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A VISCOSITY APPROXIMATION TO A SYSTEM OF CONSERVATION LAWS WITH NO CLASSICAL RIEMANN SOLUTION

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ABSTRACT: There are examples of systems of conservation laws which are strictly hyperbolic and genuinely nonlinear but for which the Riemann problem can be solved only for states which are sufficiently close together. For one such example, we introduce a particular type of artificial viscosity and show how it suggests a possible definition of "generalized" solution to the Riemann problem.

I. INTRODUCTION

The model system

$$\begin{cases} u_t + (u^2 - v)_x = 0 \\ v_t + (\frac{1}{3}u^3 - u)_x = 0 \end{cases} \tag{1}$$

$$U(x,0) = \begin{pmatrix} u \\ v \end{pmatrix}(x,0) = \begin{cases} U_L, & x < 0 \\ U_R & x \geq 0 \end{cases} \tag{2}$$

presents an example of a system of conservation laws satisfying the classical assumptions (strict hyperbolicity and genuine nonlinearity)

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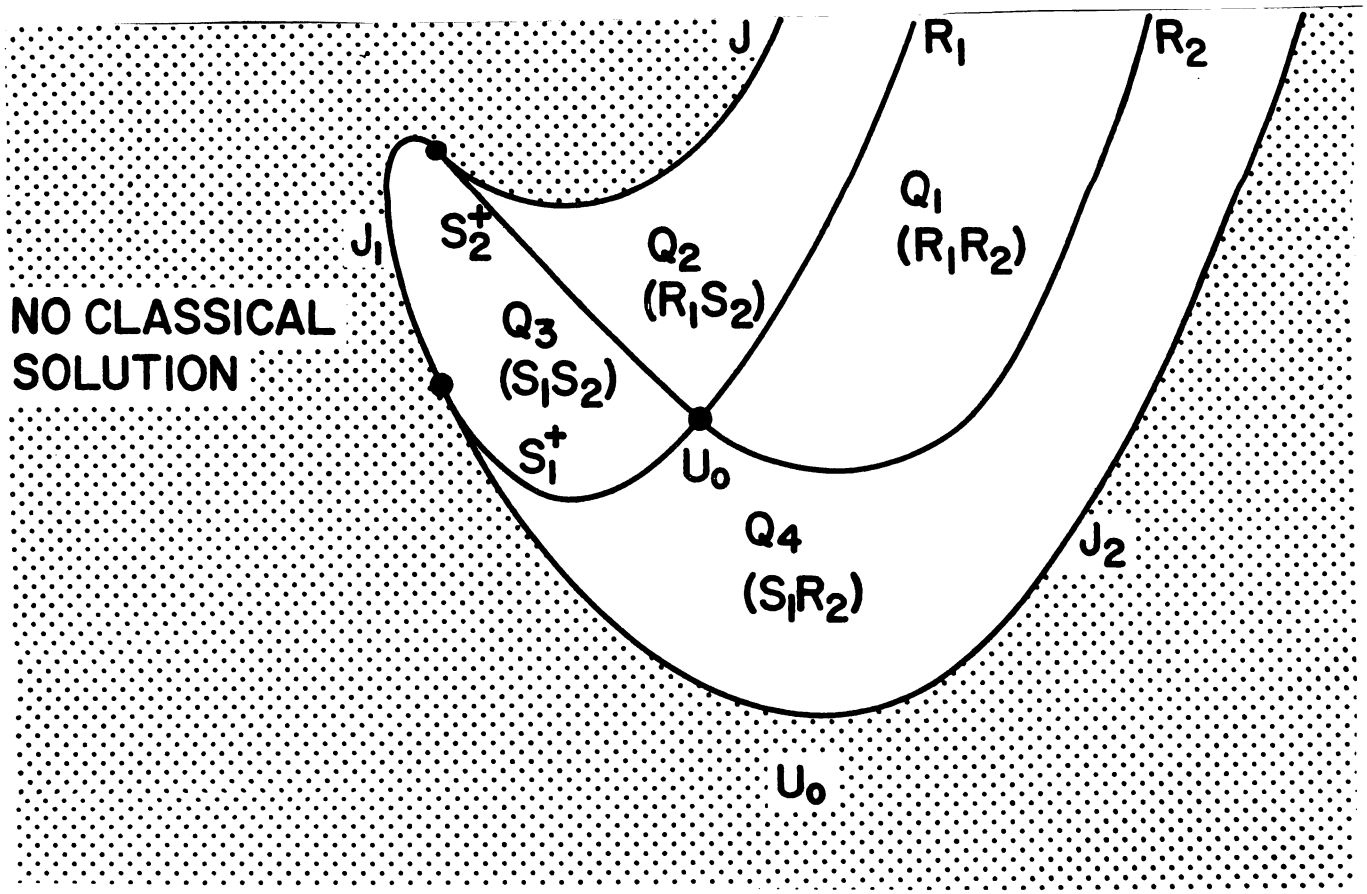

