Day 2: Linear Equations

Let's start with a linear equation

\[ 1x + 2y - 1z = 1 \]

Because we're about to run out of letters, so let's call the unknowns \( x_1, x_2, x_3 \)

\[ 1x_1 + 2x_2 - 1x_3 = 1 \]

We can write the eqn like this as

\[ a_1 x_1 + a_2 x_2 + a_3 x_3 = b \]

where \( a_1 = 1, a_2 = 2, a_3 = -1, b = 1 \).

The coef \( a_1, a_2, a_3, b \) are treated as constants not unknowns. We don't try to solve for them!

If we're real nerds, we can write \((*)\) as

\[ \sum_{i=1}^{3} a_i x_i = b. \]

We can solve for \( x_1 \) as

\[ x_1 = 1 - 2x_2 + 1x_3. \]

If we know \( x_2, x_3 \), this determines \( x_1 \). This lets us write down all solns. We'll talk about this in more generality.
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Systems

Now, we can look at systems of linear eqns where we have a number of eqns.

6x

1x_{1} + 2x_{2} - 4x_{3} = 4

(A) 3x_{1} + 4x_{2} - 6x_{3} = 10

There are two eqns in three unknowns:

(2\times3) \text{ system }

\begin{array}{c|c}
\text{# of eqns} & \text{# of unknowns} \\
\hline
2 & 3
\end{array}

(b) Or we could have

\begin{align*}
x_{1} - 3x_{2} &= 7 \\
2x_{1} + 3x_{2} &= 2 \\
x_{1} - x_{2} &= 1
\end{align*}

3 eqns in 2 unknowns (3\times2) system

A soln (x_{1}, x_{2}, x_{3}) satisfies all the eqns in the system.

For example in (A):

(x_{1}, x_{2}, x_{3}) = (2, 1, 0)

since if we sub:

\begin{align*}
1(2) + 2(1) - 4(0) &= 4 \\
3(2) + 4(1) - 6(0) &= 10
\end{align*}
So is \((x_1, x_2, x_3) = (6, -5, -2)\):

\[
\begin{align*}
1(6) + 2(-5) - 4(-2) &= 4 \\
3(6) + 4(-5) - 6(-2) &= 10.
\end{align*}
\]

There are only many solutions. We can write them as:

\[
\begin{align*}
x_1 &= 2 - 2x_3 \\
x_2 &= 1 + 3x_3
\end{align*}
\quad\text{for any } x_3.
\]

This describes a line in 3d:

\[
(x_1, x_2, x_3) = (2 - 2x_3, 1 + 3x_3, x_3)
\]

\[
= (2, 1, 0) + x_3 (-2, 3, 1)
\]

In general, we can write \(m\) eqns in \(n\) unknowns:

\[
\begin{align*}
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
&\vphantom{a_{21}} \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m
\end{align*}
\]

\((m \times n)\) system

Note: There are \(m\) rows, one for each eqn

"n columns, one for each unknown

First row, then column.
Day 2: Linear Eqns

For: \[ \begin{align*}
1^a x_1 + 2^a x_2 - 4^a x_3 &= 4^b, \\
3^a x_1 + 4^a x_2 - 6^a x_3 &= 10^b
\end{align*} \]

\[\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
\vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & a_{mn}
\end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\]

We write coeff as

\[ a_{ij} = \text{the coeff of } x_j \text{ in the } i^{th} \text{ eqn.} \]

We can write the system w/ the \( x_i \)'s

The coeff matrix of the system is

\[ A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}\end{bmatrix} \]

It is an \((m \times n)\)-matrix

\[ \# \text{ of rows} \]
\[ \# \text{ of columns} \]

If we want to include b's, we can write augmented matrix

\[ \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn} & b_m\end{bmatrix} \]

this bar helps us keep things straight.
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\[ \begin{align*}
\text{Ex} & \quad 1x_1 + 2x_2 - 4x_3 = 4 \\
& \quad 3x_1 + 4x_2 - 6x_3 = 10
\end{align*} \]

has coeff matrix \( \begin{bmatrix} 1 & 2 & -4 \\ 3 & 4 & -6 \end{bmatrix} \)

and augmented matrix \( \begin{bmatrix} 1 & 2 & -4 & | & 4 \\ 3 & 4 & -6 & | & 10 \end{bmatrix} \)

We may write \( \overline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \) and the augmented matrix \( [A|\overline{b}] \)

We'll solve linear systems by simplifying them. We have to be sure they have the same set of solutions.

Def: Two systems of linear equations are equivalent if they have the same solution set.

