Mathematical Properties of Nonhyperbolic Models for Incompressible Two-Phase Flow

Barbara Lee Keyfitz [1]

Department of Mathematics, University of Houston, Houston, Texas 77204-3476.

Abstract

Many models for multi-fluid flow result in equations which fail to be hyperbolic. One example is one-dimensional flow of an incompressible two-phase fluid. In the simplest model, the principal part of the differential operator has characteristics with nonzero imaginary part for any state of the fluid which contains both phases. Thus, the linearized equations are catastrophically unstable.

This fact has caused distrust of the equations and concern about the modeling processes.

However, these nonlinear equations behave very differently from their linearizations. Although states which are linearly unstable are also unstable in the nonlinear equations, nonlinear theory predicts jump transitions, via stable shocks, from unstable to stable states. Furthermore, the nonlinear theory eliminates both infinite-growth modes and high-frequency oscillations. The solution depends continuously on the data except at certain values where threshold or bifurcation phenomena occur. This overall stability is not affected by viscous or drag terms in the system.

Key words: Two-fluid flow modeling, loss of hyperbolicity, singular shocks, Riemann problems

1 Introduction

Systems of conservation laws (first-order partial differential equations in space and time) which are not of classical, strictly hyperbolic type have been observed by mathematicians and other scientists over the past forty years. Such equations appear in models for the dynamics of complicated flow systems.

URL: www.math.uh.edu/∼blk (Barbara Lee Keyfitz).

[1] Research supported by the Department of Energy, grant DE-FG-03-94-ER25222, and by the National Science Foundation, grant DMS-9973475 (POWRE).

Typical examples are a model for three-phase convection-driven flow in porous media, described by Bell Trangenstein and Shubin [1]; and a model for two-phase elastodynamics, given by James [2].

Conservation law models for steady transonic flow also change type. There are fundamental differences between steady and unsteady change of type: not in the equations themselves, which are identical in form, but in the boundary conditions and entropy conditions for shocks, which are necessary for the formulation of a well-posed problem. Here we are concerned only with the mysterious and controversial problem of unsteady change of type.

Considerable mathematical analysis has been done on unsteady models. This work, by many researchers, solves Riemann problems for the model equations, derives admissibility and uniqueness criteria for shocks, relates the conservation-law solutions to those of viscous perturbations, and attempts to prove existence and continuous dependence theorems for Cauchy data. (For a review of the literature, see [3]. My contribution has been to relate change of type to other manifestations of nonclassical behavior, such as failure of strict hyperbolicity and nonexistence of classical Riemann solutions for some strictly hyperbolic equations.) The theory is incomplete; however, to put this in context, a theory of well-posedness for classical, strictly hyperbolic conservation laws in one space dimension is only now in progress. People use strictly hyperbolic conservation laws with great confidence. I believe a complete theory for conservation laws that change type can be constructed as well.

How is such a theory possible in the face of the linear instability? The well-known Hadamard example predicts catastrophic failure: exponential growth of unstable modes of all frequencies, with the most rapid growth at the highest frequencies. The initial-value problem for Laplace's equation gives the general idea. For any integers k and n, the Cauchy problem

$$u_{tt} + u_{xx} = 0, \quad u(x,0) = 0, \quad u_t(x,0) = n^k \sin nx$$

has the solution $u(x,t) = n^{k-1} \sin nx \sinh nt$. For negative values of k, the initial conditions are uniformly small, but the solution is exponentially large for $t \neq 0$. The conclusion is that the constant value $u = 0$ is completely unstable, and the initial-value problem meaningless.

There are three reasons this example may not carry over to nonlinear problems.

First, the initial condition is small in the uniform norm, but not in the energy norm. In a physical problem, either the perturbation is localized or boundary conditions become important. We shall see that this is significant.

Second, in a nonlinear equation, the instability does not grow exponentially, but saturates: once u takes values in the hyperbolic region, then the solution stops growing. This is primarily a matter of scale: the size of the resulting oscillations is, roughly, the width of the nonhyperbolic region in phase space. If the model is physically correct, then there is a physical reason for an instability on this scale; for example, in two-phase elasticity, a jump whose amplitude is the width of the unstable (nonhyperbolic) zone represents a propagating phase boundary.

