these were defined in [J-5].*

Proof. One begins with the (homotopy) commutative diagram:

$$\begin{array}{ccccc}
G_{\mathcal{T}}(D_{\mathcal{T}}) & \xrightarrow{w_*} & G_{\mathcal{T}}(D) & \longrightarrow & G_{\mathcal{S}}(D) \\
\simeq \uparrow i_1^* & & \simeq \uparrow i_1^* & & \\
G_{\mathcal{T} \times \mathbb{A}^1}(D_{\mathcal{T}} \times \mathbb{A}^1) & \xrightarrow{w_*} & G_{\mathcal{T} \times \mathbb{A}^1}(\hat{D}) & & \\
i_0^* \downarrow \simeq & & i_0^* \downarrow \simeq & & \\
G_{\mathcal{T}}(D_{\mathcal{T}}) & \xrightarrow{0_*} & G_{\mathcal{T}}(C_{\mathcal{T}}(\mathcal{E}_0)) & &
\end{array}$$

Recall $D = C\mathcal{E}_0 = C_{\mathcal{S}}(\tilde{\mathcal{S}}) \underset{\mathcal{S}}{\times} \mathcal{E}_0$ and $D_{\mathcal{T}} = C\mathcal{F}_0 = C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0$. The vertical maps in the first column are weak-equivalences provided by the homotopy property of G-theory and the maps in the rightmost column are the weak-equivalences provided by the usual devissage and the homotopy property in G-theory. By devissage, the G-theory with supports in $\mathcal{T}$ ($\mathcal{T} \times \mathbb{A}^1$) identifies with the G-theory of $\mathcal{T}$ (the G-theory of $\mathcal{T} \times \mathbb{A}^1$, respectively). Therefore the horizontal maps in the diagram may be identified with the identity showing that the squares commute.

The image of $\mathcal{O}_{\mathcal{T}}^{virt} \otimes \pi_0^* f_0^* \pi^*(K^{-1})$ by the map 0_* in the bottom row is described by the last theorem. We next show that the class $\mathcal{O}_C^{virt}$ lifts to the class $\mathcal{O}_{\mathcal{S}}^{virt}$ under the isomorphisms forming the right vertical maps, i.e. the class of $\mathcal{O}_{\hat{\mathcal{S}}}^{virt}$ in $\pi_0(G_{\mathcal{T} \times \mathbb{A}^1}(\hat{D})$ maps under the map i_1^* (i_0^*) to the class of $\mathcal{O}_{\mathcal{S}}^{virt}$ in $\pi_0(G_{\mathcal{T}}(D))$ (the class of $\mathcal{O}_{C_{\mathcal{T}}(\mathcal{S})}^{virt}$ in $\pi_0(G_{\mathcal{T}}(C_{\mathcal{T}}(\mathcal{E}_0)))$, respectively).

For this recall first that

$$\mathcal{O}_{\hat{\mathcal{S}}}^{virt} = (\hat{\pi} \hat{f})^*(\lambda_{-1}(\Omega_{\hat{\tilde{\mathcal{S}}}|\hat{\mathcal{S}}/\mathbb{A}^1}) \otimes \lambda_{-1}(\hat{E}^{-1})).$$

Therefore,

$$i_0^*(\mathcal{O}_{\hat{\mathcal{S}}}^{virt}) = (\hat{\pi} \hat{f})_{t=0}^*(\pi^*(\lambda_{-1}(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) \otimes \pi^*(\lambda_{-1}(E_{|\mathcal{T}}^{-1})))) = \pi_1^* \phi_1^*(\pi^*(\lambda_{-1}(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}})) \otimes \pi^*(\lambda_{-1}(E_{|\mathcal{T}}^{-1})))$$

$$= \pi_0^* f_0^*(\lambda_{-1}(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}} \otimes \lambda_{-1}(E_{|\mathcal{T}}^{-1})) = \mathcal{O}_C^{virt}$$

since $f_0 \circ \pi_0 = \pi \circ \phi_1 \circ \pi_1$. Clearly,

$$i_1^*(\mathcal{O}_{\hat{\mathcal{S}}}^{virt}) = f^* \pi^*(\lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}}) \otimes \lambda_{-1}(E^{-1})) = \mathcal{O}_{\mathcal{S}}^{virt}$$

Observe that $\mathcal{O}_{\hat{\mathcal{S}}}^{virt}$ has supports in $\hat{\mathcal{S}}$ while $\mathcal{O}_C^{virt}$ has supports in $\mathcal{T}$ and $\mathcal{O}_{\mathcal{S}}^{virt}$ has supports in $\mathcal{S}$. Recall the class $\lambda_{-1}(\hat{K}^0)$ has supports in $\mathcal{T} \times \mathbb{A}^1$ and $\lambda_{-1}(K^0)$ has supports in $\mathcal{T}$. Therefore, $\mathcal{O}_{\hat{\mathcal{S}}}^{virt} \otimes (\hat{\pi} \hat{f})^* \lambda_{-1}(\hat{K}^0)$ has supports in $\mathcal{T} \times \mathbb{A}^1 \subseteq \hat{\mathcal{S}}$; similarly $\mathcal{O}_{\mathcal{S}}^{virt} \otimes (\hat{\pi} \hat{f})_{t=1}^* i_{t=1}^*(\lambda_{-1}(\hat{K}^0))$ has supports in $\mathcal{T} \times 1 \subseteq \mathcal{S} \times 1$ while $\mathcal{O}_C^{virt}$ has supports contained in $\mathcal{T} \times 0 \subseteq C_{\mathcal{T}}(\mathcal{E}_{0|\mathcal{T}})$. (Since $\lambda_{-1}(K_{\mathcal{S}}^0)$ lifts to a class in $\pi_0(K_{\mathcal{T}}(\mathcal{S}))$ it follows that one may take the product of the lifts of the classes $\lambda_{-1}(K^0)$ and $\mathcal{O}_C^{virt}$. A corresponding reasoning shows that the remaining tensor products above are also defined at the level of G-theory.) This completes the proof of the proposition in the non-equivariant case.

