“Oh yes, I remember Clifford. I seem to always feel him near somehow.”
– Jon Hendricks
The Chabauty-Coleman method

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Let \( C \) be a curve defined over \( \mathbb{Q} \) with good reduction at a prime \( p > 2g \). This means that viewed as a curve over \( \mathbb{Q}_p \), it can be extended to \( \mathbb{Z}_p \) such that the fiber over \( p \) is smooth. Let \( \text{MWR} = \text{rank}(J(\mathbb{Q})) \) be the Mordell-Weil rank of \( C \). Computing MWR is now an industry among number theorists.
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**Theorem:** (Coleman) If $\text{MWR} < g$ and $p > 2g$ then

$$\# C(\mathbb{Q}) \leq \# C_0(\mathbb{F}_p) + 2g - 2.$$
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In the case $p \leq 2g$, there’s a small error term.
The Chabauty-Coleman method does give a bound on the number of rational points, but it doesn’t tell you anything about their height. If the bound says that there are at most 5 points, and you’ve found 4, you don’t know if there’s an additional point. So you never know when to give up your search. It’s important to get the bound as small as possible.

The bound was lowered by Stoll in the case that \( \text{MWR} \) is even smaller than \( g - 1 \):

**Theorem:** (Stoll) If \( \text{MWR} < g \) then \( \# C(\mathbb{Q}) \leq \# C_0(F_p) + 2 \text{MWR} \).
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Idea of proof of Chabauty-Coleman:

First, work $p$-adically. If $C$ has a rational point $x_0$, use it for the base-point of the Abel-Jacobi map $C \to J$. If $\text{MWR} < g$ by an argument involving $p$-adic Lie groups, we can suppose that that $J(\mathbb{Q})$ lies in an Abelian subvariety $A_{\mathbb{Q}_p} \subset J_{\mathbb{Q}_p}$ with $\dim(A_{\mathbb{Q}_p}) \leq \text{MWR} < g$. 

Eric Katz (Waterloo)  
Rank functions  
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We might expect $C(\mathbb{Q}_p)$ to intersect $A_{\mathbb{Q}_p}$ in finitely many points. In fact, there is a 1-form $\omega$ on $J_{\mathbb{Q}_p}$ that vanishes on $A$, hence on the images of all points of $C(\mathbb{Q})$ under the Abel-Jacobi map. Pull back $\omega$ to $C_{\mathbb{Q}_p}$. By multiplying by a power of $p$, can suppose that $\omega$ does not vanish on the central fiber $C_0$. 
We should view a curve over $\mathbb{Z}_p$ as a family of curves over a disc with generic fiber being the curve over $\mathbb{Q}_p$ and the central fiber being its reduction over $\mathbb{F}_p$. Each rational point of $C(\mathbb{Q}_p)$ is a zero of $\omega$. Think of zeroes of $\omega$ degenerating and slamming together as we approach the central fiber. Each residue class $\tilde{x} \in C_0(\mathbb{F}_p)$ is the reduction of a tube $[\tilde{x}]$ of $\mathbb{Q}_p$-points. The vanishing behaviour of the restriction of $\omega$ near $\tilde{x}$ tells us about the zeroes of $\omega$ in $[\tilde{x}]$. 
Outline of Coleman’s proof (cont’d)

To make this insight precise, Coleman defines a function $\eta : C(\mathbb{Q}_p) \to \mathbb{Q}_p$ by a $p$-adic integral,

$$\eta(x) = \int_{x_0}^x \omega$$

that vanishes on points of $C(\mathbb{Q})$. By a Newton polytope argument for any residue class $\tilde{x} \in C_0(\mathbb{F}_p)$, we get

$$\#(\eta^{-1}(0) \cap [\tilde{x}]) \leq 1 + \text{ord}_{\tilde{x}}(\omega|_{C_0}).$$
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Summing over residue classes $\tilde{x} \in C_0(\mathbb{F}_p)$, we get

$$\#C(\mathbb{Q}) \leq \#\eta^{-1}(0) = \sum_{\tilde{x} \in C_0(\mathbb{F}_p)} (1 + \text{ord}_{\tilde{x}}(\omega|_{C_0})) = \#C_0(\mathbb{F}_p) + \deg(\omega) = \#C_0(\mathbb{F}_p) + 2g - 2.$$
Proof of Stoll’s improvement