\( \begin{align*}
\text{(a)} & \quad 1x_1 + 2x_2 - 4x_3 = 4 \\
\text{(b)} & \quad 3x_1 + 4x_2 - 6x_3 = 10
\end{align*} \)

Ex A \( \text{(*) is equivalent to} \)

\( \begin{align*}
\text{a)} & \quad 3x_1 + 4x_2 - 6x_3 = 10 \\
\text{b)} & \quad 1x_1 + 2x_2 - 4x_3 = 4
\end{align*} \)

We just changed the order of the exes.
Ex B (*) is equivalent to:
   i) \(3x_1 + 6x_2 - 12x_3 = 12\)
   ii) \(3x_1 + 4x_2 - 6x_3 = 10\)

because we just multiplied eqn (i) by 3. This doesn't change the solution to (i).

Ex C What if I add \((-2) \times (i)\) to (ii) but keep (i) the same:
   i)' \(1x_1 + 2x_2 - 4x_3 = 4\)
   ii)' \(1x_1 + 0x_2 + 2x_3 = 2\)

The system is equivalent because we can get (*) back by adding \(2 \times (i)'\) to (ii)'.

In general:
Recall scalar is just a fancy term for a number.

Thus if one of the following is applied to a system, then the resulting system is equivalent to the original system:
1) Interchange two eqns.
2) Multiply an eqn by a nonzero scalar.
3) Add a constant multiple of one eqn to another.
These are the elementary operations and denoted:

1) $E_i \leftrightarrow E_j$ - rows $i$ and $j$ are interchanged
2) $kE_i$ - row $i$ is multiplied by a nonzero scalar $k$
3) $E_i + kE_j$ - $k \times j^{th}$ eqn is added to $i^{th}$ eqn

Example:

A: $E_1 \leftrightarrow E_2$
B: $3E_1$
C: $E_2 + (-2)E_1$.

Let's use elementary operations to simplify (*)

$E_2 + (-3)E_1$:

1) $1x_1 + 2x_2 - 4x_3 = 4$
2) $-2x_2 + 6x_3 = -2$

$E_1 + 1E_2$:

1) $1x_1 - 2x_3 = 2$
2) $-2x_2 + 6x_3 = -2$

$-\frac{1}{3}E_2$:

1) $x_1 - 2x_3 = 2$
2) $x_2 - 3x_3 = 1$

Now, we can write $x_1$ and $x_2$ in terms of $x_3$:

$x_1 = 2 + 2x_3$
$x_2 = 1 + 3x_3$.
Day 2: Linear Eqs

To describe the set of solutions, introduce a new variable $t$ and write $(x_1, x_2, x_3)$ (so $t$ takes the place of $x_3$)

\[
\begin{align*}
x_1 &= 2 + 2t \\
x_2 &= 1 + 3t \\
x_3 &= t \\
\end{align*}
\]

\[
\alpha
\]

\[
(x_1, x_2, x_3) = (2 + 2t, 1 + 3t, t)
\]

\[
= (2, 1, 0) + t (2, 3, 1)
\]

We were able to solve this because we simplified so that $x_1$ appeared in exactly one equation and

\[
\alpha
\]

Solved for $x_1$ and $x_2$ in terms of $x_3$.

Instead of working with the system, we can work with the augmented matrix.

**Def.** The elementary row operations are

1) Interchange $i$ and $j$ $(R_i \leftrightarrow R_j)$

2) Multiply row $i$ by a non-zero scalar $k$ $(kR_i)$

3) For a constant $k$, add $k \times$ row $j$ to row $i$ $R_i + kR_j$
"Algorithm" for solving a linear system:
1) Write the augmented matrix for the system
2) Use the row ops to find a "simpler" system
3) Solve the simpler system.
I'll explain simpler next class.

Here are some simpler systems:

A) \[
\begin{bmatrix}
1 & 0 & 0 & | & 2 \\
0 & 1 & 0 & | & 1 \\
0 & 0 & 1 & | & 6
\end{bmatrix}
\]
\[x_1 = 2\]
\[x_2 = 1\]
\[x_3 = 6\]

B) \[
\begin{bmatrix}
1 & 0 & 2 & | & 4 \\
0 & 1 & -3 & | & 5 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
\]
\[x_1 + 2x_3 = 4\]
\[x_2 - 3x_3 = 5\]

\[x_1 = 4 - 2x_3\]
\[x_2 = 5 + 3x_3.\]

If we choose \(x_3\), it determines \(x_1, x_2\).