Lifting Tropical Curves and Linear Systems on Graphs

Eric Katz (University of Waterloo)

September 4, 2012
What is Tropical Geometry?
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2. My answer: the combinatorial study of degenerations and stratifications of algebraic varieties.
What is Tropical Geometry?

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1. Usual answer: geometry over the tropical semifield.

2. My answer: the combinatorial study of degenerations and stratifications of algebraic varieties.

I will not precisely define all the terms in my answer but I will give you an example of it.
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Simon was Hungarian-born.
Why the word ’tropical’?

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1. Simon was Hungarian-born.
2. Simon worked in São Paulo which is south of the tropic of Capricorn.
Q: Why 'tropical' geometry?

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Problems with that:

1. Simon was Hungarian-born.
2. Simon worked in São Paulo which is south of the tropic of Capricorn and so, in fact, was not tropical.
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Note: No additive inverses, thus ‘semi’ and \( \infty \) \((\text{not} \ 0)\) is the additive identity.
Can define tropical polynomials:

\[ x^2 \oplus 1 \circ x \oplus 3 \]
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The zero-locus of the polynomial is the set of points where the minimum is achieved by at least two terms. In this case, at \( x = 1 \) and \( x = 2 \).
Can define tropical polynomials in several variables.
Tropical hypersurfaces

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Valuation-theoretic approach

There is an algebraic approach to tropical geometry due to Kapranov. Let \( \mathbb{K} = \mathbb{C}\{\{t\}\} = \mathbb{C}((t)) \), the field of formal Puiseux series. It is the algebraic closure of the field of formal Laurent series.
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$$x = \sum_{n=k}^{\infty} a_n t^{\frac{n}{N}}, \ a_n \in \mathbb{C}, a_k \neq 0$$

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Non-Archimedean: $v(x + y) \geq \min(v(x), v(y)), \ v(xy) = v(x) + v(y)$. 
The Cartesian product \((\mathbb{K}^*)^n\) is called an algebraic torus. (In complex case, \((\mathbb{C}^*)^n\) is the natural analog of \((S^1)^n\).) An algebraic variety in \((\mathbb{K}^*)^n\) is the common zero locus of a system of Laurent polynomials in \(n\) variables with coefficients in \(\mathbb{K}\).
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Tropicalization is a procedure that takes subvarieties of an algebraic torus to polyhedral complexes. The tropicalization of a variety \(X \subset (K^*)^n\) is defined to be

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\text{Trop}(X) = \overline{v(X)} \subset \mathbb{R}^n
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**Question:** Why is this even reasonable?
Tropicalization of a line

Let $f(x, y) = x + y + 1$. Let $X = V(f)$, the classical zero-locus of $f$. What is the tropicalization of $X$?
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If it comes from $x = at^r + \ldots$ then the coefficient of $t^r$ in $x$ must be cancelled by the coefficient of lowest power in $y$ or in $1$. So, if it comes only from $y$ then $y = (-a)t^r + \ldots$ and we have $v(x) = v(y) < v(1)$. 
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and, in fact, is equal by a theorem due to Kapranov.
**Kapranov’s theorem**

**Theorem (Kapranov)** If $f$ is a Laurent polynomial in $x_1, \ldots, x_n$ with support set $\mathcal{A} \subset \mathbb{Z}^n$,

$$f = \sum_{\omega \in \mathcal{A}} a_{\omega} x^{\omega}$$

$$\text{trop}(f) = \bigoplus_{\omega \in \mathcal{A}} v(a_{\omega}) \circ x^{\omega}.$$ 

Let $Z(f) \subset (\mathbb{K}^*)^n$ be the zero-locus of $f$. Then $\text{Trop}(Z(f))$ is equal to the tropical zero-locus of $\text{trop}(f)$. 

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**Theorem (Kapranov)** If $f$ is a Laurent polynomial in $x_1, \ldots, x_n$ with support set $A \subset \mathbb{Z}^n$,

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Let $Z(f) \subset (\mathbb{K}^*)^n$ be the zero-locus of $f$. Then $\text{Trop}(Z(f))$ is equal to the tropical zero-locus of $\text{trop}(f)$.

