RESEARCH STATEMENT

ERIC KATZ

1. Summary

I study the interplay between combinatorics and algebraic geometry, with applications to number theory. I use ideas from tropical geometry which transforms questions in algebraic geometry into questions in combinatorics through the combinatorial study of degenerations and stratifications. A brief introduction to tropical geometry can be found on my website at

http://www.math.uwaterloo.ca/~eekatz/whatistropical.pdf

There are two main threads of my recent research:

- Incorporating the algebraic geometric notion of positivity into combinatorics (in joint work with Karim Adiprasito and June Huh) which has already lead to the proof of a forty-five-year-old conjecture of Rota on the log-concavity of characteristic polynomials of matroids; and
- Finding uniform effective bounds on rational and torsion points on curves by studying curves of bad reduction (in joint work with Joseph Rabinoff and David Zureick-Brown);

I also describe my work in the following directions:

- Addressing the tropical lifting problem: "Which balanced weighted polyhedral complexes arise from algebraic varieties?" (in individual work and in joint work with Tristram Bogart and with Sam Payne); and
- Relating the filtration on cohomology groups induced by a degeneration to the combinatorics of the dual complex, with applications to Ehrhart theory (in joint work with David Helm and with Alan Stapledon).

In my work, I employ Hodge theory, rigid analytic geometry, p-adic integration, combinatorics of bad reduction, and intersection theory. I have written on the following topics individually and with coauthors Karim Adiprasito, Tristram Bogart, David Helm, June Huh, Hannah Markwig, Thomas Markwig, Sam Payne, Joseph Rabinoff, Alan Stapledon, and David Zureick-Brown: rational and torsion point bounds [KZB13, KRZB15a]; log-concavity of characteristic polynomials and more generally Hodge structures on matroids [HuhK12, AHK15]; mixed Ehrhart theory of subdivisions and hypersurfaces [KS14b, KS14a]; the tropical lifting problem [K12b, BKa12]; tropical varieties and mixed Hodge strucutres [HeK11, KS12]; the tropical j-invariant [KMM08, KMM09]; and tropical realization spaces [KP11, K13]; tropical intersection theory [KP08, K09b, K12a]; relative Gromov-Witten invariants [K07]. I have also written expository papers [K09b, K14b].

2. Results

2.1. Rota's Conjecture and Hodge theory on matroids. With Karim Adiprasito and June Huh [AHK15], I have resolved Rota's conjecture on the log-concavity of the characteristic polynomial of matroids:

Theorem 1 (Adiprasito-Huh-K '15)**.** *The characteristic polynomial of any matroid is log-concave.*

Rota made this conjecture in his talk at the 1970 International Congress of Mathematicians, and it was one of the central problem in enumerative combinatorics. It is a natural extension of the Unimodality conjecture for chromatic polynomials for graphs which was resolved by Huh [Huh12] through deep arguments in singularity theory. Here, unimodality means that the coefficients of the characteristic polynomial form a unimodal sequence. In contrast to [Huh12], our proof of the more general conjecture, while inspired by Hodge theory, is elementary, making use of only ring theory and linear algebra.

Matroids are the common combinatorial generalization of graphs and linear subspaces. For background, we recommend [K14b]. Let $\mathbf{k}$ be a field and $\mathbf{k}^n$ be given distinguished coordinates $x_1, \ldots, x_n$. Let V be a linear subspace in $\mathbf{k}^n$ not contained in any coordinate hyperplane. Define the *rank function* $r : 2^{\{1,\ldots,n\}} \to \mathbb{Z}_{\geq 0}$ by, for $S \subset \{1, \ldots, n\}$,

$$r(S) = \operatorname{codim}((V \cap L_S) \subset V)$$

where L_S is the coordinate flat

$$L_S = \{x \in \mathbf{k}^{n+1} \mid x_i = 0 \text{ for } i \in S\}.$$

The rank function satisfies natural properties like monotonicity on subsets and subadditivity of codimensions under intersection. When one axiomatizes such a rank function, one obtains one definition of a *matroid*, a concept introduced by Whitney. Only some matroids, the so-called *representable* ones arise from linear subspaces. Matroids have been playing an increasing role in algebraic geometry: they label certain pathological subsets of Grassmannians called thin Schubert cells [GGMS87] which are relevant to polytope theory and are employed to show that other moduli spaces have arbitrarily bad singularities [Va06]; invariants of matroids are closely connected to cohomology and K-theory [FS12]; and matroids provide a laboratory for studying questions of realizability of cohomology classes by irreducible subvarieties [Huh14].