Stoll improved the bound by picking a good choice of $\omega$ vanishing on $C(\mathbb{Q})$ for each residue class.
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Let $\Lambda \subset \Gamma(J_{\mathbb{Q}_p}, \Omega^1)$ be the 1-forms vanishing on $\overline{J(\mathbb{Q})}$. For each residue class $\tilde{x} \in C_0(\mathbb{F}_p)$, let

$$n(\tilde{x}) = \min \{ \text{ord}_{\tilde{x}}(\omega|_{C_0}) | 0 \neq \omega \in \Lambda \}.$$ 

Let the Chabauty divisor on $C_0$ be

$$D_0 = \sum n(\tilde{x})(\tilde{x}).$$

So each $\omega \in \Lambda$ vanishes on $D_0$. 
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Coleman integration works between points in the same tube, so by summing over residue classes, one gets

$$\# C(\mathbb{Q}) \leq \# C_0(\mathbb{F}_p) + \deg(D_0).$$
Now, we just need to bound \( \deg(D_0) \). Every \( \omega \in \Lambda \) extends (up to a multiple by a power of \( p \)) to a regular 1-form vanishing on \( D_0 \).
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By a semi-continuity argument together with Clifford’s theorem, one gets

$$\dim \Lambda \leq \dim H^0(C_0, \Omega_{C_0}^1 - D_0) \leq g - \frac{\deg(D_0)}{2}.$$
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Since $\dim \Lambda = g - \text{MWR}$, $\text{deg}(D_0) \leq 2 \text{MWR}$. 
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Therefore, we get

\[
\# C(\mathbb{Q}) \leq \# C_0(\mathbb{F}_p) + 2 \text{MWR}.
\]
Bad reduction case

Now, the above argument breaks down in the bad reduction case because if $C_0$ is reducible, even if replace $\Omega^1_{C_0}$ by $K_{C_0}$, $H^0(C_0, K_{C_0} - D_0)$ goes completely haywire with 1-forms vanishing on components. However,
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**Theorem:** (K-Zureick-Brown ’12) Let $C$ by a regular minimal model for $C$ over $\mathbb{Z}_p$. Suppose $\text{MWR} < g$ then

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These are the Stoll bounds. The bad reduction case of Coleman’s bound was proved independently by Lorenzini-Tucker and McCallum-Poonen. The bad reduction case of the Stoll bound was proved for hyperelliptic curves by Stoll and the general case was posed as a question in a paper of McCallum-Poonen.
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Since $C$ is a regular minimal model, the total space is regular, but there are no conditions of the types of singularities on the central fiber. They can be worse than nodes.
If you adapt Stoll’s proof and try to apply semi-continuity arguments, you end up in the following situation:
Let $C$ be a regular minimal model of a curve $C$ over a valuation field $\mathbb{K}$ with residue field $k$. Let $L$ be a line-bundle on $C$ (think $\Omega^1_C$). Let $D_0$ be a divisor on $C^{\text{sm}}_0(k)$. Let

$$|L|_{D_0} = \{D \in |L| \mid D_0 \subset \overline{D}\}$$

where $D \subset C$ is a divisor of a section of $L$ and $\overline{D}$ denotes its closure in $C$. 

Definition: We say the rank $r(L, -D_0)$ is greater than or equal to $r$ if for any rank $r$ effective divisor $E$ supported on $C(\mathbb{K})$,

$$|L(-E)|_{D_0} \neq \emptyset.$$
A natural framework

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One can prove by a specialization argument similar to Matt Baker’s specialization lemma that if $\Lambda \subset H^0(C, L)$ is a linear subspace such that for every $s \in \Lambda$, $(\overline{s}) \supset D_0$, then $\dim \Lambda \leq r(L, -D_0) + 1$. 
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**Question:** Can we prove a Clifford bound $r(\Omega^1_C, -D_0) \leq g - \frac{\deg(D_0)}{2} - 1$?
Bounding $r(L, -D_0)$

**Problem:** It is really hard to work with $|L|_{D_0}$ directly. It’s a rigid analytic subspace of projective space and it’s not even clear if its rank has nice properties. Working with it requires developing a missing theory of rigid analytic/algebraic compatibility.
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**Solution:** Instead, we’ll bound $r(L, −D_0)$ in terms of more tractable ranks involving separate obstructions to finding a section of $L$ whose zero locus contains $D_0$ in its closure.
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**Reduction step:** We can suppose that $C$ is a semistable model. All rational points of $C$ specialize to smooth points of $C_0$ and they are not messed up too badly by the operations in semistable reduction. This does require a technical lemma.
We apply a certain extension hierarchy to this question. This is very closely related to tropical lifting. The steps have technical names which are inspired by the Néron model. The steps should be reminiscent of how one thinks about tropical lifting. Let $D_0$ be a divisor supported on smooth points of $C_0(\mathbb{F}_p)$.

