Day 6: Matrix Operations

Dot Product

There's a product that inputs two vectors of the same size and outputs a number

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\left(= \sum_{i=1}^{n} u_i v_i \right)$$

Ex $\begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 7 \\ 2 \\ 9 \end{bmatrix} = 3(-1) + 1(7) + 4(2) + 0(9) = 12.$

Matrix Multiplication

We can multiply matrices when they have a certain shape. Not immediately connected to ~~the~~ dot product.

Day 6: Matrix Operations

Matrix by Vector Multiplication

(m×n)-matrix × vector in $\mathbb{R}^n$ = vector in $\mathbb{R}^m$

~~m = #~~

m = # of rows

n = # of columns

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \left.\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}\right\} m \text{ components}$$

$\boxed{\text{Explain cross and down method}}$

We can write the i^{th} component

$$\sum_{j=1}^{n} a_{ij}x_j.$$

Day 6: Matrix Operations

Ex

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 & 4 \\ 3 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}}_{(3\times 4)\text{-matrix}} \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}}_{\text{vector in } \mathbb{R}^4} = \begin{bmatrix} 2(1) + 1(2) + 0(-1) + 4(0) \\ 3(1) + (-1)(2) + 1(-1) + 0(0) \\ 0(1) + 2(2) + 0(-1) + 1(0) \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}}_{\text{vector in } \mathbb{R}^3}$$

Think $(m \times n)$ times n-vector $= m$-vector

Note If I have a linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m. \end{aligned}$$

It can be written

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

Day 6 : Matrix Operations

$\boxed{\text{Explain why that is.}}$

A is an $(m \times n)$-matrix

$\vec{x} \in \mathbb{R}^n$

$\vec{b} \in \mathbb{R}^m$

$A\vec{x} = \vec{b}$ is the matrix form of the linear system.

Note For $A\vec{x}$ to make sense, we need

# of columns of A = # of components of $\vec{x}$.

So $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ does not make sense.

Ex The linear system

$$3x_1 + 1x_2 - 7x_3 = 2$$
$$x_1 + x_2 + 3x_3 = 5$$

has matrix form

$$\begin{bmatrix} 3 & 1 & -7 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Day 6: Matrix Operations

Matrix Multiplication

Def Let $A=(a_{ij})$ be an $(m\times n)$-matrix and let $B=(b_{ij})$ be an $(r\times s)$-matrix.

If $n=r$, the product AB is the $(m\times s)$-matrix defined by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

$$\left(= \sum_{k=1}^{n} a_{ik}b_{kj}\right)$$

A - $(m\times n)$-matrix $\quad$ B - $(n\times s)$-matrix $\quad\quad$ $(m\times s)$-matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1s} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2s} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{ns} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1s} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{is} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{ms} \end{bmatrix}$$

$c_{ij} = (i^{th}$ row of $A)\cdot(j^{th}$ column of $B)$

Note For AB to make sense, we need

# of columns of A = # of rows of B.

Day 6: Matrix Operations

__Ex__ $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1(1) + 3(0) + (-1)(3) & 1(2) + 3(-1) + (-1)(0) \\ 2(1) + (0)(0) + 3(3) & 2(2) + 0(-1) + 3(0) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 \\ 11 & 4 \end{bmatrix}$$

You should practice doing matrix multiplication

$\boxed{\text{Make up example } (2\times 2) \cdot (2\times 3)}$

__Ex__ $\begin{bmatrix} 3 & 2 & 1 & -2 \\ 1 & 1 & 1 & 0 \\ 4 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_1 + 2x_2 + 1x_3 - 2x_4 \\ x_1 + x_2 + x_3 \\ 4x_1 + \quad 1x_3 + 1x_4 \end{bmatrix}$

$$= x_1 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Day 6: Matrix Operations

General Facts:

Thm Let $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$ be vectors in $\mathbb{R}^m$.

Let A be the matrix whose columns are $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_m$. ~~Let~~ So $A = \begin{bmatrix} | & | & & | \\ \vec{A}_1 & A_2 & \cdots & \vec{A}_n \\ | & | & & | \end{bmatrix}$

(In Ex, $\vec{A}_1 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$, $\vec{A}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{A}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{A}_4 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$)

Then, $A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{A}_1 + x_2\vec{A}_2 + \cdots + x_n\vec{A}_n.$

This lets us write linear systems as matrix eqns.

Ex
$$\begin{aligned} 2x_1 + 18x_2 - 4x_3 &= -6 \\ 0x_1 + 1x_2 + 2x_3 &= 15 \\ 3x_1 + 1x_2 - 1x_3 &= 0 \end{aligned}$$

We solved by putting this in augmented form.

$$\left[\begin{array}{ccc|c} 2 & 18 & -4 & -6 \\ 0 & 1 & 2 & 15 \\ 3 & 1 & -1 & 0 \end{array}\right].$$

Day 6: Matrix Operations

We can also write the system as

$$x_1\begin{bmatrix}2\\0\\3\end{bmatrix} + x_2\begin{bmatrix}18\\1\\1\end{bmatrix} + x_3\begin{bmatrix}-4\\2\\-1\end{bmatrix} = \begin{bmatrix}-6\\15\\0\end{bmatrix}$$

Or in matrix notation, the linear system is

$$\begin{bmatrix}2 & 18 & -4\\0 & 1 & 2\\3 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-6\\15\\0\end{bmatrix}$$

General Fact

The linear system

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

expresses the same information as

$$\underset{A}{\begin{bmatrix}a_{11} & a_{12} & \dots & a_{1n}\\a_{21} & a_{22} & & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & & a_{mn}\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}x_1\\x_2\\ \vdots\\x_n\end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix}b_1\\b_2\\ \vdots\\b_m\end{bmatrix}}$$

Day 6: Matrix Operations

They can be solved by forming the augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

and row reducing.

__Thm__ Let $\vec{B}_1, \ldots, \vec{B}_s$ be vectors in $\mathbb{R}^n$. Let A be an $(m \times n)$-matrix. Then

$$A\begin{bmatrix} | & | & & | \\ \vec{B}_1 & \vec{B}_2 & \cdots & \vec{B}_s \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{B}_1 & A\vec{B}_2 & \cdots & A\vec{B}_s \\ | & | & & | \end{bmatrix}.$$