A System of Non-Strictly Hyperbolic Conservation Laws Arising in Elasticity Theory

BARBARA L. KEYFITZ & HERBERT C. KRANZER

Communicated by C. DAFERMOS

Introduction

In this paper we solve the Riemann problem for a pair of conservation laws of the form

$$\begin{aligned} u_t + (\phi\, u)_x &= 0, \\ v_t + (\phi\, v)_x &= 0, \end{aligned} \tag{1}$$

where $\phi = \phi(u, v)$. This system models the propagation of forward longitudinal and transverse waves in a stretched elastic string which moves in a plane. The wave propagation problem on an idealized nonlinear string, admitting both forward and backward waves, leads to a closely related system of four conservation laws which we also solve.

The feature of interest in system (1) is that the equations are *non-strictly hyperbolic* in the following sense. Introduce vector notation $U = (u, v)$, $F = (\phi\, u, \phi\, v)$; then the system (1) can be differentiated to produce

$$U_t + A U_x = 0, \tag{2}$$

where $A = A(U) = \dfrac{\partial F}{\partial U}$.

In classical theory, (2) is called *strictly hyperbolic* if A has real, distinct eigenvalues. In the example considered here, the eigenvalues $\lambda_1(U)$ and $\lambda_2(U)$ of A may coalesce on some subset $\Sigma \subset \mathbb{R}^2$ of phase space. On Σ, A may or may not be diagonalizable. In the elastic string equations, the matrix is everywhere diagonalizable, and we may say that the system is hyperbolic but not strictly hyperbolic. If A is not diagonalizable on Σ, we may speak of a *parabolic degeneracy*. In neither case does the usual theory (see [8] and [11] for references) for nonlinear hyperbolic conservation laws apply, since this theory demands and uses, among other things, the distinctness of the characteristic speeds λ_1 and λ_2. The major contribution of this paper is to extend the theory to these non-strictly hyperbolic cases and prove the existence of a weak solution to the Riemann problem for (1). Specifically, we solve (1) with initial data

$$U(x,0)=\begin{cases} U_l, & x<0,\\ U_r, & x>0,\end{cases} \tag{3}$$

in the class of functions containing appropriately generalized shock and rarefaction waves.

System (1) is of a particularly simple form among non-strictly hyperbolic 2×2 systems in that one of the characteristic families corresponds to a contact discontinuity and hence is essentially linear. Because this is a property also of the elastic string equations, we were motivated to consider this system first. In Section 1 we analyse the general properties of (1) and discuss its admissible discontinuities and simple wave solutions. Section 2 solves the diagonalizable case and Section 3, the nondiagonalizable case; Section 4 covers the application to an elastic string problem.

1. Model Equations

In system (1), ϕ is in general a function of u and v; for example, in the elastic string problem we model a nonlinear stress-strain relation by

$$\phi=\phi(r)=1+\delta\frac{(r-1)^2}{r}, \qquad \text{where } r^2=u^2+v^2. \tag{4}$$

(*Cf.* Section 4 for a brief discussion of the model as well as for relevant references.)

For general ϕ, if we let $\tan\theta=v/u$ and write $\phi=\phi(r,\theta)$ in polar coordinates, we find from the differentiated form of (2), with

$$A=\begin{pmatrix}\phi+u\,\phi_u & u\,\phi_v\\ v\,\phi_u & \phi+v\,\phi_v\end{pmatrix}$$

that the eigenvalues of A (characteristic speeds of the system) are

$$\lambda_1=\phi,$$
$$\lambda_2=\frac{\partial}{\partial r}(r\,\phi)=\phi+r\frac{\partial\phi}{\partial r}.$$

Then $\Sigma=\{(u,v)|\lambda_1=\lambda_2\}=\{(u,v)|r\,\phi_r=0\}$.

In the elastic string problem, solutions with $r=0$ are physically inadmissible, and we shall in general look at solutions of (1) in the punctured plane $\mathbb{R}^2-\{0\}$. Thus Σ is the set of points for which $\partial\phi/\partial r=0$. To simplify the situation, assume that Σ is a simple closed curve given by $r=f(\theta)$; that is, $\phi_r=0$ for just one point on each radial line, L_θ. This is the case, for example, if $\phi(r,\theta)$ is a convex function of r for each fixed θ and $|\phi(r,\theta)|\to\infty$ as $r\to 0$ and as $r\to\infty$, for each fixed θ. Each point $U\in L_\theta$ has a *reciprocal point* $U^*\in L_\theta$ on the opposite side of Σ with $\phi(U)=\phi(U^*)$; U^* is defined to be 0 or ∞ if no finite reciprocal point exists.

The eigenvector w_1 corresponding to λ_1 is parallel to $(-\phi_v,\phi_u)$, so that $w_1\cdot\nabla\lambda_1\equiv 0$ and hence every shock of this family is a *contact discontinuity*.

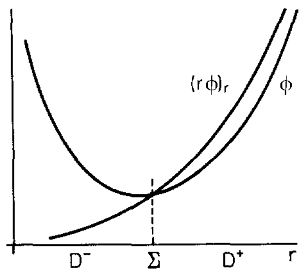

Figure 1

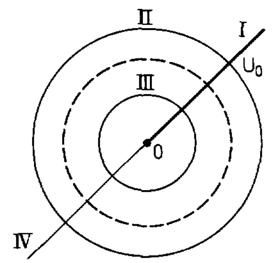

Figure 2

The eigenvector w_2 corresponding to λ_2 is parallel to (u, v), and $w_2 \cdot \nabla\lambda_2 = r(r\phi)_{rr}$, and so this family is *genuinely nonlinear* if $(r\phi)_{rr} = 2\phi_r + r\phi_{rr} \neq 0$. For definiteness, we may assume $(r\phi)_{rr} > 0$; then, since $\phi_r = 0$ on Σ, we see that $\phi_{rr} > 0$ in a neighborhood of Σ. The case $(r\phi)_{rr} < 0$ similarly implies $\phi_{rr} < 0$ near Σ, and many other inequalities are reversed; when necessary statements corresponding to this case will be displayed in parentheses [ ]. See Figure 1. The conventional normalization $w_2 \cdot \nabla\lambda_2 > 0$ suggests defining $w_2 = (\cos\theta, \sin\theta)$ [$w_2 = -(\cos\theta, \sin\theta)$].

The Rankine-Hugoniot equations for (1),

$$s(U - U_0) = F(U) - F(U_0), \tag{5}$$

can be solved to find the *Hugoniot locus* $H(U_0) \subset \mathbb{R}^2$ of states U with the property that the function

$$U(x,t) = \begin{cases} U_0, & x < st, \\ U, & x > st \end{cases} \tag{6}$$

is a weak solution of (1). For a given $U_0 = (r_0 \cos\theta_0, r_0 \sin\theta_0)$, it can be verified that the Hugoniot locus is the union of four sets (see Figure 2):

I. $U = (r\cos\theta_0, r\sin\theta_0), r > 0$, and $s = \dfrac{\phi(U)\,r - \phi(U_0)\,r_0}{r - r_0}$

II. U lies on the curve $\phi(U) = \phi(U_0)$ through U_0 and $s = \phi(U_0)$.

III. U lies on a curve $\phi(U) = \phi(U_0)$ which does not contain U_0. (III is empty if $U_0^* \in$ II. Otherwise III is the ϕ-curve through U_0^*.)

IV. $U = -(r\cos\theta_0, r\sin\theta_0)$, $r > 0$, and $s = \dfrac{\phi(U)\,r + \phi(U_0)\,r_0}{r + r_0}$.

It is well known that for hyperbolic conservation laws some restriction is necessary on the discontinuities allowed in solving the Riemann problem; otherwise there will be in general more than one solution. One criterion for admissibility is the entropy condition introduced by LAX [11] for genuinely nonlinear systems: a discontinuity must be of the k^{th} family for some k, which means, for (6),

$$\begin{array}{lll} \lambda_k(U) < s < \lambda_k(U_0), & & \\ \lambda_j(U) < s, & \lambda_j(U_0) < s, & j < k, \\ s < \lambda_j(U), & s < \lambda_j(U_0), & j > k \end{array} \tag{7}$$

where the eigenvalues are ordered: $\lambda_1 < \lambda_2 \ldots < \lambda_n$. If the k^{th} family is linearly degenerate everywhere the first relation reads: $\lambda_k(U) = s = \lambda_k(U_0)$ and the others are unchanged. LIU [13] extended this condition to systems with local linear degeneracies. The difficulty for non-strictly hyperbolic conservation laws is that there is no well-defined ordering of the eigenvalues in the domain of the problem. However, a statement that is equivalent to (7) in the strictly hyperbolic case and applies to non-strictly hyperbolic systems is that either $(n+1)$ characteristics enter the shock and $(n-1)$ leave it or $(n-1)$ characteristics enter and leave while the remaining two, which in this case must belong to linearly degenerate families, are tangent to the discontinuity. We shall refer to this rule as the *Lax Entropy Condition.* For non-strictly hyperbolic conservation laws we shall extend this criterion to include points of the Hugoniot locus $H(U_0)$ which are limits of points satisfying the above condition. Such points will be said to satisfy the *Generalized Lax Entropy Condition;* they constitute the *shock set*, $S(U_0)$.