So the valuation definition generalizes the min-plus definition in the case of hypersurfaces. This lets you talk about the tropicalization of higher codimensional subvarieties.
Tropicalization map:

\[ \text{Trop} : \{ \text{curves } C \subset (\mathbb{K}^*)^n \} \rightarrow \{ \text{tropical graphs } \Sigma = \text{Trop}(C) \subset \mathbb{R}^n \} \]
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Tropical graphs are \textit{balanced}, \textit{weighted}, \textit{integral} graphs

\textbf{Integral}: Each edge is a line-segment or a ray parallel to \( \vec{u} \in \mathbb{Z}^n \).
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**Integral**: Each edge is a line-segment or a ray parallel to $\vec{u} \in \mathbb{Z}^n$.

**Weighted**: Each edge has a weight (multiplicity) $m(E) \in \mathbb{N}$. 
Balanced: For \( v \), a vertex of \( \Sigma \) and adjacent edges \( E_1, \ldots, E_k \) in primitive \( \mathbb{Z}^n \) directions, \( \vec{u}_1, \ldots, \vec{u}_k \) then

\[
\sum m(E_i) \vec{u}_i = \vec{0}.
\]

Example:
An elliptic curve in the plane

All multiplicities are 1.
An elliptic curve in space

All multiplicities are 1. Note that the cycle in the graph is contained in the plane of the screen.
More generally...

Tropicalizations of general subvarieties are balanced, weighted, integral polyhedral complexes (by results of Bieri-Groves and Speyer).
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Can think of varieties in \((\mathbb{K}^*)^n\) as families. Their coefficients are formal Puiseux series and so are formal Laurent series in some \(\mathbb{C}(\!(t^{\frac{1}{N}})\!))\). Set \(u = t^{\frac{1}{N}}\).

Ignoring issues of convergence, if we fix a particular value of \(u\), we get a variety in \((\mathbb{C}^*)^n\). So by including all values of \(u\) in a punctured neighborhood of \(u = 0\), we get a family of varieties in \((\mathbb{C}^*)^n\) over a punctured disc. So in a certain sense we are tropicalizing a family of varieties.
Q: What does Trop(\(X\)) know about \(X\)?
Q: What does $\text{Trop}(X)$ know about $X$?

A: Some intersection theory, some topology of $X$, some of the Hodge theory of $X$ by K., Sturmfels-Tevelev, Hacking, Helm-K., K.-Stapledon, Osserman-Payne.
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Q: How are tropicalizations special among balanced weighted integral polyhedral complexes?
Natural questions

**Q:** What does $\text{Trop}(X)$ know about $X$?

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**Q:** How are tropicalizations special among balanced weighted integral polyhedral complexes?

**A:** Today’s talk.
Lifting Problem: Which tropical (that is, balanced, weighted, integral) graphs are tropicalizations of curves?

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Speyer: Elliptic Curves, necessary and sufficient conditions in genus 1.

Nishinou and Brugallé-Mikhalkin: Generalization of Speyer’s result in one-bouquet case.
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Speyer: Elliptic Curves, necessary and sufficient conditions in genus 1.

Nishinou and Brugallé-Mikhalkin: Generalization of Speyer’s result in one-bouquet case.

The condition we’ll talk about today implies the necessity of these previously known conditions.
There are tropical curves that are not tropicalizations, telling the difference is subtle.
Why?

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- The problem is combinatorial, but what kind of combinatorics even encodes this?
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The problem is combinatorial, but what kind of combinatorics even encodes this?

