Hodge theory in combinatorics

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“But Hodge shan’t be shot; no, no, Hodge shall not be shot.”
– Samuel Johnson
The characteristic polynomial of a subspace

Let $k$ be a field. Let $V \subset k^{n+1}$ be an $(r+1)$-dim linear subspace not contained in any coordinate hyperplane. Would like to use inclusion/exclusion to express $[V \cap (k^*)^{n+1}]$ as a linear combination of $[V \cap L_I]$'s where $L_I$ is the coordinate subspace given by

$$L_I = \{x_{i_1} = x_{i_2} = \cdots = x_{i_l} = 0\}$$

for $I = \{i_1, i_2, \ldots, i_l\} \subset \{0, \ldots, n\}$.
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Example: Let \( V \) be a generic subspace (intersecting every coordinate subspace in the expected dimension). Then

\[
[V \cap ((k^*)^{n+1})] = [V \cap L_{\emptyset}] - \sum_{i} [V \cap L_i] + \sum_{|I|=2} [V \cap L_I] - \sum_{|I|=3} [V \cap L_I] + \ldots.
\]

If you're fancy, you can say that this is a motivic expression.
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If you’re fancy, you can say that this is a motivic expression.
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**Definition**

A subset \( I \subset \{0, \ldots, n\} \) is said to be a **flat** if for any \( J \supset I \), \( V \cap L_J \neq V \cap L_I \).
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The rank of a flat is

$$\rho(I) = \text{codim}(V \cap L_I \subset V).$$
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We can now write for some choice of \( \nu_I \in \mathbb{Z} \),

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[V \cap (k^*)^{n+1}] = \sum_{\text{flats } I} \nu_I [V \cap L_I].
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**Fact:** \((-1)^{\rho(I)} \nu_V\) is always positive.
Definition

The characteristic polynomial of $V$ is

$$
\chi_V(q) = \sum_{i=0}^{r+1} \left( \sum_{\text{flats } I, \rho(I)=i} \nu_I \right) q^{r+1-i}
$$

$$
\equiv \mu_0 q^{r+1} - \mu_1 q^r + \cdots + (-1)^{r+1} \mu_{r+1}
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We can think of $\chi$ as an evaluation of the classes $[V \cap L_I]$ of the form

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**Example:** In the generic case subspace case, we have

$$
\chi_V(q) = q^{r+1} - \binom{r+1}{1} q^r + \binom{r+1}{2} q^{r-1} - \cdots + (-1)^{r+1} \binom{r+1}{r+1}
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A polynomial with coefficients $\mu_0, \ldots, \mu_{r+1}$ is said to be log-concave if for all $i$,

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A polynomial with coefficients \( \mu_0, \ldots, \mu_{r+1} \) is said to be unimodal if the coefficients are unimodal in absolute value, i.e. there is a \( j \) such that

\[ |\mu_0| \leq |\mu_1| \leq \cdots \leq |\mu_j| \geq |\mu_{j+1}| \geq \cdots \geq |\mu_{r+1}|. \]
Original Motivation: Let $\Gamma$ be a loop-free graph. Define the chromatic function $\chi_{\Gamma}$ by setting $\chi_{\Gamma}(q)$ to be the number of colorings of $\Gamma$ with $q$ colors such that no edge connects vertices of the same color.
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Read’s Conjecture ’68 (Huh ’10): $\chi_\Gamma(q)$ is unimodal.
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4. $\rho(\{0,\ldots,n\}) = r + 1$. 

Note: Item (3) abstracts codim($((V \cap L_I) \cap (V \cap L_J)) \subset (V \cap L_I \cap J)$) $\leq$ codim($((V \cap L_I) \subset (V \cap L_I \cap J)) + codim((V \cap L_J) \subset (V \cap L_I \cap J))$. This is one of the definitions of matroids.

Eric Katz (Waterloo)

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For matroids, \( \nu_I \) and hence \( \chi(q) \) can be defined combinatorially by Möbius inversion without reference to any linear space. This leads us to
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**Conjecture:** For any matroid, $\chi(q)$ is log-concave.

We think we have it! We’re writing it up now.
Another problem

Today, I’m going to relate the log-concavity question to the lower bound theorem in polyhedral combinatorics.