Theorem 1. *The shock set $S(U_0)$ consists of the part of* I *with $r \leqq r_0$ $[r \geqq r_0$, if $(r\phi)_{rr} < 0]$, the part of* II *on the same side of Σ as U_0; and possibly a part of* IV.

Proof. We state the arguments for the convex case, $(r\phi)_{rr} > 0$. If $U \in$ I, then $s = \dfrac{[\phi r]}{[r]}$ lies between the values of $(r\phi)_r = \lambda_2$ at U_0 and U, hence if $r < r_0$, $\lambda_2(U) < s < \lambda_2(U_0)$, while if $r > r_0$, the inequalities are reversed. In the notation of [8] we call the first segment $S_2(U_0)$, the second $S_2^*(U_0)$. For the characteristic speeds of the opposite family,

$$s - \lambda_1(U) = s - \phi(U) = \frac{r_0}{r - r_0}(\phi(U) - \phi(U_0))$$

and

$$s - \lambda_1(U_0) = \frac{r}{r - r_0}(\phi(U) - \phi(U_0)),$$

and these quantities clearly have the same sign. Thus if $r < r_0$, we have three characteristics entering the shock and one leaving, but the shock speed may be *faster* or *slower* than the opposite family, according as $\phi(U) < \phi(U_0)$ or $\phi(U) > \phi(U_0)$. The point U_0^*, with $\phi(U_0) = \phi(U_0^*)$, satisfies the generalized entropy condition. If U belongs to II or III, the discontinuity is contact, with $\lambda_1(U) = s = \lambda_1(U_0)$. As long as U and U_0 are on the same side of Σ, $s - \lambda_2(U)$ and $s - \lambda_2(U_0)$ have the same sign. The arc of II so obtained is part of the shock set, and its endpoints, if any, on Σ satisfy the generalized entropy condition. We call this arc $R(U_0)$. On the other hand, for points U on the opposite side of Σ from U_0 (*i.e.*, points of II - $R(U_0)$ or of III), $s - \lambda_2(U)$ and $s - \lambda_2(U_0)$ have opposite signs, violating the Lax entropy condition.

There also may be points in IV which satisfy the entropy conditon. Since s lies between $\phi(U)$ and $\phi(U_0)$, we have $\lambda_1(U_0) > s > \lambda_1(U)$ if $\phi(U_0) > \phi(U)$, that is for all $U \in$ IV between the points IV $\cap$ II $= \bar{U}_0$ and IV $\cap$ III $= \bar{U}_0^*$, if such points exist. Thus a shock of the first family will occur if in addition $s-\lambda_2$ has the same sign at U and U_0. This will happen for U in a subinterval $(\bar{U}_0, W)$, as can be seen by letting U move from $\bar{U}_0$ to $\bar{U}_0^*$. Now $s-\lambda_2(\bar{U}_0)$ and $s-\lambda_2(U_0)$ do have the same sign, so we have a shock for points near $\bar{U}_0$, while $s-\lambda_2(\bar{U}_0^*)$ and $s-\lambda_2(U_0)$ have opposite signs. But $\frac{\partial}{\partial r}(s-\lambda_2(U))=-(r\phi)_{rr}-\frac{1}{r+r_0}(s-\lambda_2(U))<0$ if $s=\lambda_2$, so $s-\lambda_2(U)$ changes sign only once, while $\frac{\partial}{\partial r}(s-\lambda_2(U_0))=-\frac{(s-\lambda_2(U))}{r+r_0}$, so $s-\lambda_2(U_0)$ does not change sign at all. Thus we get a single interval $(\bar{U}_0, W)$, which we call $\bar{S}_2(U_0)$, in which an anomalous entropy shock of the first family occurs.

The proof of Theorem 1 enables us to classify the entropy shocks in I as *fast* if $s>\phi(U_0)$, and *slow* if $s<\phi(U_0)$. If U_0 is inside Σ [outside if $(r\phi)_{rr}<0$], this shock is always slow [fast], but when U_0 is outside [inside] Σ the shock is fast [slow] for U between U_0 and U_0^* and slow [fast] for U between U_0^* and the origin [infinity].

The set $R_2(U_0)$ of states which can be joined on the right to U_0 by a *rarefaction wave* is also a radial line proceeding from U_0, but in the direction opposite to $S_2(U_0)$. In fact, $R_2(U_0)=S_2^*(U_0)$, and system (1) is therefore *non-interacting* in the sense of SMOLLER & JOHNSON [14]: if $U_1\in S_2(U_0)$ and $U_2\in S_2(U_1)$ for sufficiently weak shocks, then $U_2\in S_2(U_0)$.

To establish the nature of the degeneracy at Σ, it is necessary to know whether A is diagonalizable there. At Σ, $A-\lambda_1=A-\lambda_2=0$ if and only if

$$u\phi_u=v\phi_u=u\phi_v=v\phi_v=0 \quad \text{on } \Sigma;$$

that is, $\phi_u=\phi_v=0$ there, or in polar coordinates $\phi_r=\phi_\theta=0$. But Σ is defined by the relation $\phi_r=0$, so that we have proved

Theorem 2. *For the system* (1) *with $\phi=\phi(r,\theta)$ in polar coordinates, Σ is defined by $\{(u,v)|r\phi_r=0\}$. The matrix $A=\frac{\partial F}{\partial U}$ is diagonalizable on Σ if and only if $\frac{\partial\phi}{\partial\theta}=0$ on Σ.*

Corollary. *The characteristic speed $\lambda=\phi$ is constant on Σ if and only if A is diagonalizable there.*

Remarks. 1. The origin is always a singular point in the sense that all S_2 [R_2] curves meet there.

2. The eigenvectors are $w_2=(\cos\theta, \sin\theta)$ and $w_1=(-\phi_v, \phi_u)$ away from Σ. On Σ, w_1 is parallel to w_2 in the nondiagonalizable case, and in the diagonalizable case w_1, when defined on Σ by continuity, is tangent to Σ.

3. Away from Σ, $w_1\cdot w_2=0$ if and only if $v\phi_u-u\phi_v=0$ or $\phi_\theta=0$. However, at Σ, where ϕ_θ is always zero in the diagonalizable case, w_1 need not be orthogonal to w_2. An example is the function $\phi(u,v)=(u^2+2v^2-1)^2$, for which Σ is an ellipse centered at the origin, whereas $w_1\cdot w_2=0$ would force Σ to be a circle.

4. Since ϕ is a 2π-periodic function of θ, there must be at least two points on Σ at which $\phi_\theta=0$ and A therefore diagonalizable. One of these points may coincide with the origin.

5. If A is diagonalizable at Σ, then every contact curve of type II is on the same side of Σ as U_0. Hence $R(U_0)$ is all of II.

2. The Diagonalizable Case

If we assume that Σ is given by $r=f(\theta)$, then since $\phi_r \neq 0$ away from Σ, we see that all contact discontinuity curves $\phi=$constant are given by functions $r=f(\theta, c)$ for different values of c. To get a global existence theorem we assume that $\phi(r, \theta)$ is a convex function of r for each fixed θ and that $|\phi(r, \theta)| \to \infty$ as $r \to 0$ and $r \to \infty$. Then Σ is bounded away from the origin. We also assume genuine nonlinearity, that is, $(r\phi)_{rr}>0\,[<0]$ and hence $\phi_{rr}>0\ [<0]$. Note that $\lambda_1=\phi$ has a minimum [maximum] at Σ, where $\phi_r=0$, and that λ_2 is strictly increasing [decreasing] with r from 0 to ∞, as in Figure 1.

To construct solutions to the Riemann problem we note that, by our convexity assumptions on ϕ, every point (r, θ) has a unique reciprocal point (r^*, θ) on L_θ such that $\phi(r, \theta)=\phi(r^*, \theta)$.

Theorem 3. *Under the assumptions on system* (1) *that Σ is given by $r=f(\theta)$ and $\phi_\theta=0$ on Σ, and that $(r\phi)_{rr} \neq 0$, that ϕ is a convex function of r for fixed θ, and that $|\phi(r, \theta)| \to \infty$ as $r \to 0$ and ∞, there is a centered solution to the Riemann problem* (3) *consisting of at most four states separated by entropy shocks, rarefaction waves or contact discontinuities. This solution is unique provided U_l and U_r do not lie on diametrically opposite rays through the origin.*

Proof. Assume $\phi_{rr}>0$ for definiteness. Let D^- be the region of U-space in which $\lambda_1>\lambda_2$, D^+ the region where $\lambda_1<\lambda_2$. Our assumptions guarantee that every contact discontinuity curve $R(U_0)$ intersects every L_θ at a unique point.

Case I. $U_l \in D^-$. If also $U_r \in D^-$, then $R(U_r) \subset D^-$ and $U_m=R_2(U_l) \cap R(U_r)$ or $S_2(U_l) \cap R(U_r)$ is uniquely defined. The solution contains the intermediate state U_m joined to U_l by a slow shock or rarefaction wave and joined to U_r by a contact discontinuity.