Figure 1

which has no solution for some pairs of states U_L and U_R. On carrying out the standard construction of a solution to the Riemann problem (a shock or rarefaction of the slower family followed by a wave of the second), one finds that for a given left state, $U_L = U_0$, the classical type of solution exists in the four curvilinear quadrants Q_1, Q_2, Q_3, and Q_4 pictured in Figure 1. The defining equations are simple polynomial or algebraic curves. If U_R is outside the union, Q, of these sets, then no classical solution of (1), (2) exists.

If one is searching for conditions sufficient to guarantee existence for large data of solutions to systems of conservation laws, then it is interesting to try to understand how the solution to the Riemann problem breaks down for a system like (1). This motivates the study of systems that approximate (1) in some sense. One idea, first developed by Tupciev [9] and by Dafermos [2], is to approximate a system of conservation laws by the system

$$U_t + F_x = \varepsilon t U_{xx} . \tag{3}$$

The *initial*-value problem (3), (2) becomes a two-point boundary-value problem for a nonautonomous system of ordinary differential equations in the variable $\xi = x/t$ (we let $\dot{} = d/d\xi$):

$$\varepsilon U = (A(U) - \xi)\dot{U} , \tag{4}$$

$$U(\xi) \to \begin{cases} U_L & \text{as } \xi \to -\infty \\ U_R & \text{as } \xi \to +\infty \end{cases} . \tag{5}$$

In this note, we study the solutions of (4), (5) for small ε using perturbation methods; we find that this system admits solutions with the structure of "singular shocks": boundary layers in which are embedded more singular solutions which are unbounded as $\varepsilon \to 0$. In a future paper [5], we will show, using the construction of Dafermos [2], that solutions of (4), (5) actually exist in a rigorous and not just an asymptotic sense, and that they have the qualitative properties exhibited here. By means of the construction presented here, we can identify "shock speeds" and "shock strengths", which we use to define a "generalized" Riemann solution.

Another source of interest in (1) is that it arises in some model equations for a problem in elastoplasticity considered by Colombeau and Le Roux in [1], when these equations are put in conservation form. Colombeau and Le Roux solve (numerically) the system

$$\begin{cases} u_t + uu_x = \sigma_x \\ \sigma_t + u\sigma_x = u_x \end{cases} \tag{1'}$$

which is related to (1) by the change of variables

$$\sigma = v - \frac{u^2}{2} .$$

The weak solutions of (1) and (1′), even if of classical type in both cases, would be inequivalent. Le Floch, in [6], establishes a theoretical framework for the study of "nonconservation laws" of this type, and formulates admissibility conditions for generalized shocks. Le Floch's method, applied to (1′), also gives a solution only in one

quadrant of the plane, while the solution obtained by Colombeau and Le Roux is highly dependent on the specific form of the numerical scheme. However, the theory of generalized functions developed by Colombeau and used in [1] provides a tool whereby the relation of the approximation (4), (5) to the limiting system (1), (2) might be studied. This will be the subject of a future study.

II. THE SINGULAR SOLUTION

We consider the system (3). This approximation was designed for looking at solutions to the Riemann problem (initial data (2)), because it has solutions of the form $U = U^{\varepsilon}(x/t) = U^{\varepsilon}(\xi)$. In fact, Dafermos [2] proved a convergence result, as $\varepsilon \to 0$, for Riemann problems for 2×2 systems with enough restrictions that a global, bounded solution to the inviscid problem exists, and Dafermos and DiPerna [3] extended the convergence result to include the case of isentropic gas dynamics where the solution may be unbounded. The example we are considering in this paper does not fit into either of these categories. The existence of the approximate solutions and their convergence will be discussed in [5].