Let $P \subset \mathbb{R}^d$ be a full-dimensional convex polytope. For the sake of convenience, let us suppose that $P$ is simplicial (every proper face is a simplex). Let $f_k(P)$ be the number of $k$-dimensional faces of $P$. We can ask how the $f_k$'s are constrained and which $f_k$'s are possible. McMullen gave a conjectural description. This was proven by Billera-Lee and Stanley. We will talk only about the necessity part of the lower bound theorem.

We make a linear change of variables for the packaging of the $f_k$'s: define $h_k$ by

$$d \sum_{i=0}^{d} f_i - 1 (t - 1) d - i = d \sum_{k=0}^{d} h_k t^{d-k}.$$ 

Here the Dehn-Sommerville relations say that the $h_k$'s form a symmetric sequence: $h_k = h_{d-k}$. 

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Stanley-Reisner rings

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This statement is implied by a statement in commutative algebra about Stanley-Reisner rings. Let $\Delta$ be the boundary of $P$, considered as a simplicial complex. Let $v_1, \ldots, v_n$ be the vertices of $P$. Introduce variables $x_1, \ldots, x_n$. For a field $k$, let

$$I_\Delta \subset k[x_1, \ldots, x_n]$$

be the non-face ideal. This is defined as follows: for $S \subset \{1, \ldots, n\}$ let

$$x^S = \prod_{i \in S} x_i,$$

then

$$I_\Delta = \langle x^S \mid S \text{ is not a face of } P \rangle.$$

The Stanley-Reisner ring is

\[ k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta. \]

Because \( I_\Delta \) is a homogeneous ideal, \( k[\Delta] \) is a graded ring.
Now let \( l_1, \ldots, l_d \) be generic degree 1 elements of \( k[\Delta] \). Then

\[ \dim (k[\Delta]/(l_1, \ldots, l_d))_i = h_i. \]
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The lower bound theorem is reduced to the existence of a weak Lefschetz element \( \omega \in k[\Delta] \) for which the multiplication map
\[ \cdot \omega : (k[\Delta]/(l_1, \ldots, l_d))_{i-1} \to (k[\Delta]/(l_1, \ldots, l_d))_i \]
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Lefschetz elements

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Note here that the unimodality of \( h_i \)'s is different from the unimodality of the characteristic polynomial as the characteristic polynomial is not symmetric. We have no idea where the mode is supposed to be.
The existence of the Lefschetz element comes from identifying the quotient $k[\Delta]/(l_1, \ldots, l_d)$ with the cohomology of a projective algebraic variety $X \subset \mathbb{P}^n$, that is $h_i = \dim H^{2i}(X)$. This variety, a toric variety, is mildly singular, but the Hard Lefschetz theorem gives a Lefschetz element. So the result relies on hard algebraic geometry, but
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McMullen’s proof uses an alternative presentation of the Stanley-Reisner ring. Then, he applies flip moves to transform $P$ into a simplex where the Hard Lefschetz theorem is known to hold, checking that the Hard Lefschetz theorem is preserved by these moves.
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Incidentally, the presentations should be thought of in the following way: the Stanley-Reisner presentation is homology under intersection product; the Minkowski weight ring (used by McMullen) is cohomology; the conewise polynomial ring (used by Karu) is a quotient of equivariant cohomology.
I should mention that there is recent, related work by Ben Elias and Geordie Williamson proving the Hard Lefschetz theorem in a synthetic context. They are interested in questions involving the positivity of Kazhdan-Lusztig polynomials and the Kazhdan-Lusztig conjecture in the context of Coxeter systems.
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These theorems were proven in the case of Weyl groups by studying the intersection cohomology of a Schubert variety.