In the equivariant case the proof is exactly the same after localization; the key point is that after tensoring the above candidates for the virtual structure sheaves with the classes $(\hat{\pi} \hat{f})_t^*(\lambda_{-1}(\hat{K}^0))$, the resulting complexes all live in the appropriate Grothendieck groups localized at the prime ideal $\mathfrak{p}$, and hence in the above localized

Grothendieck groups with supports in $\mathcal{T} \times \mathbb{A}^1$; therefore they identify under the isomorphisms defined by i_0^* and i_1^*.

The formula ii) in the proposition follows from the first by applying the Riemann-Roch theorem and making use of the property (vii) in Theorem 1.1 of [J-5] (which relates the Todd homomorphism and the Chern character with values in Bredon-style homology and cohomology, respectively). □

Theorem 4.9. *(Pushforward formula) Assume the above hypotheses. Now one obtains the formulae*

i) $u_*(\bar{\mathcal{O}}_{\mathcal{T}}^{virt} \otimes \lambda_{-1}(K^{-1})) = \bar{\mathcal{O}}_{\mathcal{S}}^{virt} \otimes \lambda_{-1}(K_{\mathcal{S}}^0)$ *in* $\pi_0(G_{\mathcal{T}}(\mathcal{S}))$ *and hence in* $\pi_0(G(\mathcal{S}))$ *in the non-equivariant case and in the above groups localized at the prime ideal* $\mathfrak{p}$ *in the equivariant case.*

ii) $u_*(\tau(\bar{\mathcal{O}}_{\mathcal{T}}^{virt}) \otimes ch(\lambda_{-1}(K^{-1}))) = \tau(\bar{\mathcal{O}}_{\mathcal{S}}^{virt}) \otimes ch(\lambda_{-1}(K_{\mathcal{S}}^0))$ *in* $H_*^{Br}(\mathcal{S}, \Gamma(*))$.

The last formula also holds in equivariant forms of homology (and cohomology) (as in [J-5] Definition 5.12) in the equivariant case .

Proof. It suffices to interpret the formula of the last theorem in the form stated. For that, we recall the cartesian squares:

(4.0.50)
$$\begin{array}{ccc} T\tilde{\mathcal{S}}_{|\mathcal{S}} & \longrightarrow & C_{\mathcal{S}}(\tilde{\mathcal{S}})\underset{\mathcal{S}}{\times}\mathcal{E}_0 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ \mathcal{S} & \xrightarrow{0_{\mathcal{E}_1}} & \mathcal{E}_1 \end{array}$$

and

(4.0.51)
$$\begin{array}{ccc} T\tilde{\mathcal{T}}_{|\mathcal{T}} & \longrightarrow & D_{\mathcal{T}} \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle g} \\ \mathcal{T} & \xrightarrow{0_{\mathcal{F}_1}} & \mathcal{F}_1 \end{array}$$

Let $\pi_{\mathcal{E}}$ be the projection $\mathcal{E}_1 \to \mathcal{S}$, $z : \mathcal{S} \to T\tilde{\mathcal{S}}_{|\mathcal{S}}$ be the zero section and $i : T\tilde{\mathcal{S}}_{|\mathcal{S}} \to C_{\mathcal{S}}(\tilde{\mathcal{S}})\underset{\mathcal{S}}{\times}\mathcal{E}_0$ the map in top row of the first square. (Let $\pi_{\mathcal{F}}$ is the projection $\mathcal{F}_1 \to \mathcal{T}$, $z_{\mathcal{T}} : \mathcal{T} \to T\tilde{\mathcal{T}}_{|\mathcal{T}}$ be the zero section and $i_{\mathcal{T}} : T\tilde{\mathcal{T}}_{|\mathcal{T}} \to D_{\mathcal{T}}$ the map in the top row of the second square, respectively). Then one observes that the composition $i \circ z$ ($i_{\mathcal{T}} \circ z_{\mathcal{T}}$) is a section to the composite map $\pi_{\mathcal{E}} \circ f : C_{\mathcal{S}}(\tilde{\mathcal{S}})\underset{\mathcal{S}}{\times}\mathcal{E}_0 \to \mathcal{E}_1 \to S$ (to the composite map $\pi_{\mathcal{F}} \circ g : D_{\mathcal{T}} \to \mathcal{F}_1 \to \mathcal{T}$, respectively). Recall $\bar{\mathcal{O}}_{\mathcal{T}}^{virt} \varepsilon \pi_0 G(\mathcal{T}) \cong \pi_0 G_{\mathcal{T}}(D_{\mathcal{T}})$ and in fact $i_{\mathcal{T}*}(z_{\mathcal{T}*}(\bar{\mathcal{O}}_{\mathcal{T}}^{virt}))$ identifies with $\mathcal{O}_{\mathcal{T}}^{virt}$ under the above isomorphism. Similarly $i_* z_*(\bar{\mathcal{O}}_{\mathcal{S}}^{virt}) \varepsilon \pi_0 G(\mathcal{S})$ identifies with $\mathcal{O}_{\mathcal{S}}^{virt}$ under the isomorphism $\pi_0 G(\mathcal{S}) \cong \pi_0 G_{\mathcal{S}}(C_{\mathcal{S}}(\tilde{\mathcal{S}})\underset{\mathcal{S}}{\times}\mathcal{E}_0)$. Therefore we obtain:

$$\mathcal{O}_{\mathcal{S}}^{virt} \otimes f^* \pi_{\mathcal{E}}^*(\lambda_{-1}(K_{\mathcal{S}}^0)) = i_* z_*(\bar{\mathcal{O}}_{\mathcal{S}}^{virt}) \otimes f^* \pi_{\mathcal{E}}^*(K_{\mathcal{S}}^0)$$
$$= i_* z_*(\bar{\mathcal{O}}_{\mathcal{S}}^{virt} \otimes z^* i^* f^* \pi_{\mathcal{E}}^*(K_{\mathcal{S}}^0)) = i_* z_*(\bar{\mathcal{O}}_{\mathcal{S}}^{virt} \otimes K_{\mathcal{S}}^0).$$

Recall the isomorphism $\pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}})) \cong \pi_0(G(\mathcal{S}))$ and $\pi_0(K_{\mathcal{T}}(\tilde{\mathcal{T}})) \cong \pi_0(G(\mathcal{T}))$. Therefore, the above tensor products define well-defined classes in G-theory. This provides the required identification of the right-hand-side of the formula in Theorem 4.9 i) with the right-hand-side of the formula in Proposition 4.8 i).