If $U_r \in D^+$, then let $U_m=\Sigma \cap R_2(U_l)$, $U_n=\Sigma \cap S_2(U_r)$. Then U_l joins U_m by a rarefaction wave, U_m joins U_n by a contact discontinuity and U_n joins U_r by another rarefaction wave. (Note that here we are using the non-interacting nature of the shock curves to facilitate finding U_n. We should really draw a "backward rarefaction curve" from U_r to intersect Σ.)

Case II. $U_l \in D^+$. For each point $U \in R(U_l)$ the reciprocal point is $U^*=S_2(U) \cap R(U_l^*)$, and $R(U_l^*)$ is a closed curve. If U_r and U_l are on the same side of $R(U_l^*)$, then $U_m=R(U_l) \cap R_2(U_r)$ and U_l joins U_m by a contact discontinuity, U_m joins U_r by a fast shock.

If U_r and U_l are on opposite sides of $R(U_l^*)$, then $U_m=S_2(U_l) \cap R(U_r)$ and U_l joins U_m by a slow shock, U_m joins U_r by a contact discontinuity. Note that when U_r is near $R(U_l^*)$, U_m is not a continuous function of U_r.

If U_r lies on $R(U_l^*)$, U_m may have either of the above forms, but the solution $U(x,t)$ is nevertheless unique because the shock and contact discontinuity then have the same speed of propagation and so the sector of the x,t-plane in which $U=U_m$ has zero width.

Uniqueness. With one exception the solutions obtained above are the only centered solutions which satisfy the Lax entropy conditions at discontinuities, for it

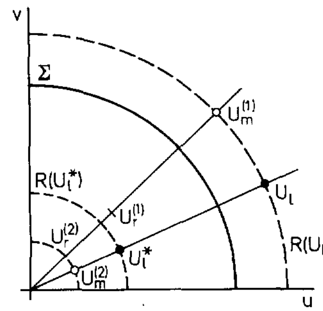

Figure 3

may be verified that in each case there is no other sequence of states U_l, U_1, $U_2, \ldots, U_j$, U_r that can be joined by a sequence of waves with increasing speeds. For example, if U_l joins U_1 by a contact discontinuity whose speed is $\phi = \phi_l = \phi_1$, then U_1 can join U_2 only by a fast shock or rarefaction wave, and then only if $\lambda_2 > \lambda_1$, *etc.*

The exception is that if $U_l \in L_\theta$, $U_r \in L_{\pi+\theta}$ and is such that $U_r \in \bar{S}_2(U_l)$, there may be a second type of solution, in which U_l is joined to a point $U_m \in \bar{S}_2(U_l)$ by an anomalous entropy shock, while U_m either equals U_r or is joined to U_r by a rarefaction wave. Since a state $U_m \in \bar{S}_2(U_l)$ is continuable only to another point in $L_{\theta+\pi}$, anomalous shocks can appear only for pairs of points U_l, U_r that are diametrically opposite. For the elastic string, this corresponds to a string completely bent back on itself. Such a Riemann problem is thus ill-posed.

3. Local Solution in the Nondiagonalizable Case

If the matrix A is not diagonalizable at Σ, various possibilities present themselves. An example which we have considered in detail is $\phi(u,v) = [(u+1)^2 + v^2]^p$, where p is a positive or negative exponent. Here Σ is the circle $(u+\frac{1}{2})^2 + v^2 = \frac{1}{4}$. The matrix is diagonalizable at the points $(0,0)$ and $(-1,0)$, both of which are singular for the system. For the general equation of this type, a global solution cannot be described without a detailed knowledge of the behavior of ϕ away from Σ, and also of the character of the (at least two) diagonalizable points of Σ.

In this section, we construct a solution to the Riemann problem in the neighborhood of Σ. Specifically, we take an open domain Γ of $\mathbb{R}^2$ which is divided into two connected subsets D^- and D^+ by a segment of Σ on which $\phi_\theta \neq 0$. For definiteness we bound Γ by two radial line segments, so that in Γ $\theta_{\min} < \theta < \theta_{\max}$. We assume, as in the previous sections, that Σ is given by $r = f(\theta)$ in Γ. The assumption of genuine nonlinearity on the λ_2 family, as we saw, implies $\phi_{rr} \neq 0$ at Σ. We also have $\phi_\theta \neq 0$ at Σ. Thus the curves $\phi =$ constant have second-order contact with lines L_θ at Σ, and the curve $\phi =$ constant going through $(f(\theta_{\max}), \theta_{\max})$ or $(f(\theta_{\min}), \theta_{\min})$ bounds a convex region within the sector $(\theta_{\min}, \theta_{\max})$ and intersects the other radial line in two points. Denote the interior of this region by Γ. See Figure 4.

Without loss of generality, we can take the bounding curve to pass through $\theta_{\max}$, and assume $\phi_\theta > 0$, $\phi_{rr} > 0$. The cases $\phi_\theta < 0$, $\phi_{rr} < 0$ and the two cases involving $\theta_{\min}$

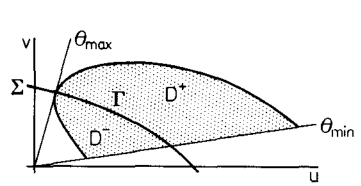

Figure 4

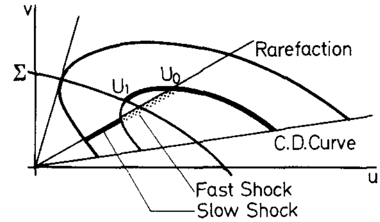

Figure 5

are similar. Thus ϕ attains its maximum in Γ on the curved arc, its minimum at θ_{min}. On each L_θ, ϕ attains a minimum at Σ, and λ_2 increases with r as in Figure 1. As before, $D^- = \{U|\lambda_1 > \lambda_2\}$ is inside Σ (*i.e.*, smaller values of r) and D^+ is outside Σ.

The part of the Hugoniot locus $H(U_0)$ in Γ is the union of the two curves $\phi(U) = \phi(U_0)$ and $\theta(U) = \theta(U_0)$.

Define the *wave set* $W(U_0)$ to be the set of states U which can be connected on the right to the state U_0 on the left, either by a rarefaction wave or by a shock or contact discontinuity satisfying an entropy condition on both families. In the present problem this set is a subset of $H(U_0)$, since the system is non-interactive.

The *continuable set* $C(U_0)$ is the subset of points U of $W(U_0)$ which can also be joined to a third state on the right by a rarefaction wave, shock or contact discontinuity whose minimum speed is at least as great as the maximum speed of the wave joining U_0 to U.

The *two-related set* $W_2(U_0)$ is the set of states which can be joined to U_0 via two waves of the listed types: *i.e.*, the union of the appropriate portions of the sets $W(U)$ for all U in $C(U_0)$, or the set of continuations of continuable points.

Higher-order continuable sets $C_j(U_0)$ and j-related sets $W_j(U_0)$ can be similarly defined.

Now, as in the diagonalizable case, $U_0 = (r_0, \theta_0)$ can be joined to any point $U = (r, \theta_0)$ on L_{θ_0} by a shock of speed $s = [r\phi]/[r]$ if $r < r_0$, and this is a fast shock if $\phi(r) < \phi(r_0)$ and a slow shock if $\phi(r) > \phi(r_0)$. The same point U_0 can be joined to U by a rarefaction if $r > r_0$. The head of the rarefaction has speed $\lambda_2(r)$ and the tail has speed $\lambda_2(r_0)$.

Also, U_0 can be joined to a point U by a contact discontinuity if $\phi(U_0) = \phi(U)$ and U lies on the same side of Σ as U_0.

Thus in Figure 5 where $U_0 \in D^+$, $W(U_0)$ consists of the segment of the line $\theta = \theta_0$ inside Γ and the segment of the curve $\phi(U) = \phi(U_0)$ inside $D^+ \cup \Sigma$. The continuable set $C(U_0)$ is the "slow shock" portion of the shock curve and the contact-discontinuity curve. Denote the point of intersection of $\phi = \phi(U_0)$ with Σ as U_1. It can be seen that the set $W_2(U_0)$ consists of the entire D^- region and the part of D^+ between θ_{min} and $\theta(U_1)$; this is obtained by continuing the slow shock by a contact discontinuity and the contact discontinuity by a fast shock or rarefaction wave.

Note that if U^* is on Σ with $\phi(U^*) > \phi(U_1)$, then U_0 can be joined to U^* through an intermediate state $\bar{U} \in D^-$ by a slow shock of speed $s < \phi(\bar{U})$ and a contact discontinuity of speed $\phi(\bar{U}) = \phi(U^*)$. But now U^* can join any point on $L_{\theta(U^*)}$ with $r > r(U^*)$ by a rarefaction wave of tail speed $\lambda_2(U^*) = \phi(U^*)$, and hence $U^* \in C_2(U_0)$.