In general, there may be no Schubert variety, so certain modules act as an abstract avatar. They prove that these modules have the required Hodge theoretic properties.
Let us delve into the hard algebraic geometry. I will discuss two theorems, the Hard Lefschetz theorem, and the Hodge Index theorem, and will explain how they are implied by an even deeper theorem, the Hodge-Riemann-Minkowski relations.
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Let $X \subset \mathbb{P}^n$ be a smooth projective $d$-dimensional algebraic variety. The cohomology ring $H^*(X)$ is a graded ring in degrees $0, 1, \ldots, 2d$. It’s an algebra over $\mathbb{C}$. We think of $H^i(X)$ as the group of codimension $i$ cycles in $X$. Now $H^{2d}(X) \cong \mathbb{C}$ is generated by the class of a point.
Now some hard algebraic geometry

Let us delve into the hard algebraic geometry. I will discuss two theorems, the Hard Lefschetz theorem, and the Hodge Index theorem, and will explain how they are implied by an even deeper theorem, the Hodge-Riemann-Minkowski relations.

Let \( X \subset \mathbb{P}^n \) be a smooth projective \( d \)-dimensional algebraic variety. The cohomology ring \( H^\ast(X) \) is a graded ring in degrees \( 0, 1, \ldots, 2d \). It’s an algebra over \( \mathbb{C} \). We think of \( H^i(X) \) as the group of codimension \( i \) cycles in \( X \). Now \( H^{2d}(X) \cong \mathbb{C} \) is generated by the class of a point.

There is a Hodge decomposition:

\[
H^k(X) = \bigoplus_{p+q=k} H^{p,q}(X)
\]
Hard Lefschetz theorem

If $H$ is a generic hyperplane in $\mathbb{P}^n$, $H \cap X$ gives a codimension 2 cycle in $X$, hence an element of $H^2(X)$. The Hard Lefschetz Theorem shows that $H$ is a strong Lefschetz element:

**Theorem (Hodge)**

Let $L : H^k(X) \to H^{k+2}(X)$ be given by multiplication by $H$. Then for all $k \leq d$,

$$L^{d-k} : H^k(X) \to H^{2d-k}(X)$$

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This implies the unimodality of $h_{2i}$'s.
The Hard Lefschetz theorem gives the Lefschetz decomposition of cohomology: define primitive cohomology $P^k \subset H^k(X)$ by

$$P^k = \ker(L^{d-k+1} : H^k(X) \to H^{2d-k+2}(X)).$$

Then

$$H^k(X) = P^k \oplus LP^{k-2} \oplus L^2 P^{k-4} \oplus \ldots.$$
The Hodge index theorem

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Let $X$ be a projective complex surface (2 complex dimensions, 4 real dimensions). Consider $H^2(X)$ equipped with intersection product

$$H^2(X) \otimes H^2(X) \to H^4(X) \cong \mathbb{C}.$$ 

**Theorem (Hodge)**

*The intersection product restricted to $H^{1,1}(X)$ is non-degenerate with a single positive eigenvalue.*
This implies the Hodge inequality:

**Corollary**

Let $\alpha, \beta \in H^{1,1}(X)$ be given by pulling back a hyperplane class from two embeddings $i_1, i_2 : X \to \mathbb{P}^{n_i}$. Then

$$(\alpha^2)(\beta^2) \leq (\alpha \cdot \beta)^2.$$
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This comes from the intersection product being indefinite on $\text{Span}(\alpha, \beta)$ so the discriminant is negative. Note we can replace $\alpha$ and $\beta$ by positive multiples (ample classes). Or look at classes that can be approximated by hyperplane classes (nef).
An even stronger theorem holds for algebraic varieties in all dimensions.

Theorem

Let $\alpha$ be an ample class. Let $P^*$ be the primitive cohomology with respect to $\alpha$. Then the pairing $Q_{p,q}$ on

$$H_{\text{prim}}^{p,q} = P^{p+q}(X) \cap H^{p,q}(X)$$

given by

$$Q_{p,q}(\beta, \gamma) = (-1)^{(p+q)(p+q-1)/2} i^{p-q-k}(\beta \cdot \gamma \cdot \alpha^{d-(p+q)})$$

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This is deep and analytic.
Hodge-Riemann-Minkowski Relations

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In the sequel, we will restrict to $H^{p,p}$ so

$$Q_{p,p}(\beta, \gamma) = (-1)^p (\beta \cdot \gamma \cdot \alpha^{d-2p}).$$
Consequences

The Hodge-Riemann-Minkowski relations immediately imply the Hard Lefschetz theorem. They also imply the Hodge index theorem:
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**Proof.**

