Hyperbolic Partial Differential Equations and Conservation Laws

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How to Look at PDE

- Modelling
- Analysis
- Simulation

Objective at this meeting: describe these aspects of PDE and show how they motivate our research

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About PDE

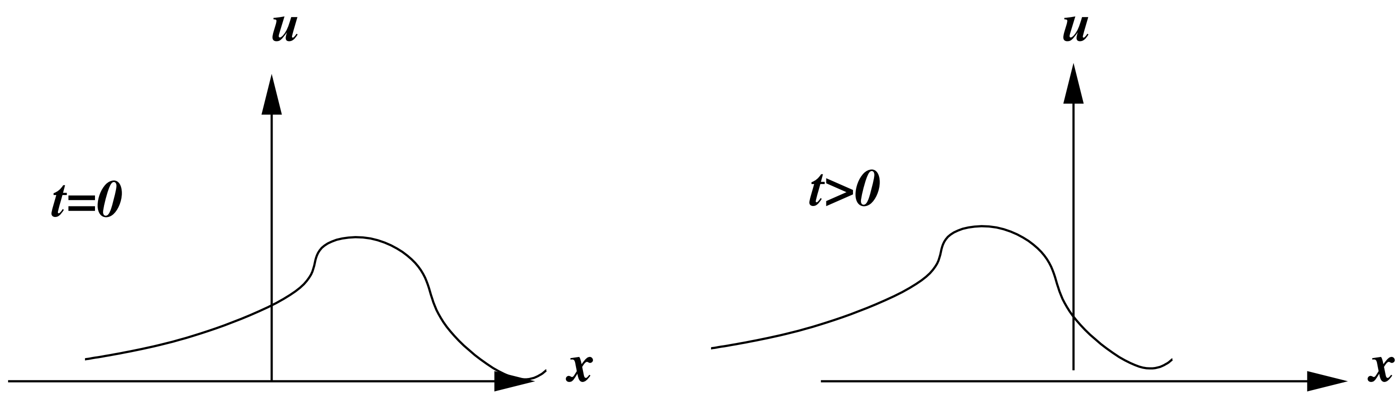
- Essential idea: local information $\Rightarrow$ Global conclusions
- $u(x,t)$ function of space (x) and time (t) (say x is one horizontal direction)
- u function of x defined at every point and every instant – eg, temperature at x
- $u_x = \frac{\partial u}{\partial x}$ is spatial gradient (slope) of u
- $u_t = \frac{\partial u}{\partial t}$ is rate of change in time of u at x
- suppose "u_t and u_x are proportional" (local information)

$$u_t = a u_x$$

- PDE, first order, two independent variables, linear
- Conservation Law: can be written as a space-time divergence: $\text{div}_{(x,t)}(-au, u) = 0$

What about it?

- Fact: every u satisfying $u_t - au_x = 0$ can be written $u(x,t) = f(x+at)$ for a function f of 1 variable (pf later)
- only moderately helpful unless we know f
- suppose $u(x,0) = u_0(x)$, then $f = u_0$ & $u(x,t) = u_0(x+at)$



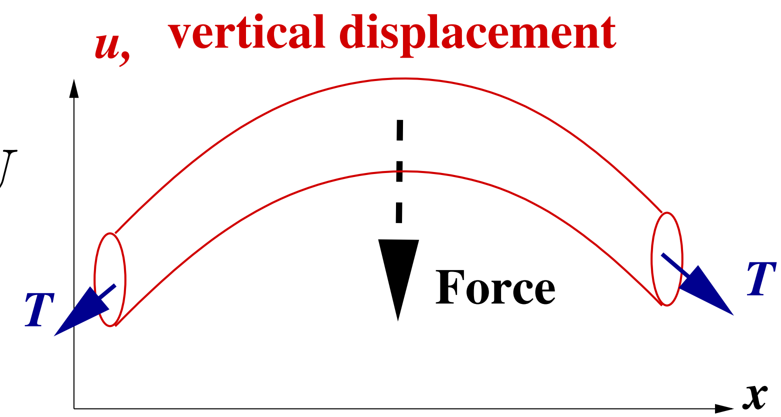
- Whether or not we know f, solution is a wave moving to the left with speed a.
- Could one guess that from $u_t = au_x$?
- Note that $a < 0$ means wave moves to the right

Why Conservation Laws?

- express physical basis for equation
- conservation of mass, momentum, etc.
- need constitutive relation to complete equation
- usual form (one dimension) $U_t + F(U)_x = 0$

Wave Eq: $(\rho u_t)_t = (T u_x)_x$
Newton's law (conservation of momentum) and
Hooke's law (elasticity)
Suppose ρ and T constant: $c^2 = \frac{T}{\rho}$, then $u_{tt} = c^2 u_{xx}$
Define $v = u_t$ and $w = c u_x$

$$U = \begin{pmatrix} u_t \\ c u_x \end{pmatrix}, F(U) = AU = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} U$$



Example of a 1-D System of Conservation Laws

Basic Conservation Laws

- Standard simplification: ignore viscosity, dispersion
  Physics is “mono-scale”
- Multi-D (“more than one space dimension”): elastic membrane or solid

$$u_{tt} - c^2 \Delta u = 0, \quad u_{tt} - \nabla \cdot (c^2 \nabla u) = 0$$

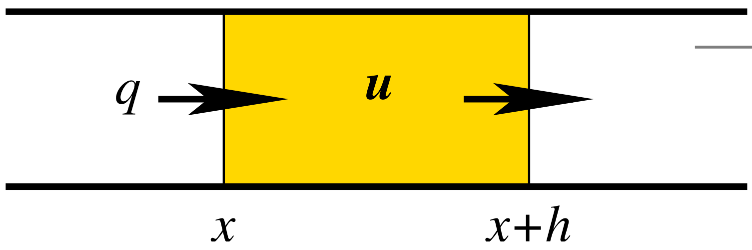
$$\Delta = \partial_x^2 + \partial_y^2 (+\partial_z^2)$$

- Quasilinear if $c = c(u)$ for example
- Standard multidimensional form:

$$U_t + \sum \partial_{x_i} F_i(U) = 0; \quad U = (u_1, \ldots, u_n) \in^n, \quad F_i \in^n$$

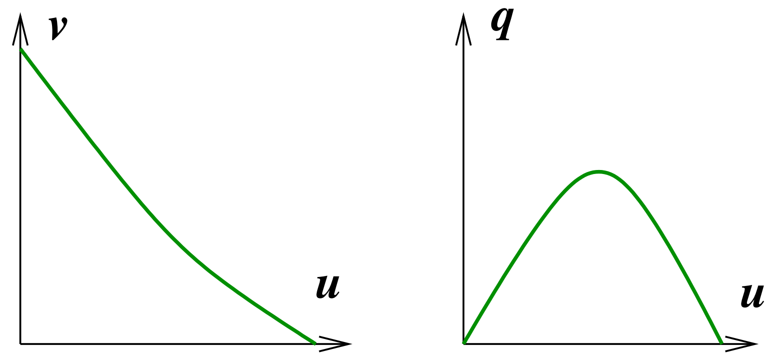
Modelling (Example)

Continuum model for traffic flow



- x = distance down one-way street
- $u(x,t)$ = density of traffic (cars per mile)
- Amount of traffic in control length is $T = \int_x^{x+h} u(y,t)\,dy$
- Evolution of T: $\frac{\partial T}{\partial t} = q(x,t) - q(x+h,t)$
- $q(x,t)$ is flux at (x,t) (cars per minute)
- So $\frac{\partial}{\partial t}\int_x^{x+h} u(y,t)\,dy = q(x,t) - q(x+h,t)$ for any h.
- Apply MVT for integrals: $\int_x^{x+h} u(y,t)\,dy = hu(x^*,t)$
- So $\frac{\partial u(x^*,t)}{\partial t} = \frac{q(x,t)-q(x+h,t)}{h}$
- Now let $h \to 0$, so $x^* \to x$ and we have $u_t + q_x = 0$

Completing the Model

- $u_t + q_x = 0$ is not like $u_t - au_x = 0$ because there are two unknown functions, u and q
- Study q. Notice $q = uv$ (density $\times$ speed)
- Study v
- Case 1: $v =$ const $= a$: $u_t + au_x = 0$ wave moves right
- 

Case 2: $v = v(u)$, say: depends on density with given relation
- $q = uv(u)$ = flux across line = function of u
- $\frac{\partial u}{\partial t} + \frac{\partial q(u)}{\partial x} = 0 \qquad u_t + q'(u)u_x = 0$
- Now it's a PDE, but not the one we solved

Conservation Law Modelling

Advantage

- Physics is trusted

Disadvantage

- Conservation principles are not sensitive to some subtle but important details, such as multiscale physics (viscosity, dispersion)

Good question: what aspects of physical behaviour are well-modelled by conservation laws, and what is left out?

