Hyperbolic Conservation Laws

Past and Future

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Introduction

Systems of quasilinear hyperbolic PDE (conservation laws)

"... the umbilical cord joining Continuum Physics with the theory of partial differential equations should not be severed ..." *C. Dafermos*

This talk:

- A mathematical view of conservation laws
- Some analytical results
- Some puzzling challenges

Mathematics vs Physics

View from physics vs View from mathematics

"trust conservation principles" vs "quasilinear hyperbolic eqns in divergence form have advantages (well-defined weak solutions)"

Adiabatic, compressible, ideal gas dynamics as an example

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y &= 0 \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y &= 0 \\ (\rho E)_t + (\rho u H)_x + (\rho v H)_y &= 0 \end{aligned}$$

Variables ρ (density), (u, v) (velocity), and p (pressure)

$$E = \frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2), \qquad H = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2)$$

Form and Features

General form

$$\partial_t u + \sum \partial_{x_i} f_i(u) + b(u) = \partial_t u + \sum A_i(u)\partial_{x_i} u + b(u) = 0$$

or

$$\text{div}_{(t,x)}(u, f(u)) + b(u) = 0$$

Hyperbolicity: EV of $\sum A_i \mu_i$ real

- Discontinuities arise from smooth data; are not on chars but on a nonlinear version; are not invariant under (nonlinear) mappings
- Time-symmetry is broken in a system that is formally time-reversible
- Admissibility defined via “entropy” or “dissipation”
- Analysis of conservation laws uses nonlinear tools (eg, compactness)

Quasilinear Hyperbolic Equations

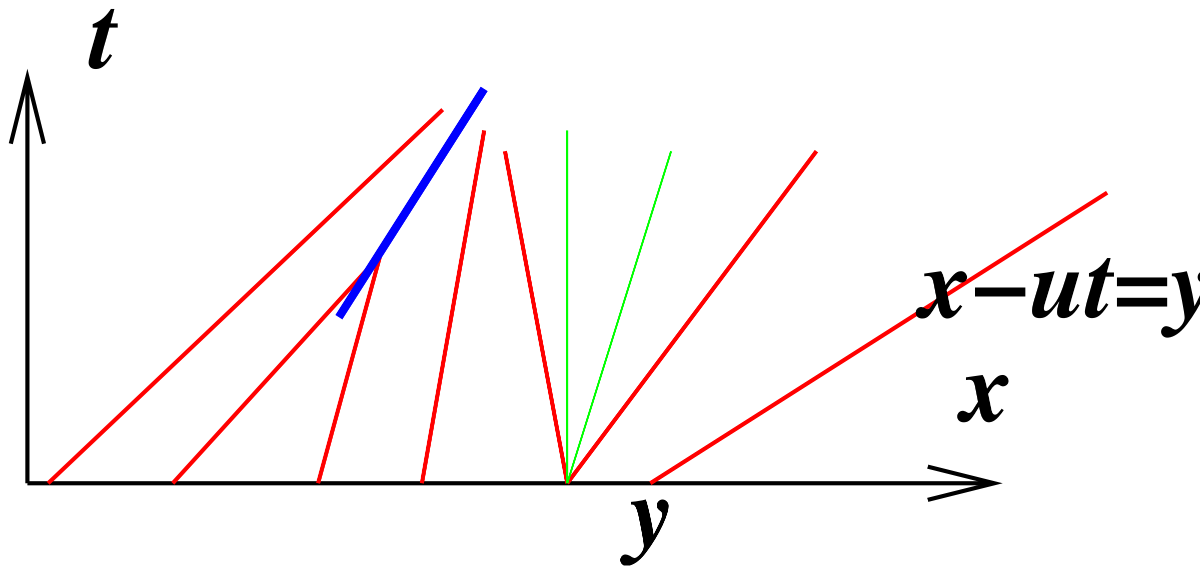
Burgers Equation $u_t + uu_x = 0$

Solution $u(y + u(y)t, t) = u(y)$

Converging characteristics: form shock, weak solution

(weak solution: $\int u\varphi_t + f(u)\varphi_x = 0$)

Diverging characteristics: form rarefaction



Loss of time reversibility: information is lost in forward time

QL eqns are irreversible – are they really hyperbolic?