We may identify the left-hand-side of the formula in i) using similar arguments applied to the second square above:

$$w_*(\mathcal{O}_{\mathcal{T}}^{virt} \otimes g^* \pi_{\mathcal{F}}^*(\lambda_{-1}(K^{-1}))) = w_*(i_{\mathcal{T}*} z_{\mathcal{T}*}(\bar{\mathcal{O}}_{\mathcal{T}}^{virt}) \otimes g^* \pi_{\mathcal{F}}^* \lambda_{-1}(K^{-1})) = w_*(i_{\mathcal{T}*} z_{\mathcal{T}*}(\bar{\mathcal{O}}_{\mathcal{T}}^{virt} \otimes \lambda_{-1}(K^{-1}))$$
$$= i_* z_* u_*(\bar{\mathcal{O}}_{\mathcal{T}}^{virt} \otimes \lambda_{-1}(K^{-1}))$$

The last identification uses $w \circ i_{\mathcal{T}} \circ z_{\mathcal{T}} = i \circ z \circ u$.

The second formula in the theorem follows from the first by applying Riemann-Roch. □

Remark 4.10. One would have liked to prove the equality in the first formula of the last theorem in $\pi_0 G_{\mathcal{T}}(\mathcal{S}, \mathcal{O}_{\mathcal{S}}^{virt})$; however, this does not seem to hold because the class of the virtual structure sheaf $\mathcal{O}_{\tilde{\mathcal{S}}}^{virt}$ does not seem to specialize to the classes of the other virtual structure sheaves unless one uses G-theory in the usual sense.

4.0.52. **Proofs of Theorems 1.4 and 1.7.** The last theorem readily proves the first two formulae in Theorem 1.4 and the first formula in Theorem 1.7. Observe the maps from Bredon homology and cohomology to smooth homology and cohomology are compatible with the pairings between cohomology and homology: this is proved in Theorem 1.2 of [J-5]. Moreover, it is observed in the same theorem that the Chern character maps into Bredon cohomology and smooth cohomology are related by the above map from Bredon cohomology to smooth cohomology. Therefore, denoting the map from Bredon homology to smooth homology (Bredon cohomology to smooth cohomology) by ϕ_* (ψ^*, respectively) one obtains the following formula by applying ϕ_* to the formula 1.1.2:

$$u_*(\phi_*(\tau(\mathcal{O}_{\mathcal{T}}^{virt}).\psi^*(ch(\lambda_{-1}(K^{-1})))) = \phi_*(\tau(\mathcal{O}_{\mathcal{S}}^{virt})).\psi^*(ch(\lambda_{-1}(K^0_{\mathcal{S}}))) \tag{4.0.53}$$

(Recall from Theorem 1.2 of [J-5] that ϕ_* is compatible with push-forwards by representable proper maps.) Now we multiply both sides of the above equation by $(Td(T\mathcal{S})^{virt})^{-1} = Td(E_0)^{-1}.Td(E_1)$ where $E_i = E^{\vee}_{-i}$. Making use of the projection formula, the left-hand-side of (4.0.53) multiplied by $(Td(T\mathcal{S})^{virt})^{-1}$ becomes:

$$u_*(\phi_*(\tau(\mathcal{O}_{\mathcal{T}}^{virt})).Td(E_{0|\mathcal{T}})^{-1}.Td(E_{1|\mathcal{T}}).\psi^*(ch(\lambda_{-1}(K^{-1})))).$$

At this point the short exact sequences $0 \rightarrow F_i \rightarrow E_{i|\mathcal{T}} \rightarrow K_i \rightarrow 0$ with $K_i = (K^{-i})^{\vee}$ provides the relations $Td(E_{i|\mathcal{T}}) = Td(F_i).Td(K_i)$. Substituting these in the last formula and observing that $K^0_{\mathcal{S}}$ restricts to K^0, we obtain the formula

$$u_*(\phi_*(\tau(\mathcal{O}_{\mathcal{T}}^{virt})).(Td(T\mathcal{T})^{virt})^{-1}.Ch(\lambda_{-1}(K^{-1})).Td(K_1)) = \phi_*(\tau(\mathcal{O}_{\mathcal{S}}^{virt})).(Td(T\mathcal{S})^{virt})^{-1}.Ch(\lambda_{-1}(K^0_{\mathcal{S}})).Td((K^0_{\mathcal{S}})^{\vee}).$$

Now the definition of the virtual fundamental classes in smooth homology as in Definition 2.3 and the observation that for a vector bundle V, the Euler class of $V^{\vee}$ in smooth homology, $e(V^{\vee}) = Td(V^{\vee}).Ch(\lambda_{-1}(V))$ (see [FL] p.22) completes the proof of the formula 1.1.3 in Theorem 1.4. The second formula in Theorem 1.7 follows by similar reasoning.

Examples 4.11. a) Observe that the when the stacks $\mathcal{S}$ and $\mathcal{T}$ are smooth we recover the usual pushforward formula for the structure sheaves. i.e. We may let the obstruction theories for $\mathcal{S}$ and $\mathcal{T}$ to be given by $\Omega_{\mathcal{S}}[0]$ and $\Omega_{\mathcal{T}}[0]$. Then we recover the familiar formulae: $u_*(\mathcal{O}_{\mathcal{T}}) = \mathcal{O}_{\mathcal{S}}.\lambda_{-1}(N^{\vee})$ where $N^{\vee}$ denotes the conormal sheaf associated to the closed immersion $u : \mathcal{T} \rightarrow \mathcal{S}$. Applying the usual Riemann-Roch to this formula, then provides $u_*([\mathcal{T}]) = [\mathcal{S}].e(N)$ where $e(N)$ denotes the Euler class of N, $[\mathcal{T}]$ ($[\mathcal{S}]$ denotes the fundamental class of $\mathcal{T}$ ($\mathcal{S}$, respectively). All the remaining examples will fit into the second class of examples considered in 4.7.