The third difference with linear theory is that the domination of a mixed-mode solution by high-frequency oscillations has no counterpart in the nonlinear problem. This is due to the mechanism of shock formation. Shocks cause decay of energy even in the absence of an explicit dissipative mechanism such as viscosity. Again, this is a matter of scale, in this case the time scale. On a given time scale, oscillations of a particular frequency grow and form shocks, owing to the nonlinearity of the system. A periodic pattern of shocks is established; the shocks interact, resulting in decay. Oscillations of higher frequency develop into shocks more rapidly; since the shock wave speeds are independent of the frequency, higher frequency oscillations interact and decay more rapidly than those of lower frequency. Eventually, only oscillations of some characteristic frequency will remain.

I conjecture that the total variation of the solution satisfies a bound of the form K/t, where K is a constant which depends on the initial data: as t decreases to zero, the variation of the solution grows without bound, as in some hyperbolic systems. However, there would be no catastrophic Hadamard instability for t positive.

While nonlinear nonhyperbolic systems are not catastrophically unstable, they do not enjoy the same kind of well-posedness as hyperbolic systems. Can their solutions be of any use?

Here is a sense in which one can use the solutions exhibited later in this paper: Model systems which change type are only approximate descriptions of the complete physics. Such model systems are often regularized by adding a dissipative mechanism, to produce a system which is parabolic and well-posed. One reason scientists view the regularized systems with suspicion is that they fear that there is no reasonable sense in which their solutions converge as the dissipation is reduced.

However, if one can prove that solutions to the dissipative equations converge as the dissipation tends to zero, then the limits represent, in the usual weak sense, solutions to the nonhyperbolic model system. Existence of limits does not prove that this system contains the correct physics; however, it does give predictions about dynamics, based on the physics which is contained in the model system. The predictions can be tested against experiment, common sense or solutions of models which are known to be more complete.

2 The Simplest Model

The remainder of this note focuses on the simplest model for two-phase, one-dimensional incompressible flow, exhibited under this name in Drew and Passman's book, [4, page 248]. Call the phases 1 and 2, and their (constant) densities ρ_1 and ρ_2. Let α_i be the volume fraction of the i-th phase, u_i the velocity and p_i the pressure of the phase. Then conservation of mass and balance of momentum yield the four averaged equations:

$$\partial_t(\alpha_i) + \partial_x(\alpha_i u_i) = 0, \tag{1}$$
$$\alpha_i \partial_t(\rho_i u_i) + \alpha_i u_i \partial_x(\rho_i u_i) + \alpha_i \partial_x p_i = G_i, \tag{2}$$

where G_1 and $G_2 = -G_1$ are balance terms involving the interfacial force density. This lower-order term does not influence the type of the equation. We shall proceed as though the G_i were both zero.

We make two standard scaling assumptions: $\alpha_1 + \alpha_2 = 1$ and $\rho_1 - \rho_2 = 1$, and we make the "single pressure" hypothesis,

$$p_1 \equiv p_2, \tag{3}$$

which is often blamed for the controversial type of the equation. However, in at least one case, that of separated flow, where α_1 and α_2 represent the fraction of a channel filled by each fluid, then the pressures in each phase are equal, except for the effects of surface tension, which we ignore.

The two equations in (1) both govern the time-evolution of a single volume fraction, while there is no term at all for the time-evolution of the pressure. Drew and Passman describe this by saying that the quasilinear system (1), (2) has two infinite-speed characteristics. (The incompressible Euler equation, governing the evolution of a single ideal fluid, also features an infinite speed.) An interpretation of this feature is that two of the four variables adjust instantaneously to changes in the other two, and therefore one can reduce (1), (2) to a system of two equations for two variables, and solve by integration for the other two. The following procedure was suggested by Constantine Dafermos.