There is a natural invariant of matroids called the *characteristic polynomial* $\chi_M(q)$ which generalizes the chromatic polynomial of graphs. It is defined by Möbius inversion, which is a generalization of inclusion-exclusion. In the representable case, χ_M can be thought of as the class of

$$[V \cap (\mathbf{k}^*)^{n+1}] \in K_0(\mathrm{Var}_{\mathbf{k}}),$$

where $K_0(\mathrm{Var}_{\mathbf{k}})$ is the Grothendieck group of varieties over $\mathbf{k}$. Consequently, it is closely related to the Poincaré polynomial and $\mathbb{F}_q$-point enumeration of hyperplane arrangement complements. For a number of reasons, many motivated by probabilistic considerations [Br14], combinatorialists are interested in the structure of coefficients of the characteristic polynomials. Theorem 1 asserts that the characteristic polynomial is log-concave, that is, the logarithms of the absolute values of its coefficients form a concave sequence.

Our proof draws on an early special case jointly with Huh [HuhK12]. The earlier proof, itself based on [Huh12], required that the matroids be representable. We take a wonderful compactification $\tilde{V}$ of the intersection $V \cap (\mathbf{k}^*)^n$. By tropical geometric techniques, we show that the coefficients of the reduced characteristic polynomial $\frac{\chi(q)}{q-1}$ are intersection numbers coming from taking mixed powers of two nef classes on $\tilde{V}$. These numbers are shown to be log-concave by applying the Khovanskii-Teissier inequality [Laz04] which is a generalization of the Hodge index theorem.

To prove the general case, we define a *matroidal intersection ring* $A^*(M)$, a combinatorial avatar for the cohomology of the wonderful compactification. It is naturally a quotient of the Stanley-Reisner ring of the order complex of the lattice of flats of the matroid, but it has a very different character and many surprising properties. By using deep ideas originating in the work of McMullen and Karu [McM93, Kar04] on the combinatorial proof of the lower bound theorem in polytope theory and refined in the work of de Cataldo and Migliorini [dM05], we are able to show that $A^*(M)$ behaves as if it were the cohomology ring of an algebraic variety: it has a natural combinatorial ample cone with respect to which it has desirable Hodge theoretic properties. This is to say that it satisfies the *Kahler package*: the Hard Lefschetz theorem and the Hodge-Riemann bilinear relations. From this, we are able to deduce inequalities of Hodge-type strong enough to imply log-concavity.

We expect our proof to have consequences similar to those of Stanley's work on the g-theorem. In fact, our result may point to a deep theory of "combinatorial positivity" analogous to the theory of positivity in algebraic geometry as exposited in Lazarsfeld's textbook [Laz04]. Our proof has been announced on Gil Kalai's blog:

http://gilkalai.wordpress.com/2015/08/14/updates-and-plans-iii/

and it is likely that he will have an expository post on it by the time that this research statement is read. This theory of combinatorial positivity was the subject of a workshop for graduate students and postdocs at the University of Oregon in August 2015 in which I gave a lecture series on our proof.

2.2. **Bounds on rational points on algebraic curves.** Algebraic varieties are the solution sets to systems of polynomial equations. One may ask how many rational points lie on an algebraic variety, that is, what are the rational solutions to a system of polynomial equations? This is the central question in Diophantine geometry. The one-dimensional case of algebraic curves is already very deep. The major development in this area was the proof by Faltings in 1983 that an algebraic curve of genus $g \geq 2$ has only finitely many rational points. However, his work does not given an explicit bound. The work discussed in this section attempts to find effective, explicit bounds. Even more ambitiously, we hope to find *uniform* bounds, that is, bounds depending on the genus of the curve and not its arithmetic structure. It is the subject of the uniformity conjecture that such bounds exist. I have work in this direction with collaborators using the Chabauty-Coleman method [MP12]. This method gives an explicit bound on the number of rational points on curves subject to bounds on the Mordell-Weil rank of the curve, that is the rank of the Abelian group of rational points of its Jacobian, J.