We have

\[ H^{1,1}(X) = LH^{0,0}_{\text{prim}}(X) \oplus H^{1,1}_{\text{prim}}(X). \]

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More generally, we get the Khovanskii-Teissier inequality: for \( \alpha, \beta \) nef

\[ (\alpha^{r-i+1}\beta^{i-1})(\alpha^{r-i-1}\beta^{i+1}) \leq (\alpha^{r-i}\beta^i)^2. \]
Proof of Log-concavity

Now, let us outline the proof of log-concavity in the realizable case. First, we use the reduced characteristic polynomial:

From the fact $\chi(1) = 0$, we can set $\chi(q) = \chi(q^r - 1)$.

The log-concavity of $\chi$ implies the log-concavity of $\chi$.

Coefficients of $\chi$ have a combinatorial description:

$\chi(V)(q^r - 1) = \mu_0 q^r - \mu_1 q^r - 1 + \cdots + (-1)^r \mu_r q^0$.

Then $\mu_i = (-1)^i \sum_{\text{flats } I} \rho(I) = i^0 \not\in I \nu I$. 
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The log-concavity of $\bar{\chi}$ implies the log-concavity of $\chi$.

Coefficients of $\bar{\chi}$ have a combinatorial description:

$$\bar{\chi}_V(q) = \mu^0 q^r - \mu^1 q^{r-1} + \cdots + (-1)^r \mu^r q^0.$$

Then

$$\mu^i = (-1)^i \sum_{\text{flats } l \quad \rho(l) = i \quad 0 \not\in l} v_l.$$
A new Stanley-Reisner ring

We define a Stanley-Reisnerish ring attached to the matroid:

Definition

Let $x_F$ be indeterminates indexed by proper flats. Let $I_M$ be the ideal in $k[x_F]$ generated by

1. For each $i, j \in \{0, 1, \ldots, n\}$,
   \[
   \sum_{F \ni i} x_F - \sum_{F \ni j} x_F,
   \]

2. For incomparable flats $F, F'$,
   \[
   x_Fx'_F.
   \]

Let $R_M = k[x_F]/I_M$.

This is the Stanley-Reisner ring of the order complex of the lattice of flats of the matroid quotiented by a linear ideal. Henceforth, let us take $k = \mathbb{C}$. 

Eric Katz (Waterloo)
Properties of the ring

There is a canonical isomorphism

\[ \text{deg} : (R_M)_r \to \mathbb{C} \]

that takes the value 1 on an ascending chain of flats \( x_{F_1} \ldots x_{F_r} \).
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There are two important elements of \( R_M \): pick \( i \in \{0, 1, \ldots, n\} \), and set

\[ \alpha = \sum_{F \ni i} x_F \]
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**Lemma**

We have the equality

$$\mu^i = \text{deg}(\alpha^i \beta^{r-i}).$$
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**Lemma**

*We have the equality*

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Aside: We proved this using tropical intersection theory. You can give a direct proof in this presentation.
Theorem

If $M$ is realizable over $\mathbb{C}$, there is an algebraic variety $\widetilde{V}$ with $H^{2*}(\widetilde{V}) = R_M$. The classes $\alpha$ and $\beta$ are nef on $\widetilde{V}$ and the Hodge-Riemann-Minkowski relations hold for suitably perturbed $\alpha$ and $\beta$. 

So HRM implies the log-concavity of the $\mu_i$'s by the Hodge inequality. This implies the log-concavity of the $\mu_i$'s.

The same argument holds over fields besides $\mathbb{C}$. One has to use a different derivation of the Khovanskii-Teissier inequality making use of Kleiman's transversality.
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**Theorem**

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**Theorem**

If $M$ is realizable over $\mathbb{C}$, there is an algebraic variety $\tilde{V}$ with $H^{2\ast}(\tilde{V}) = R_M$. The classes $\alpha$ and $\beta$ are nef on $\tilde{V}$ and the Hodge-Riemann-Minkowski relations hold for suitably perturbed $\alpha$ and $\beta$.