Adding the two equations in (1) and using $\alpha_2 = 1-\alpha_1$ gives $\partial_x(\alpha_1 u_1 + \alpha_2 u_2) = 0$, and so $\alpha_1 u_1 + \alpha_2 u_2 = f(t)$. By rescaling u_1, u_2 and x, we can set $f \equiv 0$ and deduce the identity

$$\alpha_1 u_1 + \alpha_2 u_2 = 0 \tag{4}$$

in a coordinate system moving with weighted average flow speed. This eliminates one variable, and we can eliminate the pressure by subtracting the two equations in (2), and then use one of those equations to solve for p. Hence, we can write (1), (2) as a system of two equations in two conserved quantities, from which we can recover the others. Define

$$\beta = \rho_2\alpha_1 + \rho_1\alpha_2, \tag{5}$$
$$v = \rho_1 u_1 - \rho_2 u_2. \tag{6}$$

Then

$$\alpha_1 = \frac{\beta-\rho_1}{\rho_2-\rho_1}, \quad \alpha_2 = \frac{\beta-\rho_2}{\rho_1-\rho_2}, \quad u_1 = \frac{(\beta-\rho_2)v}{\beta(\rho_1-\rho_2)}, \quad u_2 = \frac{(\rho_1-\beta)v}{\beta(\rho_2-\rho_1)} \tag{7}$$

expresses α_1, α_2, u_1 and u_2 in terms of β and v. Furthermore, β and v are affine functions of the conserved quantities in (1), (2), so conservation equations for β and v are equivalent to the original equations, (1) and (2).

Carrying out the calculations leads to a system for β and v:

$$\beta_t + \Big(vB_1(\beta)\Big)_x = 0 \tag{8}$$
$$v_t + \Big(v^2B_2(\beta)\Big)_x = 0, \tag{9}$$

where

$$B_1(\beta) = \frac{(\beta-\rho_1)(\beta-\rho_2)}{\beta}, \qquad B_2(\beta) = \frac{\beta^2-\rho_1\rho_2}{2\beta^2}, \tag{10}$$

and we have used the normalization $\rho_1 - \rho_2 = 1$. This is a system of conservation laws,

$$U_t + F_x = 0, \tag{11}$$

with state variable $U = (\beta, v)$ and flux function $F = (vB_1(\beta), v^2B_2(\beta))$. The physical range for β is $\rho_2 \le \beta \le \rho_1$.

3 Mathematical Analysis

The characteristics of the system (8), (9) are

$$\lambda = 2vB_2(\beta) \pm v\sqrt{B_1B_2'} \tag{12}$$

and since $B_1 \le 0$ and $B_2' > 0$ on the physical range of β, the system is never strictly hyperbolic. The eigenvalues have nonzero imaginary part except when $\beta = \rho_1$, $\beta = \rho_2$ or $v = 0$. On this subset of state space, shaped like the letter 'H', which we will call H, there is a real double characteristic speed.

Each segment of H is an *invariant set* for the system (8), (9); that is, if initial data $(\beta_0(x), v_0(x))$ are given in the set, then the solution remains in the set for all $t > 0$. On the sets $\{\beta = \rho_1\}$ and $\{\beta = \rho_2\}$, the system reduces to the scalar equation

$$v_t + B_2(\rho_i)(v^2)_x = 0$$

with quadratic flux function, convex up at ρ_1 and convex down at ρ_2. Along the β-axis, the system reduces to a trivial linear system, $\beta_t = v_t = 0$, with zero characteristic speed. On the vertical sides of H, solutions in the form of shock and rarefaction waves can be found, while the horizontal line admits weak solutions in the form of contact discontinuities with speed zero. One can pose Riemann problems with data in H, and if the local wavespeeds are ordered so that they increase from left to right in physical space, then a stable solution, completely contained in H, exists. This is the only case in which classical Riemann solutions exist.

3.1 Singular Shocks in the Model System

Bounded, piecewise smooth weak solutions to systems of conservation laws satisfy Rankine-Hugoniot equations at discontinuities. In this example, the Rankine-Hugoniot equations are

$$s[\beta] = [vB_1], \quad s[v] = [v^2B_2],$$

where $[\cdot]$ represents the jump in a quantity across a shock of speed s. The Rankine-Hugoniot equations have no real solutions for states in the interior of the physical region. Instead, we find the partial differential equation admits *singular shock* solutions, which satisfy the first Rankine-Hugoniot relation but not the second. Singular shocks were studied in [5] and [6]; there they appeared in strictly hyperbolic problems. The theory in [6] shows that singular shocks are limits of approximate solutions which can be obtained in a number of ways, including by the addition of viscous regularization, and that the limits are independent of the type of approximation used. Furthermore, approximations to singular shocks are approximate solutions with small residuals.