b) Next we consider the following situation, in preparation for the general case of the setting of the conjecture of Cox, Katz and Lee as in Theorem 1.6. Accordingly X is a smooth projective variety and Y is a closed sub-variety. We will further assume that X is *convex*, for example, X is a flag variety. Let V be a *convex* vector bundle on X, so that $H^1(C, f^*(V)) = 0$ for all genus 0 stable maps $f : C \rightarrow X$ and let s be a section of V so that Y identifies with the zeros of s. $\beta \varepsilon CH_1(X, \mathbb{Z})$, $\gamma \varepsilon CH_1(Y, \mathbb{Z})$ are cycle classes so that γ maps to β under the map $Y \rightarrow X$. We consider the moduli stacks $\mathfrak{M}_{0,n}(X, \beta)$ and $\mathfrak{M}_{0,n}(Y, \gamma)$. Let $e_k : \mathfrak{M}_{0,n}(X, \beta) \rightarrow X$ be the obvious map sending the stable map $f : (C, p_1, ..., p_n) \rightarrow X$ to $f(p_k)$. The *universal stable curve* over $\mathfrak{M}_{0,n}(X, \beta)$ is $\pi_{n+1} : \mathfrak{M}_{0,n+1}(X, \beta)$ which ignores the last marked point and contracts any components which have become unstable. Let $\mathcal{V}_{\beta,n} = \pi_{n+1*}e^*_{n+1}(V)$ which is a vector bundle on $\mathfrak{M}_{0,n}(X, \beta)$ by the convexity of V. Observe that the section s defines a section σ of the bundle $\mathcal{V}_{\beta,n}$ such that $\sqcup_{i_*(\gamma)=\beta}\mathfrak{M}_{0,n}(Y, \gamma)$ identifies with the zeros of the section σ.

Now we obtain the cartesian square:

$$\begin{array}{ccc} \sqcup_{i_*(\gamma)=\beta}\mathfrak{M}_{0,n}(Y, \gamma) & \xrightarrow{s'} & \mathfrak{M}_{0,n}(X, \beta) \\ \downarrow & & \downarrow 0 \\ \mathfrak{M}_{0,n}(X, \beta) & \xrightarrow{\sigma} & \mathcal{V}_{\beta,n} \end{array} \tag{4.0.54}$$

Observe that this is a diagram as in 4.0.26, with $\tilde{\mathcal{S}} = \mathcal{V}_{\beta,n}$, $\tilde{\mathcal{T}} = \mathcal{S} = \mathfrak{M}_{0,n}(X, \beta)$ and $\mathcal{S}_0 = \sqcup_{i_*(\gamma)=\beta}\mathfrak{M}_{0,n}(Y, \gamma)$. Now we let $E^{-1} = \Gamma(\mathcal{V}_{\beta,n})$ = the sheaf of sections of $\mathcal{V}_{\beta,n}$, $E^0 = 0^*(\Omega_{\mathcal{V}_{\beta,n}})$ with the obvious map $E^{-1} \rightarrow E^0$. We also let

$F^{-1} = s'^*(E^{-1})$ and $F^0 = s'^*\Omega_{\mathcal{S}}$ with the map $F^{-1} \to F^0$ defined as dual to the following map. The differential of the section σ defines a map $T\tilde{\mathcal{T}} \to T\mathcal{V}_{\beta,n}$: we compose this with the projection $T\mathcal{V}_{\beta,n} \to \mathcal{V}_{\beta,n}$ to obtain a map $T\tilde{\mathcal{T}} \to \mathcal{V}_{\beta,n}$. Now observe that the $F^{-1} = s'^*(E^{-1})$ so that $K^{-1} = 0$ and $K^0 = kernel(s'^*0^*(\Omega_{\mathcal{V}_{\beta,n}}) \to s'^*(\Omega_{\mathcal{S}})$ which identifies with $s'^*(E^{-1})$ again.

Theorem 4.9 shows that with the above obstruction theories, one obtains the formula:

$$\oplus_{i_*(\gamma)=\beta} i_{\gamma *}(\mathcal{O}_{\mathfrak{M}_{0,n}(Y,\gamma)}) = \lambda_{-1}(\Gamma(\mathcal{V}_{\beta,n})).\mathcal{O}_{\mathfrak{M}_{0,n}(X,\beta)}$$

c). Next we consider a generalization of the case in the previous example, where X is no longer required to be *convex*, but only smooth. We will also require that V satisfy the following hypotheses:

i) V is generated by global sections and ii) the exact sequence $\Gamma(X,V)\otimes\mathcal{O}_X \to V \to 0$ defines a closed immersion of X in the Grassmanian of r-planes in $\mathbb{A}^n$, where $n = dim(\Gamma(X,V))$.

In this situation we may first assume that X is imbedded in the Grassmanian, $G(r,k)$. Moreover, the section σ induces a section σ_Q of the universal quotient bundle Q on $G(r,k)$ via the tautological quotient mapping $H^0(X,V)\otimes \mathcal{O}_{G(r,k)} \to Q$. Let $G \subseteq G(r,k)$ be the zero locus of σ_Q. It follows that σ_Q is a regular section of Q, that $G \cong G(r,k-1)$ and that $Y = X \cap G$. Let $\beta \varepsilon H_2(X)$ be fixed and let β map to $d\varepsilon H_2(G(r,k)) \cong H_2(G) \cong \mathbb{Z}$. We let the vector bundle on $\mathfrak{M}_{0,n}(G(r,k),d)$ defined by Q be denoted $\mathcal{V}_{d,n}$. Therefore, we obtain the cartesian diagram as in 4.0.26 with $\mathcal{T} = \underset{i_*(\gamma)=\beta}{\sqcup} \mathfrak{M}_{0,n}(Y,\gamma)$, $\mathcal{S} = \mathfrak{M}_{0,n}(X,\beta)$, $\tilde{\mathcal{T}} = \mathfrak{M}_{0,n}(G,d)$ and $\tilde{\mathcal{S}} = \mathfrak{M}_{0,n}(G(r,k),d)$. Moreover $\tilde{\mathcal{T}}$ is defined by the vanishing of a section of $\mathcal{V}_{d,n}$.