To simplify the notation, we drop the superscript ε when we look at solutions to the system (4), (5) of ordinary differential equations. Note that, unlike the systems generally used in studying viscous approximation to a single shock, this system depends explicitly on ε and on ξ.

The classical solutions to (4) are approximations either to <u>rarefactions</u>, in which U, $\dot{U}$ and U are bounded as ε 0, or to <u>shocks</u>, in which U is bounded but $\dot{U}$ and U are not. We do not expect (4), (5) to have a classical solution unless $U_R \in Q(U_L)$. We consider the possibility that <u>singular solutions</u> of (4) exist, in which U is unbounded for ξ near some value, s. Thus, let

$$\tilde{U}(\xi) = \begin{pmatrix} \frac{1}{\varepsilon^p}\tilde{u}\left(\frac{\xi-s}{\varepsilon^q}\right) \\ \frac{1}{\varepsilon^r}\tilde{\psi}\left(\frac{\xi-s}{\varepsilon^q}\right) \end{pmatrix} . \tag{6}$$

If we let $\eta = \frac{\xi-s}{\varepsilon^q}$, $' = \frac{d}{d\eta}$, then (4) becomes

$$\varepsilon^{1-q-p}\,\tilde{u}'' = (2\tilde{u}\,\varepsilon^{-p} - \varepsilon^{q}\eta - s)\tilde{u}'\varepsilon^{-p} - \varepsilon^{-r}\tilde{v}'$$

$$\varepsilon^{1-q-r}\,\tilde{v}'' = (\tilde{u}^2\,\varepsilon^{-2p} - 1)\tilde{u}'\varepsilon^{-p} - (\varepsilon^{q}\eta + s)\varepsilon^{-r}\tilde{v}' \ .$$

For nontrivial solutions to exist we must balance at least two terms in each equation. Thus we set $1 - q - r = -3p$ in the second and either $1 - q - p = -2p$ or $1 - q - r = -r$ in the first; either implies the other and yields $r = 2p$, $q = 1 + p$ and hence

$$\begin{cases} \tilde{u}'' = 2\tilde{u}\tilde{u}' - \tilde{v}' - \varepsilon^{p}(s\tilde{u}' + \varepsilon^{p+1}\eta\tilde{u}') \\ \tilde{v}'' = \tilde{u}^2\tilde{u}' \qquad - \varepsilon^{p}(s\tilde{v}' + \varepsilon^{p}\tilde{u}' + \varepsilon^{p+1}\eta\tilde{v}' \ ) \end{cases} . \tag{7}$$

Now we expand $\tilde{u}$, $\tilde{v}$ as series in ε:

$$\tilde{u} = \tilde{u}_0(\eta) + o(1) \qquad\qquad \tilde{v} = \tilde{v}_0(\eta) + o(1) \ ,$$

to obtain

$$\begin{cases} \tilde{u}_0'' = 2\tilde{u}_0\tilde{u}_0' - \tilde{v}_0' \\ \tilde{v}_0'' = \tilde{u}_0^{\,2}\tilde{u}_0' \end{cases} . \tag{8}$$

We note that from (6) we must have $\tilde{u}_0$, $\tilde{v}_0 \to 0$ as $|\eta| \to \infty$, under the assumption that the singular behavior is confined to a neighborhood of $\xi = s$ (or $\eta = 0$); hence $\tilde{u}_0'$, $\tilde{v}_0' \to 0$ as $|\eta| \to \infty$ and when we integrate (8) once we obtain

$$\begin{cases} \tilde{u}_0' = \tilde{u}_0^{\,2} - \tilde{v}_0 \\ \tilde{v}_0' = \frac{1}{3}\tilde{u}_0^{\,3} \end{cases} . \tag{9}$$