Singular shocks are self-similar solutions, of the form $U(x/t) = U(\xi)$, to equation (11), which are concentrated at a particular speed $\xi = s$, the singular shock speed. One way to approximate singular shocks is through the self-similar viscosity approximation to the conservation law system,

$$U_t + F_x = \epsilon t U_{xx}. \tag{13}$$

Approximate singular shocks are solutions to (13) which, at distances $\mathcal{O}(\epsilon)$

from s, can be described with regular shock profiles, of the form $U(\xi) = \overline{U}((\xi - s)/\epsilon)$. However, closer to the shock they have singular behavior:

$$\widetilde{U} = \begin{pmatrix} \beta \\ v \end{pmatrix} = \begin{pmatrix} \tilde{\beta}\left(\frac{\xi - s}{\epsilon^2}\right) \\ \frac{1}{\epsilon}\tilde{v}\left(\frac{\xi - s}{\epsilon^2}\right) \end{pmatrix}.$$

The singular part of the shock, $\widetilde{U}$, and the regular part, $\overline{U}$, can be found using asymptotic expansions as in [5]. Details can be found in [7]. In the limit $\epsilon \to 0$, a singular shock disappears: Because, for small ϵ, the unbounded part of the v component is very narrow compared to its height, the singular part has mass zero in the limit (in this respect it differs from, for example, an approximation to a Dirac delta-function, which has unit mass). Thus, the singular shock appears merely as a discontinuity in the exact solution, as drawn in Figure 2; in approximations it may appear as a sharp spike.

If we let U^- and U^+ be the end states on the left and right sides, respectively, of a singular shock, then we have the following proposition. (Here $\Re$ means the real part of a complex number, and $\lambda^\pm$ refer to the characteristics, equation (12), evaluated at $U^\pm$.)

Proposition 3.1 *If a singular shock connects U^- and U^+, then both states are in the same vertical half-plane. The end states satisfy the generalized Rankine-Hugoniot conditions,*

$$s(\beta^+ - \beta^-) = v^+ B_1(\beta^+) - v^- B_1(\beta^-) \tag{14}$$
$$s(v^+ - v^-) = (v^+)^2 B_2(\beta^+) - (v^-)^2 B_2(\beta^-) + C \tag{15}$$

where C may have any finite value and is positive for end states in the upper-half-plane, negative in the lower. Singular shocks satisfy an admissibility condition ,

$$\Re(\lambda^-) \geq s \geq \Re(\lambda^+). \tag{16}$$

Strict inequalities in (16) yield *overcompressive* singular shocks, which are locally isolated transitions. On the other hand, when equality holds in one of the conditions in (16), then a singular shock may form part of a complex wave pattern, as it may lie at the head or tail of a rarefaction. A calculation gives

Corollary 3.2 *For a strictly overcompressive singular shock with left state U^-, the right state U^+ lies in the interior of a cusped triangular region $Q(U^-)$ bounded by the curves*

$$v^+ = v^- \left(\frac{2B_2(\beta^-)(\beta^+ - \beta^-) + B_1(\beta^-)}{B_1(\beta^+)} \right) \tag{17}$$