Ideal Gas Dynamics (Important)

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0 \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0 \\ (\rho E)_t + (\rho uH)_x + (\rho vH)_y &= 0 \end{aligned}$$

$$E = \frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2), \qquad H = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2)$$

Variables ρ (density), (u, v) (velocity), and p (pressure)

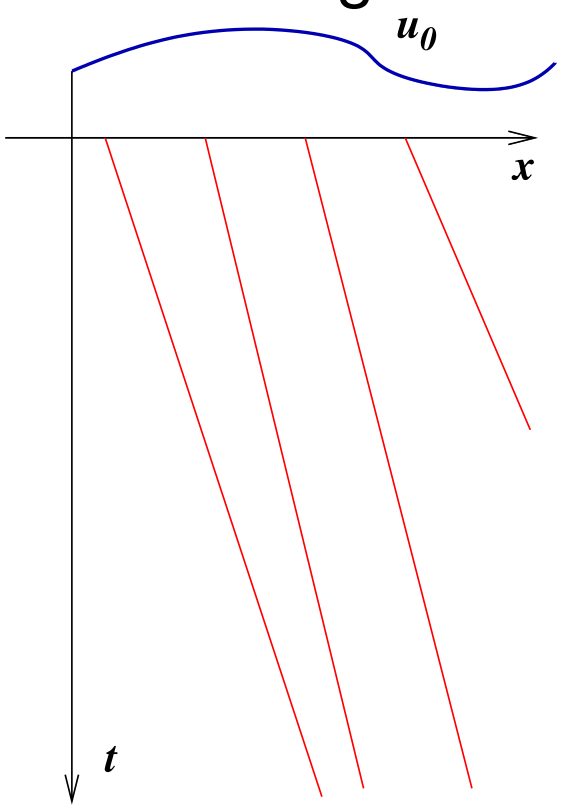
First equation is analogue of traffic flow model

State $U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}$ Flux $F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{pmatrix}$ $G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vH \end{pmatrix}$

Method of Characteristics

Traffic flow model: $u_t + q'(u)u_x = 0$ $u_t + a(u)u_x = 0$

Seek distinguished curves in (x,t) (not an assumption)



- $(x(s), t(s))$:
- $\dot{t} = 1$
- $\dot{x} = a(u)$
- $\dot{u}(x(s), t(s)) = u_x\dot{x} + u_t\dot{t} = a(u)u_x + u_t$
- $u =$ constant on curve: $u = u_0(x_0)$
- $\dot{x} =$ constant on curve
- curve is line $x = a(u_0(x_0))t + x_0$

Implicit solution $u(x_0 + q'(u_0(x_0))t, t) = u_0(x_0)$

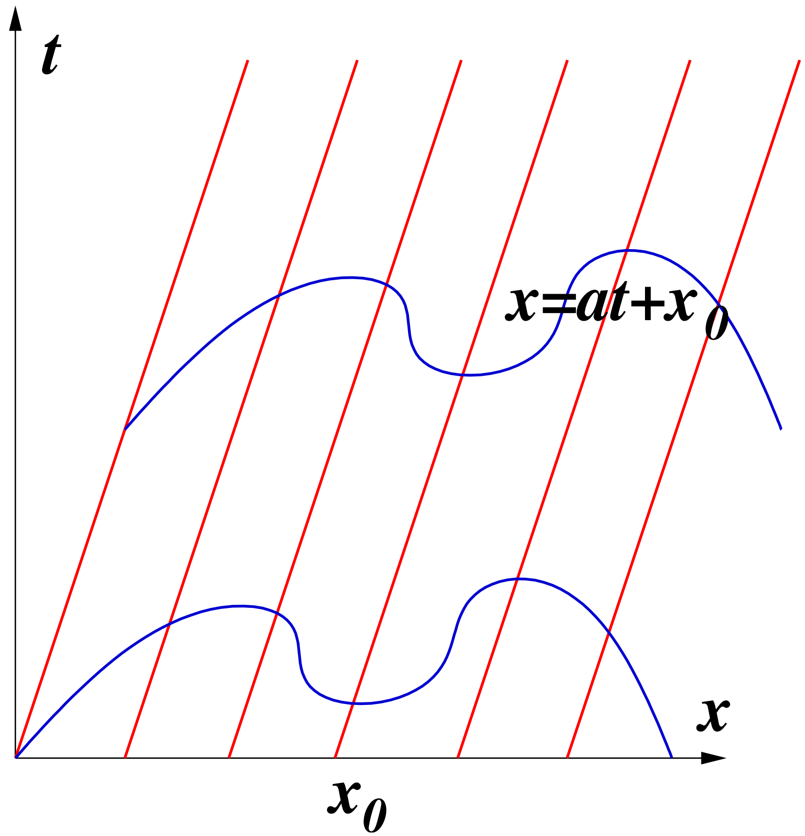
Invert (okay at $t = 0$):

$$\frac{dx}{dx_0} = \frac{d}{dx_0}\left(x_0 + q'(u_0(x_0))t\right) = 1 + q''(u_0)u_0't \neq 0$$

Linear Equation

$u(x_0 + q'(u_0(x_0))t, t) = u_0(x_0)$
$u_t + au_x = 0$, a constant

- $x = x_0 + at$
- $u(x_0 + at, t) = u_0(x_0)$ or $u(x,t) = u_0(x - at)$



Solution for all t for any u_0 (even discontinuous u_0)

Weak Solutions

Another advantage of conservation law form: it allows generalizing the notion of “weak solution” to quasilinear equations.

