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The hypersurface $V(f) \subset \mathbb{C}^n$ is the zero locus of $f$.

**Example:**

1. $x + y + 1 = 0$ is a line.
2. $y^2 - x^3 - x - 1 = 0$ is an elliptic curve.
3. $z^2 - x^2 - y^2 - 1 = 0$ is a conic surface.
There’s a pretty good invariant of hypersurfaces when you view them as living in $\mathbb{P}_\mathbb{C}^n \supset \mathbb{C}^n$, the degree.
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The degree can be used to compute generic intersection numbers:

**Bézout's Theorem:** Let $f, g$ be generic polynomials of two variables of degrees $d$ and $e$ respectively. Then $V(f), V(g) \subset \mathbb{P}^2_{\mathbb{C}}$ intersect in $d \cdot e$ points.

Here, generic means, for generic choice of coefficients. This theorem has a generalization for intersecting $n$ hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$. 
Newton polytope

What if we don’t want to compactify $\mathbb{C}^n$ to $\mathbb{P}^n_{\mathbb{C}}$? Instead, say, we want to study hypersurfaces in $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$, that is $\mathbb{C}^n$ with the coordinate hyperplanes removed.
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The Newton polytope of $y^2 - x^3 - x - 1$ is

![Newton polytope diagram]

Eric Katz (Waterloo) Tropicalization October 25, 2012 4 / 27
Bernstein’s Theorem

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In the two-dimensional case, for two generic 2-variable polynomials $f, g$ with given Newton polytopes, the intersection number of $V(f)$ and $V(g)$ in $(\mathbb{C}^*)^2$ is

$$\text{Vol}(P(f) + P(g)) - \text{Vol}(P(f)) - \text{Vol}(P(g))$$

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where the addition of polytopes is Minkowski sum.

By results of Danilov-Khovanskii, one can compute the Euler characteristic \(\chi_c(V(f))\) for generic hypersurfaces for a given Newton polytope. More specifically, one can compute the Hodge polynomial for the mixed Hodge structure on \(H^*_c(V(f))\).
Another motivating example for this talk is projective subspaces.
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Let $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ be projective space with a choice of basis $\vec{e}_0, \ldots, \vec{e}_n \in \mathbb{C}^{n+1}$. Let $V^r \subset \mathbb{P}^n$ be a projective subspace not contained in any coordinate subspace. Consider the hyperplane arrangement complement

$$V \setminus (H_0 \cup \cdots \cup H_n),$$

where $H_0, \ldots, H_n$ are the coordinate hyperplanes. We may want to compute its Euler characteristic or some of its Hodge-theoretic invariants. The compactly supported cohomology of this space is determined by a combinatorial encoding of the projective subspace called a matroid.
Let $L_I$ be the coordinate subspace given by

$$L_I = \{x_{i_1} = x_{i_2} = \cdots = x_{i_l} = 0\}$$

for $I = \{i_1, i_2, \ldots, i_l\} \subset \{0, \ldots, n\}$. 

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Matroids

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4. $\rho(\{0, \ldots, n\}) = r + 1.$
Note: Item (3) abstracts

\[ \text{codim}(((V \cap L_I) \cap (V \cap L_J)) \subset (V \cap L_{IJ})) \leq \]
\[ \text{codim}((V \cap L_I) \subset (V \cap L_{IJ})) + \text{codim}((V \cap L_J) \subset (V \cap L_{IJ})). \]
Note: Item (3) abstracts
\[
\text{codim}(\langle V \cap L_I \cap (V \cap L_J) \rangle) \leq \text{codim}(\langle V \cap L_I \rangle) + \text{codim}(\langle V \cap L_J \rangle).
\]
This is one of the definitions of matroids. There are many others.
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1. One can construct matroids that are only representable over fields in which certain algebraic equations have solutions.

2. Over $\mathbb{Q}$, an algorithm to determine representability is equivalent to Diophantine decidability algorithm over $\mathbb{Q}$ which is open but thought to be impossible.
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2. Over $\mathbb{Q}$, an algorithm to determine representability is equivalent to Diophantine decidability algorithm over $\mathbb{Q}$ which is open but thought to be impossible.