Weak Solutions

Linear theory:

1. Sobolev spaces are useful – $W^{s,p}$, often with $p = 2$
2. "weak convergence" is useful, and is a different concept from "weak solution"
3. combine with regularity to get classical solutions (especially for elliptic equations)

Three difficulties with nonlinear equations:

1. need to define $f(u)$; in 1-D, most useful space is BV
2. weak convergence does not preserve nonlinear relations
3. hyperbolic and elliptic theory very different

Irreversibility and Entropy

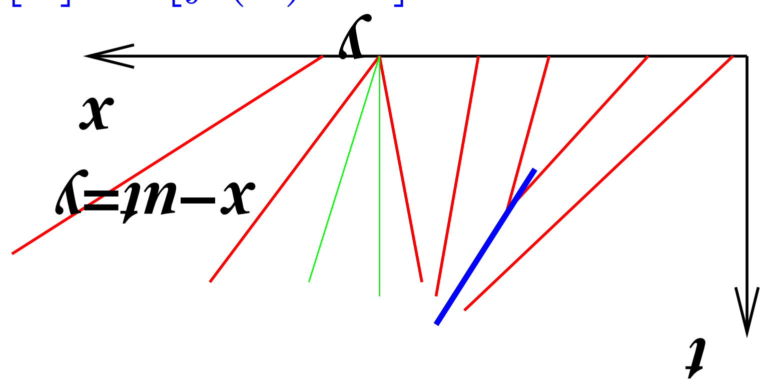
Complete defn of weak solution known in one case: scalar eqn $\partial_t u + \sum \partial_{x_i} f_i(u) = 0$ (Kruzkov): $\forall k \in \mathbf{R}, \varphi \in C^\infty$,

$$\int_0^T \int_{\mathbf{R}^n} |u-k|\partial_t \varphi + \sum_1^n \text{sgn}\,(u-k)(f_i(u) - f_i(k))\partial_{x_i}\varphi \, dx\, dt$$
$$+ \int_{\mathbf{R}^n} |u_0(x) - k|\varphi(x,0)\, dx \geq 0$$

Quasilinear system $\partial_t u + \sum \partial_{x_i} f_i(u) + b(u) = 0$
Discontinuity must satisfy RH $s[u] = [f(u) \cdot \nu]$ and more:

Vanishing viscosity

$$\partial_t u + \sum A_i(u)\partial_{x_i} u + b(u) = \varepsilon \Delta u$$



Entropy: convex function $\eta(u)$, with $\partial_t \eta(u) + \sum \partial_{x_i} q_i(u) \leq 0$