I also discuss bounds on torsion points on curves. An Abel-Jacobi map is a particular canonical (up to a choice of base-point) map $\iota : C \hookrightarrow J$ of the curve into its Jacobian. The torsion points on the curve with respect to ι is the pre-image, $\iota^{-1}(J_{\mathrm{tors}})$ of the torsion points of the Jacobian. One can consider the rational torsion points, that is, torsion points defined over a number field, or the geometric torsion points which are defined over an algebraically closed field. They are finite by the Manin-Mumford conjecture as proved by Raynaud [Ray83]. There are effective bounds due to Buium [Bui96] but no uniform bounds.

2.2.1. *Uniform bounds on rational and torsion points.* With Joseph Rabinoff and David Zureick-Brown [KRZB15a], I have proved an important case of the uniformity conjecture. We give uniform bounds on the number of rational points on a curve C over a number field K when its Mordell-Weil rank r satisfies $r \leq g-3$, extending a result of Stoll on hyperelliptic curves [Sto13]. We also provide unconditional bounds on rational torsion, and bounds on geometric torsion subject to conditions on the reduction type of the curve. To our knowledge, no uniformity result was previously known even in this case for non-hyperelliptic curves.

Theorem 2. *Let $d \geq 1$ and $g \geq 3$ be integers. There exist explicit constants $N(g,d)$, $N_{\mathrm{tors}}(g,d)$. $N_{\mathrm{tors},\dagger}(g,d)$ such that for any number field K of degree d and any smooth, proper, geometrically connected genus g curve C/K, we have*

(1) *If C has Mordell–Weil rank at most $g-3$, then*
$$\#C(K) \leq N(g,d).$$
(2) *For any Abel–Jacobi embedding $\iota\colon C \hookrightarrow J$ into its Jacobian (defined over K), we have*
$$\#\iota^{-1}\big(J_C(K)_{\mathrm{tors}}\big) \leq N_{\mathrm{tors}}(g,d).$$
(3) *Let $\mathfrak{p}$ be a prime of F, and let $\mathscr{C}$ be the stable model of C over an algebraic closure of the completion $K_{\mathfrak{p}}$. Suppose that C satisfies $g > 2g(C_i) + n_{C_i}$ for each component C_i of the closed fiber $\mathscr{C}_{\mathfrak{p}}$, where $g(C_i)$ is the geometric genus of C_i and n_{C_i} is the number points of the normalization of C_i mapping to nodal points of $\mathscr{C}_{\mathfrak{p}}$. Then, we have*
$$\#\iota^{-1}\big(J(\bar{K})_{\mathrm{tors}}\big) \leq N_{\mathrm{tors},\dagger}(g,d).$$

For (2), our bound is unconditional The constant $N(g,d)$ is quadratic in genus as is $N_{\mathrm{tors}}(g,d)$ while $N_{\mathrm{tors},\dagger}(g,d)$ is exponential but explicit. In the case of $K = \mathbb{Q}$, we may take
$$N_{\mathrm{tors}}(g,1) = N(g,1) = 76g^2 - 82g + 22.$$

Our proof works by studying curves at primes of possibly bad reduction, that is, primes p for which when the equations of the curve is taken mod p, the curve may become singular. Previous effective bounds usually depend on the first prime of good reduction. By working in the bad reduction case, we are able to circumvent this restriction. We develop the theory of p-adic integration in the sense of Coleman [Col85] and extended by Berkovich [Be07] to apply this method in the bad reduction case. In fact, we completely describe the correction terms (which depend on monodromy) between the Berkovich-Coleman integral which can be performed locally and the Abelian integral which is a Lie group p-adic logarithm. We employ techniques from Berkovich spaces, a branch of rigid analytic geometry that allows us to deal with curves defined over p-adic fields. We also make use of non-Archimedean potential theory and the Baker-Norine theory of linear systems on graphs to understand how the combinatorics of the reduction of the curve influence rational points.

