VANISHING OF ODD DIMENSIONAL INTERSECTION COHOMOLOGY FOR SPHERICAL VARIETIES IN POSITIVE CHARACTERISTICS

Roy Joshua

Abstract. In this note we show that the odd dimensional intersection cohomology sheaves vanish for all spherical varieties in all characteristics, extending the results of Michel Brion and the author in characteristic 0.

0. Introduction. Both the global intersection cohomology and the cohomology sheaves of the intersection cohomology complexes are known to vanish in odd degrees for a large number of varieties. One of the successes of equivariant intersection cohomology (introduced in [Bryl] and [J-1] independently) was to provide a geometric proof of this phenomenon. These results were extended and strengthened in [B-J]. One of the results established there was that the above vanishing phenomenon extends to all complex spherical varieties. In the present paper, we extend this to spherical varieties over an algebraically closed field of arbitrary positive characteristic.

In order to keep the paper rather self-contained, we begin by recalling the relevant facts about equivariant derived categories and equivariant intersection cohomology complexes in positive characteristics. (One may consult [J-1] and [J-2] for further details.) This concludes with a result (see Proposition (1.6)) about the direct images of the equivariant intersection cohomology complexes under finite maps. We discuss the local structure and weak-resolution of singularities for spherical varieties in positive characteristics in the next section. The last section is devoted to a proof of the above vanishing result, which may be summarized as follows:

Main Theorem. Let G denote a connected reductive group, X a G-spherical variety and $\mathcal{L}$ a G-equivariant local system on the open dense orbit. Now $\mathcal{H}^i(IC(X;\mathcal{L})) = 0$ for all *odd* i. In case X is also projective, $IH^i(X;\mathcal{L}) = 0$ for all *odd* i as well. (Here $IC(X;\mathcal{L})$ denotes the intersection cohomology complex with the middle perversity whose restriction to the open G-orbit is $\mathcal{L}$ and $IH^*(X;\mathcal{L})$ denotes the corresponding equivariant intersection cohomology groups.) $\square$

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Throughout the paper, we will let k denote an algebraically closed field of characteristic $p > 0$ and will only consider schemes of finite type over k. l will denote a prime different from p.

1. Equivariant derived categories and equivariant intersection cohomology in positive characteristics.

(1.0) We begin by recalling the definition of equivariant derived categories and equivariant intersection cohomology from [J-1] and [B-J]. The main difference from the characteristic 0 situation will be that we adopt the definition of EG, BG and $EG\underset{G}{\times}X$ as simplicial schemes defined in the usual manner. (See [Fr] pp. 8-9.) Observe that now $BG_0 = Spec(k)$ and we will call this the base point of $BG_\bullet$. The above simplicial schemes will be provided with the following étale topology. Let $X_\bullet$ denote a simplicial scheme. Now $Et(X_\bullet)$ will denote the category whose objects are étale maps $u : U \to X_n$ for some n; a morphism $u : U \to X_n$ to $v : V \to X_m$ will denote a map $w : U \to V$ lying over some structure map $X_n \to X_m$. Now a sheaf F on $ET(X_\bullet)$ is given by a collection of sheaves $\{F_n|n\}$, with F_n a sheaf on $Et(X_n)$ provided with maps $\phi_\alpha : \alpha^*(F_m) \to F_n$ for any structure map $\alpha : X_n \to X_m$. (These maps are required to satisfy an obvious compatibility condition. See [Fr] p.14.) This definition applies to abelian sheaves as well as l-adic sheaves. $D_b(X_\bullet)$ will denote the derived category of complexes of l-adic sheaves that are bounded.

(1.1) *Equivariant derived categories (positive characteristics).* Now assume $X_\bullet$ is the simplicial scheme $EG\underset{G}{\times}X$ associated to the action of G on X. A sheaf F on $EG\underset{G}{\times}X$ will be called *equivariant* if the above structure maps $\{\phi_\alpha|\alpha\}$ are isomorphisms. The category of equivariant sheaves is an abelian sub-category closed under extensions in the category of all sheaves on $EG\underset{G}{\times}X$; therefore one defines $D_b^G(X)$ to be the full sub-category of $D_b(EG\underset{G}{\times}X)$ consisting of complexes K whose cohomology sheaves are all equivariant.

(1.2) *G-equivariant local systems in positive characteristics.* A G-equivariant local system on X is a G-equivariant l-adic sheaf $F = \{F_n|n\}$ on $EG\underset{G}{\times}X$ so that F_0 is a *lisse* sheaf on X. These correspond to l-adic representations of the étale fundamental group $\pi_1(EG\underset{G}{\times}X, \bar{x})$ (where $\bar{x}$ is a geometric point of X): this correspondence sends a G-equivariant local system to its stalk at the geometric point $\bar{x}$. To keep the notation uniform, we will identify geometric points with points: i.e. x will be denote $\bar{x}$ as well.