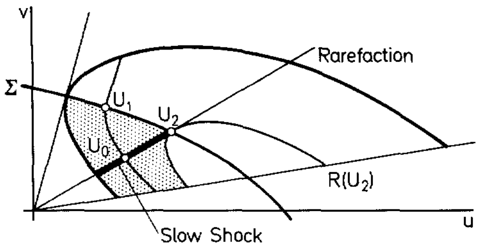

Figure 6

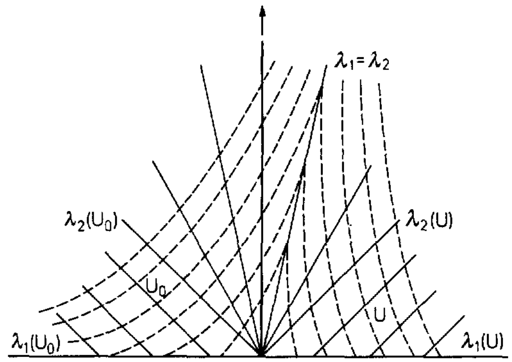

Figure 7

Thus the remainder of Γ consists of points in $W_3(U_0)$. Since none of the regions covered by different solutions overlap, the solution is unique.

If $U_0 \in D^-$, as in Figure 6, the analogous construction is as follows. $W(U_0)$ consists of the line $\theta=\theta_0$ and the segement of $\phi=\phi(U_0)$ in D^-. Define U_1 as before, and let U_2 be the intersection of Σ with L_{θ_0}. Only the (slow) shock and the portion of the rarefaction curve inside $D^- \cup \Sigma$ are continuable, together with the isolated point U_1. The rarefactions with $U \in D^+$ have the odd feature that the so-called cross-flow characteristic family (in x, t-space) does not cross the wave but instead flows into it from both front and back (see Figure 7 where cross-flow is in dotted lines). Such waves are neither continuable nor continuations of other waves, and hence may be expected to appear only for very special initial conditions U_l and U_r.

The set $W_2(U_0)$ consists of the part of D^- with $\phi \geqq \phi(U_2)$ and the line $\theta=\theta(U_1)$ in D^+ of points joined to U_1 by a rarefaction wave, together with the curve $R(U_2)$ in D^+, points of which can be joined to U_2 by a contact discontinuity of speed $\phi(U_2)$ equal to the head speed of the rarefaction wave joining U_0 and U_2. The set $C_2(U_0)$ consists of points of Σ with $\phi>\phi(U_2)$ and the curve $R(U_2) \cap D^+$. From the latter fast shocks and rarefaction waves, from the former rarefaction waves alone may be drawn, so that $W_3(U_0)$ just fills the remainder of Γ.

Thus we have proved

Theorem 4. *For the system* (1), *assume that Σ is given by $r=f(\theta)$ in the sector $\theta_{\min} \leqq \theta \leqq \theta_{\max}$ and that in this sector $(r\phi)_{rr} \neq 0$ everywhere and $\phi_\theta \neq 0$ on Σ. Let Γ*

denote the subset of this sector bounded by one of its radial bounding lines and by the $\phi = \text{const.}$ *curve tangent to the other bounding line at* Σ. *Then the Riemann problem* (3) *has for all* U_l, U_r *in* Γ *a unique centered entropy solution* $U(x,t) \subset \Gamma$. *The solution consists of at most four constant states separated by discontinuities or simple waves.*

It is noteworthy that in this problem a solution cannot be found by restricting attention to one or the other side of Σ where the equations are strictly hyperbolic. Intermediate states on the opposite side of Σ may be necessary to join two points on the same side.

It is also clear that the above construction can be carried out in any region containing only non-diagonalizable points of Σ, and such regions may be pieced together. However, singular behavior will occur at the diagonalizable points on Σ.

4. Application to a Problem of Shocks on an Elastic String

We consider an elastic string of infinite length which is constrained to move in the (w, v) plane. See [2], [7]. Let x be arc-length along the string measured from some origin when it is in a reference configuration of uniform tension T_0 and density ρ_0, and let $w(x,t)$ and $v(x,t)$ be the horizontal and vertical components at time t of the point which was at distance x from the origin in the reference state. This normalization is convenient for studying the Riemann problem.

The *strain* is defined as $\varepsilon = \sqrt{w_x^2 + v_x^2} - 1$, and we let

$$r = 1 + \varepsilon = \sqrt{w_x^2 + v_x^2}, \qquad \theta = \tan^{-1}\left(\frac{v_x}{w_x}\right).$$

Assuming a stress-strain relation $T = T(\varepsilon)$, we can write the equations of motion of the string (see CRISTESCU [5]) in the form

$$\rho_0 \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{T}{1+\varepsilon}\,\frac{\partial w}{\partial x}\right), \tag{9}$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{T}{1+\varepsilon}\,\frac{\partial v}{\partial x}\right).$$

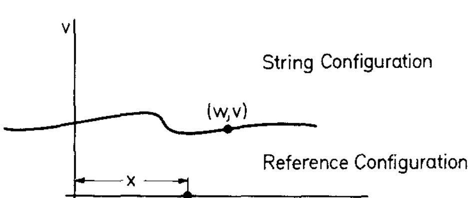

Figure 8

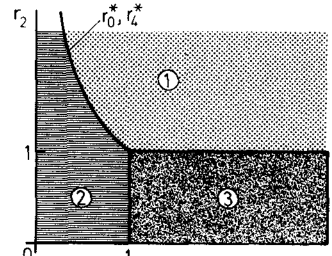

Figure 9

To avoid proliferation of symbols, we let $\rho_0 = 1$ and $T_0 = 1$, and let $\phi = \phi(r) = \frac{T}{1+\varepsilon}$. Define vector functions $U = (w_x, w_t, v_x, v_t)$ and $F(U) = (-w_t, -\phi\, w_x, -v_t, -\phi\, v_x)$. Then we have a system of four conservation laws of the form of equation (2). The eigenvalues of A are $\pm\sqrt{\phi} = \pm\lambda_1$ and $\pm\sqrt{\frac{dT}{d\varepsilon}} = \pm\sqrt{\frac{d}{dr}(r\,\phi)} = \pm\lambda_2$. Thus the equations are hyperbolic if T is positive and monotone, but non-strictly hyperbolic if, for some value of r, $\phi = \frac{d}{dr}(r\,\phi)$, or $r\,\phi_r = 0$. Positive tension is a necessary hypothesis, and it is also physically reasonable that stress increase with strain. For a stress-strain relation that is linear near the reference position, $T = 1 + \sigma\,\varepsilon$, so $\phi(r) = \sigma + \frac{1-\sigma}{r}$ and $\phi_r = -\frac{1-\sigma}{r^2}$ is zero if and only if $\sigma = 1$. Thus in the linear approximation we have a characteristic equation with roots of constant multiplicity if $\sigma = 1$ and distinct roots if $\sigma \neq 1$. If, however, T is nonlinear, then near the defining state we may write $T(\varepsilon) = 1 + \sigma\,\varepsilon + \delta\,\varepsilon^2 + O(\varepsilon^3)$, and $(1+\varepsilon)\frac{dT}{d\varepsilon} - T = \sigma - 1 + 2\delta\,\varepsilon + O(\varepsilon^2)$ will be equal to zero for some $\varepsilon \sim \frac{1-\sigma}{2\delta}$ which is near zero for σ near 1. Since the choice of the reference state was arbitrary, we choose it to correspond to this critical value: that is, $\sigma = \frac{dT}{d\varepsilon}(0) = 1$. Thus A will have an eigenvalue of multiplicity two if $\varepsilon = 0$ $(r = 1)$.

For example, if $T(\varepsilon)$ is exactly $1 + \varepsilon + \delta\,\varepsilon^2$, then $\phi = 1 + \frac{\delta(r-1)^2}{r}$.

It is stated in CRISTESCU [5] that the speeds $\pm\sqrt{\phi}$ are characteristic of the propagation of transverse waves, or of changes in the shape of the string without changes in tension. This family is linearly degenerate; that is, a discontinuity in U, which is, physically, a corner in the string, propagates as a contact discontinuity with the same speed as the common value of $\pm\sqrt{\phi}$ ahead of or behind the corner. The speeds $\pm\sqrt{\frac{dT}{d\varepsilon}}$ characterize longitudinal or tension waves, in which there is no change of shape in the string. A discontinuity in this family is a tension jump, analogous to the pressure jump in the equations of gas dynamics [3], and such discontinuities are shocks in the sense of this work. The condition for genuine nonlinearity is reduced to $\frac{d^2}{dr^2}(r\,\phi) \neq 0$, or $\frac{d^2 T}{d\varepsilon^2} \neq 0$, a condition we shall assume in the region of interest. We shall also assume that the only physically relevant or admissible discontinuities are those which satisfy the Lax entropy condition (or a weakened condition with non-strict inequalities). In Appendix B we present an argument based on the total energy in the string to support this mathematically reasonable assumption. However, it must be stated that the equations (9) for the string are an idealized system which describes an infinitely thin string. We shall demonstrate in the theorem of this section that the generalized Lax entropy condition makes the Riemann Problem for (9) mathematically well-posed, but the solutions we find are physically meaningful only if it can be shown that they are indeed the limits of solutions of the more complicated mathematical system containing *all* the physically relevant quantities. This limiting procedure would serve to define a length scale on which shock thickness could be measured. As far as we know, this framework does not exist at present; however, shocks in elastic strings are observed experimentally and are important in some engineering applications (see IOSUE [7], CRISTESCU [5]). It would be interesting to know the circumstances under which the solutions found here describe the observed phenomena.