Divergence form equations (conservation laws):

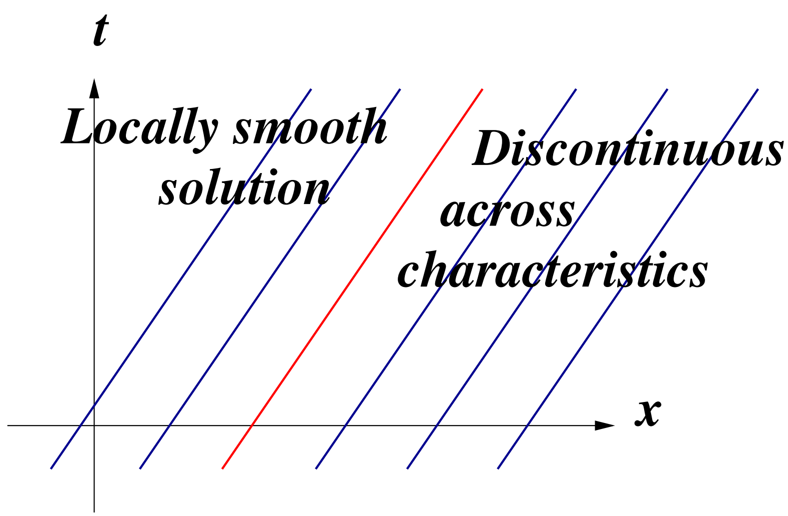
$$\nabla \cdot F(U) = 0 \Rightarrow \iint F(U) \cdot \nabla\theta = 0, \quad \forall\theta$$

$\exists$ weak solutions that are not differentiable: plausible

$$u_t + au_x = 0$$

$$u = f(x - at)$$

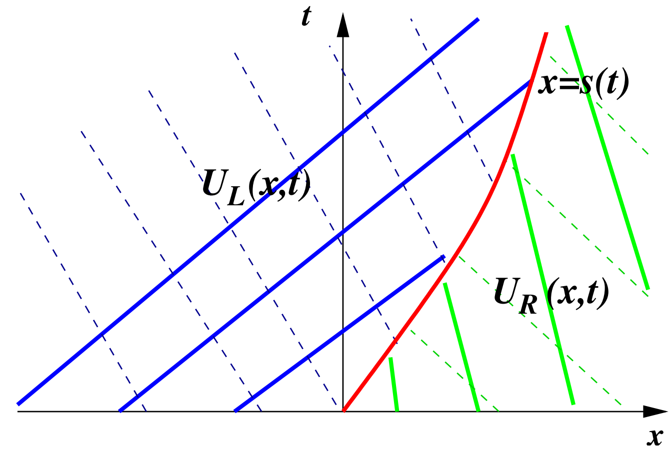
f not differentiable



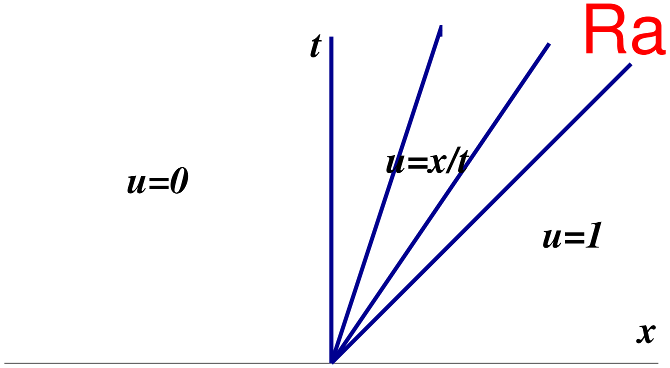
Weak solutions for nonlinear equation

Apply $\int u\theta_t + f(u)\theta_x = 0$ across a discontinuity at $x = st$:
Find $s[U_R - U_L] = f(U_R) - f(U_L)$ (Rankine-Hugoniot rel)
Discontinuities on shocks
& RH relation holds (from integral)
Burgers equation, $u_t + (u^2/2)_x = 0$

$$s[u] = \left[\frac{u^2}{2}\right] \quad \text{or} \quad s = \frac{u_L + u_R}{2}$$



Rarefaction wave (Burgers Equation)



$$u(x,t) = \begin{cases} u = 0, & x < 0 \\ u = x/t, & x < t \\ u = 1, & x > t \end{cases}$$

Interesting continuity properties: $u_x(\cdot, t)$ not bdd in L^p, $p > 1$

Analysis of PDE

- We've seen: Local information $\Rightarrow$ Global conclusions

$$u_t + au_x = 0 \qquad \Rightarrow \qquad u(x,t) = f(x - at)$$

Eq'n $+$ enough initial, boundary conditions (Cauchy P)

- "Applied analysis": study of properties of solutions of PDE, "well-posedness"
- Function spaces in action: Functions and Mappings

$$L^p({}^n) \quad L^p(\Omega) \quad W^{m,p}({}^n) \quad W^{m,p}(\Omega)$$

(typically not C^∞, C^ω or C^k)

- Type of analysis depends on type of equation, (another story): Equations with t typically hyperbolic
- Conservation laws or quasilinear hyperbolic equations

$$u_t + uu_x = 0 \qquad \text{Burgers equation}$$

- Story could be called 'horrible functions' (rarefaction wave)

Analysis

Finding solutions or proving they exists usually involves approximation

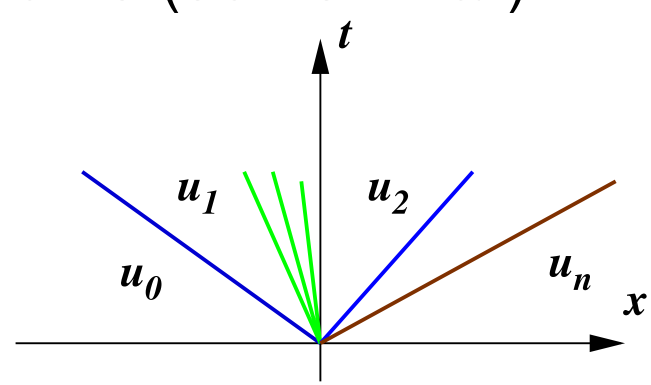
When does a sequence of approximations converge?

Compactness an important tool

One space dimension $u_t + f(u)_x = 0$, $u \in \mathbf{R}^n$
Approximate by Riemann problems (self-similar)

Riemann Data

$$u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x \geq 0 \end{cases}$$

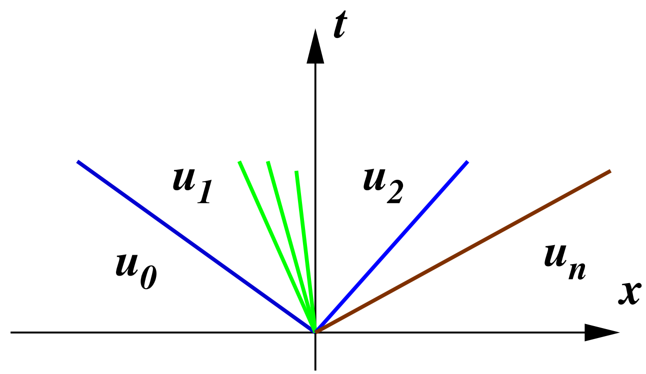


Development of CL Theory

One space dimension $u_t + f(u)_x = 0$, $u \in \mathbf{R}^n$

Riemann Problem (Lax, 1956)

$$u(x,0) = \begin{cases} u_\ell, & x < 0 \\ u_r, & x \geq 0 \end{cases}$$



Form of Riemann solutions suggests right space is BV

Approximate initial data by piecewise constant data

Approximate solution by (local) Riemann solutions

For convergence use Compactness

Random Choice and Wave Front Tracking

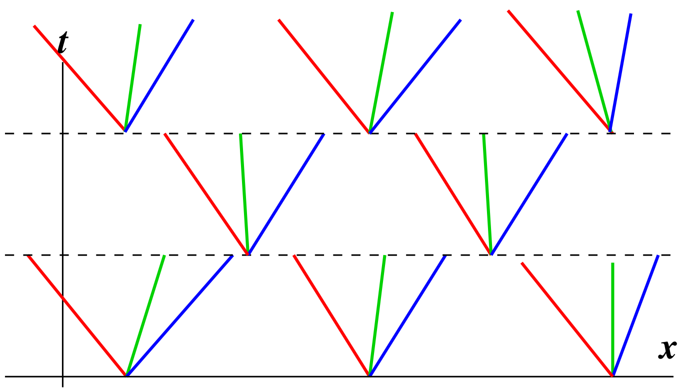
Weak solutions defined for $u \in L^\infty$ (bdd, mble)

1-D, more regularity: $u(x,0) \in BV \Rightarrow$ sol'n in BV

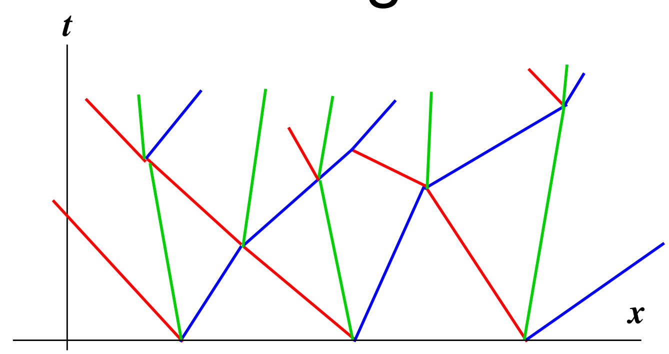
"Outside a set of 1-D Hausdorff measure 0, a BV fn is either approx continuous or has an approx jump discont."