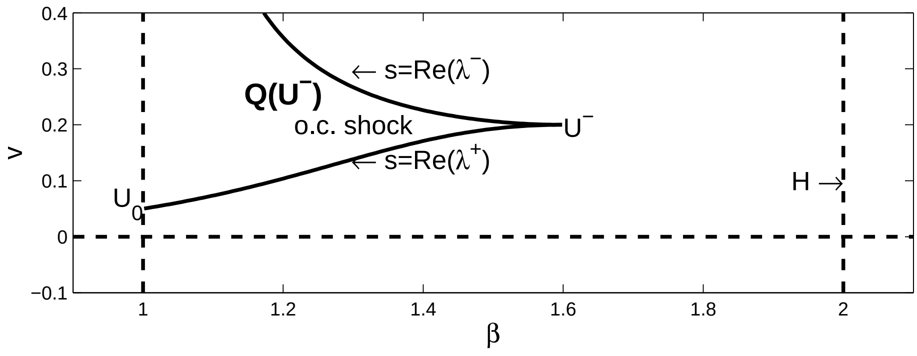

Fig. 1. The Singular Shock Region

and

$$v^+ = v^- \left(\frac{B_1(\beta^-)}{B_1(\beta^+) - 2B_2(\beta^+)(\beta^+ - \beta^-)} \right). \tag{18}$$

On the boundary segment (17), $s = \Re(\lambda^-)$, *and on* (18), $s = \Re(\lambda^+)$.

The curve (18) meets H at a point U_0 defined by

$$U_0(U^-) = \left(\rho_2, -\frac{v^- B_1(\beta^-)}{2B_2(\rho_2)(\rho_2 - \beta^-)}\right) \quad \text{or} \quad \left(\rho_1, -\frac{v^- B_1(\beta^-)}{2B_2(\rho_1)(\rho_1 - \beta^-)}\right)$$

according as v is positive or negative. The significance of this point is that the singular shock from U^- to U_0 can form part of a composite wave, with a rarefaction on the right. There is an analogous description of overcompressive shocks from the viewpoint of a fixed state U^+ on the right; in this case, there is a unique point $U_1(U^+)$ to which U^+ can be joined by a singular shock preceded by a rarefaction. The curves (17) and (18) and the region Q where overcompressive shock solutions exist are illustrated in Figure 1.

3.2 Riemann Problems

In conservation law theory and computation, Riemann problems are the building blocks for solving initial and boundary value problems. Riemann data,

$$U(x,0) = \begin{cases} U_L, \ x < 0 \\ U_R, \ x > 0, \end{cases} \tag{19}$$

give rise to self-similar solutions $U(x,t) = U(x/t) = U(\xi)$. Four types of waves occur in our model: besides singular shocks, we see rarefactions and regular shocks (between states on the vertical sides of H) and contact discontinuities (along the horizontal line in H). When multiple types of wave occur in a

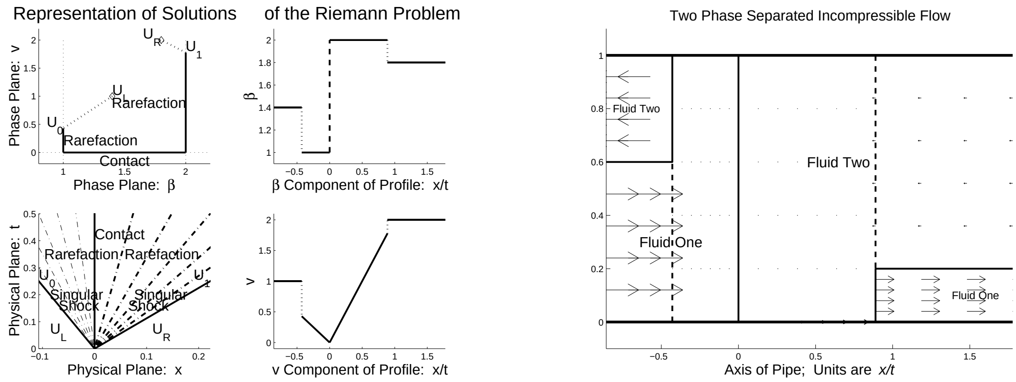

Fig. 2. Nonhyperbolic Data and Pipe Flow Interpretation

solution connecting two states, the waves abut to form a single composite transition between the states. We have the following proposition.

PROPOSITION 3.3 *In the class of self-similar solutions, with the admissibility condition* (16) *for singular shocks and the standard Lax admissibility condition for regular shocks, a nontrivial solution of the Riemann problem exists for any pair of states* U_L, U_R *in the physical region.*

We illustrate a typical case. When U_L and U_R are both in the interior of the nonhyperbolic region, and v_L and v_R have the same sign, there is a solution containing two singular shocks, with a composite wave between them consisting of a rarefaction followed by a contact discontinuity followed by another rarefaction. Several views of the solution are given in Figure 2: On the left are illustrated the phase portrait, showing the location of the intermediate states; a sketch in physical x,t space, showing the relative wave speeds; and profiles of the β and v components of the solution.