Try to construct a rational section $s_0$ on the central fiber whose vanishing behaviour is controlled by $D_0$.

(a) numerical: Is there an extension $L$ of $L$ to $C$ such that $L|_{C_0}(D_0)$ has non-negative degree?

(b) Abelian: For each component $C_v$ of the central fiber, is there a section $s_v$ on $C_v$ of $L|_{C_v}(D_0)$?

(c) toric: Can the sections $s_v$ be chosen to agree on nodes?

Use deformation theory to extend the glued together section $s_0$ to $C$. We will concentrate on the first step.
Extension hierarchy for sections

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The rank hierarchy

This hierarchy lets us define new rank functions following Baker-Norine. We say a pair \((L, D_0)\) where \(L\) is a line-bundle on \(C\) and \(D_0\) is a divisor on \(C_{0}^{\text{sm}}\) has \(i\)-rank \(\geq r\) if for any effective divisor \(E_0\) on \(C_{0}^{\text{sm}(k)}\) of degree \(r\), steps (1) – (i) are satisfied: for an extension \(\mathcal{L}\) of \(L\),
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3. **toric**: The sections $s_v$ be chosen to agree across nodes.
New rank functions

So we have rank functions $r_{\text{num}}, r_{\text{Ab}}, r_{\text{tor}}$. 

$r_{\text{num}}(L, D_0)$ depends only on the multi-degree of $L$ and $D_0$, that is $\deg(L C_v(D_0))$ for all $v$. It does not depend on the geometry of the components. It is, in fact, identical to the Baker-Norine rank. In fact, a divisor on a graph is the same thing as a multi-degree. $r_{\text{Ab}}$ depends on the geometry of the components and the location of the points of $D_0$, but not the location of the nodes. This is identical to the rank function independently introduced by Amini-Baker.

The rank functions $r_{\text{Ab}}, r_{\text{tor}}$ are sensitive to the residue field $k$ since bigger $k$ allows for more divisors $E$. But they eventually stabilize.
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$r_{\text{num}}(L, D_0)$ depends only on the multi-degree of $L$ and $D_0$, that is $\deg(\mathcal{L}_{C_v}(D_0))$ for all $v$. It does not depend on the geometry of the components. It is, in fact, identical to the Baker-Norine rank. In fact, a divisor on a graph is the same thing as a multi-degree.
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So now, we have ways to bound \( r(\Omega_C, -D_0) \).
The appropriate bound would follow from an analogue of Clifford’s theorem: let $D_0$ be an effective divisor supported on points of $C_{0}^{\text{sm}}(k)$; then we have

$$r(\Omega^1, -D_0) \leq g - \frac{\deg(D_0)}{2} - 1.$$
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Now, the multi-degree of its restriction to the central fiber is (considered as a divisor on the dual graph $\Gamma$),

$$\deg(K_{C_0}) = \sum_v (2g(C_v) - 2 + \deg(v))(v) = K_\Gamma + \sum_v 2g(C_v)(v)$$

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If all components are rational, then $\deg(K_\Gamma) = 2g - 2$ and the Baker-Norine’s Clifford bounds for $r_{\text{num}}$ are sufficient.
In general, we have

**Theorem:** (Clifford-Brown-Amini-Baker-K) Let $D_0$ be a divisor supported on smooth $k$-points of $C_0$ then

$$r_{Ab}(K_{C_0} - D_0) \leq g - \frac{\deg D_0}{2} - 1.$$ 

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Our proof uses the Baker-Norine version of Clifford’s theorem, classical Clifford’s theorem, and a general position argument. We cook up a divisor $E_0$ of degree at most $g - \frac{\deg D_0}{2}$ such that for any $\varphi$, there is some component $C_v$ such that the line bundle $\mathcal{L}(\varphi)|_{C_v}(D_0 - E_0)$ on $C_v$ has no non-zero sections.
Further Questions

1. Because Clifford’s bounds are usually strict, in any given case, one can probably do better by bounding the Abelian rank by hand. Is there a general statement that incorporates the combinatorics of the dual graph?

2. What can we say about the number of rational points specializing to different components of the central fiber?

3. What about \(r_{tor}\)? Does that help us improve the bounds?

4. What about passing from the special fiber to the generic fiber? This should give even better bounds. We can use deformation-theoretic obstructions from tropical lifting here. Probably really need to understand the bad reduction analogue of the Coleman integral which is the Berkovich integral.
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5. $r(L, -D_0)$?
O. Amini, M. Baker, *Linear series on metrized complexes of algebraic curves.*