3. It is a conjecture of Rota to characterize $\mathbb{F}_q$-representable matroids in terms of forbidden minors ($\mathbb{F}_2$ due to Tutte; $\mathbb{F}_3$ due to Seymour; $\mathbb{F}_4$ due to Geelen-Gerards-Kapoor).
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Let \(X \subset (\mathbb{C}^*)^n\) be an algebraic variety, that is, a common zero set of a system of polynomials. We can define a weighted polyhedral complex in \(\mathbb{R}^n\) that simultaneously generalizes Newton polytopes (for hypersurfaces) and matroids (for linear subspaces).
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Define $\text{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ by

$$\text{Log}(z_1, \ldots, z_r) = (\log(|z_1|), \ldots, \log(|z_n|)).$$
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The set $\text{Log}(X)$ is said to be the amoeba of $X$. 
Figure: The amoeba of the line \( \{ z_1 + z_2 - 1 = 0 \} \subset (\mathbb{C}^*)^2 \).

The tentacles correspond to

1. \( z_1 \to 0, \ z_2 \to 1 \),
2. \( z_2 \to 0, \ z_1 \to 1 \),
3. \( |z_1| \to \infty \).
To get something combinatorial, we need to look at the tropicalization which is the limit set

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where the limit is taken in the Hausdorff sense.
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Tropicalizations

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In practice, the logarithmic limit set definition is mostly unusable, and it’s more pleasant to use a purely algebraic definition.
We may also consider the tropicalization of a family of varieties $X_t$ parameterized by $t \in \mathbb{C} \setminus \{0\}$. In this case,

$$\text{Trop}(X) = \lim_{t \to 0} \frac{1}{\log(t)} \text{Log}(X_t).$$
Tropicalizations of Families

We may also consider the tropicalization of a family of varieties $X_t$ parameterized by $t \in \mathbb{C} \setminus \{0\}$. In this case,

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Example: Consider a family of cubic curves $V(f_t) \subset (\mathbb{C}^*)^2$ where

$$f_t = \sum_{0 \leq i, j \leq 3 \atop i+j \leq 3} a_{ij} x^i y^j$$

for $a_{ij} \in \mathbb{C}[t, t^{-1}] \setminus \{0\}$. The limit may have many different combinatorial types but below is one possibility.
A cubic curve in the plane
Tropicalizations of general subvarieties are balanced, weighted, integral polyhedral complexes (by results of Bieri-Groves and Speyer).
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The real dimension of Trop($X$) is equal to the complex dimension of $X$.

**Integral**: Each polyhedral cell is cut out by linear inequalities with rational coefficients.

**Weighted**: Each top-dimensional cell has a weight $w(P) \in \mathbb{N}$. (in almost all of our examples, it will be 1.)
Balanced: For 1-dimensional varieties, it’s easy to state For \( \nu \), a vertex of \( \Sigma \) and adjacent edges \( E_1, \ldots, E_k \) in primitive \( \mathbb{Z}^n \) directions, \( \vec{u}_1, \ldots, \vec{u}_k \) then

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Example:

For higher dimensions, the balancing condition is analogous.
Tropicalization compared to Newton polytope

How is tropicalization a generalization of Newton polytopes?
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**Theorem (Kapranov):** If \( f = \sum_{\omega \in \mathbb{Z}^n} a_\omega x^\omega \) is a polynomial, then \( \text{Trop}(V(f)) \) is the codimension 1 skeleton of the normal fan to \( P(f) \).
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The normal fan is made up of cones dual to the faces of the polytope. A cone dual to a face \( F \) is the set of all linear functionals on \( \mathbb{R}^n \) that achieve their minimum on \( F \). The codimension 1 skeleton means that we look at cones dual to positive dimensional faces.
How is tropicalization a generalization of matroids?