(1.3) Let $\mathcal{L}$ be a G-equivariant l-adic local system on $EG\underset{G}{\times}X$ such that the action of $\pi_1(EG\underset{G}{\times}X, x_o)$ on the stalk $\mathcal{L}_{x_o}$ is through a finite quotient group F. Then the local system $\mathcal{L}$ corresponds to a representation of F on the $\mathbb{Q}_l$-vector space associated to $\mathcal{L}_{x_o}$ which splits up into the sum of irreducible representations of F on $\mathbb{Q}_l$-vector spaces. Since the group F is finite, one may show by standard arguments that each of the summands corresponds to an irreducible l-adic representation of F and therefore to a G-equivariant irreducible local system on $EG\underset{G}{\times}X$.

(1.3.*)*It follows that, under the hypothesis that $\pi_1(EG\underset{G}{\times}X, x_o)$ acts on the stalk $\mathcal{L}_{x_o}$ through a finite quotient group, any G-equivariant local system $\mathcal{L}$ is semi-simple.*

We will presently recall the definition of equivariant intersection cohomology from [J-1] p. 242. Let X be an equi-dimensional G-variety of dimension d. Let $\phi = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_d = X$ denote a filtration by closed G-invariant sub-varieties so that each X_i is closed in X and each $X_i - X_{i-1}$ is smooth, $i = 0, \ldots, n$. Next one considers the complementary filtration $U_1 \xrightarrow{j_1} U_2 \xrightarrow{j_2} \ldots U_d \xrightarrow{j_d} U_{d+1} = X$ where $U_i = X - X_{d-i}$ and j_i denotes the obvious G-equivariant open immersion. Now one applies the construction of in (1.1) to this filtration to obtain the following diagram:

(1.4.1)
$$\begin{array}{ccccccc}
EG\underset{G}{\times}U_1 & \xrightarrow{j_1^G} & EG\underset{G}{\times}U_2 & \xrightarrow{j_2^G} & \ldots & \xrightarrow{j_d^G} & EG\underset{G}{\times}X \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \ldots & & \downarrow{\scriptstyle \pi_d} \\
BG & \xrightarrow{id} & BG & \xrightarrow{id} & \ldots & \xrightarrow{id} & BG
\end{array}$$

Let $\mathcal{L}$ denote a G-equivariant local system on $EG\underset{G}{\times}U_1$. We extend $\mathcal{L}$ to $EG\underset{G}{\times}X$ to obtain a complex $IC^G(X;\mathcal{L})$ in $D_b^G(EG\underset{G}{\times}X)$, defined by $IC^G(X;\mathcal{L}) = \tau_{\leq d-1}Rj_{n*}^G \cdots \tau_{\leq 0}Rj_{0*}(\mathcal{L})$ (see [J-1] for details). This is the *equivariant intersection cohomology complex (with respect to the middle perversity)* obtained from $\mathcal{L}$. In case $\mathcal{L}$ is the constant sheaf $\underline{\mathbb{Q}}$ ($\underline{\mathbb{Q}_l}$ in positive characteristics), we will denote the corresponding complex by $IC^G(X)$. (Starting with [J-1], we have used the cohomology notation for perverse sheaves. This differs from the one adopted in [B-B-D] as follows: a complex of sheaves K on a variety X of dimension d is perverse if the dimensions of the supports of the sheaves $\mathcal{H}^n(K)$ and $\mathcal{H}^n(D(K)[2n])$ are $\leq d-n$ for all n. (In [B-B-D] a complex K is defined to be perverse if the dimensions of the supports of the sheaves $\mathcal{H}^n(K)$ and $\mathcal{H}^n(D(K))$ are $\leq -n$ for all n.) We define

(1.4.2)
$$IH_G^*(X;\mathcal{L}) = \mathbb{H}_G^*(X; IC^G(X;\mathcal{L})) = \mathbb{H}^*(EG\underset{G}{\times}X; IC^G(X;\mathcal{L}))$$

This is a module over $H^*(BG;\mathbb{Q}_l)$.

(1.5.1) **Theorem**. (*Degeneration of the spectral sequence in equivariant intersection cohomology*). Let X be a projective equi-dimensional G-variety, where G is connected. Let $\mathcal{L}$ denote a G-equivariant local system on an open dense smooth sub-variety of X such that $\mathcal{L}$ is semi-simple as a local system. Let $IC^G(X;\mathcal{L})$ denote the corresponding equivariant intersection cohomology complex. Now the spectral sequence:

$$E_2^{s,t} = H^s(BG; R^t\pi_*(IC^G(X;\mathcal{L}))) \Rightarrow IH_G^{s+t}(X;\mathcal{L})$$

degenerates, where $\pi : EG\underset{G}{\times}X \to BG$ is the obvious map. Thus, $IH_G^*(X;\mathcal{L}) \cong H^*(BG) \otimes IH^*(X;\mathcal{L})$.