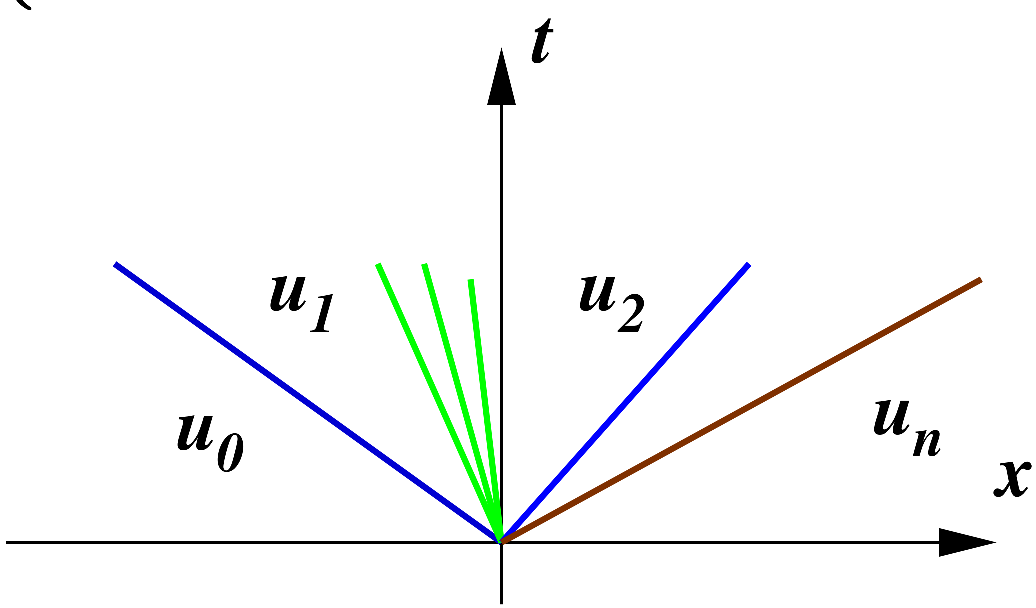
Riemann Solutions

One space dimension $u_t + f(u)_x = 0$, $u \in \mathbf{R}^n$

Riemann Data

$$u(x,0) = \begin{cases} u_\ell, & x < 0 \\ u_r, & x \geq 0 \end{cases}$$

Self-similar: $\xi = \frac{x}{t}$

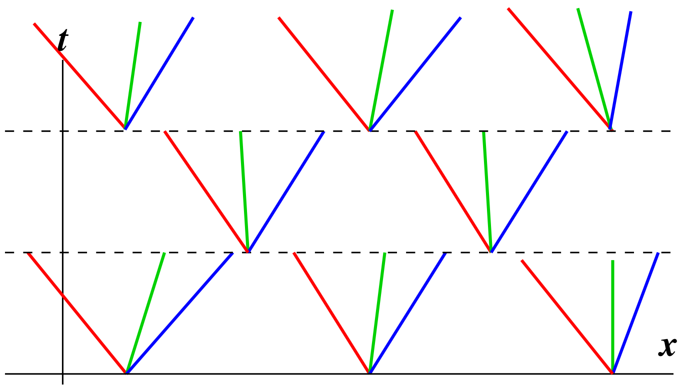


(Linear analogue: 1-D characteristic decomposition of a discontinuity)

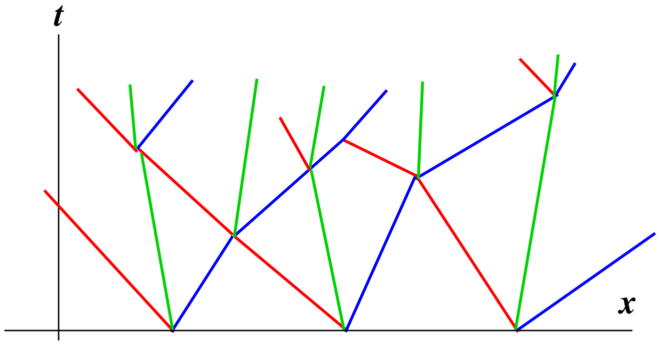
Random Choice and Front Tracking

1-D solution space is BV: $u(x,0) \in BV \Rightarrow$ sol'n in BV
"Outside a set of 1-D Hausdorff measure 0, a BV fn is either approx continuous or has an approx jump discont"
Use Riemann solutions to prove existence:

Glimm's random choice



Risebro-Bressan's wave front tracking



$\mathrm{Var}\, u(\cdot,0) \le \varepsilon \Rightarrow \mathrm{Var}\, u(\cdot,t) \le M$, $\int |u(t,x) - u(s,x)| \le L|t-s|$
Helly's theorem $\Rightarrow$ subsequence cvges ptwise to BV soln.
Bressan: SRS (Standard Riemann Semigroup) – uniqueness, well-posedness, & greater regularity (cont's except for ct'ble set of shock curves & interaction points)

Large Data

Things go wrong; here's an example (K-Kranzer)

Isentropic GD $\begin{cases} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + A\rho^\gamma)_x &= 0 \end{cases}$

Isothermal GD ($\gamma = 1$) use *specific enthalpy* $q(\rho) = A \log \rho$

$$u_t + \left(\tfrac{1}{2}u^2 + q(\rho)\right)_x = 0$$

Conserve u (velocity) and $v = \frac{1}{2}u^2 - q = \frac{1}{2}u^2 - \log \rho$, (cvx fn of u and ρ):

$$\begin{aligned} u_t + (u^2 - v)_x &= 0 \\ v_t + (\tfrac{1}{3}u^3 - u)_x &= 0 \end{aligned}$$