Closely tied to deformation theory which is often grungy, maybe there’s a combinatorial approach.
Example of non-liftable curve

Change the length of a bounded edge in the spatial elliptic curve so that it does not lie on the tropicalization of any plane (possible by dimension counting).
Example of non-liftable curve (cont’d)

This is not liftable to a curve over $\mathbb{K}$ because
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Example of non-liftable curve (cont’d)

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This is not liftable to a curve over $\mathbb{K}$ because

1. three unbounded edges in each direction in the curve shows that it must be a cubic,
2. the loop in the curve shows that any lift must have genus at least 1,
3. any classical cubic is either \textit{genus 0 and spatial} or \textit{genus 1 and planar},

no lift of the curve can be planar or genus 0, so the curve does not lift.
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A tropical parameterization of a tropical graph $\Sigma$ is a map $p : \tilde{\Sigma} \rightarrow \Sigma$ (maps vertices to vertices but may contract edges) such that

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Note: If all the multiplicities of $\Sigma$ are 1 and all vertices are trivalent, then the only parameterization of $\Sigma$ is the identity. In fact, the only parameterization used in explicit examples will be the identity.
If \( \varpi \) is a piecewise-linear function on \( \tilde{\Sigma} \) (linear on all edges),
If $\varphi$ is a piecewise-linear function on $\tilde{\Sigma}$ (linear on all edges), if $v \in \tilde{\Sigma}$, $E \ni v$, write $s(v, E)$ for the slope of $\varphi$ on $E$ coming from $v$. 
If \( \varpi \) is a piecewise-linear function on \( \tilde{\Sigma} \) (linear on all edges),

if \( v \in \tilde{\Sigma}, E \ni v \), write \( s(v, E) \) for the slope of \( \varpi \) on \( E \) coming from \( v \).

Define the \textbf{Laplacian} of \( \varpi \) by

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A **divisor** \( \Lambda \) on \( \tilde{\Sigma} \) is a \( \mathbb{Z} \)-combination of vertices of \( \tilde{\Sigma} \).
We write \( \varpi \in L(\Lambda) \) (\( \varpi \) is the linear system associated to \( \Lambda \)) if

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0 \leq \Lambda(\nu) + \Delta \varpi(\nu).
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0 \leq \Lambda(w) + \Delta \varphi(w).
\]

\( \tilde{\Sigma} \) has canonical divisor:

\[
K_{\tilde{\Sigma}} = \sum_{v}(\deg(v) - 2)(v)
\]
**Theorem:** If $\Sigma \subset \mathbb{R}^n$ is a tropicalization of a curve then there exists $p : \tilde{\Sigma} \to \Sigma$ and for all $m \in \mathbb{Z}^n$ (which will be the normal vector to a plane), there is a piecewise-linear function $\varphi_m : \tilde{\Sigma}_l \to \mathbb{R}_{\geq 0}$ ($\tilde{\Sigma}_l$ is the $l$-fold subdivision of $\tilde{\Sigma}$) with $\mathbb{Z}$-slopes such that
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$$\varphi_m \in L(K_{\tilde{\Sigma}_l}).$$
Main theorem

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2. $\varphi_m = 0$ on $E$ with $m \cdot E \neq 0$,
3. $\varphi_m$ never has slope 0 on edges $E$ with $m \cdot E = 0$,
4. $\varphi_m$ obeys the cycle-ampleness condition.
Let $H$ be a hyperplane given by $H = \{x | x \cdot m = c\}$. 
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Let $\Gamma$ be a cycle in the interior of $p^{-1}(H) \subset \tilde{\Sigma}$.

Set $h = \min_{v \in \Gamma} (\varphi_m(v))$ then,

$$D_{\varphi_m} \equiv \sum_{v \in \Gamma | \varphi_m(v) = h} \left( \sum_{E \not\in \Gamma | s(v, E) < 0} (-s(v, E)) \right) \geq 2.$$  

“sum of positive slopes coming into the cycle at min’s of $\varphi_m$ must be at least 2.”
Before we use these conditions, we need the following observation:

\( \varphi_m \in L(K_{\Sigma_i}) \) translates into

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\Delta(\varphi_m)(v) = - \sum_{E \ni v} s(v, E) \geq 2 - \deg(v).
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If \( v \in \Gamma \) is a vertex with edges \( E_1, \ldots, E_k, F_1, \ldots, F_l \) (partitioned in any way). By hypothesis \( s(v, E_i), s(v, F_j) \neq 0 \).
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\[ \sum s(v, F_j) \leq \left( \sum -s(v, E_i) \right) + (\deg(v) - 2) \]