Use Riemann solutions to prove existence:

Glimm's random choice



Risebro-Bressan's wave front tracking



$\mathrm{Var}\, u(\cdot,0) \leq \varepsilon \Rightarrow \mathrm{Var}\, u(\cdot,t) \leq M$, $\int |u(t,x) - u(s,x)| \leq L|t-s|$

Helly's theorem $\Rightarrow$ subsequence cvges ptwise to BV soln.

Bressan: SRS (Standard Riemann Semigroup) – uniqueness, well-posedness, & regularity (cont's except for countable set of shock curves & interaction points)

The Paradox of Multidimensional CL

Systems of Conservation Laws

$$U_t + F(U)_x + G(U)_y = 0,$$

- no existence theory, even for “small data”.

Why?

- smooth data lead to discontinuous solutions (need to study weak solutions)
- discontinuities in quasilinear equations propagate on shocks, not on characteristics
- Characteristics in multiD are complicated (WF sets)
- Don’t even know right function space to approximate solutions

A Blow to Generalization

$$u_t + \sum A_j(u) u_{x_j} + B(u) = 0$$

• Linear & semilinear: H^s theory for smooth data (short time for QL)

Theorem (Rauch): No BV bounds. For C^∞ data, if

$$\int_{R^n} |\nabla_x u(x,t)|\, dx \leq C \int_{R^n} |\nabla_x u(x,0)|\, dx$$

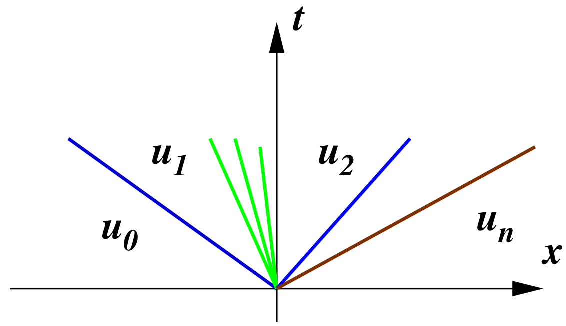
then $\qquad A_j A_k = A_k A_j \quad \forall j, k.$

What's wrong with $A_j A_k = A_k A_j$? No physically interesting system has this property.

Approach Via Self-Similar Solutions

Basic tool in 1-D: $u_t + f(u)_x = 0$, $u_t + A(u)u_x = 0$

$$u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases}$$



Solution $u = u(\xi) = u(x/t)$
1-D analogue of our work: 2-point BVP for ODE for $u(\xi)$

$$-\xi u' + A(u)u' = 0 \quad \text{or} \quad (-\xi I + A)u' = 0, u(-\infty) = u_L, u(\infty) = u_R$$

Two types of solutions (locally in ξ):

- $\xi = \lambda(u)$, $u' = \vec{r}(u)$ Rarefaction if λ increasing with u
- ODE holds weakly at a discontinuity at $\xi = s$ if

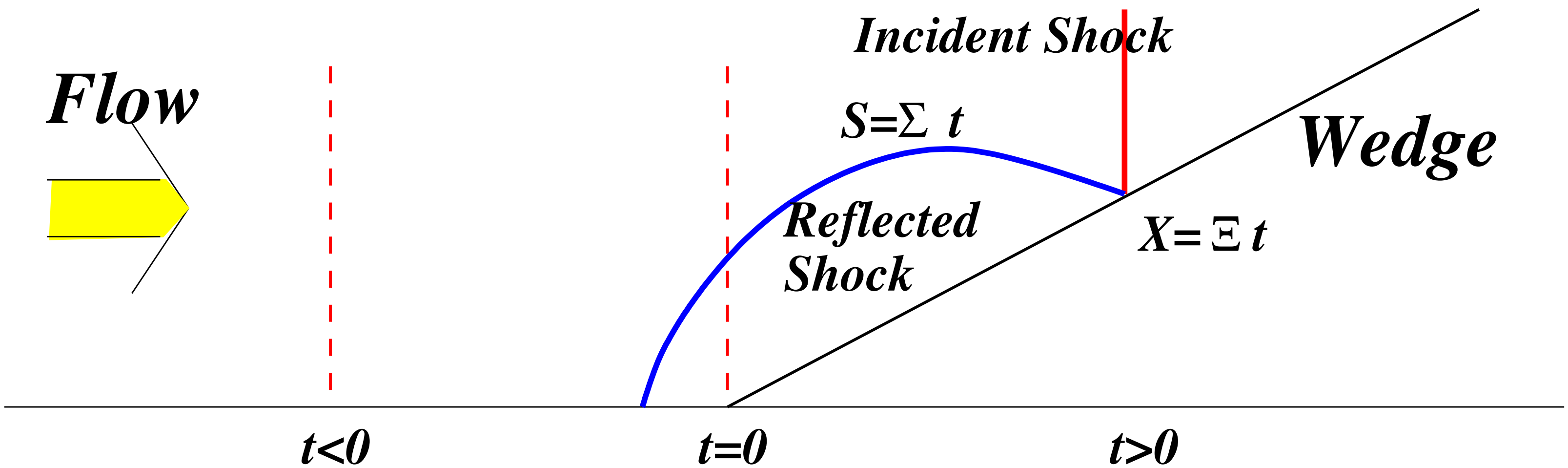
$$\big(-\xi u + f(u) \big]_{s-}^{s+} = 0 \quad \text{or} \quad s[u] = [f(u)]$$

Shock, λ decreasing across discontinuity

Do not solve ODE in conventional way

Why Study 2-D Riemann Problems?

- Analogy with 1-D
- Occurrence in physically interesting problems
  Shock reflection by a wedge



- Shock interactions
- Numerical simulations

Similarity Reduction in Two-D Systems

$$U_t + F(U)_x + G(U)_y = 0, \quad U \in^n, \quad$$ hyperbolic

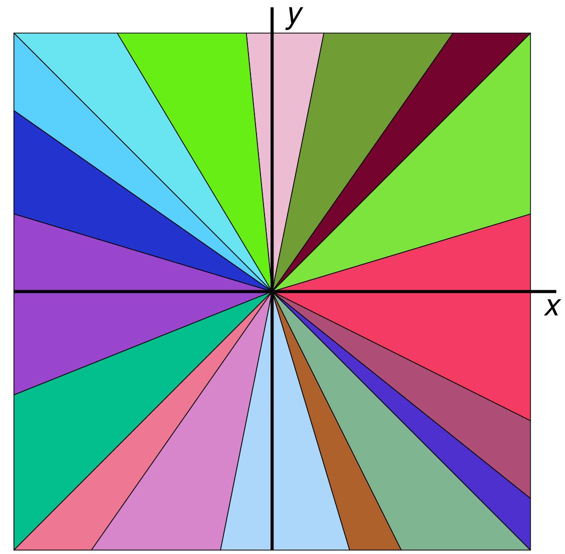
Riemann Data: $U(x, y, 0) = f\left(\frac{x}{y}\right)$