In the problem we have just outlined, we shall assume genuine non-linearity, that is $\frac{d^2}{dr^2}(r\phi)\neq 0$. Now if r is convex downward, then $r\phi$, and hence ϕ, might become negative as $r\to 0$ or as $r\to\infty$, and the equations would no longer be hyperbolic there. Also, if $(r\phi)_r$ becomes negative for some values of r the equations are not hyperbolic there. Thus we will consider only states U corresponding to values of r in an interval $r_{\min}\leqq r\leqq r_{\max}$ in which ϕ and $(r\phi)_r$ are positive. Note that a state $U=(u_1,u_2,u_3,u_4)$ is completely specified if $r=\sqrt{u_1^2+u_3^2}$, $\theta=\tan^{-1}\left(\frac{u_3}{u_1}\right)$, u_2 and u_4 are known.

The solution to the Riemann problem with initial states U_l, U_r will consist of five states $U_l=U_0, U_1,\ldots,U_4=U_r$, joined by two backward and two forward waves. As in Section 2 we define $\Sigma=\{U|\phi=(r\phi)_r\}=\{U|r=1\}$; Σ is a 3-dimensional manifold in $\mathbb{R}^4$ which separates $D^-=\{U|\phi>(r\phi)_r\}$ from $D^+=\{U|\phi<(r\phi)_r\}$. For definiteness we shall take $(r\phi)_{rr}>0$; then $D^\pm=\{U|r\gtrless 1\}$.

To find the wave curves we must solve the Rankine-Hugoniot equations $s[U]=[F]$, or

$$s^2[u_1]=[\phi u_1],\qquad s^2[u_3]=[\phi u_3].$$

Following the discussion in Section 1, with s^2 replacing s, we find that there are two types of waves: contact discontinuities with $r_+=r_-$ and $s^2=\phi(r_+)=\phi(r_-)$, and shocks with $\theta_+=\theta_-$ and $s^2=\frac{[r\phi]}{[r]}$. There are also, as in the model problem, anomalous shocks, with $\theta_+=\pi+\theta_-$, which correspond to cusps, or 180° bends in the string. We do not include these in the construction. See the discussion at the end of Section 2. The *backward* shocks, with $s=-([r\phi]/[r])^{\frac{1}{2}}<0$, satisfy an entropy condition on the characteristic speed of the same family if $-\lambda_2^->s>-\lambda_2^+$ or $r_-<r_+$; all such shocks satisfy the entropy condition on the other family, and as in Section 2, the shock is *slow* if $|s|<\lambda_1^\pm$, *i.e.* $s>-\sqrt{\phi^\pm}$, and fast otherwise. For each point r define the *reciprocal* point r^* such that $\phi(r)=\phi(r^*)$; since $r=1$ is a minimum of ϕ, such a point always exists for r near 1, and we let r^* be 0 or ∞ if no finite reciprocal point exists for $r>1$ or $r<1$, respectively.

There are the following possibilities:

If $1<r_-<r_+<\infty$, s is a fast shock.
If $r_-<r_+<r_-^*$, s is a slow shock.
If $r_-<r_-^*<r_+$, s is a fast shock.

Similarly the *forward* shocks, with $s=([r\phi]/[r])^{\frac{1}{2}}$, satisfy an entropy condition if $\lambda_2^->s>\lambda_2^+$ or $r_->r_+$ and if

$1>r_->r_+>0,\quad$ s is a slow shock;
$r_->r_-^*>r_+,\quad$ s is a slow shock;
$r_->r_+>r_-^*,\quad$ s is a fast shock.

The equations for the shock curves, S, and contact discontinuity curves, C (subscripts b and f denoting backward and forward waves) are

$$U\in C_b(U_0) \quad \text{if} \quad r=r_0 \quad \text{and}$$

$$u_2=u_2^{(0)}+r_0\sqrt{\phi(r_0)}(\cos\theta-\cos\theta_0),$$
$$u_4=u_4^{(0)}+r_0\sqrt{\phi(r_0)}(\sin\theta-\sin\theta_0);$$

$$U\in C_f(U_0) \quad \text{if} \quad r=r_0 \quad \text{and}$$

$$u_2=u_2^{(0)}-r_0\sqrt{\phi(r_0)}(\cos\theta-\cos\theta_0),$$

and a similar equation holds for u_4;

$$U\in S_b(U_0) \quad \text{if} \quad \theta=\theta_0,\ r>r_0 \quad \text{and}$$

$$u_2=u_2^{(0)}+\psi(r,r_0)(r-r_0)\cos\theta_0$$

where

$$\psi(r,r_0)=\psi(r_0,r)=\sqrt{\frac{r\phi-r_0\phi_0}{r-r_0}};$$

$$U\in S_f(U_0) \quad \text{if} \quad \theta=\theta_0,\ r<r_0 \quad \text{and}$$

$$u_2=u_2^{(0)}-\psi(r,r_0)(r-r_0)\cos\theta_0$$

with analogous equations for u_4 in each case.

The rarefaction curves R_b and R_f are the integral curves of the vector fields given by right eigenvectors of A corresponding to $-\lambda_2$ and $+\lambda_2$ respectively. It may be verified that

$$U\in R_b(U_0) \quad \text{if} \quad \theta=\theta_0,\ r<r_0 \quad \text{and}$$

$$u_2=u_2^{(0)}+\left[\int_{r_0}^{r}\sqrt{[t\,\phi(t)]'}\,dt\right]\cos\theta_0=u_2^{(0)}+\chi(r,r_0)\cos\theta_0$$

$$U\in R_f(U_0) \quad \text{if} \quad \theta=\theta_0,\ r>r_0 \quad \text{and}$$

$$u_2=u_2^{(0)}-\left[\int_{r_0}^{r}\sqrt{(t\,\phi(t))'}\,dt\right]\cos\theta_0=u_2^{(0)}-\chi(r,r_0)\cos\theta_0$$

with similar expressions for u_4.

If $r<1<r_0$ in the backward case or $r>1>r_0$ in the forward case, the rarefactions can be combined with contact discontinuities that are in the middle, as in the model problem in Section 2. In this superposition, the two waves can be treated independently.

In the following theorem and corollary we give our main result on global existence of solutions to the problem of the elastic string.

Theorem. *The Riemann Problem* (9), (3), *with* $\phi'(1)=0$, $(r\phi)''\neq 0$, *has a unique solution consisting of five states connected by two backward waves and two forward waves, for* U_l, U_r *in a region* $\Gamma\subset\mathbb{R}^4\times\mathbb{R}^4$.

Proof. We consider the case $(r\phi)''>0$, for which the curves described above have been constructed. Since U_0, U_1 and U_2 must be connected by backward waves and U_2, U_3 and U_4 by forward ones, we have the following possibilities in each case:

1. $U_1\in S_b(U_0)$ with a fast (backward) shock $\theta_1=\theta_0, r_1=r_2$; or $U_1\in R_b(U_0)$ with a rarefaction $\theta_1=\theta_0$, $r_1=r_2$;

2. $U_1\in C_b(U_0)$ with a contact discontinuity and $U_2\in S_b(U_1)$ (slow shock) or $U_2\in R_b(U_1)$ (rarefaction), $\theta_1=\theta_2$, $r_1=r_0$.