2.2.2. *Effective bounds on curves of bad reduction.* Earlier work with Zureick-Brown [KZB13] gives an effective, but not uniform bound on the number of rational points on curves of Mordell-Weil rank at most $g-1$. Here, we lower the known bounds, verifying a conjecture of Stoll [Sto06] for our main result:

Theorem 3. *Let C be a curve over a number field K with Mordell-Weil rank $r \leq g-1$. Suppose $p > 2r+2$ is a prime which is unramified in K and let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime above p. Let $\mathscr{C}$ be a proper regular model of X over $\mathcal{O}_{K_{\mathfrak{p}}}$, the ring of integers of the completion of K at $\mathfrak{p}$, Then*
$$\#C(K) \leq \#\mathscr{C}_{\mathfrak{p}}^{\mathrm{sm}}(\mathbb{F}_{\mathfrak{p}}) + 2r.$$

Here, $\mathscr{C}_{\mathfrak{p}}^{\mathrm{sm}}$ denotes the smooth points of the closed fiber of $\mathscr{C}$. This result extends Stoll's strengthening of the Chabauty-Coleman bound to the bad reduction case. Our proof introduces rank functions to give bounds on the size of linear systems on curves over valued fields and then shows that such rank functions satisfy a Clifford inequality analogous to the one in the classical theory of curves. This extends the work of Baker-Norine on linear systems on graphs. The rank functions we study was independently introduced and systematized in the work of Amini-Baker [AB14].

Our result is discussed on Matt Baker's blog:

http://mattbakerblog.wordpress.com/2014/04/11/effective-chabauty/

and summarized in Baker-Jensen's survey on linear systems on graphs [BJ15]

2.3. **The tropical lifting problem.** Central to tropical geometry is the procedure of *tropicalization* which attaches a combinatorial object called a *tropical variety* to an algebraic variety. Specifically, one begins with an algebraic variety X, the common zero set of a system of polynomial equations in an algebraic torus $(\mathbb{K}^*)^n$ defined over a valued field $\mathbb{K}$. From it, one produces a *tropical variety*, $\mathrm{Trop}(X)$, which is a polyhedral complex, as a combinatorial shadow of X. The top-dimensional cells of $\mathrm{Trop}(X)$ are equipped with a locally constant multiplicity function m satisfying a balancing condition. Tropical varieties have a very rich structure reflecting their origins in algebraic varieties. It is a natural, difficult question to determine which polyhedral complexes come from algebraic varieties. There are non-obvious combinatorial obstructions. This is called the *tropical lifting problem*.

2.3.1. *Lifting problem for curves in space.* In [K12b], I study the case of algebraic curves whose tropicalizations are graphs embedded in space. By finding a combinatorial translation of deformation theory, I give a necessary condition for a graph to be the tropicalization of an algebraic curve. My condition encompassed all known obstructions while also excluding additional cases. My proof made use of deformation theory, logarithmic geometry, and the theory of linear systems on graphs.

Theorem 4. *Let $f : C^* \to (\mathbb{K}^*)^n$ be a map of a smooth curve with parameterized tropicalization $\mathrm{Trop}(f) : \Sigma \to \mathbb{R}^n$. After a possible subdivision, for any $m \in \mathrm{Hom}((\mathbb{K}^*)^n, \mathbb{K}^*)$, there exists a non-negative piecewise-linear function φ_m on Σ with integer slopes. The function φ_m is in the linear system $L(K_\Sigma)$, satisfies the vanishing and constant slope condition with respect to m, and is $\mathscr{C}_0$-ample on m-orthogonal hyperplane intersection interiors.*

The above conditions are rather technical but can be transformed into easily-checked combinatorial criteria. This work can also be seen in the light of Berkovich spaces and gives a necessary condition for a piecewise-linear function on the skeleton to arise as the logarithm of a section of a formally metrized line bundle. While it may not seem so, this work is very closely related to my work in arithmetic geometry.