Similarity Variables:

$$\xi = \frac{x}{t}, \quad \eta = \frac{y}{t} \qquad U = U(\xi, \eta)$$

Reduced System in Two Variables

$$\partial_\xi(F - \xi U) + \partial_\eta(G - \eta U) = -2U$$



Sectorially Const Data

Method: resolve 1-D far-field discont; IV/BVP in 2-D

RP in $2+1$ dim $\Rightarrow$ CP in 2 ind. vbles. w. data at ∞

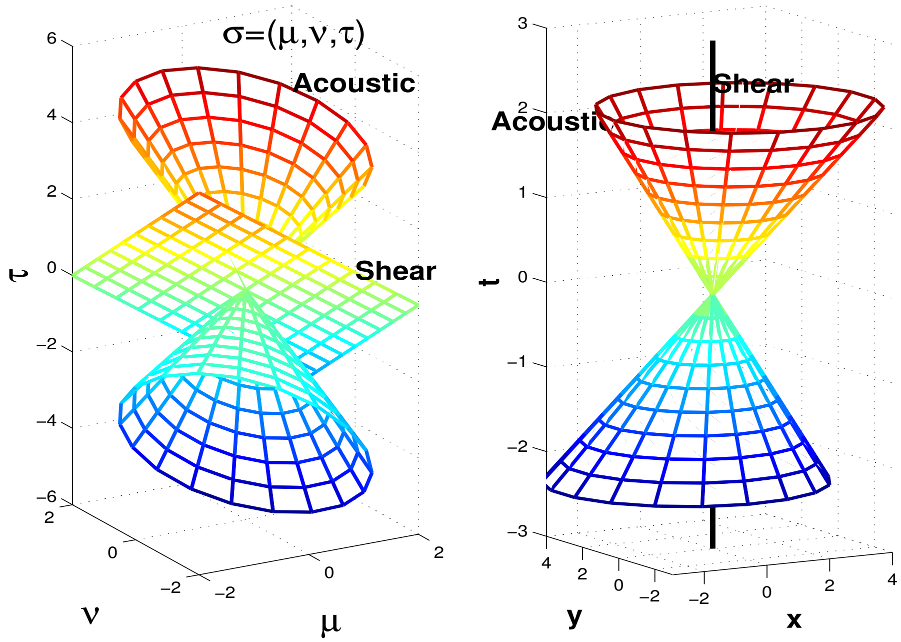
Reduced to a previously solved problem

BUT

Type Changes: hyperb in far field; 'subsonic' region near 0

Acoustic-type Structure

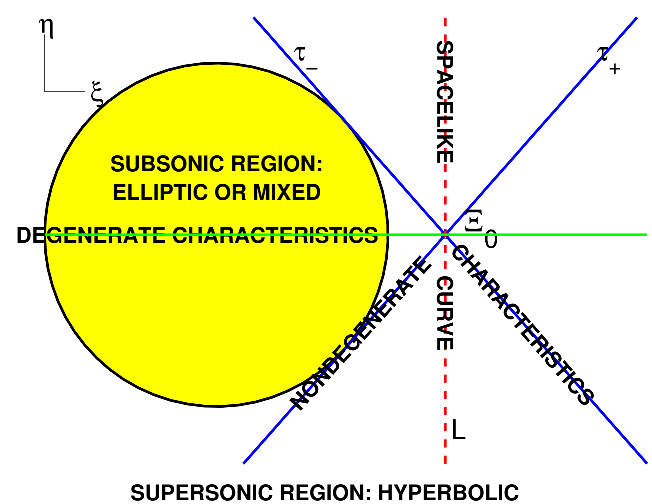
$$U_t + AU_x + BU_y = 0; \quad \det|I\tau + A\lambda + B\mu| = \left(\prod_{i=1}^{n-2} \ell_i \cdot \sigma\right)\sigma^T Q_N \sigma$$



For PDE enthusiasts: a new type of PDE system & BC

CHANGE OF TYPE THEOREM

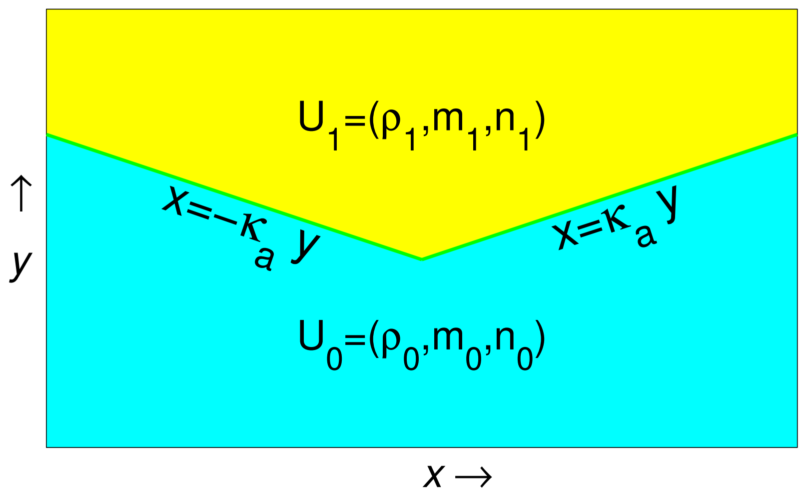
Reduced equation hyperbolic outside acoustic wave cone

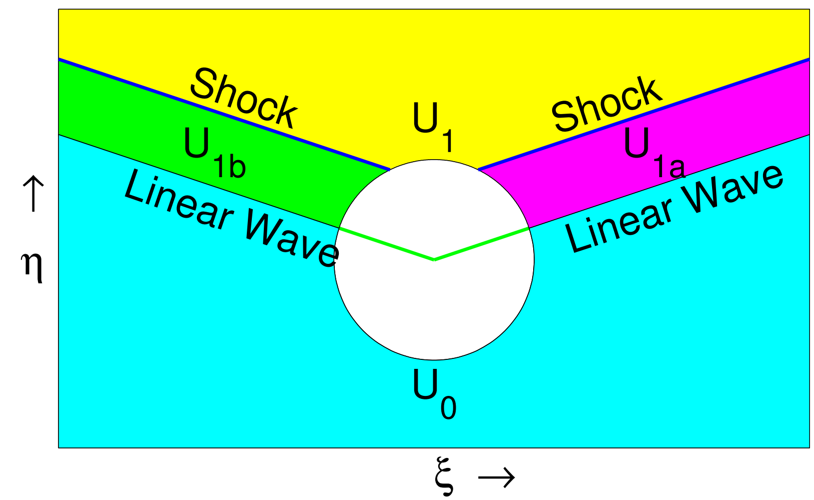


Paradoxical Problems

- Shock reflection & interaction patterns
- No ∃ or ! theorems
- von Neumann paradox

