Day 4: Echelon Algorithm

Recall We can apply row operations to a matrix:

1) Interchange rows i and j $(R_i \leftrightarrow R_j)$
2) Multiply row i by a nonzero constant k (kR_i)
3) Add $k \times$(row j) to row i $(R_i + kR_j)$

Recall A matrix is in echelon form if

1) All rows of 0 are grouped together at the bottom
2) The first nonzero entry in each nonzero row is 1 (called a pivot)
3) The pivots go from left to right as we go down the matrix

Recall A matrix is in reduced echelon form if it is in echelon form and all entries above and below a pivot is 0.

Ex $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is in reduced echelon form

$\begin{bmatrix} 1 & 3 & 7 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is in echelon form but not reduced echelon form

Day 4: Echelon Algorithm

Recall Applying row operations to a linear system does not change the set of solutions.

Thm Any matrix A can be put in reduced echelon form by row operations. The reduced echelon form is unique.

Algorithm for putting a matrix in reduced echelon form

Left to right

1) Locate the leftmost column that contains a nonzero entry
2) If necessary, interchange the first row with another row so the first nonzero column has a nonzero entry in the first row
3) Multiply the first row by a nonzero constant so the first nonzero entry is 1 (a pivot)
4) Add a multiple of row 1 to the lower rows to make the entries below the pivot equal to 0
5) Ignore the first row and repeat the process on the remaining rows until the matrix is in echelon form

To get it in reduced echelon form.

Right to left

6) Add multiples of each row to the ones above it to clear the entries above the pivots.

Day 4: Echelon Algorithm

Ex
$$\begin{bmatrix} 0 & 3 & 9 & 6 \\ 1 & 5 & 16 & 7 \\ 2 & 4 & 14 & 3 \\ 2 & 7 & 23 & 8 \end{bmatrix}$$

There's a nonzero entry in the first column. (say the one in the second row), move it to the first row ($R_1 \leftrightarrow R_2$)

$$\begin{bmatrix} 1 & 5 & 16 & 7 \\ 0 & 3 & 9 & 6 \\ 2 & 4 & 14 & 3 \\ 2 & 7 & 23 & 8 \end{bmatrix}$$

Clear the entries below the pivot

$R_3 - 2R_1$:

$$\begin{bmatrix} 1 & 5 & 16 & 7 \\ 0 & 3 & 9 & 6 \\ 0 & -6 & -18 & -11 \\ 2 & 7 & 23 & 8 \end{bmatrix}$$

$R_4 - 2R_1$:

$$\begin{bmatrix} 1 & 5 & 16 & 7 \\ 0 & 3 & 9 & 6 \\ 0 & -6 & -18 & -11 \\ 0 & -3 & -9 & -6 \end{bmatrix}$$

Day 4: Echelon Algorithm

Now, forget row 1 and look at rows 2-4.
Leftmost column w/ nonzero entry is the second
Nonzero entry is in row 2. Divide row 2 by 3
to make this entry 1.

$$\tfrac{1}{3}R_2 : \begin{bmatrix} 1 & 5 & 16 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & -6 & -18 & -11 \\ 0 & -3 & -9 & -6 \end{bmatrix}$$

Let's clear entries below this pivot.

$$R_3 + 6R_2 : \begin{bmatrix} 1 & 5 & 16 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & -9 & -6 \end{bmatrix}$$

$$R_4 + 3R_2 : \begin{bmatrix} 1 & 5 & 16 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, the matrix is in echelon form.
Let's clear the entries above the pivots starting
with the rightmost one.

$$R_1 + (-7)R_3 : \begin{bmatrix} 1 & 5 & 16 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Day 4: Echelon Algorithm

$$R_2 + (-2)R_3 : \begin{bmatrix} 1 & 5 & 16 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now 3 is not a pivot, but the 1 in the second row is.

$$R_1 + (-5)R_2 : \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in row echelon form. Not a pivot

Solving a system in reduced echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

corresponds to

$$x_1 + 3x_2 + 1x_5 = 2$$

$$x_3 + 2x_5 = 1$$

$$x_4 + 3x_5 = -2$$

There are pivots in column 1, 3, and 4.

x_1, x_3, x_4 are pivot variables

The other variables x_2 and x_5 are free.

Day 4: Echelon Algorithm

Solve for pivot variables in terms of the free ones

$$x_1 = 2 - 3x_2 - x_5$$
$$x_3 = 1 - 2x_5$$
$$x_4 = -2 - 3x_5$$

The free variables can be any real numbers.

Two free variables = Two degrees of freedom.

Def The __rank__ of a matrix in reduced echelon form is

r = # of pivots = # of nonzero rows in reduced echelon form.

__Observations__

1) # of pivots ≤ # of rows

$$r \leq m$$

2) If we have an augmented matrix representing the system of m eqns and n unknowns, then there are $n-r$ free variables

If the system is consistent, we get $n-r$ degrees of freedom | Explain: you can choose the values of the free variables

Day 4: Echelon Algorithm

Aside: If the system is consistent and there are no free variables, how many solns? 1

If 1 free variable, shape of solns? a line

" 2 " " " " " ? a plane

3) The augmented matrix is an $m \times (n+1)$-matrix b/c

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

has matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

which has m rows and n+1 columns.