2.3.2. *Relative lifting problem for curves in hypersurfaces.* With Tristram Bogart [BKa12], I study the relative lifting problem of curves in a hypersurface. This work explains a number of pathological examples of tropical curves in tropical surfaces that could not lift to curves in a surface even though the curve and the surface both lifted individually [Vi10]. These examples had frustrated researchers hoping to use tropical geometry to enumerate curves in non-toric surfaces. Our result provides a criterion that these counterexamples violate:

Theorem 5. *Let $\Gamma \subset \mathrm{Trop}(V(f))$ be a tropical curve in a unimodular tropical hypersurface. Suppose that w is a vertex or an interior point of an edge of $\mathrm{Trop}(V(f))$ and that $\mathrm{Star}_w(\Gamma)$ spans a rational plane U. If Γ lifts in $V(f)$ then one of the following must hold:*

(1) *Γ is locally equivalent to an integral multiple of the stable intersection $\mathrm{Trop}(V(f)) \cap_{\mathrm{st}} U$,*

(2) *The stable intersection* $\mathrm{Trop}(V(f)) \cap_{\mathrm{st}} U$ *contains a classical segment of weight* 1 *as a local tropical cycle summand.*

Our proof made use of obstructions to factoring polynomials in the spirit of [Stu96]. The obstruction prevents one from constructing a particular broken curve in the central fiber of a degeneration of the hypersurface. Our work was later extended by Brugallé-Shaw [BS11] and Gathman-Schmitz-Winstel [GSW13].

2.4. **Tropical geometry, Hodge theory, subdivisions, and Ehrhart theory.** I have a series of projects applying Hodge theory to tropical geometry and Ehrhart theory. Hodge theory is the study of additional structures on the cohomology of algebraic varieties, and it provides new, interesting, and valuable combinatorial invariants.

2.4.1. *Tropical geometry and Hodge theory.* In work with David Helm [HeK11] and Alan Stapledon [KS12, KS14b, KS14a], I studied the topology of tropical varieties. Here, we used mixed Hodge theory. Specifically, a family of complex varieties, degenerating over a disc, carries a limit mixed Hodge structure on the cohomology of its generic fiber, encoding the action of monodromy. The ranks of certain associated gradeds of the Hodge structure are controlled by the combinatorics of the components of the degenerate limit of the family, and in turn, the Hodge structure gives interesting combinatorial invariants. The necessary background is provided in my paper with Helm [HeK11], which is the non-constant coefficient analogue of a paper of Hacking [Ha08], and which has influenced recent work on the topology of non-Archimedean spaces as summarized in Payne's recent *Bulletin* article [P15].

My first paper with Alan Stapledon extracts data about the limit mixed Hodge structure on the generic fiber of a family over a disc $f : Y \to \mathbb{D}$ from the data of the tropical variety and initial degenerations. When the family satisfies a natural smoothness condition called schönness, we give a tropical recipe to compute Bittner's motivic nearby fiber [Bi05] which specializes to the Hodge polynomial of the limit mixed Hodge structure.

2.4.2. *Mixed Ehrhart theory.* Two more recent papers with Stapledon [KS14a, KS14b] develop mixed Ehrhart theory by studying degenerations of hypersurfaces over a disc. Classical Ehrhart theory [BS07] involves counting the number of lattice points in dilates of a lattice polytope. This data is encoded in an invariant called the Ehrhart polynomial $h^*(P;t)$. By work of Batyrev-Borisov [BB96], extending results of Danilov-Khovanskii [DK86], $h^*(P;t)$ is related to the cohomology of a hypersurface. We consider the analogous problem of studying degenerations of hypersurfaces over a disc. This in turn is related to the study of polyhedral subdivisions of lattice polytopes, which we call *mixed Ehrhart theory.* By looking at the ranks of the associated gradeds of the limit mixed Hodge structure on cohomology, we give new, effectively computable, combinatorial invariants of subdivisions of lattice polytopes that specialize to the usual Ehrhart polynomial. These polynomials include the refined limit mixed polynomial, $h^*(P,\mathcal{S};u,v,w)$ and its specializations and local analogues. Because the ranks of the gradeds are non-negative, these polynomials have many positivity properties. Moreover, the Lefschetz properties of the filtrations and the deep decomposition theorem of Beilinson, Bernstein, Deligne and Gabber [BBD82] give powerful ways of obtaining lower bound, unimodality, and symmetry results. To this extent, we are able to prove a conjecture of Nill and Schepers and answer a question of Athanasiadis.