Sample data for shock reflection problems

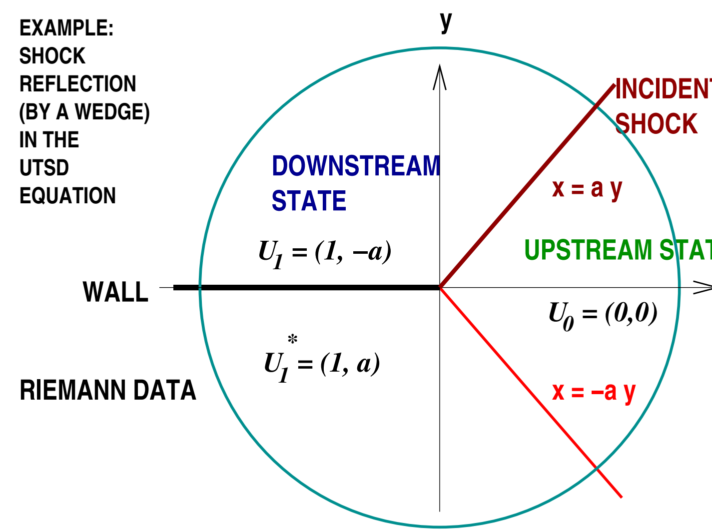


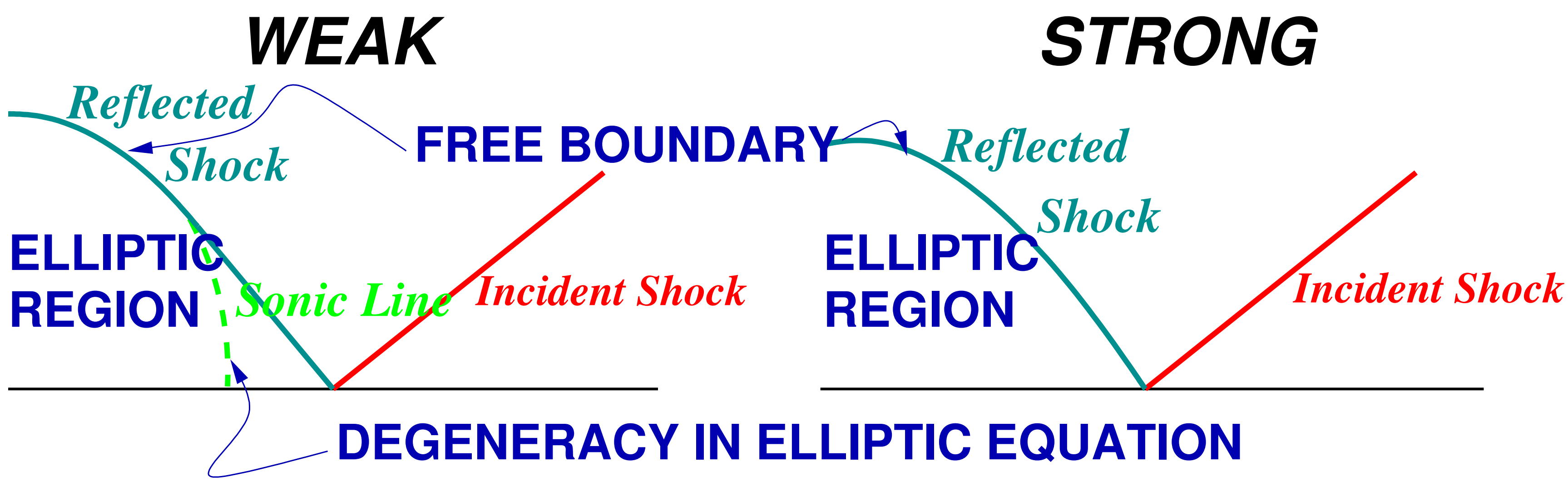


2-state data: U_0, U_1
Data give 2 shocks
Far field solution: 4 waves

Regular Reflection Patterns

- Non-paradoxical case
- Two types of solution postulated
- ‘Local’ existence (Canic, Jegdic, Kim)





Gas Dynamics, UTSD & NLWS

3 systems with char structure similar to gas dynamics:

UTSD system (no linear/degenerate waves)

$$\begin{aligned} u_t + uu_x + v_y &= 0 \\ -v_x + u_y &= 0 \end{aligned}$$

Isentropic Gas Dyn: $p = \rho^\gamma/\gamma$

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0$$
$$(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0$$

Nonlinear Wave System:

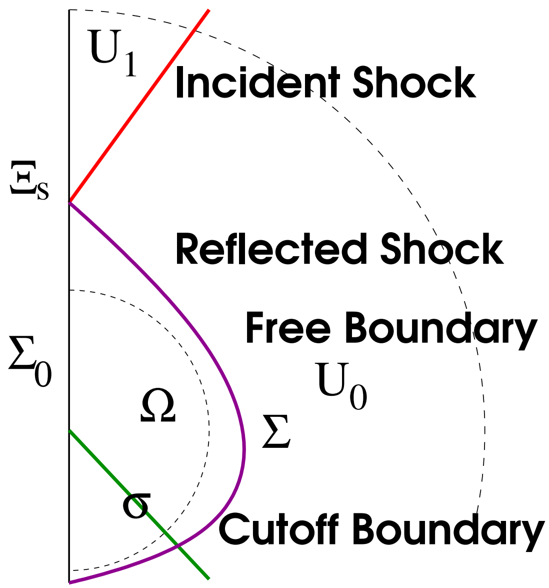
$$\rho_t + m_x + n_y = 0$$
$$m_t + p_x = 0 \qquad m = \rho u$$
$$n_t + p_y = 0 \qquad n = \rho v$$

In Y. Zheng's P-G Sys, $p(\rho) = e^\rho$, $u = c^2(\rho) = p'(\rho) = e^\rho$

NLWS: Analysis of Regular Reflection

'Strong' RR: (Nondegenerate) Elliptic Free Boundary Prob
Jegdic-K-Canic; using work on UTSD and NLWS with Kim
Existence theorem for NLWS - local result

$$Q \equiv \left((c^2(\rho)-\xi^2)\rho_\xi-\xi\eta\rho_\eta\right)_\xi+\left((c^2(\rho)-\eta^2)\rho_\eta-\xi\eta\rho_\xi\right)_\eta+\xi\rho_\xi+\eta\rho_\eta$$



$Q(\rho) = 0$ (degenerate elliptic) in Ω
$\rho = f$ on σ (cutoff boundary)
$\rho_\xi = 0$ (symmetry) on Σ_0
Free boundary from RH equations:
$N(\rho) \equiv \beta \cdot \nabla\rho = 0$ (oblique deriv) on Σ

$$\frac{dr}{d\theta} = r\sqrt{\frac{r^2}{s^2} - 1} \qquad s^2 = \frac{[p]}{[\rho]}$$