Proof is essentially the same as in [J-1] Proposition (13) where only the case G is a one dimensional torus is considered. That G be connected is necessary to ensure that all local systems on BG are in fact constant. Let U denote an open smooth G-stable sub-variety of X on which $\mathcal{L}$ is a local system. Since X is equi-dimensional, U is the disjoint union of its connected components U_i all of which are of the same dimension. Since G is connected the U_i are stable under the group action. Now let $\mathcal{L}_i$ denote the G-equivariant local system on U defined by ${\mathcal{L}_i}_{|U_j} = \mathcal{L}_{|U_i}$ if $j = i$ and $= 0$ otherwise. Then one may see that $IC^G(X;\mathcal{L}) = \oplus_i IC^G(X;\mathcal{L}_i)$. Clearly each $\mathcal{L}_i$ is a semi-simple local system; therefore each $IC(X;\mathcal{L}_i)$ and hence $IC(X;\mathcal{L})$ is a pure perverse sheaf. Therefore the Hard Lefschetz theorem holds for $IH^*(X;\mathcal{L})$ and the same proof as in [J-1] Proposition (13) applies. □

(1.5.2) **Theorem**. Let X denote a projective equi-dimensional variety provided with the action of a torus T and let $\mathcal{L}$ denote a T-equivariant local system on an open smooth T-stable sub-variety of X. Assume that $\mathcal{L}$ is semi-simple as a local system. Let $i : X^T \to X$ denote the inclusion of the fixed point sub scheme. Now one obtains the isomorphisms after inverting all non zero elements of $H^*(BT)$ (i.e. on localization at the prime ideal (0)):

(1.5.2.∗) $$IH_T^*(X;\mathcal{L})_{(0)} \cong H^*(BT)_{(0)} \otimes IH^*(X;\mathcal{L}) \cong H^*(BT)_{(0)} \otimes \mathbb{H}^*(X^T; Ri^!IC(X;\mathcal{L})).$$

In particular, if $IH^i(X;\mathcal{L}) = 0$ for all *odd* i and x is an *isolated* fixed point of T, then $IH_x^i(X;\mathcal{L}) = 0$ for all *odd* i. □

Proof. The first isomorphism follows from (1.5.2) by localizing at (0). By the localization theorem in (1.4.5), one has the isomorphism:

(1.5.2.1) $$IH_T^*(X;\mathcal{L})_{(0)} \cong \mathbb{H}_T^*(X^T; Ri^!IC^T(X;\mathcal{L}))_{(0)}.$$

Now $\mathbb{H}_T^*(X^T; Ri^!IC^T(X;\mathcal{L}))_{(0)} \cong H^*(BT)_{(0)} \otimes \mathbb{H}^*(X^T, Ri^!IC^T(X;\mathcal{L}))_{(0)}$.

Next let $i_x : x \to X^T$ be the inclusion of an isolated fixed point of T. Then $Ri^!IC^T(X;\mathcal{L})$ breaks up into the sum of two complexes one of which is

$$Ri_x^!IC^T(X;\mathcal{L}) \cong Di_x^*D(IC^T(X;\mathcal{L})) \simeq (i_x^*IC^T(X;\mathcal{L}^\vee))^\vee[-2n]$$

where n is the dimension of X. This proves the last assertion of the theorem. □

Next we consider the following proposition which will be used repeatedly in the paper.

(1.6) **Proposition** Let $\pi : Y \to X$ denote a finite G-equivariant surjective map of equi-dimensional varieties. Let X_0 denote a G-stable open subscheme of X and let $Y_0 = X_0\underset{X}{\times}Y$ be its inverse image under π. If $\mathcal{L}$ is a G-equivariant local system on Y_0 and $\pi_0 : Y_0 \to X_0$ is the map induced by π, after possibly shrinking X_0, we may assume that $\pi_{0*}(\mathcal{L})$ is a G-equivariant local system on X_0. Moreover, now $\pi_*(IC^G(Y;\mathcal{L})) \simeq IC^G(X;\pi_{0*}(\mathcal{L}))$. □

Proof. Since the map π is finite, one may readily show that $\pi_*(IC^G(Y;\mathcal{L}))$ satisfies all the axioms of an intersection cohomology complex on X except possibly for the axiom that says there exists a dense smooth open subscheme V of X so that $\pi_*(IC^G(Y;\mathcal{L}))_{|V}$ is a local system. We proceed to show this presently.