“At \( v \), sum of outgoing slope along edges \( F_j \) is less than sum of incoming slopes along edges \( E_i \) plus \( (\deg(v) - 2) \).”
Before we use these conditions, we need the following observation: \( \varphi_m \in L(K_{\Sigma_l}) \) translates into

\[
\Delta(\varphi_m)(v) = - \sum_{E \ni v} s(v, E) \geq 2 - \deg(v).
\]

If \( v \in \Gamma \) is a vertex with edges \( E_1, \ldots, E_k, F_1, \ldots, F_l \) (partitioned in any way). By hypothesis \( s(v, E_i), s(v, F_j) \neq 0 \).

\[
\sum s(v, F_j) \leq \left( \sum -s(v, E_i) \right) + (\deg(v) - 2)
\]

“At \( v \), sum of outgoing slope along edges \( F_j \) is less than sum of incoming slopes along edges \( E_i \) plus \( (\deg(v) - 2) \).”

If \( \deg(v) = 2 \), then the slope is non-increasing through \( v \) (\( \varphi_m \) is concave at \( v \)).
Elliptic curve example

Note: This is $p^{-1}(H)$ where $H$ is the plane of the screen.
Elliptic curve example (cont’d)

Need to pay attention to positive incoming slope coming into the cycle.

1. Direct edges towards cycle.
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6. There is positive incoming slope at \( \leq 3 \) points on the cycle. At those points, \( \varphi_m \) is equal to distance to \( \partial(p^{-1}(H)) \)
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7. For \( \deg(D\varphi_m) \geq 2 \), the minimum distance must be achieved at least twice.
In summary, minimum distance from $\Gamma$ to $\tilde{\Sigma} \setminus p^{-1}(H)$ must be achieved by at least two paths.
Elliptic curve example (concluded)

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Also get generalization to higher genus as given by Nishinou and Brugallé-Mikhalkin. This requires strong conditions on combinatorics of $\Sigma$. 
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**Theorem:** Let $\Sigma \subset \mathbb{R}^n$ be a tropicalization. Then there exists $p : \tilde{\Sigma} \to \Sigma$ that satisfies the following property:

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if $H \subset \mathbb{R}^n$ is a hyperplane and $\Gamma'$ is any component of $p^{-1}(H) \subset \tilde{\Sigma}$ with $h^1(\Gamma') > 0$ then $\partial \Gamma'$ is not a single trivalent vertex of $\tilde{\Sigma}$. 
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Add unbounded edges pointing out of the plane to ensure that is globally balanced. Give every edge multiplicity 1. Can ensure that only parameterization is the identity.
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There does not exist the desired $\varphi_m$, so it does not lift.
Direct edges towards cycle.
A new example (cont’d)

1. Direct edges towards cycle.
2. \(\varphi_m\) is equal to 0 on \(\partial(p^{-1}(H))\) and has slope at most 1 there.
1 Direct edges towards cycle.

2 $\varphi_m$ is equal to 0 on $\partial(p^{-1}(H))$ and has slope at most 1 there.

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\( \deg(D\varphi_m) \leq 1 \) on one cycle.
Quick Outline of proof

1. Suppose $\Sigma$ lifts. By Nishinou-Siebert, $C \hookrightarrow (\mathbb{K}^*)^n$ extends to a stable map $f : C \to P$ from a complete semi-stable curve to a toric scheme. These are families of object over an unpunctured disc.
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3. Obtain 1-forms \( \omega_m = f^* \frac{dz_m}{z_m} \), a section of log cotangent bundle \( \Omega^1_{C^\dagger/O^\dagger} \).
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5. Cycle-ampleness condition comes from $\omega_m$ being “almost” exact on the cycle and the fact that a non-constant rational function on a (possibly degenerate) elliptic curve must have (counted with multiplicity) at least two poles.
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Asides and future directions

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6. Possible applications to number theory? Further refinement of Chabauty in bad reduction case?


Nishinou, Takeo. *Correspondence Theorems for Tropical Curves*, arXiv:0912.5090