Simplifying the notation again, we see that we wish to study solutions of

$$\begin{cases} x' = x^2 - y \\ y' = \frac{1}{3}x^3 \end{cases} \tag{10}$$

which approach $(0,0)$ as $|\eta|$ ∞. The linearization of (10) at $(0,0)$ gives the matrix

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

which is nilpotent (double zero eigenvalue). This apparently nongeneric behavior is stable: if we had begun with

$$\begin{cases} u_t + (f(u) - v)_x = 0 \\ v_t + \quad (g(u))_x = 0 \end{cases}$$

and had assumed $f \to u^2$, $g \to \frac{1}{3} u^3$ as $|u| \to \infty$, in following the derivation of equations (6), (7), and (8), we would have ended up with (9). The system is invariant under the group action $x \to -x$, $\eta \to -\eta$.

Now, it happens that we can integrate (10). Let

$$z = y - kx^2$$

where k is either root of

$$k^2 - k + \frac{1}{6} = 0 \ .$$

Then

$$z' = 2kxz$$

and

$$\frac{d}{d\eta}\left(x^2 z^m + \frac{z^{m+1}}{2k-1} \right) = 0 \ ,$$

where $m = \frac{k-1}{k}$. Hence, along trajectories,

$$z^m\left(z + (2k - 1)x^2 \right) = \text{constant.}$$

We see that the two parabolas $y = k_1 x^2$,

$$k_1 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}} \right) \qquad\qquad k_2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}} \right) ,$$

are trajectories of the flow $(z = 0)$. By choosing $k = k_1$, we see that if $z(0) = y(0) - k_1(x(0))^2 > 0$ then the trajectory is bounded: it is, in fact, a homoclinic orbit in the upper half plane, making second-order contact with $y = k_1 x^2$. For all these orbits, we may choose the symmetric solution: $x(0) = 0$, $x(-\eta) = - x(\eta)$; then each orbit is characterized by $y(0) = z(0) = y_{max} > 0$. We also have