$\rho = \rho_F$ at Ξ_s

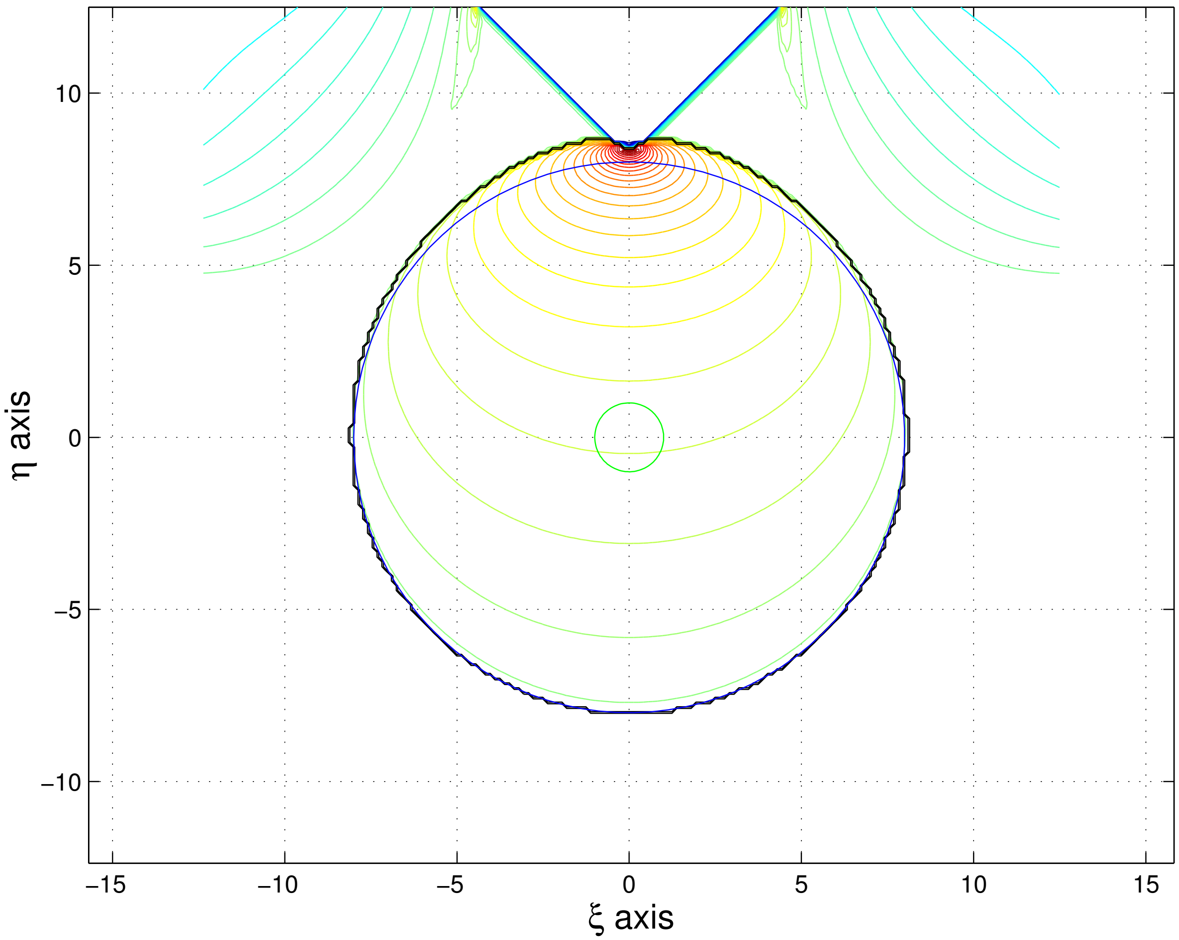
Approach: Fixed Point Theorem (CK & Lieberman, CKK)

- Nonlinear evolution eqn may not be well-defined
- Compare with work of Zheng, Chen-Feldman, etc.

Some Simulations

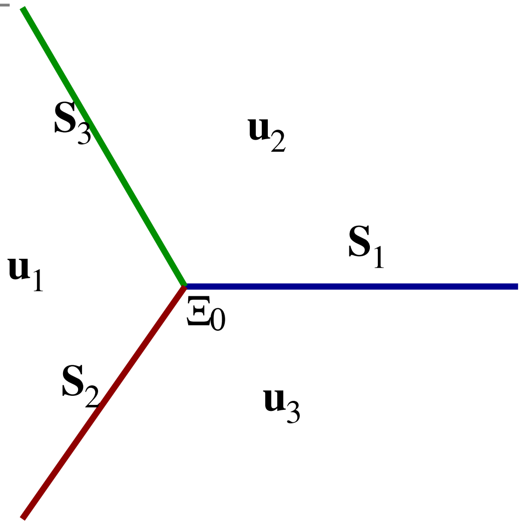
Regular Reflection Not Possible

Contour Plot of Density ρ. Data $U_0 = (64,0,507.9222)$; $U_1 = (1,0,0)$; $\kappa_a = 1$; $\kappa_b = -1$



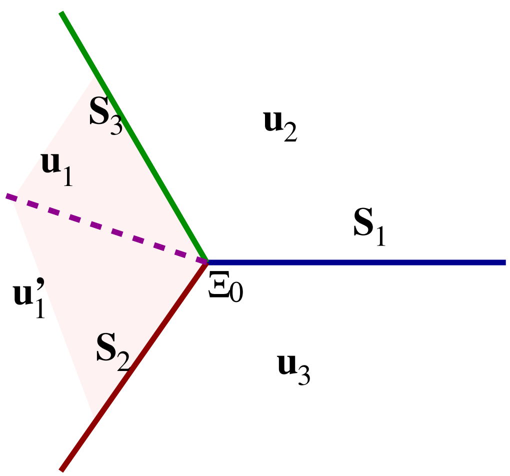
Alex Kurganov (high-order centered scheme)
focus on 'triple point'

Weak Shock Reflection: Triple Points



Theorem (Serre) $\nexists$ triple points: $\nexists$ states u_1,u_2, u_3, shock angles κ_1, κ_2, κ_3 and meeting point Ξ_0

Resolution of paradox for strong shocks: slip line

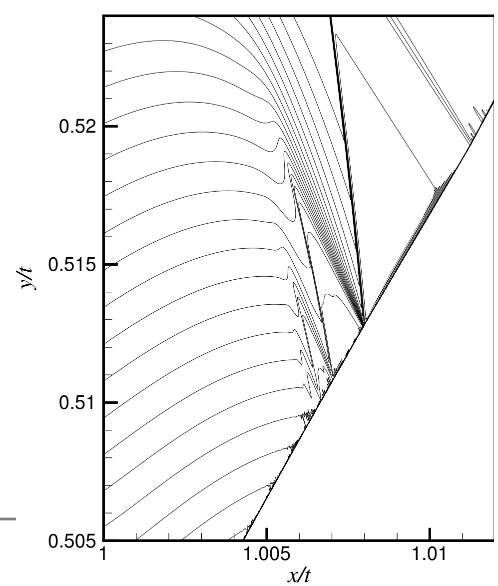


For weak shocks, slip line cannot occur

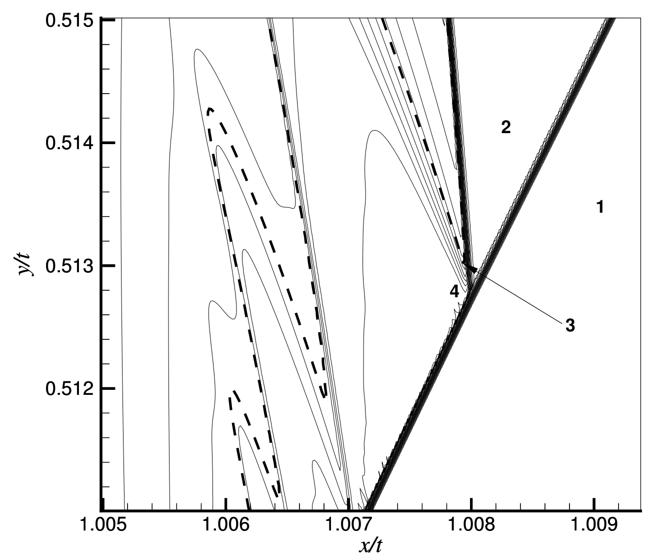
What happens in weak shock reflection?

Guderley Mach Reflection

- Classical: mention of ‘rarefactions’ (Guderley)
- UTSD model for weak shock refl: only wave available is a rarefaction (Canic-K conjecture, 1998, no evidence)
- Evidence: simulations of Tesdall & Hunter on UTSD (2003)
- Quasi-steady simulation
  $U_\tau + (F(U) - \xi U)_\xi + (G(U) - \eta U)_\eta = -2U$, $\tau = \log t$