We also develop a formalism for studying subdivision and introduce new invariants of subdivisions of convex, not-necessary-lattice polytopes. In fact, we are able to find multivariable generalizations of the h-polynomial. Our formalism systematizes a number of results on subdivisions scattered throughout the literature and extends many of them beyond the simplicial case. Our work here builds on Stanley's theory of local h-vectors [Sta92]. Our formalism is related to work of Brenti and is motivated by and well-adapted to Kazhdan-Lusztig theory although we have not yet applied it in that case.

2.5. **Other work.** I have other projects that do not fit into the above categories.

2.5.1. *Work related to enumerative geometry.* I do not write much about my older work on tropical intersection theory [K09b, K12a], my work with Payne on piecewise polynomials [KP08], or my thesis work on relative Gromov-Witten invariants [K07]. Developing Gromov-Witten theory relative to a non-smooth divisor was my original motivation to study tropical geometry. Now that this dream is being realized by Abramovich, Chen, Gross, Siebert, and others [ACGS], I hope to revisit this area of math.

2.5.2. *Realization spaces for tropical varieties.* With Payne [KP11] and individually [K13], I studied the set of all varieties with given tropicalization proving that it forms a moduli space (over fields with trivial valuation) and a rigid analytic parameter space (in the discretely valued case). As a consequence, I show that tropical realization spaces obey *Murphy's Law* in the sense of Vakil [Va06] which proves that lifting questions can be arbitrarily complicated with regard to field of definition and deformability of realizations. I also prove a sort of Artin approximation theorem: if a polyhedral complex is the tropicalization of a formal family of varieties, then it is the tropicalization of an algebraic family of varieties, allowing one to transfer information between the algebraic and analytic points of view.

2.5.3. *Anabelian graph theory.* I also have a paper in graph theory graph [K14a] which explains how a graph is determined (up to contraction of bridges) by a certain unipotent extension of its cycle space. This result is the anabelian analogue of Whitney's 2-isomorphism theorem, and I wrote the paper to understand, by analogy, some aspects of the iterated integrals behind Kim's non-abelian Chabauty. There is a connection between this work and the monodromy correction terms for the two types of iterated p-adic integrals.

References

[ACGS] D. Abramovich, Q. Chen, M. Gross, and B. Siebert. Decomposition of degenerate Gromov-Witten invariants. In preparation.

[AHK15] K. Adiprasito, J. Huh, and E. Katz. Hodge theory on combinatorial geometries. In preparation.

[AB14] O. Amini and M. Baker. *Linear series on metrized complexes of algebraic curves*, *Math Ann.*, to appear.

[AK06] F. Ardila and C. Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Combin. Theory Ser. B* 96:38–49, 2006.

[BH15] F. Babaee and J. Huh. A tropical approach to the strongly positive Hodge conjecture. arXiv:1502.00299

[Ba08] M. Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory* 2:613–653, 2008.

[BJ15] M. Baker and D. Jensen. Degeneration of Linear Series From the Tropical Point of View and Applications. arXiv:1504.05544

[BN07] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.* 215:766–788, 2007.

[BBM15] J. Balakrishnan, A. Besser, and J. S. Müller. Quadratic Chabauty: p-adic heights and integral points on hyperelliptic curves. *J. Reine Agnew Math.*, to appear, 2015.

[BB96] V. Batyrev and L. Borisov, Mirror duality and string-theoretic Hodge numbers. *Invent. Math.* 126:183–203, 1996.

[BS07] M. Beck and S. Robins, *Computing the continuous discretely.* Springer-Verlag, New York, 2007.

[BBD82] Alexander A. Beilinson, Joseph N. Bernstein, Pierre Deligne. Faisceaux pervers, *Astérisque* 100:5–171, 1982.

[Be07] V. Berkovich. *Integration of one-forms on p-adic analytic spaces*, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 2007.

[Bi05] F. Bittner. On motivic zeta functions and the motivic nearby fiber. *Math. Z.* 249:63–83, 2005.

[BKa12] T. Bogart and E. Katz. Obstructions to Lifting Tropical Curves in Hypersurfaces. *SIAM J. Discrete Math.* 26: 1050–1067, 2012.

[Br14] P. Brändén. Unimodality, log-concavity, real-rootedness and beyond. *The Handbook of Enumerative Combinatorics.* CRC Press, 2015.