**Theorem (Sturmfels, Ardila-Klivans):** Let \( V \subseteq \mathbb{P}^n \) be a projective subspace. Then \( \text{Trop}(V \cap (\mathbb{C}^*)^n) \) is determined by the matroid \( M \) of \( V \).
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There is an explicit recipe for constructing the tropicalization from $\mathcal{M}$. It works over fields besides $\mathbb{C}$ by using the algebraic definition of Trop.
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There is a sort of converse to this theorem saying that if the tropicalization of a variety looks like the tropicalization of a subspace, then the variety is a subspace. I like calling it the duck theorem. It was written down by K.-Payne but also announced by Mikhalkin-Ziegler.
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Now let’s look at some pictures.
Tropicalization of a family of lines in the tropicalization of a plane in space
An elliptic curve in a plane in space

All multiplicities are 1. There are arrows pointing into and out of the screen to ensure balancing.
Properties encoded in tropicalization

What does the tropicalization know about the original variety?
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What does the tropicalization know about the original variety?

Some Intersection Theory:
What does the tropicalization know about the original variety?

**Some Intersection Theory:**
It knows the degree of the variety.
Given two varieties $X, Y \subset (\mathbb{C}^*)^n$ with $\dim(X) + \dim(Y) = n$, we can also read off an expected intersection number under genericity assumptions. This is a generalization of Bernstein's theorem due to K., Osserman-Payne, Rabinoff in different degrees of generality.
Some Hodge Theory: For $X \subset (\mathbb{C}^*)^n$ satisfying genericity assumptions, we can look at $H^*(X)$. This has a mixed Hodge structure. The lowest weight bit is described by $H^*(\text{Trop}(X))$ by a theorem of Hacking. For families, the analogous result is due to Helm-K.
Some Hodge Theory: For $X \subset (\mathbb{C}^*)^n$ satisfying genericity assumptions, we can look at $H^*(X)$. This has a mixed Hodge structure. The lowest weight bit is described by $H^*(\text{Trop}(X))$ by a theorem of Hacking. For families, the analogous result is due to Helm-K.

Under certain assumptions, the tropical variety knows much much more about the original variety. This is when the tropical variety locally looks like the tropicalization of a linear subspace. These are the so-called smooth tropical varieties. Results due to Itenberg-Kazarkov-Mikhalkin-Zharkov and K.-Stapledon.
Lifting problem

How are tropicalizations special among balanced, weighted, integral polyhedral complexes?
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Specifically, if I give you a balanced, weighted, integral polyhedral complex, how can you be sure that it comes from an algebraic variety? This is analogous to the representability problem for matroids. In fact, it contains that problem by the duck theorem so it must be subtle. This is called the lifting problem.
Lifting problem

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Here is an example of a non-liftable graph due to Mikhalkin and Speyer.
Example of non-liftable curve

Change the length of a bounded edge in the spatial elliptic curve so that it does not lie on the tropicalization of any plane (possible by dimension counting).
This is not liftable to a family of curves because
Example of non-liftable curve (cont’d)

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1 three unbounded edges in each direction in the curve shows that it must be a cubic,
Example of non-liftable curve (cont’d)

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3. any classical cubic is either genus 0 and spatial or genus 1 and planar,

no lift of the curve can be planar or genus 0, so the curve does not lift.
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It’s really subtle for surfaces. Huh has produced a two-dimensional complex that violates the Hodge index theorem and so cannot be a tropicalization. We cannot yet figure out what’s wrong with this surface, but we’re working on it. There’s lots of subtle positivity.
Lifting Problem (cont’d)

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3. It’s really subtle for surfaces. Huh has produced a two-dimensional complex that violates the Hodge index theorem and so cannot be a tropicalization. We cannot yet figure out what’s wrong with this surface, but we’re working on it. There’s lots of subtle positivity.

4. There’s an interesting example due to Vigeland of a curve $C$ and a surface $S$ in $(\mathbb{C}^*)^3$ where $\text{Trop}(C) \subset \text{Trop}(S)$ but it’s impossible to change $C, S$ to ensure $C \subset S$ without changing the tropicalizations. This makes enumerating curves on surfaces through tropical geometry tricky. This class of examples has been studied by Bogart-K., Brugallé-Shaw, Gathmann-Winstel.
Pathological curve in a surface