Since there's at most one pivot in each column

$$\# \text{ of pivots} \le \# \text{ of columns}$$

$$r \le n+1$$

4) If the system is consistent, ~~then~~ there's no row that looks like $[0 \dots 0 | 1]$

(since this means $0 = 1$)

Consistent $\Rightarrow$ no pivot in rightmost column

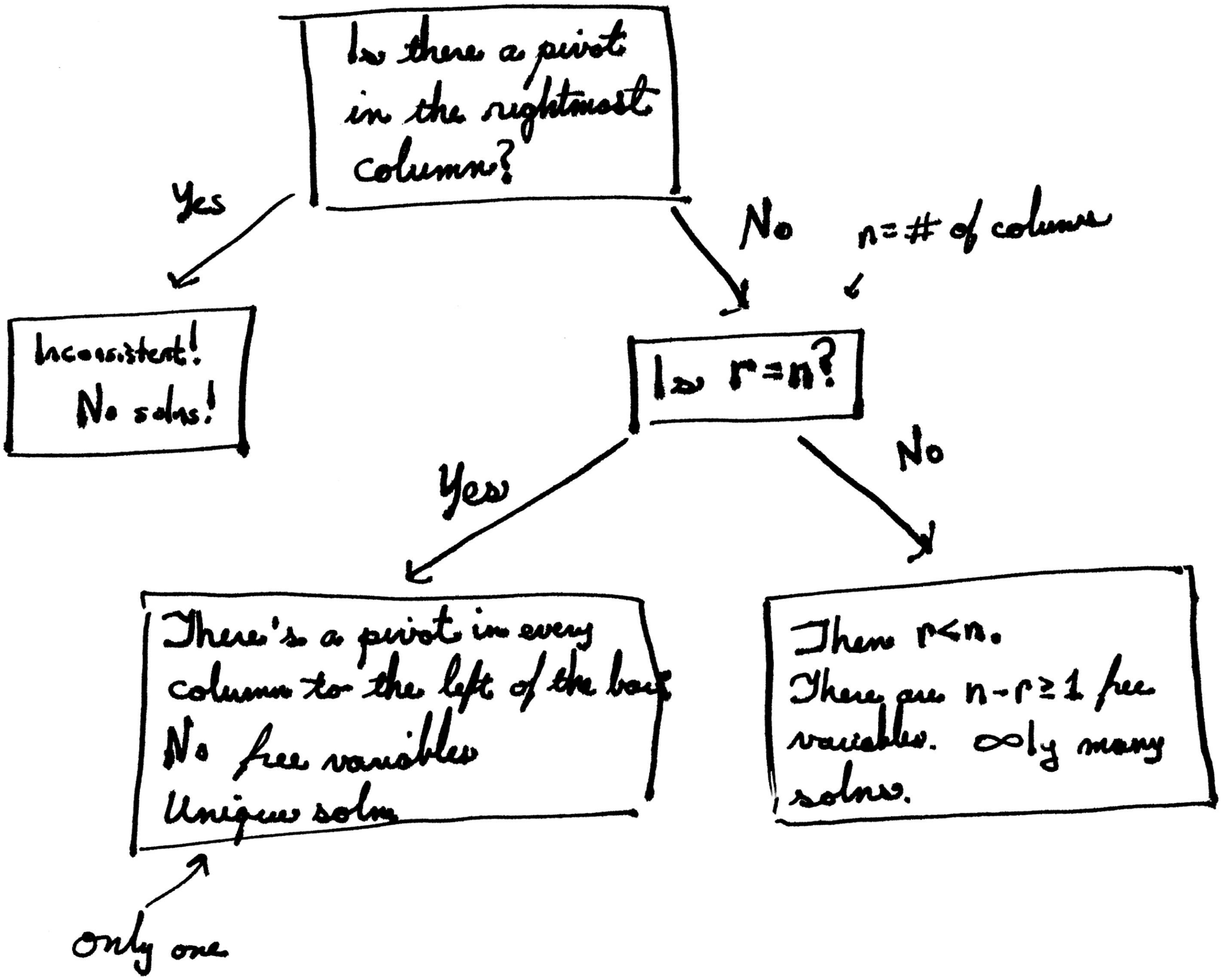
Day 4: Echelon Algorithm

If the system is consistent, then there are at most n ~~columns~~ pivots (~~one for ea~~ n = # of columns to the left of the bar)

$$r \le n$$

<u>Note</u> If $r \le n$, the system could still be inconsistent.

Flow chart for solving a reduced echelon system:



Day 4: Echelon Algorithm

Q What if we have 3 eqns in 4 unknowns?
Can we get a unique soln?
$\underbrace{\text{unique soln}}_{\text{exactly one soln}}$

A No. Why?
The system looks like
$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{array}\right]$$

If we row reduce it might look like this (there's ≤ 3 pivots)

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 7 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{or (lots more possibilities)}$$

consistent   inconsistent

In any case, there's ≥ 1 free variables.
If the system is inconsistent, there are no solns.
If " " is consistent, ∞ly many solns

Thm
Consider an $(m \times n)$-system (m eqns in n variables)
If $m < n$, then the system is inconsistent or has ∞ly many solns.

Pf $r \leq m$ (rank $\leq$ # of eqns)
And $m < n$, so $r < n$. There are r pivot variable and $n-r$ free variables. If the system is consistent, there are ∞ly many solns. $\blacksquare$