Since intersection cohomology is invariant under normalization, we normalize all the varieties and assume the varieties are integral. Let y_0 and x_0 be the generic points of Y and X. Assume the characteristic of $k(x_0)$ is p. Now either $k(y_0)$ is separable over $k(x_0)$ or there exists some positive integer N so that $k(y_0)^{p^N}.k(x_0)$ is separable over $k(x_0)$. If $k(y_0)$ is separable over $k(x_0)$, we take $N = 0$, $p^N = 1$; otherwise p^N is the inseparable degree of $k(y_0)$ over $k(x_0)$. Let $Y^{(N)}$ be the pull-back of $\pi : Y \to X$ along the Frobenius $F^N : X \to X$ and let $Y \to Y^{(N)}$ be the map induced by $F^N : Y \to Y$ and $\pi : Y \to X$. Now the function field of $Y^{(N)}$ is $k(y_0)^{p^N}.k(x_0)$ which is separable over X. Thus the projection $Y^{(N)} \xrightarrow{\bar{\pi}} X$ induces an étale map of the generic point of $Y^{(N)}$ to the generic point of X; therefore there exist open subsets U of $Y^{(N)}$ and V of X so that the map $\bar{\pi}_{|U} : U \to V$ is étale. We proceed to show that we may take $U = \bar{\pi}^{-1}(V) = V \underset{X}{\times} Y^{(N)}$.

Let $F = \{z \,\epsilon\, Y^{(N)} | \bar{\pi} \quad \text{is not étale at} \quad z\}$. This is a proper closed subset of $Y^{(N)}$. Since $\bar{\pi}$ is induced by base-change from π, it is also finite. Therefore $\bar{\pi}(F)$ is a proper closed subset of X. Let $V = X - \bar{\pi}(F)$ and $U = \bar{\pi}^{-1}(V)$. Now $\bar{\pi}_{|U}$ is étale. Moreover, since $\bar{\pi}_{|U}$ is also obtained by base-change from π, it is also a *finite* map.

Finally let the inverse image of U in Y by the induced map $Y \to Y^{(N)}$ be W. i.e. $W = Y \underset{Y^{(N)}}{\times} U$. Now the map π restricted to W factors as the composition of the purely inseparable map $W \to U$ and the finite étale surjective map $U \to V$. Therefore, if $\mathcal{L}$ is a local system on W, the direct image $\pi_{|W*}(\mathcal{L})$ is a local system on V. The G-equivariance is clear from the above argument; therefore $\pi_*(IC^G(Y;\mathcal{L})) \simeq IC^G(X; \pi_{|W*}(\mathcal{L}))$. This completes the proof. $\square$

2. Spherical varieties in positive characteristics: local structure and weak-resolution of singularities

(2.1) In positive characteristic, the *local structure of spherical varieties* is due to Knop (see [Kn]), which we recall presently. Let G denote a connected reductive group and let X denote a G-spherical variety and let $x \,\epsilon\, X$. After replacing X by an open G-stable sub-scheme we may assume Gx is the unique closed G-orbit in X. Now one can find a G-linearized ample line bundle L, and a global section s of L which is an eigenvector of a Borel subgroup B of G. Let X_s be the open subset of X where s is non zero; let P be the subgroup of G which stabilizes X_s. Then P is a parabolic subgroup of G and $X_s = X_{Bx} = \{By | y \,\epsilon\, X, \overline{By} \supseteq Bx\}$ and $Gx \cap X_{Bx} = Bx$. Knop shows the following: the unipotent radical P_u of P acts properly on X_s, the quotient X_s/P_u exists and there exists a subvariety Z of X_s such that the natural maps $\rho : P_u \times Z \to X_s$ and $\bar{\rho} : Z \to X_s/P_u$ are finite and surjective. Moreover, Z is stable under a maximal torus T of P. We may further assume that T is contained in B.

In characteristic zero, there exists Z such that the above maps are isomorphisms and so that Z is stable under a Levi subgroup L of P. This may fail in positive characteristics, but X_s/P_u still has an action of P/P_u and the latter is isomorphic to L. Moreover, if X is spherical, then X_s contains a dense B-orbit and hence X_s/P_u contains a dense orbit of B/P_u so that X_s/P_u is an affine spherical L-variety. Moreover the the image of Lx is closed in X_s/P_u.

(2.2.1) Now this closed orbit is a torus ($\cong \mathbb{G}_m^r$, for some $r > 0$) by the choice of L. This follows from the observation that $Px = Bx$ (since $Gx \cap X_{Bx} = Bx$) and that therefore the stabilizer of x in L contains the derived group of L. Let f_1, ..., $f_r \,\epsilon\, k[Lx]$ be the eigen-vectors of L (actually of L/L_x) that provide the isomorphism $Lx \cong \mathbb{G}_m^r$. By standard arguments from [GIT], see for example [MFK] p. 195 there exists a large enough q so that f_i^q, $i = 1, ..., r$ extend to maps $\phi_i : X_s/P_u \to \mathbb{G}_m$ which are also eigen-vectors of L/L_x. Let $\phi = (\phi_1, ..., \phi_r) : X_s/P_u \to \mathbb{G}_m^r$ be the corresponding induced map. Now the composition $Lx \to X_s/P_u \to \mathbb{G}_m^r$ is the identity map raised to the q-th power. Let $\mathcal{S} = \phi^{-1}(1)$, $1 = (1, ..., 1) \,\epsilon\, \mathbb{G}_m^r$. Now the dimension of $\mathcal{S}$ = dimension of $X_s/P_u - r$ = the codimension of the G-orbit of x.