3. The contact discontinuity is in the middle of the rarefaction wave and there is no clearly defined U_1, but instead two states U_1 and $\bar{U}_1$ with

$$r(U_1)=r(\bar{U}_1)=1,\qquad \theta(U_1)=\theta_0,\qquad \theta(\bar{U}_1)=\theta_2,$$

and

$$U_1\in R_b(U_0),\qquad \bar{U}_1\in C_b(U_1),\qquad U_2\in R_b(\bar{U}_1).$$

In all cases, the situation is determined once we know r_2 and θ_2, and we obtain in each case:

1b. $$u_2^{(2)}=u_2^{(0)}+\psi(r_2,r_0)(r_2-r_0)\cos\theta_0+r_2\sqrt{\phi(r_2)}(\cos\theta_2-\cos\theta_0)$$

or

$$u_2^{(2)}=u_2^{(0)}+\chi(r_2,r_0)\cos\theta_0+r_2\sqrt{\phi(r_2)}(\cos\theta_2-\cos\theta_0);$$

2b. $$u_2^{(2)}=u_2^{(0)}+r_0\sqrt{\phi(r_0)}(\cos\theta_2-\cos\theta_0)+p(r_2,r_0)\cos\theta_2$$

where

$$p(r_2,r_0)=\begin{cases}\psi(r_2,r_0)(r_2-r_0) & \text{if } r_2>r_0,\\ \chi(r_2,r_0) & \text{if } r_2<r_0;\end{cases}$$

3b. $$u_2^{(2)}=u_2^{(0)}+\chi(1,r_0)\cos\theta_0+1\cdot(\cos\theta_2-\cos\theta_0)+\chi(r_2,1)\cos\theta_2.$$

In general, letting $S_i=r_i\sqrt{\phi(r_i)}$ gives

$$u_2^{(2)}=u_2^{(0)}+T_0\cos\theta_0+T_2\cos\theta_2$$

where

$$T_0=\begin{cases}p(r_2,r_0)-S_2 & \text{if}\quad r_0>1, r_2>1\quad\text{or}\quad r_0<1, r_2>r_0^*\ \text{(region ①)},\\ -S_0 & \text{if}\quad r_0<1\quad\text{and}\quad 0<r_2<r_0^*\ \text{(region ②)},\\ \chi(1,r_0)-1 & \text{if}\quad r_0>1>r_2\ \text{(region ③)}\end{cases}$$

and

$$T_2=\begin{cases} S_2 \text{ in region ① above,} \\ S_0+p(r_2,r_0) \quad \text{in region ②,} \\ 1+\chi(r_2,1) \quad \text{in region ③.} \end{cases}$$

These regions are illustrated in Figure 9. In an analogous fashion, we find

$$u_4^{(2)}=u_4^{(0)}+T_0\sin\theta_0+\mathrm{T}_2\sin\theta_2.$$

Similarly in joining U_2 to U_4 by two forward waves, we have

$$u_2^{(2)}=u_2^{(4)}+\bar{T}_2\cos\theta_2+T_4\cos\theta_4$$

and

$$u_4^{(2)}=u_4^{(4)}+\bar{T}_2\sin\theta_2+T_4\sin\theta_4$$

where

$$\bar{T}_2=\begin{cases} -S_2, \\ -S_4-p(r_2,r_4), \\ -1-\chi(r_2,1), \end{cases} \qquad T_4=\begin{cases} -p(r_2,r_4)+S_2 & \text{in ①,} \\ S_4 & \text{in ②,} \\ 1-\chi(1,r_4) & \text{in ③.} \end{cases}$$

Now, eliminating $u_2^{(2)}$ and $u_4^{(2)}$ from these equations, we get

$$u_2^{(0)}+T_0\cos\theta_0+T_2\cos\theta_2=u_2^{(4)}+\bar{T}_2\cos\theta_2+T_4\cos\theta_4,$$
$$u_4^{(0)}+T_0\sin\theta_0+T_2\sin\theta_2=u_4^{(4)}+\bar{T}_2\sin\theta_2+T_4\sin\theta_4.$$

Let

$$R^2=[u_2^{(4)}-u_2^{(0)}]^2+[u_4^{(4)}-u_4^{(0)}]^2, \qquad \omega=\tan^{-1}\left(\frac{u_4^{(4)}-u_4^{(0)}}{u_2^{(4)}-u_2^{(0)}}\right).$$

Then defining

$$A=T_4\cos\theta_4-T_0\cos\theta_0+R\cos\omega, \tag{10}$$
$$B=T_4\sin\theta_4-T_0\sin\theta_0+R\sin\omega,$$

we can eliminate θ_2 to obtain a single equation for r_2:

$$(T_2-\bar{T}_2)^2=A^2+B^2. \tag{11}$$

Define

$$G(r_2)=G(r_2;\, r_0,r_4,\theta_0,\theta_4,R,\omega)=(T_2-\bar{T}_2)^2-A^2-B^2. \tag{12}$$

We now look for conditions on U_0 and U_4 which guarantee a solution to $G=0$. First we shall show that there can be no more than one solution, because at a root of $G=0$, G is an *increasing* function of r_2. Now G is not differentiable at some values of r_2, but G is continuous and one-sided derivatives do exist everywhere. Hence we can calculate

$$\frac{1}{2}\frac{dG}{dr_2}=(T_2'-\bar{T}_2')(T_2-\bar{T}_2)-A'A-B'B,$$

where $'=\dfrac{d}{dr_2}$.

It may be verified that $T_2'>0$ and $\bar{T}_2'<0$ for all cases, while* $T_0'\leqq 0$ and $T_4'\geqq 0$; in fact

$$0\geqq T_0'>-T_2' \quad \text{and} \quad 0\leqq T_4'<-\bar{T}_2'.$$

Thus at a root of $G=0$,

$$(T_2'-\bar{T}_2')(T_2-\bar{T}_2)>|A'A+B'B|,$$

for $T_2-\bar{T}_2$ is always positive (by the Schwarz inequality on χ) and is greater than $\max(|A|,|B|)$ if $G=0$ by (11), while

$$\begin{aligned}|A'A+B'B|&=|T_4'(A\cos\theta_4+B\sin\theta_4)-T_0'(A\cos\theta_0+B\sin\theta_0)|\\ &\leqq T_4'\max(|A|,|B|)-T_0'\max(|A|,|B|)\\ &<(T_2'-\bar{T}_2')\max(|A|,|B|).\end{aligned}$$

Thus $G'>0$ at any root of $G=0$ so that the solution U_2, if it exists, must be unique.

Now when does such a state U_2 actually exist? A necessary and sufficient condition for its existence is that $G(r_{\min})<0<G(r_{\max})$, where $0\leqq r_{\min}<r<r_{\max}\leqq+\infty$ is the r-interval in which the assumptions $\phi>0$, $(r\phi)_r>0$, $(r\phi)_{rr}\neq 0$ of our theorem hold. We must therefore investigate the signs of $G(r_{\min})$ and $G(r_{\max})$.

We look first at $G(r_{\max})$. The simplest behavior occurs when $r_{\max}=\infty$ and $\phi(r_{\max})=\lim_{r\to\infty}\phi(r)=+\infty$. Then $r^*\to 0$ as $r\to\infty$, so that (r_0,r_2) and (r_4,r_2) are always in region ① of Figure 9 when r_2 is large. Hence for $r_2=r\to\infty$

$$\begin{aligned}T_0&=\psi(r,r_0)(r-r_0)-r\sqrt{\phi(r)}=r\sqrt{\phi(r)}\left[\sqrt{\left(1-\frac{r_0\phi_0}{r\phi}\right)\left(1-\frac{r_0}{r}\right)}-1\right]\\ &=-\frac{1}{2}r_0\sqrt{\phi(r)}\left[\left(1+\frac{\phi_0}{\phi(r)}\right)+O\left(\frac{1}{r}\right)\right],\end{aligned}$$

which approaches $-\infty$ as $r\to\infty$, while $T_2=S_2=r\sqrt{\phi(r)}$ approaches $+\infty$ at a faster rate. Similarly $T_4\simeq+\frac{1}{2}r_4\sqrt{\phi(r)}\to+\infty$, while $\bar{T}_2=-S_2\to-\infty$. Thus as $r\to\infty$ we have $(T_2-\bar{T}_2)^2=O(r^2\phi)$ while $A^2+B^2=O(\phi)$ only. Hence $G(r)>0$ for r sufficiently large; *i.e.*, $G(r_{\max})=G(+\infty)>0$ in this case.

Next, we consider $r_{\max}=\infty$, but $\phi(r_{\max})$ finite; this can happen, for example, with a ϕ which behaves like const. $-r^{-\frac{1}{2}}$ for large r. In this case $r^*_{\max}$ may be positive, so that the possibility exists of an r_0 (or r_4) between $r_{\min}$ and $r^*_{\max}$; for such an r_0 or r_4 the corresponding r_2 will lie in region ② of Figure 9. But whether we are region ① or ② we find that $T_2-\bar{T}_2$ still approaches $+\infty$, while now T_0 and T_4 approach finite limits as $r\to\infty$. Hence again $G(r_{\max})>0$.

* For, in region ①,

$$T_0'=\begin{cases}(2\psi\lambda_1)^{-1}(\lambda_1-\psi)(\lambda_2^2-\lambda_1\psi)<0,\ r_2>r_0,\\ -\dfrac{1}{2\sqrt{\phi}}(\lambda_1-\lambda_2)^2<0,\ r_2<r_0,\end{cases}$$

and $T_0'=0$ in ② and ③. Similar formulas hold for T_4'.

Finally, suppose $1<r_{\max}<\infty$. We may then take r_0 and r_4 less than $r_{\max}$. Then taking $r_2=r_{\max}$ places (r_0, r_2) and (r_4, r_2) in regions ① or ② of Figure 9; in these regions we have

$$\begin{aligned} T_2 &= -T_0 + p(r_2, r_0), \\ \bar{T}_2 &= -T_4 - p(r_2, r_4) \end{aligned}$$

and $0<p(r_2, r_i)=\psi(r_2, r_i)(r_2-r_i)<S_2$ if $r_2>r_i$, so that $T_0<0<T_4$. But from (10),

$$A^2+B^2<(|T_4|+|T_0|+R)^2;$$

hence

$$G=(T_2-\bar{T}_2)^2-A^2-B^2 \geqq (T_4-T_0+p(r_2, r_0)+p(r_2, r_4))^2-(T_4-T_0+R)^2>0$$

if

$$R<p(r_2, r_0)+p(r_2, r_4)=\psi(r_2, r_0)(r_2-r_0)+\psi(r_2, r_4)(r_2-r_4),$$

where $r_2=r_{\max}$. Thus if R is smaller than some positive value R_{crit} (which depends on r_0 and r_4), $G(r_{\max})>0$.