- Strictly hyperbolic, but no Riemann solutions for large data
- Generalize with singular shocks
- Obstructions to solving Cauchy problem:

  – oscillations producing blow-up in BV (Sever)

Large Data; and Hyperbolicity

- Counterexamples also in systems of 3 equations
- Related to ‘nonlinear resonance’ between characteristic families (‘non-strict hyperbolicity’)?
- Bressan’s SRS theory requires ‘tame oscillation condition’ for uniqueness
- Def’n of a wk soln for a system is still incomplete
- If hyperbolic does not imply “well-posed”, then does loss of hyperbolicity imply “catastrophically ill-posed”?
- Examples from two-fluid model for two-phase flow; three-phase porous media flow; continuum model for traffic flow
- Phase-changing shocks stable in ways similar to hyperbolic shocks

Multidimensional problems

Quasilinear systems in more than one space dimension

$$u_t + \sum A_j(u) u_{x_j} + b(u) = 0$$

Linear & semilinear: $W^{s,2}$ theory for smooth data (short time for QL)

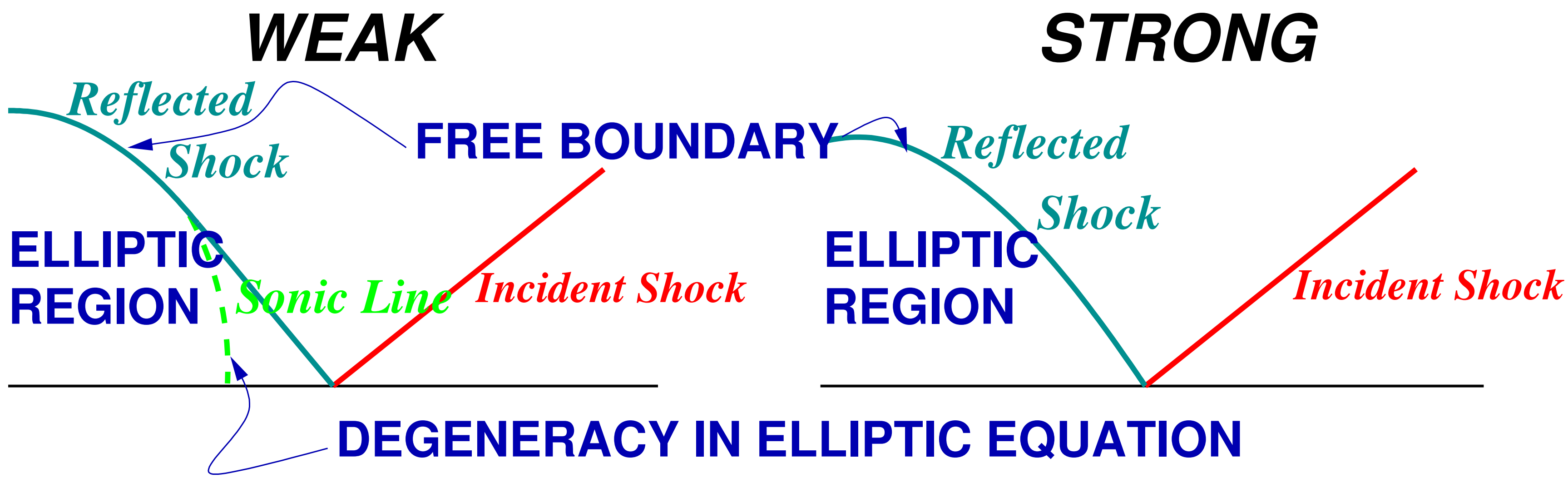
Theorem (Rauch): No BV bounds. For C^∞ data, if

$$\int_{R^n} |\nabla_x u(x,t)|\, dx \le C \int_{R^n} |\nabla_x u(x,0)|\, dx$$

then $\qquad A_j A_k = A_k A_j \quad \forall j,k.$

What's wrong with $A_j A_k = A_k A_j$? No physically interesting system has this property.