(2.2.2) The observation that the $\{\phi_i | i\}$ are eigen-vectors of L/L_x shows that $\mathcal{S}$ is stable under the action of L_x. Now one obtains an induced L-equivariant map $r : L \times \mathcal{S} \to X_s/P_u$. The same observation that the $\{\phi_i|i\}$ are eigen-vectors of L/L_x shows that the resulting induced map $\bar{r} : L \underset{L_x}{\times} \mathcal{S} \to X_s/P_u$ is bijective and therefore purely inseparable.

(2.3) *Weak resolution of singularities.* In positive characteristic, we have the following statement, weaker than resolution of singularities. Using the embedding theory of spherical homogeneous spaces (which also works in positive characteristics), one can construct a spherical variety $\tilde{X}$ along with a proper equivariant birational map $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is covered by open subsets $\tilde{X}_s$ as above, where the $(\tilde{X})_s/P_u$ are affine toric varieties with quotient singularities. In characteristic zero, this gives a resolution of singularities. In the general case, one obtains a rationally smooth $\tilde{X}$. To see this, consider the finite surjective map $\rho : P_u \times Z \to X_s$. The variety Z is rationally smooth since it is a simplicial toric variety (i.e. a toric variety whose fan is simplicial). Therefore $P_u \times Z$ is rationally smooth and the local cohomology groups of X_s with supports in any fixed geometric point are trivial in all degrees except the top degree. One may readily show that these are equal to $\mathbb{Q}_l$ as well proving X_s is rationally smooth.

3. Vanishing of odd dimensional intersection cohomology.

(3.1)**Lemma**. Let X denote a not-necessarily normal spherical variety and let $\mathcal{L}$ denote a local system on the open G-orbit. Let $\pi : \tilde{X} \to X$ denote the normalization and let $\pi^*(\mathcal{L}) = \tilde{\mathcal{L}}$. Let $IC(X;\mathcal{L})$ ($IC(\tilde{X};\tilde{\mathcal{L}})$) denote the intersection cohomology complex of X with respect to $\mathcal{L}$ (of $\tilde{X}$ with respect to $\mathcal{L}$, respectively). Now the intersection cohomology sheaves $\mathcal{H}^i(IC(X;\mathcal{L}))$ vanish for all odd i if and only if the sheaves $\mathcal{H}^i(IC(\tilde{X};\tilde{\mathcal{L}}))$ vanish for all odd i.

Proof. First observe that the map π is an isomorphism on the dense G-orbit. Therefore one may readily show that $R\pi_*(IC(\tilde{X};\tilde{\mathcal{L}})) \cong IC(X;\mathcal{L})$. Now consider the Leray spectral sequence:

$$E_2^{s,t} = R^s\pi_*\mathcal{H}^t(IC(\tilde{X};\tilde{\mathcal{L}}))) \Rightarrow R^{s+t}\pi_*(IC(\tilde{X};\tilde{\mathcal{L}})) \cong \mathcal{H}^{s+t}(R\pi_*(IC(\tilde{X};\tilde{\mathcal{L}}))) \cong \mathcal{H}^{s+t}(IC(X;\mathcal{L})).$$

Since π is a finite map, $E_2^{s,t} = 0$ for all $s > 0$ in this spectral sequence; therefore one obtains the isomorphism $\pi_*\mathcal{H}^t(IC(\tilde{X};\tilde{\mathcal{L}})) \cong E_2^{0,t} \cong E_\infty^{0,t} = \mathcal{H}^t(IC(X;\tilde{\mathcal{L}}))$. Now the lemma follows readily. $\square$

(3.2) **Proposition**. Let X denote a projective G-spherical variety and let $\mathcal{L}$ denote a G-equivariant local system on the open G-orbit. Then $IH^i(X;\mathcal{L}) = 0$ for all odd i. The same holds for all T-equivariant local systems.

Proof. Exactly the same proof as in characteristic 0 applies here in view of the discussion on local systems in positive characteristics as in (1.2) and (1.3). However, we sketch the details, for the sake of completeness. We will first use the following technique to reduce to the case of the constant local system. We may first assume that X is normal by (3.1). Next let $Gx_o \cong G/G_{x_o}$ denote the open G-orbit in X. Let $\tilde{X}$ denote the normalization of X in the function field $k(G/G^0_{x_o})$. Now we obtain the cartesian square

$$\begin{array}{ccc} G/G^0_{x_o} & \longrightarrow & \tilde{X} \\ {\scriptstyle \pi_o}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/G_{x_o} & \longrightarrow & X \end{array}$$