To summarize, we have $G(r_{\max})>0$ always when $r_{\max}=\infty$ and for sufficiently small R when $r_{\max}$ is finite.

To complete the existence proof we must consider $G(r_{\min})$. Now $0 \leqq r_{\min}<1$, so that (r_0, r_2) and (r_4, r_2) are in regions ② or ③ in Figure 9. In any case, $r_2<r_0$ and $r_2<r_4$, so

$$\begin{aligned} T_2 &= -T_0+\chi(r_2, r_0), \\ \bar{T}_2 &= -T_4-\chi(r_2, r_4) \end{aligned}$$

and

$$\chi(r_2, r_i)<0.$$

Therefore $(T_2-\bar{T}_2)^2=(-T_0+T_4+\chi(r_2, r_0)+\chi(r_2, r_4))^2<(T_4-T_0)^2$. To find the sign of $G(r_{\min})$ we must compare $(T_2-\bar{T}_2)^2$ with A^2+B^2. Now the two-vector $\langle A, B\rangle$ defined in (10) is the vector sum of three vectors of lengths T_4, $|T_0|$ and R, as indicated in Figure 10. It is clear that under some circumstances (for example $\theta_0=\theta_4$ and $|w-\theta_0| \leqq \pi/2$), we will have $A^2+B^2 \geqq (T_4-T_0)^2$ and therefore $G(r_{\min})<0$. In other circumstances (*e.g.*, r_0 and r_4 close to $r_{\min}$, $\theta_4-\theta_0 \simeq \pi$, R small), the functions $\chi(r_2, r_i)$ will be close to zero and $(T_2-\bar{T}_2)^2 \sim (T_4-T_0)^2$ but $A^2+B^2 \sim (T_4-|T_0|)^2$ and so $G(r_{\min})>0$. Some bounds on Γ, the region in which U_0 and U_4 should lie for the Riemann problem to have a solution, can be found by setting $R=0$. Then it is found that $G(r_{\min})<0$ if

$$\cos(\theta_0-\theta_4) \geqq 1-\frac{\chi(\chi-2T_0+2T_4)}{2T_0T_4},$$

where $\chi=\chi(r_{\min}, r_0)+\chi(r_{\min}, r_4)<0$, and

$$T_0=\begin{cases}-S_0, r_0<1, \\ \chi(1, r_0)-1, r_0>1,\end{cases} \qquad T_4=\begin{cases}S_4, r_4<1, \\ 1-\chi(1, r_4), r_4>1.\end{cases}$$

By the Schwarz inequality,

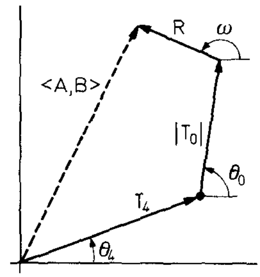

Figure 10

$$|\chi(r, r_i)|^2 \leqq (r_i - r_0)(r_i\,\phi(r_i) - r\,\phi(r)) \leqq r_i^2\,\phi(r_i) = S_i^2,$$

so for r_0 and r_4 larger than $r_{\min}$, a positive sector, $|\theta_0 - \theta_4| < C_{r_0, r_4}$, exists in which $G(r_{\min}) < 0$. By the usual continuity argument, $G(r_{\min})$ will remain negative for small values of $R < R_{\text{crit}}$, where R_{crit} may now depend on $\theta_0 - \theta_4$ and on ω.

Since $G(r_{\max}) > 0$ for sufficiently small R independent of θ_0, θ_4 and ω, we see that there is a subset Γ of $\mathbb{R}^4 \times \mathbb{R}^4$ in which the Riemann data U_0, U_4 must lie in order for a U_2 satisfying the above equations to exist; whenever U_2 exists it is unique. Lower bounds on Γ are given by $r_{\min} \leqq r_0, r_4 \leqq r_{\max}$; $|\theta_0 - \theta_4| < C_{r_0, r_4}$ and $R < R_{\text{crit}}(r_0, r_4, \theta_0 - \theta_4, w)$.

Corollary. *A pair of initial states U_0, U_4 belong to Γ if and only if $G(r_{\max}) > 0$ and $G(r_{\min}) < 0$ where G is the function defined in equation* (12).

In the proof we have considered the case $(r\,\phi)'' > 0$ explicitly. The case $(r\,\phi)'' < 0$ requires that the roles of ψ and χ be reversed, and the regions defined in Figure 9 are reflected in the diagonal; then the properties of T_0, T_4, T_2 and $\tilde{T}_2$ are the same and the analogous calculation on G leads to the same conclusion on the existence of solutions.

It should be mentioned that r_2 is a continuous function of the data U_0 and U_4. However, for values of r_2 near r_0^* or r_4^* (solutions with these values can easily be constructed), the states U_1 or U_3 may suddenly change, just as did the intermediate state in the model problem in Section 2.

5. Conclusions

In this paper we have looked at a class of non-strictly hyperbolic conservation laws characterized by a relatively simple flux $F(U)$ which enabled us to find the subspaces on which equal characteristic speeds occurred and to construct solutions to the Riemann problem, at least in a neighborhood of the non-strictly hyperbolic points. In an application of interest, shocks on an elastic string, we found conditions for the existence of solutions to the Riemann problem but could not give *a priori* bounds to determine which intermediate states would occur. All the solutions we found have the property, though, that for certain states U_l and U_r, the intermediate states are extremely sensitive to small changes in U_l and U_r. This implies, among other things, that the Glimm difference scheme for solving the Cauchy problem cannot be applied without some changes, including perhaps some modification to the total variation norm used in establishing convergence in [6] and other papers. This, in turn, may affect estimates on asymptotic decay of solutions to such systems. It would be interesting to obtain results along these lines.

In another direction, our results relate to other equations displaying "pathological" behavior. KORCHINSKI [10] studied a non-strictly hyperbolic model equation in which the "linear" family had solutions containing δ-functions. We also demonstrated in Section 3 that a problem which has no solution in a hyperbolic region may be solvable if the definition of the problem is extended to include both sides of the parabolic line. Some examples of hyperbolic problems for which the Riemann problem has no solution were given by BOROVIKOV in [1]; it would be interesting to see whether many such problems arise because of a parabolic degeneracy that is artificially used to limit the domain of the solutions. To study this problem it will be necessary to extend the construction of the present paper to systems which are genuinely nonlinear in both families. Model problems [9] we have considered indicate that this is possible in some cases.

Appendix A: Condition for Evolutionary Shock Solutions

To distinguish physically meaningful weak solutions it is often required that they be evolutionary. The solution

$$U(x,t)=\begin{cases} U_0, & x<st, \\ U_1, & x>st \end{cases} \tag{A.1}$$

of the Riemann problem

$$U_t+F(U)_x=0, \tag{A.2}$$

$$U(x,0)=\begin{cases} U_0, & x<0, \\ U_1, & x>0 \end{cases} \tag{A.3}$$

is called *evolutionary* if the perturbed problem obtained by adding a small viscous damping term εU_{xx} to the right-hand side of (A.2) has a solution which is close to (A.1) except in a narrow band around the line $x=st$. A system of two genuinely nonlinear, strictly hyperbolic conservation laws is evolutionary if and only if the Lax entropy condition holds. Here we extend this equivalence to our system (1). Specifically, we prove that, under appropriate conditions on the function ϕ, both the ordinary ($U_1 \in \mathrm{I}$) and anomalous ($U_1 \in \mathrm{IV}$) shocks described in Section 1 are evolutionary.

Theorem. *In the notation of Section* 1, *assume that* $U_1 \in H(U_0)$ *satisfies either*

$$U_1 \in \mathrm{I} \quad \text{with } r_1<r_0 \, [r_1>r_0 \text{ if } (r\phi)_{rr}<0] \tag{A.4}$$

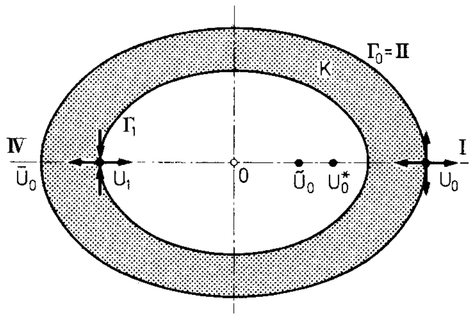

Figure A-1

or

$$U_1 \in (\bar{U}_0, W) \subset \mathrm{IV} \tag{A.5}$$

and the R-curves $\phi = \text{const.}$ *through* U_0 *and* U_1 *are convex.*

Then for $\varepsilon > 0$ *the equation* $U_t + F_x = \varepsilon U_{xx}$ *possesses a traveling wave solution* $U(x, t; \varepsilon) = w\left(\frac{x - s\,t}{\varepsilon}\right)$ *with* $w(-\infty) = U_0$ *and* $w(+\infty) = U_1$.