When U_R is in the cusped region $Q(U_L)$ defined in Proposition 3.1, then a second solution consisting of a single overcompressive singular shock can also be constructed. We note that the simpler solution is the one seen in calculations, but no uniqueness criterion exists at present.

If the equations are used to model the flow of two separated phases in a pipe, then reconstructing α_i and u_i for the two phases gives another representation of the Riemann solution. This is shown on the right in Figure 2. One sees a clear alternation between the two phases. The local flow speeds, indicated with arrows, differ from the speeds of the discontinuities, and the entire picture is expanding in time. The instantaneous expansion of the flow at the transitions from mixed to pure phases cannot be captured realistically in a one-dimensional model. It would necessarily be associated with large velocity gradients; however, whether this has any relation to the large velocities predicted by the singular shock approximations is unclear.

In addition to the nonuniqueness, which awaits further study, solutions to the Riemann problem exhibit some threshold behavior. For example, there is a bifurcation at the boundary of the region Q.

Another feature of the Riemann solution given in Proposition 3.3 is that if $U_L = U_R$ is a state in the interior of the physical region, then there is a solution which looks like the illustration in Figure 2. That is, even for constant data, the solution is nonconstant. In the context of self-similar solutions, this does not necessarily violate uniqueness or continuous dependence.

3.3 Numerical Approximation of Singular Shock Solutions

Finite difference approximations to the partial differential equation provide another approximation to singular shocks. Numerical simulations were done using a first-order Lax-Friedrichs scheme, and, for comparison, numerical solutions to the parabolic system (13) were also calculated. The simulations bear out our claim that the nonlinear problem has a computable shock structure. Details will be found in [7].

4 Other Initial-Boundary Problems

The other two claims in the Introduction are that high frequency oscillations will be rapidly damped and that a localized perturbation will decay. We present the following argument based on the Riemann solutions we have constructed. Suppose that a constant state U in the interior of the physically feasible region is subjected to a small localized perturbation, $\epsilon V \sin(x/\epsilon)$, for $|x| \leq M$. As stated earlier, only if the perturbation is localized can it be considered small in, say, an energy norm. Now, this perturbation will grow, and within a time of order ϵ will produce a shock solution consisting of a periodic sequence of Riemann solutions, each as described in the previous section. Suppose that we start with this periodic train. If the spatial period is $h = 2\pi\epsilon$, then the Riemann solutions begin to interact at time $T_0 = h/(s_2 - s_1)$, where s_2 and s_1 are the speeds of the leading and trailing singular shock waves respectively; these speeds differ from zero by an order one quantity, and so T_0 is a time of order ϵ. As the singular shocks interact with each other and with the rarefaction waves at their edges, the amplitude of the shocks decreases: the first component, β, remains constant while the second component, v, has an amplitude of order t^{-1} at time t. Thus, the nonlinear structure inside the wave decays to zero. See Figure 3. Localization of the perturbation on an interval $[-M, M]$ plays a key role in this argument.