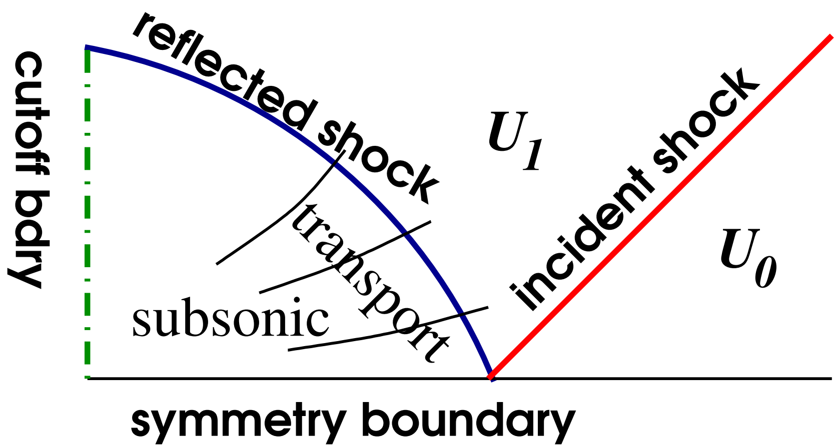
Self-similar Approach to 2-D CL



Simplified data: 2-D RP
von Neumann paradox
Self-similar problems
Model equations
Hölder & Sobolev spaces
Existence results

- Canic, K, Kim, Jegdic, Tesdall
- T Chang (D Zhang), J-Q Liu
- S-X Chen
- Y Zheng, K-W Song
- G-Q Chen & M Feldman
- T-P Liu, V Elling

Nature of the Analysis



Isentropic gas dynamics

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0$$
$$(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0$$

$$(\xi, \eta) = (x/t, y/t)$$
$$(U, V) = (u - \xi, v - \eta)$$

Elliptic equation for ρ (where $(u-\xi)^2 + (v-\eta)^2 < \rho^2$):

$$(U^2 - c^2)\rho_{\xi\xi} + 2UV\rho_\eta + (V^2 - c^2)\rho_{\eta\eta} + \ldots = 0$$

Transport system for (U, V):

$$(U, V) \cdot \nabla U + p_\xi/\rho + U = 0 \quad (U, V) \cdot \nabla V + p_\eta/\rho + V = 0$$

Free boundary (reflected shock): shock evolution + OD BC

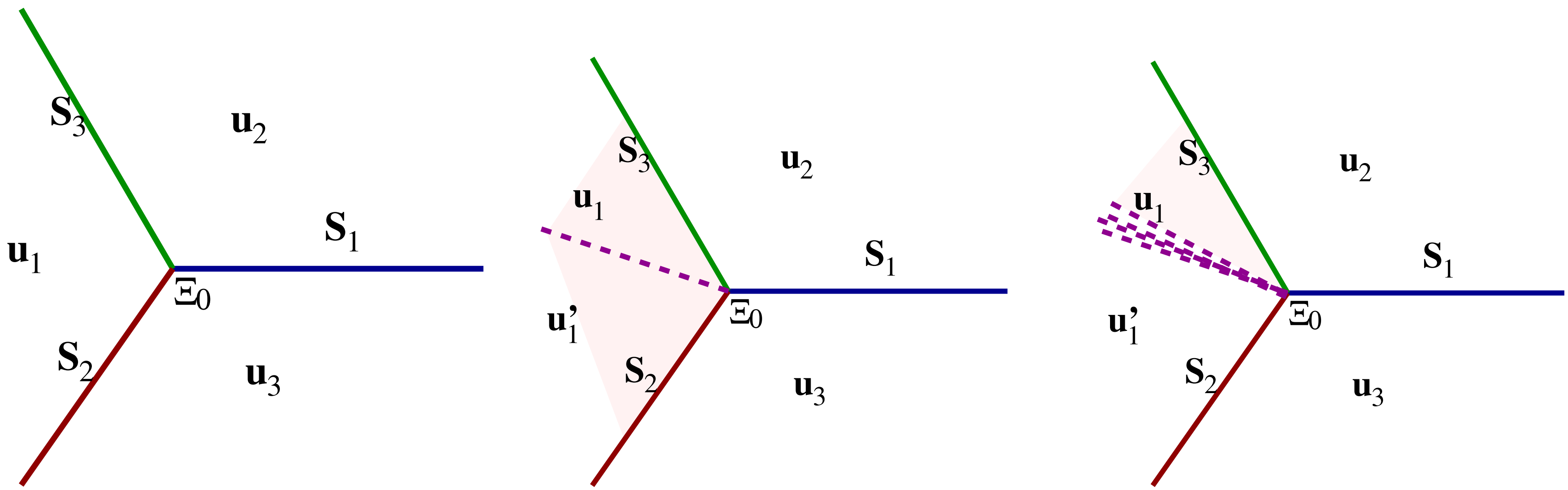
$$\frac{d\eta}{d\xi} = \frac{U_0V_0 - \sqrt{s^2(U_0^2 + V_0^2 - s^2)}}{U_0^2 - s^2} \qquad \beta(\rho, U, V) \cdot \nabla\rho = F(\rho, U, V)$$