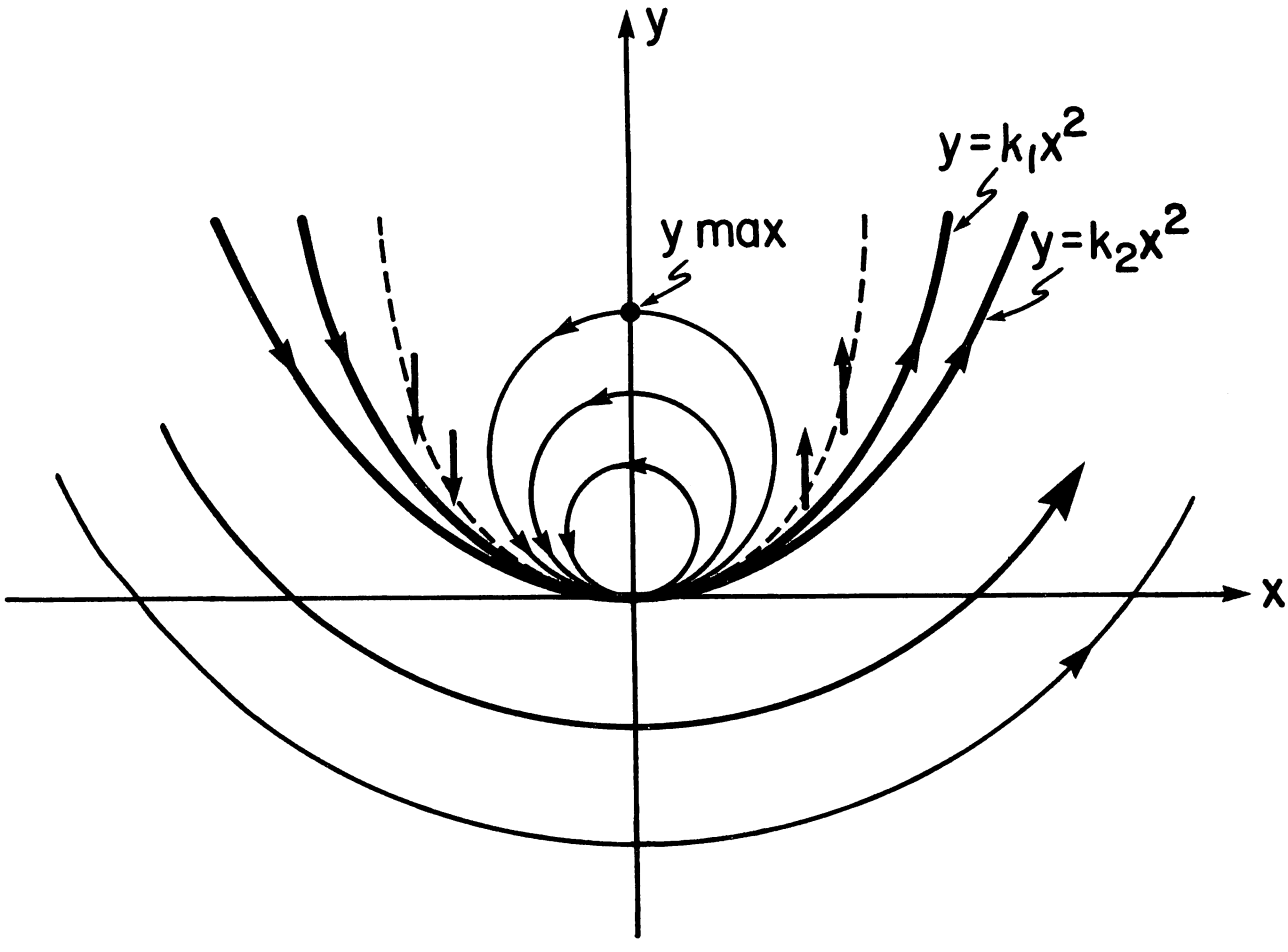

Figure 2

$y(-\eta) = y(\eta)$. The asymptotic behavior is

$$x = \frac{c}{\eta} + O\left(\frac{1}{\eta^2}\right)$$

$$y = \frac{d}{\eta^2} + O\left(\frac{1}{\eta^3}\right)$$

where $c = -6k_1 = -3 - \sqrt{3}$ and $d = c^2 + c = 9 + 5\sqrt{3}$ for all the orbits. In the sectors between the parabolas $y = k_1x^2$ the flow is radially inward (if $x < 0$) or outward (if $x > 0$); below $y = k_2x^2$ the flow follows unbounded trajectories from left to right. This is summarized in Figure 2. Finally, we identify $(\tilde{u}_0, \tilde{v}_0)$ with (x,y) for

one of the homoclinic orbits. We see that there are many singular solutions of (9) and hence of type (6).

III. COMPLETION OF THE BOUNDARY LAYER SOLUTION

The singular solution constructed in Section II has its essential support in a layer of width $|\xi - s| = \mathcal{O}(\varepsilon^q) = \mathcal{O}(\varepsilon^{p+1})$. Since $p > 0$, this solution is narrower than a classical shock (which has width $|\xi - s| = \mathcal{O}(\varepsilon)$); also, far away from $\xi = s$, it tends to zero. Thus, by itself it does not solve any Riemann problems. We are led to the idea of embedding a singular shock in a shock profile of the usual type: a solution $\bar{U}(\tau) = \bar{U}(\frac{\xi-s}{\varepsilon})$ of (4) which is bounded in a layer $|\xi - s| = \mathcal{O}(\varepsilon)$ outside the singular layer and whose derivatives are $\mathcal{O}(\frac{1}{\varepsilon})$ outside the singular layer. We shall call this two-component region, $\varepsilon^{p+1} < |\xi - s| < \varepsilon$, the <u>boundary layer</u>. Now, in terms of $\tau = \frac{\xi-s}{\varepsilon}$, equation (4) can be written

$$\frac{d^2\bar{U}}{d\tau^2} = (A(\bar{U}) - \varepsilon\tau - s)\frac{d\bar{U}}{d\tau},$$

or

$$\frac{d}{d\tau}\left(\frac{d\bar{U}}{d\tau} - F(\bar{U}) + s\bar{U}\right) = -\varepsilon\tau\frac{d\bar{U}}{d\tau}. \tag{11}$$