[Bui96] A. Buium, Geometry of p-jets. *Duke J. Math.* 82:349–367, 1996.

[BS11] E. Brugallé and K. Shaw. Obstructions to approximating tropical curves in surfaces via intersection theory. *Canadian J. Math.* arXiv:1110.0533

[Ca15] D. Cartwright. Combinatorial tropical surfaces. arXiv:1506.02023

[Col85] R. F. Coleman, Torsion points on curves and p-adic abelian integrals. *Ann. of Math.* 121:111-168, 1985.

[Colm92] P. Colmez. Périodes p-adiques des variétés abéliennes. *Math. Ann.* 292:629–644, 1992.

[CFPR] F. Cools, J. Draisma, S. Payne, and E. Robeva. A tropical proof of the Brill-Noether theorem. *Adv. Math.* 230:759–776, 2012.

[DK86] V. Danilov and A. Khovanskiĭ. Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers. *Izv. Akad. Nauk SSSR Ser. Mat.* 50:925–945,1986.

[dM05] M. A. de Cataldo and Luca Migliorini, The Hodge theory of algebraic maps. *Ann. Sci. Éc. Norm. Supér. (4)* 38:693–750, 2005.

[FS12] A. Fink and D. Speyer, K-classes of matroids and equivariant localization. *Duke Math. J.* 161:2699–2723, 2012.

[GSW13] A. Gathmann, K. Schmitz, and A. L. Winstel. The Realizability of Curves in a tropical plane. arXiv:1307.5686

[GGMS87] I. M. Gelfand, M. Goresky, R. MacPherson, and V. Serganova, Combinatorial geometries, convex polyhedra, and Schubert cells. *Adv. Math.* 63:301–316, 1987.

[GM10] A. Gibney and D. Maclagan. Lower and upper bounds on nef cones. *Int. Math. Res. Not.* 14:3224–3255, 2012.

[Ha08] P. Hacking. The Homology of Tropical Varieties *Collect. Math.* 59:263–273, 2008.

[HeK11] D. Helm and E. Katz. Monodromy Filtrations and the Topology of Tropical Varieties. *Canadian J. Math.* 64:845–868, 2011.

[HLS] H. van der Holst, L. Lovàsz, and A. Schrijver, The Colin de Verdière Graph Parameter *Bolyai Soc. Math. Stud.* Vol. 7, 1999, 29–85.

[Huh12] J. Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs. *J. Amer. Math. Soc.* 25:907–927, 2012.

[Huh14] J. Huh, *Rota's conjecture and positivity of algebraic cycles in permutohedral varieties,* PhD thesis, University of Michigan, 2014.

[HuhK12] J. Huh and E. Katz. Log-concavity of characteristic polynomials and the Bergman fan of matroids, *Math. Ann.* 354:1103–1116, 2012.

[Kar04] K. Karu, Hard Lefschetz theorem for nonrational polytopes, *Invent. Math.* 157:419–447, 2004.

[K07] E. Katz. An algebraic formulation of symplectic field theory. *J. Symplectic Geom.*, 5:385–437, 2007.

[K09b] E. Katz. A Tropical Toolkit. *Expos. Math.* 27:1–36, 2009.

[K12a] E. Katz. Tropical Intersection Theory from Toric Varieties. *Collect. Math.* 63:29–44, 2012.

[K12b] E. Katz. Lifting Tropical Curves in Space and Linear Systems on Graphs. *Adv. Math.* 230:853–875, 2012.

[K13] E. Katz. Tropical Realization Spaces for Polyhedral Complexes. *Proceedings of the Workshop on Tropical Geometry.* Contemp. Math. 589:235–251, 2013.

[K14a] E. Katz A non-abelian analogue of Whitney's 2-isomorphism theorem. *J. Algebraic Combinatorics.* 39:683–690, 2014.

[K14b] E. Katz Matroid theory for algebraic geometers, *Simons Symposium volume*, accepted, arXiv:1409.3503.

[KMM08] E. Katz, H. Markwig, and T. Markwig. The j-invariant of a plane tropical cubic. *J. Algebra* 320:3832–3848, 2008.