Solution (local) in weighted Hölder space

Schauder FP thm for ρ (cpct); Contraction Mapping for (U, V)

Types of Shock Reflection

Regular reflection not always possible (nonlinear effect)
"Mach Stem" configurations are seen



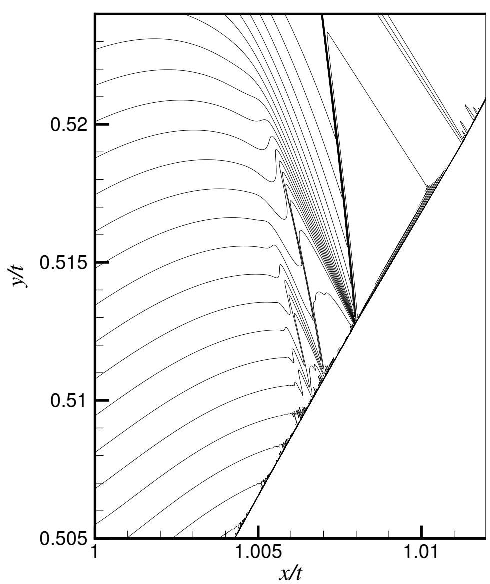
No Triple Points: the pictured configuration is mathematically impossible (Serre)

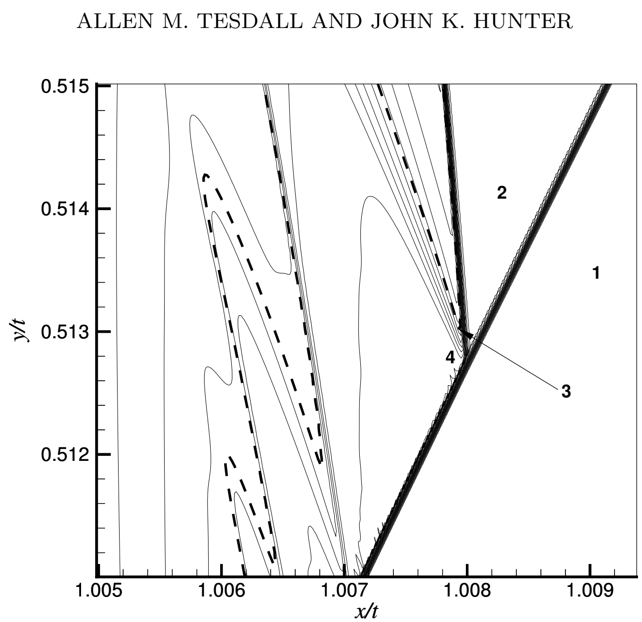
Slip line in "large shock" regime is not mathematically possible for small shocks (von Neumann paradox)

For smaller shocks or systems without linear waves, rarefaction is "mathematically possible"