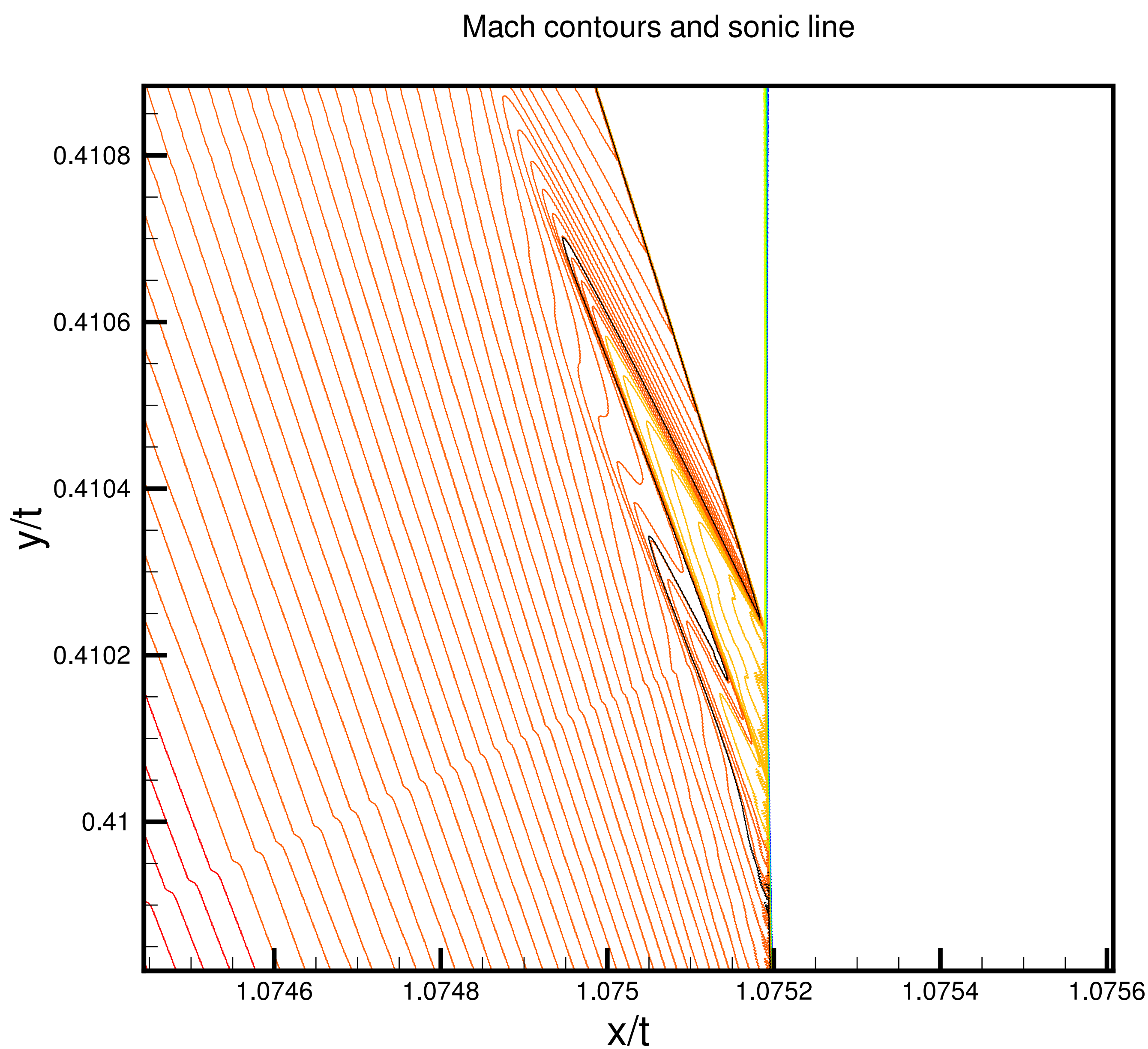


ALLEN M. TESDALL AND JOHN K. HUNTER



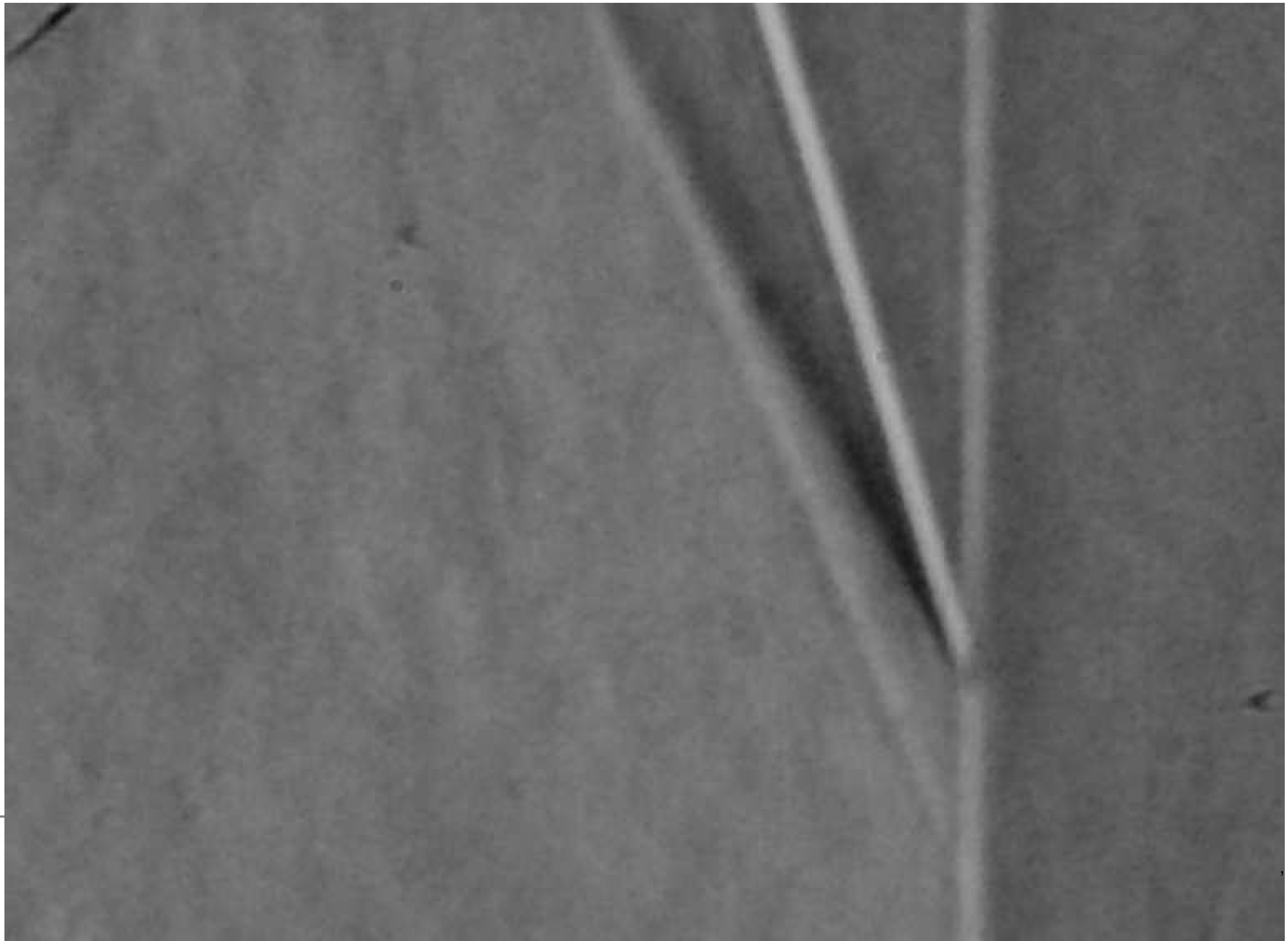
Guderley Mach Reflection in Gas Dynamic

Simulations by Allen Tesdall, Gas Dynamics



Experimental Evidence

Experimental data of B. W. Skews & J. T. Ashworth (JFM) following Tesdall-Hunter calculation



Future Directions

- Extend UTSD & NLWS results to Gas Dynamics: technical difficulties
- Large Data: obstructions to existence of weak solutions
- "Resonances" among different wave families (exploring the nature of hyperbolicity in the large for quasilinear systems)
- Relation to kinetic theory and other "more physical" continuum mechanics theories
- Multidimensional problems:
  - BV not the correct space: what are good candidates?
  - what are good model problems?
  - what information can numerical simulations give?

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