where the maps π_o and π are finite. Note that G acts on $\tilde{X}$ and that $\tilde{X}$ is a spherical G-variety. Let $\underline{\mathbb{Q}_l}$ denote the constant G-equivariant local system on $G/G^0_{x_o}$. Now $R\pi_{o*}(\underline{\mathbb{Q}_l}) = \pi_{o*}(\underline{\mathbb{Q}_l})$. The stalk of this sheaf at x_o is the l-adic regular representation of the finite group $G_{x_o}/G^0_{x_o}$. Therefore, by (1.3), the G-equivariant local system $\pi_{o*}(\underline{\mathbb{Q}_l})$ can be written as a sum $\underset{\chi}{\oplus}\dim(\chi)\mathcal{L}_\chi$, where $\mathcal{L}_\chi$ is the local system corresponding to the irreducible character χ of the finite group $G_{x_o}/G^0_{x_o}$ and the sum varies over all such characters. Therefore,

$$\pi_*IC(\tilde{X},\underline{\mathbb{Q}_l}) = \underset{\chi}{\oplus}\dim(\chi)IC(X,\mathcal{L}_\chi).$$

Taking the hyper-cohomology, it follows that

$$IH^i(\tilde{X}) = \underset{\chi}{\oplus} \dim(\chi) IH^i(X, \mathcal{L}_\chi)$$

for all i. Thus, it suffices to consider X with the constant local system.

Next, let $\pi : \tilde{X} \to X$ denote a G-equivariant weak-resolution of singularities as in (2.3). Now $\tilde{X}$ is a projective rationally smooth spherical variety. Now T acts on $\tilde{X}$ with only finitely many fixed points. The fixed point formula for torus actions (i.e. (1.5.2)) now gives the isomorphism:

$$H_T^*(\tilde{X}; \underline{\mathbb{Q}}_l)_0 \simeq H^*(BT) \otimes H^*(\tilde{X}^T; Ri^!(\underline{\mathbb{Q}}_l))$$

Since $\tilde{X}$ is rationally smooth, it follows that $Ri^!(\underline{\mathbb{Q}}_l) \simeq \underline{\mathbb{Q}}_l[-2n]$ where n is the dimension of $\tilde{X}$. It follows as in (1.5.2) that $H^n(\tilde{X}; \underline{\mathbb{Q}}_l) = 0$ for all odd n. Now the decomposition theorem in intersection cohomology shows that $IH^i(X; \underline{\mathbb{Q}}_l)$ is a split summand of $H^i(\tilde{X}; \underline{\mathbb{Q}}_l)$ for any i. The latter is trivial for all odd i; this completes the proof of the Proposition for all G-equivariant local systems. The assertion about the T-equivariant local systems follows from proposition (3.3) below. □

(3.3) **Proposition**. Let G denote a connected reductive group, T a fixed maximal torus, B a Borel subgroup containing T and P a parabolic subgroup containing B. Let X denote a scheme with an action by P. Now the restriction functor from the category of P-equivariant local systems on X to T-equivariant local systems is an equivalence of categories. If X is G-scheme, the restriction functor from the category of G-equivariant local systems on X to B (or T)-equivariant local systems is also an equivalence of categories.

Proof. Consider the diagram $B/T \to EB\underset{B}{\times}(B\underset{T}{\times}X) \to EB\underset{B}{\times}X$. (This is induced by the B-equivariant map $B\underset{T}{\times}X \to B\underset{B}{\times}X \cong X$ which is a smooth map with fibers B/T.) We choose a geometric point for X and for B; this choice provides base-points for all of the above simplicial schemes. If Z is a pointed simplicial scheme, we will let Z_{et} denote the *pointed étale topological type* of Z as in [Fr] chapter 4: recall this is an inverse system of pointed simplicial sets. Now Corollary 10.8 of [Fr] shows readily that in each degree n, $(EB\underset{B}{\times}B\underset{T}{\times}X)_{n,et} \simeq B_{et}^{\overset{n}{\times}} \times (B\underset{T}{\times}X)_{et}$ and similarly $(EB\underset{B}{\times}X)_{n,et} \simeq B_{et}^{\overset{n}{\times}} \times X_{et}$. Here we may assume that the above inverse systems are indexed by the same indexing set and that therefore the $\simeq$ denotes a weak-equivalence of each term of the corresponding inverse systems. As n varies, $\{(EB\underset{B}{\times}(B\underset{T}{\times}X))_{n,et}|n\}$, $\{(EB_{et}\underset{B_{et}}{\times}(B\underset{T}{\times}X)_{et})_n|n\}$, $\{(EB\underset{B}{\times}X)_{n,et}|n\}$ and $\{B_{et}^{\overset{n}{\times}} \times X_{et}|n\}$ form inverse systems of pointed bisimplicial sets. Since taking the diagonal of pointed bisimplicial sets sends degree-wise weak-equivalences of pointed bisimplicial sets to weak-equivalences, it follows that we obtain the weak-equivalences:

$(EB\underset{B}{\times}B\underset{T}{\times}X)_{et} \simeq EB_{et}\underset{B_{et}}{\times}(B\underset{T}{\times}X)_{et}$ and

$(EB\underset{B}{\times}X)_{et} \simeq EB_{et}\underset{B_{et}}{\times}X_{et}$

of inverse systems of pointed simplicial sets. Clearly $(B/T)_{et} \to (EB\underset{B}{\times}(B\underset{T}{\times}X))_{et} \to (EB\underset{B}{\times}X)_{et}$ is now an inverse system of fibrations. It follows that $(B/T)_{et} \to (EB\underset{B}{\times}B\underset{T}{\times}X)_{et} \to (EB\underset{B}{\times}X)_{et}$ provides a long-exact sequence of *homotopy groups*:

(3.3.*) $\ldots \to \pi_1((B/T)_{et}) \to \pi_1((EB\underset{B}{\times}B\underset{T}{\times}X)_{et}) \to \pi_1((EB\underset{B}{\times}X)_{et}) \to \pi_0((B/T)_{et}) \to \ldots$

Observe that $(B/T)_{et} \simeq *$ and therefore that the map $\pi_1(EB\underset{B}{\times}(B\underset{T}{\times}X))_{et}) \to \pi_1(EB\underset{B}{\times}X_{et})$ is an isomorphism. $\pi_1(Z_{et}) = \pi_{1,et}(Z)$ is the étale fundamental group classifying l-adic local systems on any pointed connected simplicial scheme Z as shown in [Fr] Corollary 5.8. (See also [J-2].) Observe that all the simplicial schemes appearing above are connected. Therefore we have proven the first assertion for when P-equivariant local systems are replaced by B-equivariant local systems.

Next observe that $(G/T)_{et} \simeq (G/B)_{et}$ which is simply connected. Therefore an entirely similar argument shows that the map $\pi_1(EG\underset{G}{\times}(G\underset{T}{\times}X)) \to \pi_1(EG\underset{G}{\times}X)$ is an isomorphism proving the last assertion. Now it suffices to prove the first assertion for P-equivariant local systems. Observe that $P/B \cong L/(L \cap B)$ where L is a Levi-subgroup of P. Since $L \cap B$ is a Borel subgroup of L, one observes that P/B is also simply connected. Therefore an argument as above proves the first assertion for P-equivariant local systems. □

Remarks. In view of the above discussion, it suffices to consider T-equivariant local systems in the rest of the paper: observe that the variety Z appearing in (2.1) is only stable by T and not by G. Observe that, in the above proposition, we only need the existence of étale fundamental groups of simplicial schemes and the exactness of only the part of the long-exact sequence appearing in (3.3.*). If one is willing to assume this, one can avoid the use of the étale topological type which may be somewhat unfamiliar.

(3.4) **Proposition**. Let X denote a G-spherical variety and let $x \in X$ be a fixed point for T. Now $\mathcal{H}^i(IC(X;\mathcal{L}))_x = 0$ for all *odd* i and all T-equivariant local systems $\mathcal{L}$ on the open G-orbit.

Proof. By Lemma 3.1, we may assume that X is normal. Then x admits an open G-stable quasi-projective neighborhood U_x (see [Su]). Thus, we may replace X by the closure of U_x, and assume that X is projective. Now we conclude by Theorem (1.5.2) applied to $\mathcal{L}^\vee$ in the place of $\mathcal{L}$, together with Proposition (3.2). (Observe that the fundamental group of the open orbit, $\pi_1(Gx_o)$ acts on the stalks of $\mathcal{L}$ through its image in $\pi_1(EG\underset{G}{\times}Gx_o)$ which is finite. Therefore $\mathcal{L}$ is semi-simple as a local system, and Theorem (1.5.2) applies.) □

(3.5) **Proposition**. Let X denote a G-spherical variety and let $\mathcal{L}$ denote a T-equivariant local system on the open dense orbit. Now $\mathcal{H}^i(IC(X;\mathcal{L})) = 0$ for all *odd* i.

Proof. The proof proceeds by ascending induction on the dimension of the G-spherical variety for any connected reductive group G. Since a spherical variety of dimension 1 may be assumed to be normal and hence non-singular, we may start the induction with spherical varieties of dimension 1.