In this case there is a proof of the required formula in [CKL] using prior work of [Gat]. However, we will show that Theorem 4.9 provides a quick independent proof. Let I define the sheaf of ideals defining $\mathcal{S}$ in $\tilde{\mathcal{S}}$. Since $\tilde{\mathcal{S}}$ is smooth, the complex $I/I^2 \to \Omega^1_{\tilde{\mathcal{S}}|\mathcal{S}}$ is an obstruction theory for $\mathcal{S}$. Now we claim, $F^{-1} = \Gamma(\mathcal{V}_{d,n|\mathcal{T}}) \oplus u^*(I/I^2) \to \Omega^1_{\tilde{\mathcal{S}}|\mathcal{T}} = u^*(\Omega^1_{\tilde{\mathcal{S}}|\mathcal{S}}) = F^0$ defines an obstruction theory for $\mathcal{T}$. First observe the short exact sequence:

$$\Gamma(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{\mathcal{O}_{\tilde{\mathcal{T}}}}{\otimes} \mathcal{O}_{\mathcal{T}} \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}})) \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{T}}) \to 0$$

where we have used $C_X(Y) =$ the normal cone of a closed substack X in Y and $\Gamma(C_X(Y))$ denotes its sheaf of sections, which is the conormal sheaf. Next observe that $u^*(I/I^2)(= u^*\Gamma(C_{\mathcal{S}}(\tilde{\mathcal{S}})))$ maps to $\Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}}))$ so that the composition into $\Gamma(C_{\mathcal{T}}(\tilde{\mathcal{T}}))$ is a surjection. Moreover there is a natural surjection $\Gamma(\mathcal{V}_{d,n|\mathcal{T}}) = u^*(\Gamma(\mathcal{V}_{d,n}) \to u^*\Gamma(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))$. It follows that one has an induced surjection $F^{-1} \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}}))$.

The differential $F^{-1} \to F^0$ is defined by the surjection $F^{-1} \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}}))$ followed by the obvious map of the latter to $\Omega^1_{\tilde{\mathcal{S}}|\mathcal{T}}$. The fact that the map $F^{-1} \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}}))$ is a surjection also shows that the sequence $F^{-1} \to \Omega^1_{\tilde{\mathcal{S}}|\mathcal{T}} \oplus \Gamma(C_{\mathcal{T}}\tilde{\mathcal{S}}) \to \Omega^1_{\tilde{\mathcal{S}}|\mathcal{T}} \to 0$ is *exact*. Therefore $F^{-1} \to F^0$ defines a perfect obstruction theory for $\mathcal{T}$. (See 4.0.19.)

Next one observes that there is a distinguished triangle $u^*(E^{-1}) \to F^{-1} \to \Gamma(\mathcal{V}_{d,n|\mathcal{T}})$ and that the map $u^*(E^0) \to F^0$ is an isomorphism. One views the map $u^*(E) \to F$ as double complex of sheaves and takes the total complex to obtain the mapping cone; one follows this by the shift $[-1]$ to obtain the homotopy fiber which is the complex K. These observations readily show that $K^{-1} = 0$ and that $K^0 = \Gamma(\mathcal{V}_{d,n|\mathcal{T}})$. Therefore, Theorem 4.9 provides the required formula directly.

4.1. **Proof of the conjecture of Cox, Katz and Lee.** (See Theorem 1.6.)

Finally we consider the most general case of the above examples, where X is still required to be smooth, but there are no other hypotheses on V except that it is convex. The required result will follow from the general pushforward formula and the examples 4.7 once we show that it is possible to choose weakly-compatible obstruction theories with $K^{-1} = 0$ and $\mathcal{K}^0_{\mathcal{S}} = \mathcal{V}_{\beta,0} = \pi_{n+1*}e^*_{n+1}(V)$ the vector bundle induced by V on $\mathcal{M}(X,\beta)_{0,n}$.

In this case let the base stack $B = \mathcal{M}_{0,n} =$ the stack of pre-stable curves with n-marked points. Clearly there is a forgetful map $F : \mathcal{M}_{0,n}(X,\beta) \to B$ which forgets the map but does not stabilize. Now one may make the following choice for a perfect relative obstruction theory for the stack $\mathcal{S} = \mathcal{M}_{0,n}(X,\beta)$: $E^\bullet = R\pi_{n+1*}e^*_{n+1}(\sigma_{\geq -1}L_X)[1]$ where L_X is the cotangent complex of X and $\sigma_{\geq -1}L_X$ its naive truncation to degrees ≥ -1. (Observe that the fibers of the map π_{n+1} are curves so that $R\pi_{n+1}$ has cohomological dimension at most 1.) In fact one has the following more explicit description of $\sigma_{\geq -1}L_X$: choose a closed immersion i of X into a smooth convex variety, $\mathbb{P}$, and let $\Gamma(C_X(\mathbb{P}))$ denote the corresponding co-normal sheaf. Then $\sigma_{\geq -1}L_X = \Gamma(C_X(\mathbb{P})) \rightarrow i^*(\Omega_{\mathbb{P}})$ as in (4.0.18).

This choice works even when X is not smooth, so that the same choice would be give us a relative obstruction theory for $\mathcal{T} = \mathcal{M}_{0,n}(Y, \gamma)$. However, to obtain a relative obstruction theory $F^\bullet$ weakly compatible with $E^\bullet$, one may make the following alternate choice: let $Ob_Y^\bullet$ be the two-term complex $V \oplus \Gamma(C_X(\mathbb{P}))_{|Y} \rightarrow \Omega_{\mathbb{P}|Y}$ in degrees -1 and 0 where the differential is defined as in the last example above. (As shown in the last example above, this in fact defines a perfect obstruction theory for Y.) Now a straight-forward spectral sequence computation will show that $F^\bullet = R\pi_{n+1*}ev_{n+1}^*(Ob_Y^\bullet)[1]$ is also a perfect obstruction theory for $\mathcal{M}_{0,n}(Y, \gamma)$.