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The space \( \tilde{V} \) is natural. Start with \( V \subset \mathbb{C}^{n+1} \). Projectivize to get \( \mathbb{P}(V) \subset \mathbb{P}^n \). The coordinate hyperplanes of \( \mathbb{P}^n \) induce a hyperplane arrangement on \( \mathbb{P}(V) \). We blow-up the 0-dimensional strata, and then the proper transforms of the 1-dimensional strata, and so on to produce \( \tilde{V} \).
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The space $\tilde{V}$ lives in a blown-up projective space $\tilde{\mathbb{P}}^n$ which has two natural maps to $\pi_1, \pi_2 : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$. Think: it resolves a Cremona transform. Then $\alpha = \pi_1^*H$, $\beta = \pi_2^*H$. 
The space $\tilde{\mathcal{V}}$ is natural. Start with $\mathcal{V} \subset \mathbb{C}^{n+1}$. Projectivize to get $\mathbb{P}(\mathcal{V}) \subset \mathbb{P}^n$. The coordinate hyperplanes of $\mathbb{P}^n$ induce a hyperplane arrangement on $\mathbb{P}(\mathcal{V})$. We blow-up the 0-dimensional strata, and then the proper transforms of the 1-dimensional strata, and so on to produce $\tilde{\mathcal{V}}$.

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We perturb $\alpha$ and $\beta$ so that they are ample. We get an inequality and then take limits.
Every time I’ve given a talk about log-concavity, I’ve asked if this result can be made purely combinatorial and thus prove Rota-Heron-Welsh. Every time, I’ve suggested some approach. I’ve even made jokes about the failures of these approaches.
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Well, this time is different. We have a lot of details to check, but we’re very confident that we did it!
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Well, this time is different. We have a lot of details to check, but we’re very confident that we did it!

Our idea is to start with projective space and do each blow-up one-by-one in a purely combinatorial fashion to produce intermediate Stanley-Reisner rings. We also have intermediate analogues of $\alpha, \beta$. We have to show that the Hodge-Riemann-Minkowski relations (with respect to a “combinatorial ample cone”) are preserved by our blow-ups. We have a geometric picture in mind of slicing faces off of a simplex to get a permutohedron.
Outline of proof

The proof has several steps making use of an inductive argument used by McMullen and Karu and elevated to a cornerstone of Hodge theory by de Cataldo and Migliorini:

1. Define a combinatorial analogue of an ample cone sitting in $(\mathbb{R}^M)^1$.
2. Show that the intermediate Stanley-Reisner rings satisfies Poincaré duality of dimension $r$.
3. Show that if two intermediate Stanley-Reisner rings satisfy Hodge-Riemann-Minkowski, their "skew tensor product" also does.
4. Show that if all skew tensor products of rank $r-1$ satisfy Hodge-Riemann-Minkowski than all intermediate Stanley-Reisner rings of rank $r$ satisfy Hard Lefschetz.
5. Show that if an intermediate Stanley-Reisner ring satisfies Hodge-Riemann-Minkowski with respect to one ample class, it satisfies it with respect to all of them.
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It is exactly as difficult as giving a purely (linear) algebraic proof of the following:

**Theorem**  
Let $X$ is a smooth projective variety with ample divisor $H$. Let $Z$ be a smooth subvariety. Suppose that $X$ and $Z$ satisfy the Hodge-Riemann-Minkowski relations. Then $\text{Bl}_Z X$ satisfies the Hodge-Riemann-Minkowski relations with respect to $H - \epsilon E$ where $E$ is the exceptional divisor and $\epsilon > 0$.  

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And the tropical geometry?

Since this conference is tropical geometry in the tropics, where’s the tropical geometry in this talk?

A Stanley-Reisner ring modulo a linear ideal, $R_{\Delta}/(l_1, \ldots, l_d)$, is said to have an $r$-dimensional fundamental class if there is an isomorphism $\deg: (R_{\Delta}/(l_1, \ldots, l_d))^r \to R$. To every degree 1 generator is associated a ray. To every square-free monomial not in $I_{\Delta}$ (thus a face) is associated a cone. The top-dimensional cones are given a weight by looking at the value of their corresponding monomial under $\deg$. The linear ideal generated an embedding into $R^d$ for which the fan is balanced.

This procedure produces the face fan from the S-R ring of a polytope. It produces the Bergman fan from the S-R ring of a matroid.
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Thanks!


Huh, June. *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs.*