Guderley Mach Reflection

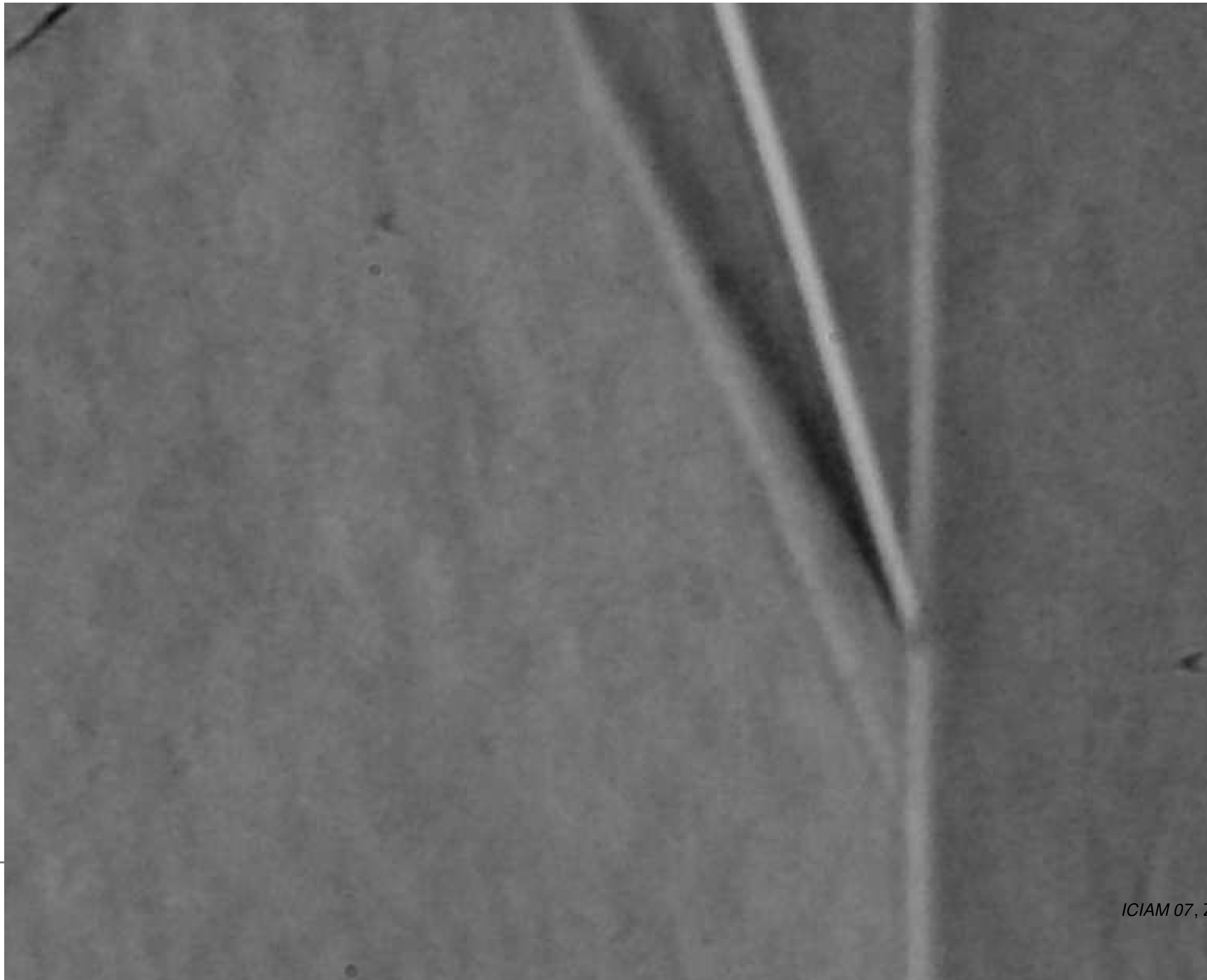
- Classical: mention of ‘rarefactions’ (Guderley)
- UTSD model for weak shock refl: only wave available is a rarefaction (Canic-K conjecture, 1998, no evidence)
- Evidence: simulation by Tesdall & Hunter on UTSD, 2003
- Quasi-steady simulation
  $U_\tau + (F(U) - \xi U)_\xi + (G(U) - \eta U)_\eta = -2U$, $\tau = \log t$