To verify that these are weakly-compatible, one first needs to observe that the square

$$\begin{array}{ccc} M_{0,n+1}(Y,\gamma) & \xrightarrow{v} & M_{0,n+1}(X,\beta) \\ \downarrow{\scriptstyle \pi^Y_{n+1}} & & \downarrow{\scriptstyle \pi^X_{n+1}} \\ M_{0,n}(Y,\gamma) & \xrightarrow{u} & M_{0,n}(X,\beta) \end{array}$$

is cartesian. Moreover using the observation that π_{n+1} is flat of relative dimension 1, one may make use of Grothendieck duality and flat-base-change to conclude that the base-change map $u^*(R\pi^Y_{n+1*}) \rightarrow R\pi^X_{n+1*}v^*$ is an isomorphism of derived functors. Therefore, one observes that for the two obstruction theories, $E^\bullet$ and $F^\bullet$ defined above, $u^*(E^0) \simeq F^0$ and $F^{-1} = R\pi_{n+1*}ev_{n+1}^*(V) \oplus u^*(E^{-1})$. One may also observe using the convexity of the bundle V that $R^1\pi_{n+1*}ev_{n+1}^*(V) = 0$ so that $F^{-1} = \pi_{n+1*}ev_{n+1}^*(V) \oplus u^*(E^{-1})$. Now an argument as in the last two examples shows $K^{-1} = 0$ and $K^0_{\mathcal{S}} = \pi_{n+1*}ev_{n+1}^*(V)$. Therefore, Theorem 4.9 provides the required formula in Bredon homology.

Recall the definition of the virtual fundamental class as a term of weight $= d=$ the virtual dimension and degree $=$ twice the weight in $\tau(\mathcal{O}_{\mathcal{S}}^{virt})$: see 2.3. If c is the virtual codimension of the stacks, i.e. the difference between the virtual dimensions of the ambient stack $\mathcal{S}$ and the sub-stack $\mathcal{T}$, one defines the Euler class $e(\Gamma(\mathcal{E})^\vee)$ to be the term of weight c and degree $2c$ in $ch(\lambda_{-1}(\Gamma(\mathcal{E})))$. Observe that since the vector bundle $\mathcal{E}$ is obtained by pull-back from X, it descends to a vector bundle $\bar{\mathcal{E}}$ on the coarse moduli space of $\mathfrak{M}_{0,n}(X, \beta)$. Therefore, one can see from the definition of our Riemann-Roch transformation in [J-5] that the Chern-character is essentially the usual Chern-character of the corresponding complex $\lambda_{-1}(\Gamma(\bar{\mathcal{E}}))$ on the moduli space, so that it makes sense to take terms of a certain degree. Then one obtains the corresponding formula involving the virtual fundamental classes by taking the terms of appropriate degree. The last formula in smooth homology is obtained as in the proof of Theorem 1.4. This completes the proof of Theorem 1.6.

4.2. **Proof of Theorem 1.8.** Next assume the situation of Theorem 1.8. We first let $\mathcal{O}_{\tilde{S}}^{virt}$ be the complex of sheaves of $\mathcal{O}_{\tilde{S}}$ modules obtained as extension by zero of $\mathcal{O}_{\mathcal{S}}^{virt}$; similarly $\mathcal{O}_{\tilde{T}}^{virt}$ will be the extension by zero of $\mathcal{O}_{\mathcal{T}}^{virt}$ to $\tilde{T}$. We proceed to define a Gysin map $u_* : \pi_0(K_{\mathcal{T}}(\tilde{T}, \mathcal{O}_{\tilde{T}}^{virt}, T))_{(\mathfrak{p})} \rightarrow \pi_0(K_{\mathcal{S}}(\tilde{S}, \mathcal{O}_{\tilde{S}}^{virt}, T))_{\mathfrak{p}}$ where the Grothendieck groups are the Grothendieck groups of $Perf_{\mathcal{S}}(\tilde{S}, \mathcal{O}_{\tilde{S}}^{virt}, T)$ and of $Perf_{\mathcal{T}}(\tilde{T}, \mathcal{O}_{\tilde{T}}^{virt}, T)$.

Recall from (5.0.4) that an object $P \varepsilon Perf_{\mathcal{T}}(\tilde{T}, \mathcal{O}_{\tilde{T}}^{virt}, T)$ has a finite increasing filtration $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n$ so that for each $0 \leq i \leq n$, $F_i(P)/F_{i-1}(P) \simeq \mathcal{O}_{\tilde{T}}^{virt} \underset{\mathcal{O}_{\tilde{T}}}{\otimes} Q_i$, where $Q_i \varepsilon Perf_{\mathcal{T}}(\tilde{T}, \mathcal{O}_{\tilde{T}}, T)$ and is a complex of flat $\mathcal{O}_{\tilde{T}}$-modules. Therefore, it suffices to define u_* on a class of the form $\mathcal{O}_{\tilde{T}}^{virt} \underset{\mathcal{O}_{\tilde{T}}}{\otimes} Q$, where Q is a complex as one of the Q_is above.

Next observe the isomorphism $u_* : \pi_0(K_{\mathcal{T}}(\tilde{T}, T))_{(\mathfrak{p})} \rightarrow \pi_0(K_{\mathcal{S}}(\tilde{S}, T))_{(\mathfrak{p})}$. Moreover u^* is also an isomorphism, though not the inverse of u_*. Therefore, for each class $Q_{\mathcal{T}} \varepsilon \pi_0(K_{\mathcal{T}}(\tilde{T}, T))_{(\mathfrak{p})}$, there exists a unique class $Q_{\mathcal{S}} \varepsilon \pi_0(K_{\mathcal{S}}(\tilde{S}, T))_{(\mathfrak{p})}$ such that $u^*(Q_{\mathcal{S}}) = Q_{\mathcal{T}}$. Observe that there is a natural pairing $\pi_0(K_{\mathcal{S}}(\tilde{S}, T)) \otimes \pi_0(K_{\mathcal{S}}(\tilde{S}, \mathcal{O}_{\tilde{S}}^{virt}, T))$ induced by the tensor product. We define

$$u_*(\mathcal{O}_{\tilde{T}}^{virt} \underset{\mathcal{O}_{\tilde{T}}}{\otimes} Q_{\mathcal{T}} \underset{\mathcal{O}_{\tilde{T}}}{\otimes} \lambda_{-1}(K^{-1})) = \mathcal{O}_{\tilde{S}}^{virt} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(K^0_{\mathcal{S}}) \otimes Q_{\mathcal{S}} \tag{4.2.1}$$

Observe also that the classes $\lambda_{-1}(K^0_{\mathcal{S}})\varepsilon\pi_0(K_{\mathcal{S}}(\tilde S, T))_{(\mathfrak{p})}$ and $\lambda_{-1}(K^{-1})\varepsilon\pi_0(K_{\mathcal{T}}(\tilde T, T))_{(\mathfrak{p})}$ are invertible. Therefore, the above formula defines u_* on $\pi_0(K_{\mathcal{T}}(\tilde T, \mathcal{O}^{virt}_{\tilde T}, T))_{(\mathfrak{p})}$.