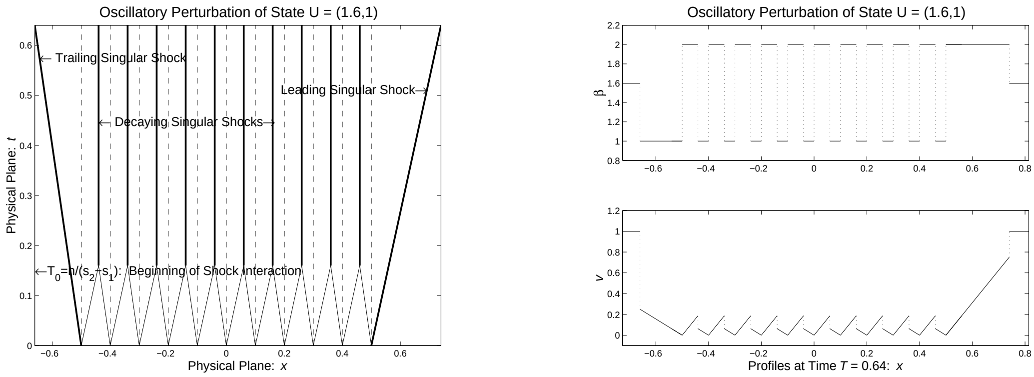

Fig. 3. The Decay of a Localized Periodic Perturbation

5 Interpretation of the Solutions

We have used the simple model for incompressible two-phase flow to illustrate how one can construct a predictive mathematical theory for equations which are not hyperbolic. Using the explicit solution to the Riemann problem, we have illustrated the three points made in the Introduction: the role of localization of perturbations of a nonhyperbolic state; the finite amplitude of oscillations; and the fixed frequency of oscillations (specifically, the survival of a single oscillatory mode). Further work remains: for example, the balance terms G_i in equation (2) represent essential physics which could be included without changing the mathematical framework. In addition, a theory for compressible two-phase flow was partially worked out in [8].

We close with a comment on the unstable "mixed-phase" states in the incompressible model. Figure 2 offers a plausible-looking illustration of separated flow; however, the left-most shock, whose speed is negative, will eventually move out of any finite region. Now, setting $f(t) = 0$ to obtain the identity (4) is done by choosing a spatial coordinate system that moves with the average (weighted) velocity of the fluid mixture. Relative to this velocity, the backward shock moves upstream; however, it would move downstream relative to an inlet where mixture is introduced with positive velocities.

If the initial-boundary data consisted of a constant, mixed state in the pipe initially, with the same constant mixture being fed in at the upstream end, then we would still expect to see the solution of Figure 2. That is, the mixture would separate into the stable configuration given by the Riemann solution. The nonconstant solution corresponding to constant initial data thus is physically correct. However, to fix an origin for the self-similar solution one has to appeal to some notion of localization. At this level, it is clear that one cannot completely eliminate the ill-posedness of the problem, and solutions constructed by the methods of this paper remain only first approximations.

Acknowledgments

It is a pleasure to acknowledge the hospitality of Brown University, where this research was initiated while the author was on leave. I benefited especially from discussions with my hosts, Walter Strauss and Constantine Dafermos. In addition, I would like to thank Theo Theofanous, Andrea Prosperetti, and their colleagues at IMuST for their encouragement to describe the mathematical features of change of type as they appear in two-fluid models.

References

[1] J. B. Bell, J. A. Trangenstein, G. R. Shubin, Conservation laws of mixed type describing three-phase flow in porous media, SIAM Journal on Applied Mathematics 46 (1986) 1000–1023.

[2] R. D. James, The propagation of phase boundaries in elastic bars, Archive for Rational Mechanics and Analysis 73 (1980) 125–158.

[3] B. L. Keyfitz, C. A. Mora, Prototypes for nonstrict hyperbolicity in conservation laws, in: J. Bona, K. Saxton, R. Saxton (Eds.), Nonlinear PDE's, Dynamics and Continuum Physics, Vol. 255 of Contemporary Mathematics, American Mathematical Society, Providence, 2000, pp. 125–137.

[4] D. A. Drew, S. L. Passman, Theory of Multicomponent Fluids, Springer, New York, 1999.

[5] B. L. Keyfitz, H. C. Kranzer, A viscous approximation to a system of conservation laws with no classical Riemann solution, in: C. Carasso, et al. (Eds.), Nonlinear Hyperbolic Problems, Springer, Lecture Notes in Mathematics 1402, Berlin, 1989, pp. 185–197.

[6] B. L. Keyfitz, H. C. Kranzer, Spaces of weighted measures for conservation laws with singular shock solutions, Journal of Differential Equations 118 (1995) 420–451.

[7] B. L. Keyfitz, Lack of hyperbolicity in the simplest model for two-phase incompressible flow, in preparation.

[8] B. L. Keyfitz, Change of type in simple models of two-phase flow, in: M. Shearer (Ed.), Viscous Profiles and Numerical Approximation of Shock Waves, Society for Industrial and Applied Mathematics, Philadelphia, 1991, pp. 84–104.