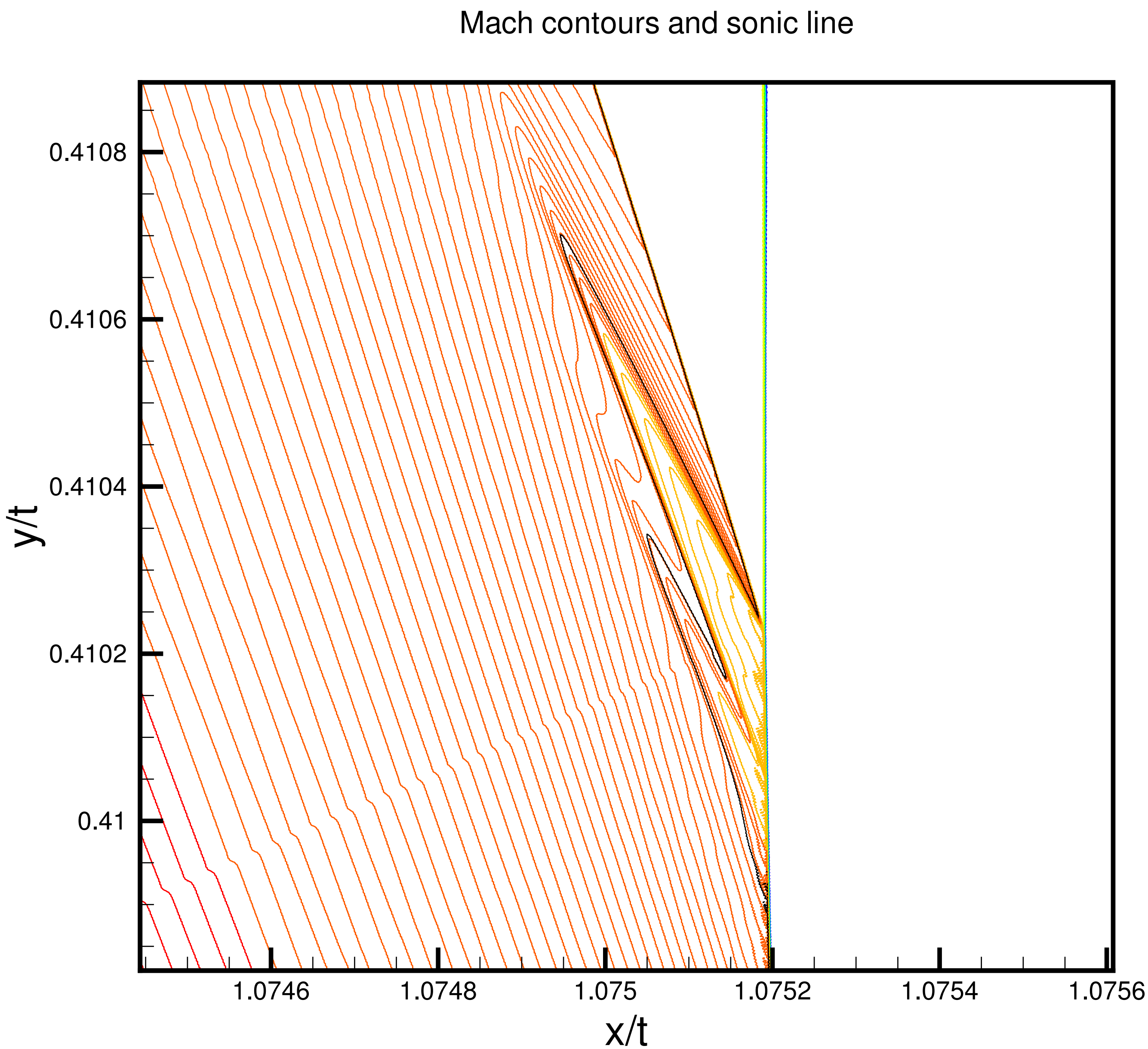
Experimental Evidence

Experimental data of B. W. Skews & J. T. Ashworth (JFM) following Tesdall-Hunter calculation



GMR in Compressible GD Equations

Recent simulation by Allen Tesdall, Gas Dynamics



Future Directions

- Large Data: obstructions to existence of weak solutions
- "Resonances" among different wave families (exploring the nature of hyperbolicity in the large for quasilinear systems)
- Relation to kinetic theory and other "more physical" continuum mechanics theories
- Multidimensional problems:
  - BV not the correct space: what are good candidates?
  - what are good model problems?
  - what information can numerical simulations give?

Slides for talk
http://www.math.uh.edu/∼blk

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