Observe that a pull-back $u^* : \pi_0(K_{\mathcal{S}}(\tilde S, \mathcal{O}^{virt}_{\tilde S}, T)) \to \pi_0(K_{\mathcal{T}}(\tilde T, \mathcal{O}^{virt}_{\tilde T}, T))$ is always defined. In view of the formula for u_* above, we see that the composition $u^* \circ u_*$ is given by:

$$u^*u_*(F) = F \underset{\mathcal{O}_{\tilde T}}{\otimes} \lambda_{-1}(K^0) \underset{\mathcal{O}_{\tilde T}}{\otimes} \lambda_{-1}(K^{-1})^{-1}, F\varepsilon\pi_0((K_{\mathcal{T}}(\tilde T, \mathcal{O}^{virt}_{\tilde T}, T)))_{(\mathfrak{p})} \tag{4.2.2}$$

(To obtain this first apply u^* to both sides of (4.2.1). This provides the formula: $u^*u_*(F \otimes \lambda_{-1}(K^{-1})) = F \otimes \lambda_{-1}(K^0)$. Now replace F by $F \otimes (\lambda_{-1}(K^{-1}))^{-1}$ in this formula. This is possible since the class $\lambda_{-1}(K^{-1})$ is invertible in the above localized K-groups.) We have thereby proven all but the last formula in Theorem 1.8.

We proceed to consider this next. By the hypotheses on the complexes $\Gamma^h(*)$ restricted to the smooth sites of schemes, we may identify the latter with $\mathbb{H}^*_{smt, ET\underset{T}{\times}\mathcal{S}}(ET\underset{T}{\times}\tilde{\mathcal{S}}, \Gamma(*))$. We will denote this by $H^*_T(\mathcal{S}, \Gamma(*))$. In view of this we obtain a restriction map $u^* : H^T_*(\mathcal{S}, \Gamma(*)) \to H^T_*(\mathcal{T}, \Gamma(*))$. Moreover one has a localization isomorphism $H^T_*(\mathcal{S}, \Gamma(*))_{(\mathfrak{p})} \cong H^T_*(\mathcal{T}, \Gamma(*))_{(\mathfrak{p})}$ induced by both u_* and hence u^*. Therefore the Euler-class $e(K_0)$ is the restriction of the class $e(K^{\mathcal{S}}_0)\varepsilon H^T_*(\mathcal{S}, \Gamma(*))_{(\mathfrak{p})}$ where $K^{\mathcal{S}}_0 = (K^0_{\mathcal{S}})^\vee$. Therefore, the projection formula applied to the formula (1.1.6) in Theorem 1.7 provides the formula:

$$u_*([\mathcal{T}]^{virt}.e(K_1).e(K_0)^{-1}) = u_*([\mathcal{T}]^{virt}.e(K_1).u^*(e(K^{\mathcal{S}}_0)^{-1})) = u_*([\mathcal{T}]^{virt}.e(K_1)).e(K^{\mathcal{S}}_0)^{-1} = [\mathcal{S}]^{virt}$$

Observe that our hypotheses imply that the the class $e(K_0)$ is invertible in the localized equivariant homology: $H^T_*(\mathcal{T}, \Gamma(*))_{(\mathfrak{p})} \cong H^T_*(\mathcal{S}, \Gamma(*))_{(\mathfrak{p})}$. Hence so is the class $e(K^{\mathcal{S}}_0)$. This proves the last formula in Theorem 1.8 and completes the proof of Theorem 1.8.

5. **Appendix: G-theory and K-theory of DG-stacks, Equivariant homology for algebraic stacks**

For the convenience of the reader, we summarize some of the key definitions and properties of dg-stacks and their G-theory and K-theory. Further details may be found in [J-5], [J-6] and [J-7].

Definition 5.1. A DG-stack is an algebraic stack $\mathcal{S}$ of Artin type which is also Noetherian provided with a sheaf of commutative dgas, $\mathcal{A}$, on $\mathcal{S}_{smt}$, so that $\mathcal{A}^i = 0$ for $i > 0$ $\mathcal{H}^i(\mathcal{A}) = 0$ for $i << 0$, $\mathcal{A}^0 = \mathcal{O}_{\mathcal{S}}$ and each $\mathcal{A}^i$ is a coherent $\mathcal{O}_{\mathcal{S}}$-module. We will further assume that each $\mathcal{H}^*(\mathcal{A})$ is a sheaf of graded Noetherian rings. (The need to consider such stacks should be clear from from the applications to virtual structure sheaves and virtual fundamental classes considered in this paper. See [J-7] for a comprehensive study of such stacks from a K-theory point of view.) For the purposes of this paper, we will define a *DG*-stack $(\mathcal{S}, \mathcal{A})$ to have property P if the associated underlying stack $\mathcal{S}$ has property P: for example, $(\mathcal{S}, \mathcal{A})$ is *smooth* if $\mathcal{S}$ is smooth.Often it is convenient to also include disjoint unions of such algebraic stacks into consideration.

5.0.3. *Morphisms of dg stacks.* A 1-morphism $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ of *DG*-stacks is a morphism of the underlying stacks $\mathcal{S}' \to \mathcal{S}$ together with a map $\mathcal{A} \to f_*(\mathcal{A}')$ compatible with the map $\mathcal{O}_{\mathcal{S}} \to f_*(\mathcal{O}_{\mathcal{S}'})$. Such a morphism will have property P if the associated underlying 1-morphism of algebraic stacks has property P. Clearly *DG*-stacks form a 2-category. If $(\mathcal{S}, \mathcal{A})$ and $(\mathcal{S}', \mathcal{A}')$ are two *DG*-stacks, one defines their *product* to be the product stack $\mathcal{S} \times \mathcal{S}'$ endowed with the sheaf of DGAs $\mathcal{A} \boxtimes \mathcal{A}'$. An *action* of a group scheme G on a *DG*-stack $(\mathcal{S}, \mathcal{A})$ will mean morphisms $\mu, pr_2 : (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A}) \to (\mathcal{S}, \mathcal{A})$ and $e : (\mathcal{S}, \mathcal{A}) \to (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A})$ satisfying the usual relations.