Proof. We consider explicitly the case $(r\,\phi)_{rr} > 0$; the other case is proved in similar fashion. As in CONLEY & SMOLLER's work [4], we let $\xi = (x - s\,t)/\varepsilon$ and find

$$w'(\xi) = V(w) = F(w) - s\,w + C \tag{A.6}$$

where $C = s\,U_0 - F(U_0) = s\,U_1 - F(U_1)$. Now $V(U_0) = V(U_1) = 0$ and if (A.4) holds, the solution of (A.6) reduces to $w = \tau(\xi)\,U_0$, where τ satisfies the scalar equation

$$\frac{d\tau}{d\xi} = \tau\,\phi(\tau\,U_0) - \phi(U_0) - s(\tau - 1),$$

with $\tau(-\infty) = 1$, $\tau(+\infty) = r_1/r_0$. The convexity of $r\,\phi$ on each radial line is assurance that the right-hand side is negative for $r_1/r_0 < \tau < 1$ and thus guarantees a solution.

When (A.5) holds, we must look at the singularities of $V(w)$ in $\mathbb{R}^2 - \{0\}$. Any singularities must be points U on the Hugoniot locus $H(U_0)$ with the additional restriction that $s^* \equiv s(U, U_0)$ is equal to s. We consider the case $U_0 \in D^+ (r_0 > r_0^*)$ illustrated in Figure A-1; the case $U_0 \in D^-$ is treated similarly. Since $s^* = \phi(U_0)$ for U on II and III while $s < \phi(U_0)$, we see that no singularities occur on II or III. On I, $s^* = (r\,\phi - r_0\,\phi_0)/(r - r_0)$ is monotonic, so there is besides U_0 at most one other singularity $\tilde{U}_0$ on I, which must lie in $(0, U_0^*)$, since $s^*(U_0^*) = \phi_0 > s$. On IV, $s^* = (r\,\phi + r_0\,\phi_0)/(r + r_0)$, so $\partial s^*/\partial r = (\lambda_2 - s^*)/(r + r_0) > 0$ in $[U_1, W)$. Since $s^* = s$ at U_1 and $s^* = \phi_0 > s$ at $\bar{U}_0^*$, there are singularities at U_1 and at some point $\tilde{U}_1 \in (W, \bar{U}_0^*)$.

Consider now the closed annulus K bounded by the R-curves $\Gamma_0 = \mathrm{II}$ through U_0 and Γ_1 through U_1. The only singularities of V in K are U_0 and U_1. Since

$\partial V/\partial w = \partial F/\partial U - s = A - s$ has eigenvalues $\mu_1 = \lambda_1 - s$ and $\mu_2 = \lambda_2 - s$, U_0 is an unstable node with $\mu_2 > \mu_1 > 0$, while U_1 is a saddle with $\mu_2 > 0 > \mu_1$. Furthermore, we may calculate from (A.6) that $V(w)$ is a positive scalar multiple of $w - U_0$ on Γ_0 and of $U_1 - w$ on Γ_1. Because of the assumed convexity of Γ_0 and Γ_1, this means that the vector field V points out of the region K along its entire boundary except perhaps at U_0 and U_1. Thus every solution curve of equation (A.6) within K must have entered at U_0 or U_1. Since only one trajectory enters K from the saddle U_1, infinitely many solution curves must enter K from U_0, and these fall into two classes according as they leave K through Γ_0 or Γ_1. By continuity, there must be at least one trajectory T originating at U_0 which separates the two classes, and T must leave K at a singularity. Since U_0 is an unstable node, T must run to U_1. Thus T represents a solution of (A.6) with $w(-\infty) = U_0$, $w(+\infty) = U_1$. ∎

Appendix B: An "Entropy" Inequality for the Nonlinear Elastic String

Another criterion for evolutionarity of weak solutions is the existence of a concave functional of the solution, usually called an entropy, which is constant for smooth solutions but increases in time when discontinuities are present. See [11]. We can associate with the problem of the elastic string of Section 4 the *total energy* (kinetic and strain), which is non-increasing for any of the discontinuities we have allowed. Hence its negative will serve as an "entropy." The strain energy is a function of the stress; for a given strain ε which determines, in our problem, a value of $r = \sqrt{w_x^2 + v_x^2}$, we may define

$$\Phi(r) = \int T(\varepsilon)\, d\varepsilon = \int r\, \phi(r)\, dr$$

to be the stored energy function, and

$$E(t) = \int_{-L}^{L} e(x,t)\, dx = \int_{-L}^{L} \{1/2(w_t^2 + v_t^2) + \Phi(\sqrt{w_x^2 + v_x^2})\}\, dx$$

to be the total energy in a length $2L$ of the string. If the motion has compact support in $(-L, L)$, we find that

$$\frac{dE(t)}{dt} = 0$$

for smooth solutions. If there is a shock at $x = \zeta(t)$, then

$$E(t) = \int_{-L}^{\zeta(t)} e(x,t)\, dx + \int_{\zeta(t)}^{L} e(x,t)\, dx$$

and

$$\frac{dE}{dt} = -[e]\, s - [\phi(w_t w_x + v_t v_x)]$$

where $s = \dot{\zeta}$ and $[f]$ means the jump across the shock, $f(\zeta^+) - f(\zeta^-)$; here $f(\zeta^+) = \lim_{x \to \zeta(t)+0} f(x,t)$, and so on.

Thus decrease of energy is equivalent to $[e]s+[\phi(w_t w_x+v_t v_x)]\geqq 0$ for a shock of speed s.

Across a contact discontinuity, $\phi^- = \phi^+ = s^2$, $[w_x^2+v_x^2]=0$, so $[e]=\frac{1}{2}[w_t^2+v_t^2]$; since $[w_x]=r[\cos\theta]$, $[v_x]=r[\sin\theta]$ and $w_t^+=w_t^- - s\,r[\cos\theta]$, it can be verified by calculation that $\dfrac{dE}{dt}=0$.

Across a shock, θ is constant and we select $\theta=0$ for convenience; then

$$s^2=\frac{[\phi r]}{[r]}, \quad v_x^+=v_x^-=0, \quad w_x^\pm=r_\pm, \quad [w_t]=-s[w_x]$$

and

$$[v_t]=-s[v_x]=0.$$

Hence dE/dt is calculated to be

$$-s[r]\left\{\frac{1}{r_+-r_-}\int_{r_-}^{r_+} r\,\phi(r)\,dr-\frac{\phi_+ r_+ +\phi_- r_-}{2}\right\}.$$

Now if $r\phi$ is convex (that is, $(r\phi)_{rr}>0$), the quantity in braces is negative; if $r\phi$ is concave, the quantity is positive. Thus the sign of dE/dt is the sign of $s[r]$ $(r\phi)_{rr}$ and we see that $E(t)$ is decreasing precisely in the cases we have called entropy shocks.

It is also interesting to investigate the sign of dE/dt for anomalous shocks which are constructed as in Section 1. Here $[\theta]=\pm\pi$, and for simplicity we select $\theta_+=0$ and $\theta_-=\pi$. We then have

$$s^2=\frac{\phi_+ r_+ +\phi_- r_-}{r_+ + r_-} \quad \text{and} \quad w_x^\pm=\pm r_\pm;$$

all else is as above. We find

$$\begin{aligned}\frac{1}{s}\frac{dE}{dt}&=\frac{1}{2}(r_+ + r_-)(r_+\phi_+ - r_-\phi_-)-\int_{r_-}^{r_+} r\,\phi(r)\,dr\\ &=\left\{\frac{1}{2}(r_+ - r_-)(r_+\phi_+ + r_-\phi_-)-\int_{r_-}^{r_+} r\,\phi(r)\,dr\right\}+r_+ r_-[\phi]\end{aligned}$$

Now for an anomalous forward shock $\phi_- > s > \phi_+$, so $[\phi]<0$. If $[r](r\phi)_{rr}<0$, however, the expression in braces is also negative, for it is identical with the value of $\dfrac{1}{s}\dfrac{dE}{dt}$ obtained in the preceding paragraph for an ordinary shock joining $(r_-,0)$ to $(r_+,0)$. If, on the other hand, $[r](r\phi)_{rr}>0$, so that $(r_-,0)$ and $(r_+,0)$ would normally be joined by a rarefaction wave, the expression in brackets, though positive, is only of third order in $[r]$, while the term $r_+ r_-[\phi]$ is negative and of first order, and so predominates. Thus in both cases $dE/dt<0$, and the anomalous shock on the elastic string is evolutionary.

B. L. KEYFITZ's research was partially supported by the U.S. National Science Foundation under Grant MCS-77-04164 at Princeton University; that of H. C. KRANZER, by a summer research grant from Adelphi University.

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Department of Mathematics
Arizona State University
and
Department of Mathematics
Adelphi University

(Received June 29, 1979)