[KMM09] E. Katz, H. Markwig, and T. Markwig. The tropical j-invariant. *LMS J. Comput. Math.* 12:275–294, 2009.

[KP08] E. Katz and S. Payne. Piecewise polynomials, Minkowski weights, and localization on toric varieties. *Algebra Number Theory* 2:135–155, 2008.

[KP11] E. Katz and S. Payne. Realization Spaces for Tropical Fans. *Combinatorial Aspects of Commutative Algebra and Algebraic Geometry*, vol. 6, 2011.

[KS12] E. Katz and A. Stapledon. The Tropical Motivic Nearby Fiber. *Compositio Math.*, 148:269–294, 2012.

[KS14a] E. Katz and A. Stapledon. Local h-polynomials, invariants of subdivisions, and mixed Ehrhart theory. *Adv. Math.* acceptance recommended, arXiv:1411.7736.

[KS14b] E. Katz and A. Stapledon. Tropical geometry, the motivic nearby fiber and limit mixed Hodge numbers of hypersurfaces. submitted, arXiv:1404.3000.

[KZB13] E. Katz and D. Zureick-Brown. The Chabauty-Coleman bound at a prime of bad reduction and Clifford bounds for geometric rank functions. *Compositio Math.* 149: 1818-1838, 2013.

[KRZB15a] E. Katz, J. Rabinoff, and D. Zureick-Brown. Uniform bounds for the number of rational points on curves of small Mordell-Weil rank. submitted, arXiv:1504.00694.

[Ki05] M. Kim, The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel. *Invent. Math.* 161:629–656, 2005.

[Laz04] R. Lazarsfeld, *Positivity in Algebraic Geometry. I.*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 48. Springer-Verlag, Berlin, 2004.

[LT02] D. Lorenzini and T. Tucker. Thue equations and the method of Chabauty-Coleman. *Invent. Math.* 148:47–77, 2002.

[Ma13] B. Matschke. Successive Spectral Sequences. arXiv:1308.3187

[MP12] W. McCallum and B. Poonen. The Method of Chabauty and Coleman. *Explicit methods in number theory,* Panor. Synthèses, vol. 36, Soc. Math. France, Paris, 2012, 99–117.

[McM93] P. McMullen, On simple polytopes., *Invent. Math.* 113:419–444, 1993.

[MZ08] G. Mikhalkin and I. Zharkov. Tropical curves, their Jacobians and Theta functions. *Curves and Abelian Varietes*, 203–230, 2008.

[P15] S. Payne. Topology of nonarchimedean analytic spaces and relations to complex algebraic geometry. *Bull. Amer. Math. Soc.* 52:223–247, 2015.

[Ray83] M. Raynaud. Courbes sur une variete abelienne et points de torsion. *Invent. Math.* 71:207–233, 1983.

[SW13] P. Scholze and J. Weinstein. Moduli of p-divisible groups *Cambridge Journal of Mathematics* 1:145–237, 2013.

[Sta92] Richard Stanley, Subdivisions and local h-vectors, *J. Amer. Math. Soc.* 5:805–851, 1992.

[Sta96] Richard Stanley. *Combinatorics and commutative algebra. Second edition.* Progress in Mathematics, 41. Birkhäuser Boston, Inc., Boston, MA, 1996

[Ste75] J. Steenbrink. Limits of Hodge structures, *Invent. Math.* 31:229–257, 1975.

[Sto06] M. Stoll. Independence of rational points on twists of a given curve, *Compos. Math.*, 142:1201–1214, 2006.

[Sto13] M. Stoll. Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank, arXiv:1307.1773

[Stu96] B. Sturmfels. *Gröbner Bases and Convex Polytopes*, volume 8 of *Univ. Lectures Series*. American Mathematical Society, Providence, Rhode Island, 1996.

[Te07] J. Tevelev. Compactifications of subvarieties of tori. *Amer. J. Math.* 129:1087-1104, 2007.

[Va06] R. Vakil. Murphy's Law in algebraic geometry: Badly-behaved deformation spaces. *Invent. Math.*, 164:569–590, 2006.

[Vi10] M. Vigeland. Smooth tropical surfaces with infinitely many tropical lines *Ark. Mat.*, 48:177–206, 2010.