We illustrate the scalings in Figure 3. If we expand $\bar{U} = \bar{U}_0 + o(1)$ in the boundary layer, then, since by assumption the right hand side of (11) is $\mathcal{O}(\varepsilon)$ there, we have

$$\frac{d}{d\tau}\left(\frac{d\bar{U}_0}{d\tau} - F(\bar{U}_0) + s\bar{U}_0\right) = 0$$

in each separate interval of the boundary layer, $\tau < 0$ and $\tau > 0$, and so

$$\frac{d\bar{U}_0}{d\tau} - F(\bar{U}_0) + s\bar{U}_0 = C_{\mp}; \tag{12}$$

the two constants being constants of integration in the two intervals. Furthermore, integrating (11), we have

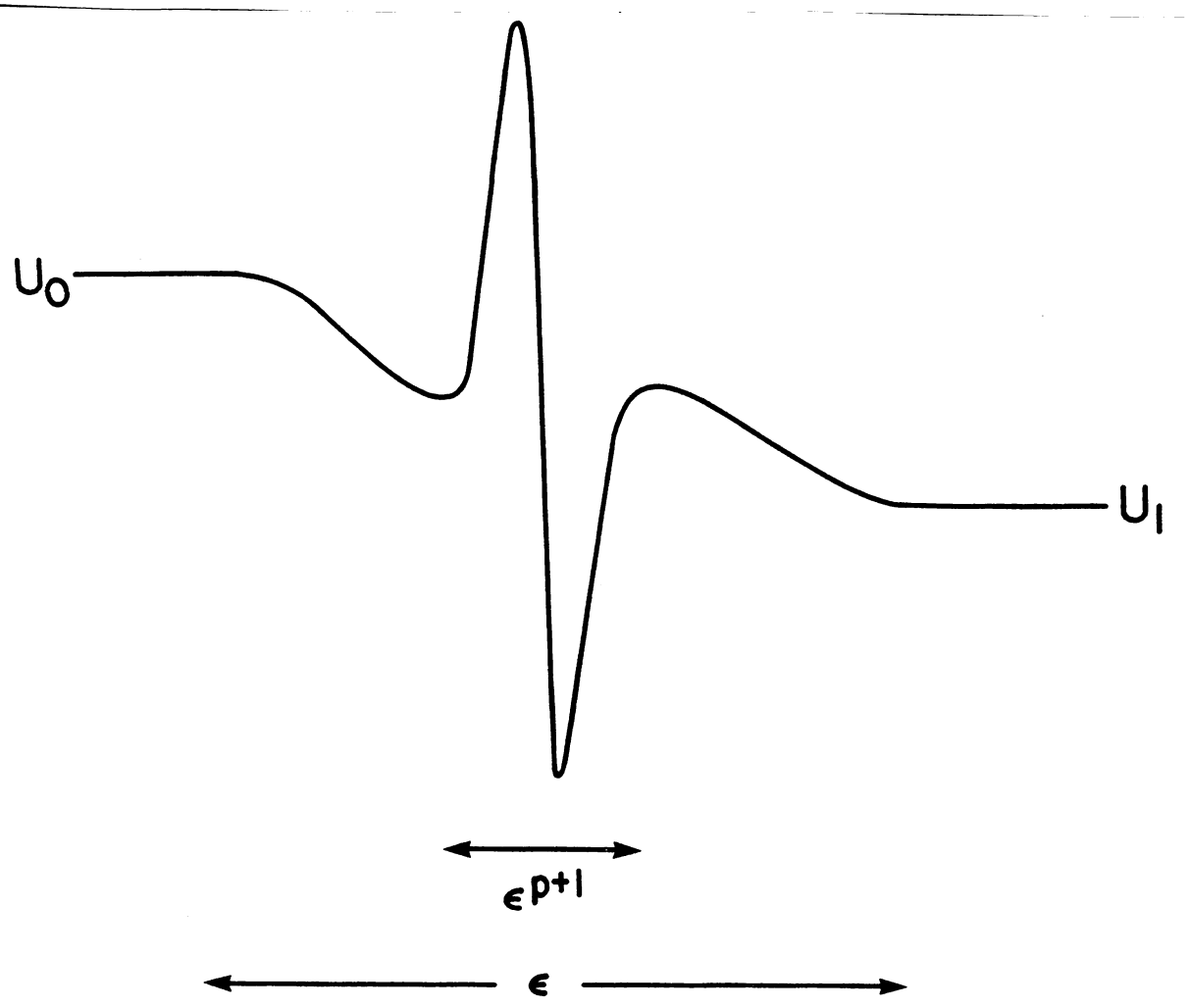

Figure 3

$$\left\{ \frac{d\bar{U}}{d\tau} - F(\bar{U}) + s\bar{U} \right|_{\tau<0}^{\tau>0} = -\varepsilon \int_{\tau<0}^{\tau>0} \tau \frac{d\bar{U}}{d\tau}\, d\tau. \tag{13}$$

Now, let $\bar{U}(\tau) = \tilde{U}(\xi)$ in the integrand on the right side of (13); then, using (12), we see that

$$C_+ - C_- = \lim_{\varepsilon\to 0}\left(-\varepsilon \int_{-\infty}^{\infty} \varepsilon^p \eta \begin{pmatrix} \frac{1}{\varepsilon^p}\tilde{u}' \\ \frac{1}{\varepsilon^{2p}}\tilde{v}' \end{pmatrix} d\eta \right) = \lim_{\varepsilon\to 0}\begin{pmatrix} -\varepsilon \int \eta \tilde{u}_0' d\eta \\ -\varepsilon^{1-p}\int \eta \tilde{v}_0' d\eta \end{pmatrix}. \tag{14}$$

Now, $\tilde{u}_0' \approx \frac{c}{\eta^2}$ when $|\eta| \to \infty$, so $\eta\tilde{u}_0'$ is not absolutely integrable. However, it is an odd function, so its PV integral is zero. On the other hand, $\int \eta \tilde{v}_0' d\eta$ exists for all the homoclinic trajectories, and

$$\int_{-\infty}^{\infty} \eta\tilde{v}_0' d\eta = \eta\tilde{v}_0\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \tilde{v}_0 d\eta = - \int_{-\infty}^{\infty} \tilde{v}_0 d\eta$$