Let $i : \mathcal{S} \to \tilde{\mathcal{S}}$ denote a closed immersion of algebraic stacks. Assume $\mathcal{S}$ is provided with a sheaf of dgas $\mathcal{A}$ making $(\mathcal{S}, \mathcal{A})$ a dg-stack. One may now define a dg-structure sheaf $\tilde{\mathcal{A}} = i_*(\mathcal{A})$. For the following discussion we consider the category of modules over $\tilde{\mathcal{A}}$: clearly this discussion reduces to the case of modules over $\mathcal{A}$ by considering the case $i =$ the identity.

5.0.4. A left $\tilde{\mathcal{A}}$-module is a complex of sheaves M of $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules, bounded above and so that M is a sheaf of left-modules over the sheaf of dgas $\tilde{\mathcal{A}}$. The category of all left $\tilde{\mathcal{A}}$-modules and morphisms will be denoted $Mod_l(\mathcal{S}, \tilde{\mathcal{A}})$. A diagram $M' \to M \to M'' \to M[1]$ in $Mod_l(\mathcal{S}, \tilde{\mathcal{A}})$ is a *distinguished triangle* if it is one in $Mod_l(\mathcal{S}, \mathcal{O}_{\tilde{\mathcal{S}}})$. We define a map $M' \to M$ in $Mod_l(\mathcal{S}, \tilde{\mathcal{A}})$ to be a quasi-isomorphism if it is a quasi-isomorphism in $Mod(\mathcal{S}, \mathcal{O}_{\tilde{\mathcal{S}}})$. Since we assume $\mathcal{A}$ is a sheaf of commutative dgas, there is an equivalence of categories between left and right modules; therefore, henceforth we will simply refer to $\tilde{\mathcal{A}}$-modules rather than left or right $\tilde{\mathcal{A}}$-modules. The derived category $D(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ is the localization of $Mod_l(\mathcal{S}, \tilde{\mathcal{A}})$ by inverting maps that are quasi-isomorphisms. An $\tilde{\mathcal{A}}$-module M is *perfect* if the following holds: there exists a non-negative integer n and distinguished triangles $F_iM \to F_{i+1}M \to \mathcal{A}\overset{L}{\underset{\mathcal{O}_S}{\otimes}} P_i \to F_iM[1]$ in $Mod(\mathcal{S}, \tilde{\mathcal{A}})$, for all $0 \le i \le n-1$ so that $F_0M \simeq \tilde{\mathcal{A}}\overset{L}{\underset{\mathcal{O}_{\tilde{\mathcal{S}}}}{\otimes}} P_0$ with each

P_i a perfect complex of $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules. (In the presence of a group-scheme action G on the stack, we define a $\tilde{\mathcal{A}}$-module M to be perfect if it has a similar filtration with each P_i a perfect complex of G-equivariant $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules.) The morphisms between two such objects will be just morphisms of $\tilde{\mathcal{A}}$-modules. This category will be denoted $Perf(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$. One may similarly define the category $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ where the complexes P_i are required to be perfect complexes of $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules with supports contained in $\mathcal{S}$. Let $Perf_{fl,\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ denote the full sub-category of $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ consisting of flat $\tilde{\mathcal{A}}$-modules. We will let $Coh(\mathcal{S}, \mathcal{A})$ ($Per(\mathcal{S}, \mathcal{A})$) denote the above category with this Waldhausen structure.

An $\tilde{\mathcal{A}}$-module M is coherent if $\mathcal{H}^*(M)$ is bounded and finitely generated as a sheaf of $\mathcal{H}^*(\tilde{\mathcal{A}})$-modules. Again morphisms between two such objects will be morphisms of $\tilde{\mathcal{A}}$-modules. This category will be denoted $Coh(\mathcal{S}, \tilde{\mathcal{A}})$.

Definition 5.2. The categories $Coh(\mathcal{S}, \tilde{\mathcal{A}})$, $Perf(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ and $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ along with quasi-isomorphisms as $\tilde{\mathcal{A}}$-modules form Waldhausen categories with fibrations and weak-equivalences. The fibrations are maps of $\tilde{\mathcal{A}}$-modules that are degree's surjections (i.e. surjections of $\mathcal{O}_{\mathcal{S}}$-modules) and the weak-equivalences are maps of $\tilde{\mathcal{A}}$-modules that are quasi-isomorphisms. The K-theory (G-theory) spectra of $(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ will be defined to be the K-theory of the Waldhausen category $Perf(\mathcal{S}, \tilde{\mathcal{A}})$ ($Coh(\mathcal{S}, \tilde{\mathcal{A}})$, respectively) and denoted $K(\mathcal{S}, \tilde{\mathcal{A}})$ ($G(\mathcal{S}, \tilde{\mathcal{A}})$, respectively). $K_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ will denote $K(Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}}))$. When $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$, $K(\mathcal{S}, \mathcal{A})$ ($G(\mathcal{S}, \mathcal{A})$) will be denoted $K(\mathcal{S})$ ($G(\mathcal{S})$, respectively).

Proposition 5.3. *(i) There exists a natural tensor-product pairing* $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}) \otimes Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}}) \rightarrow Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ *making* $K(Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}}))$ *a module-spectrum over* $K(Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}))$.

(ii) Given a distinguished triangle $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ *of* $\tilde{\mathcal{A}}$*-modules, with two of* M', M *and* M'' *in* $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$*, the third also belongs to* $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$.

(iii) Let $M \varepsilon Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$*. Then there exists a flat* $\tilde{\mathcal{A}}$*-module* $\tilde{M} \varepsilon Perf_{fl,\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ *together with a quasi-isomorphism* $\tilde{M} \rightarrow M$.

Proof. We skip the details here. One may consult [J-6] and [J-7] for details. □

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