The discussion on the local structure in (2.2.1) and (2.2.2) provides us with the commutative square:

$$\text{(3.5.1)}\qquad \begin{array}{ccc} P_u \times Z & \xrightarrow{\rho} & X_s \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\bar{\rho}} & X_s/P_u \end{array}$$

We may assume X is irreducible and that Gx denotes the open G-orbit on X; observe that $X_s \cap Gx = Bx = Px=$ the open P-orbit on X_s where B is a Borel subgroup containing T and contained in P. Now $\mathcal{L}_0 = \mathcal{L}_{|Px}$ a T-equivariant local system on Px. (Observe that, by (3.3), there is an one-to-one correspondence between T-equivariant local systems on Px, B-equivariant local systems on Px and P-equivariant local systems on Px. Therefore we may assume that $\mathcal{L}_0$ is a P-equivariant local system. The map ρ is T equivariant when T acts on $P_u \times Z$ by $t.(p,z) = (t.p.t^{-1}, t.z)$. Moreover all the other maps in (3.5.1) are T-equivariant and that the same map ρ is also P_u-equivariant when P_u acts on $P_u \times Z$ by left-translation on the factor P_u. Therefore, the pull-back $\rho^*(\mathcal{L}_0)$ is a T-equivariant local system on a T-stable sub-scheme of $P_u \times Z$ of the form $P_u \times Z_0$ where Z_0 is a T-stable open subscheme of Z. Therefore it clearly descends to a T-equivariant local system on Z_0. Call this local system $\mathcal{L}_1$.

We may replace the varieties Z and X_s/P_u by their normalizations, if necessary and assume all the varieties in the diagram in (3.5.1) are normal. Next recall that the maps ρ and $\bar{\rho}$ are finite surjective maps between normal varieties. Therefore one may invoke Proposition (1.6) to show that $\rho_*(\rho^*\mathcal{L}_0)$ is a T-equivariant local system on a T-stable open subscheme of X_s; we will denote this by $\mathcal{L}_0'$. Similarly $\bar{\rho}_*(\mathcal{L}_1)$ is a T-equivariant local system on a T-stable open subscheme of X_s/P_u and

(3.5.2) $\rho_*(IC^T(P_u \times Z; \rho^*(\mathcal{L}_0))) \simeq IC^T(X_s, \mathcal{L}'_0)$ while

(3.5.3) $\bar{\rho}_*(IC^T(Z; \mathcal{L}_1)) \simeq IC^T(X_s/P_u, \bar{\rho}_*(\mathcal{L}_1))$.

Moreover one may also observe that $\mathcal{L}_0$ is a split summand of the local system $\mathcal{L}'_0 = \rho_*(\rho^*(\mathcal{L}_0))$. Therefore it suffices to show the odd dimensional cohomology sheaves of $IC^T(X_s, \mathcal{L}'_0)$ are trivial. Since the map ρ is finite, by (3.5.2) one may identify $\mathcal{H}^i(IC^T(X_s, \mathcal{L}'_0))_{\bar{x}}$ with $\underset{\bar{y} \,\epsilon\, \rho^{-1}(\bar{x})}{\oplus} \mathcal{H}^i(IC^T(P_u \times Z, \rho^*(\mathcal{L}_0)))_{\bar{y}}$. Now observe that the map $\tilde{\pi}$ in (3.5.1) is evidently smooth with fibers $= P_u$; therefore $(IC^T(P_u \times Z; \rho^*(\mathcal{L}_0)) \simeq \tilde{\pi}^*(IC^T(Z; \mathcal{L}_1))$. Therefore one reduces to showing $\mathcal{H}^i(IC^T(Z; \mathcal{L}_1)) = 0$ for all odd i. Now the finiteness of the map $\bar{\rho}$ and (3.5.3) show: $\mathcal{H}^i(IC^T(X_s/P_u; \bar{\rho}_*(\mathcal{L}_1)))_{\bar{x}} \cong \underset{\bar{y} \,\epsilon\, \bar{\rho}^{-1}(\bar{x})}{\oplus} \mathcal{H}^i(IC^T(Z, \mathcal{L}_1))_{\bar{y}}$. Therefore it suffices to show $\mathcal{H}^i(IC(X_s/P_u; \bar{\rho}_*(\mathcal{L}_1))) = 0$ for all odd i.

Now we recall that there exists an L-equivariant retraction $r : X_s/P_u \to Lx=$ a torus with the fiber at x being isomorphic to the L_x^o-spherical variety $\mathcal{S}$. Now $IC(X_s/P_u; \bar{\rho}_*(\mathcal{L}_1))_x \simeq IC(\mathcal{S}; \bar{\rho}_*(\mathcal{L}_1)_{|\mathcal{S}})_x$. This follows from proposition (1.6) applied to the purely inseparable map $\bar{r} : L \underset{L_x}{\times} \mathcal{S} \to X_s/P_u$. Therefore it suffices to prove that $\mathcal{H}^i(IC(S; \mathcal{L})) = 0$ for any $T_x^o = T \cap L_x^o$-equivariant local system on the open L_x-orbit in $\mathcal{S}$. (Observe that T_x is a maximal torus in L_x by the choice of L and T.) Since the dimension of S is the codimension of the G-orbit at x, the inductive hypothesis applies to complete the proof. $\square$

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Department of Mathematics,
The Ohio State University,
Columbus, Ohio, 43210, USA