has a different finite value for each trajectory. Thus we get a nontrivial result in (14) if $p = 1$. Then

$$C_+ - C_- = \begin{pmatrix} 0 \\ c \end{pmatrix}, \quad \text{where} \quad c = \int_{-\infty}^{\infty} \tilde{v}_0 d\eta \ . \tag{15}$$

Finally, a shock in the boundary layer which approaches constant values, $\bar{U}_0 \to U_\pm$ as $\tau \to \pm\infty$, and $d\bar{U}_0/d\tau \to 0$ as $|\tau| \to \infty$, must satisfy, from (12),

$$sU_- - F(U_-) = C_- , \qquad\qquad sU_+ - F(U_+) = C_+$$

and hence

$$s(U_+ - U_-) - (F(U_+) - F(U_-)) = C_+ - C_- = \begin{pmatrix} 0 \\ c \end{pmatrix} . \tag{16}$$

This is the <u>Generalized</u> <u>Rankine-Hugoniot</u> <u>Condition</u> for singular shocks.

Which states (U_-,U_+) can be joined by an <u>admissible</u> singular shock? That is, when does there exist a trajectory $\bar{U}(\tau)$ joining U_- and U_+? We note that for any pair of states (u_-,v_-) and (u_+,v_+), we have solutions to (16) given by

$$s = \frac{-[-v]+[u^2]}{[u]} = u_+ + u_- - \frac{v_+-v_-}{u_+-u_-} \tag{17}$$

and

$$c = s[v] - \tfrac{1}{3}[u^3] - [u] = [v]\left(\frac{[u^2]-[v]}{[u]}\right) + [u] - \tfrac{1}{3}[u^3] \ .$$

But for trajectories to exist, we need at least one positive eigenvalue at U_- and one negative eigenvalue at U_+ in the linearized matrix $A(U) - sI$. We conjecture [5], that trajectories exist if and only if there are two such eigenvalues: that is,

$$\lambda_2(u_-) > \lambda_1(u_-) \geq s \geq \lambda_2(u_+) > \lambda_1(u_+). \tag{18}$$

Since s depends on $[v]$, this defines a sector, $S_\delta(U_-)$, for each U_-, by

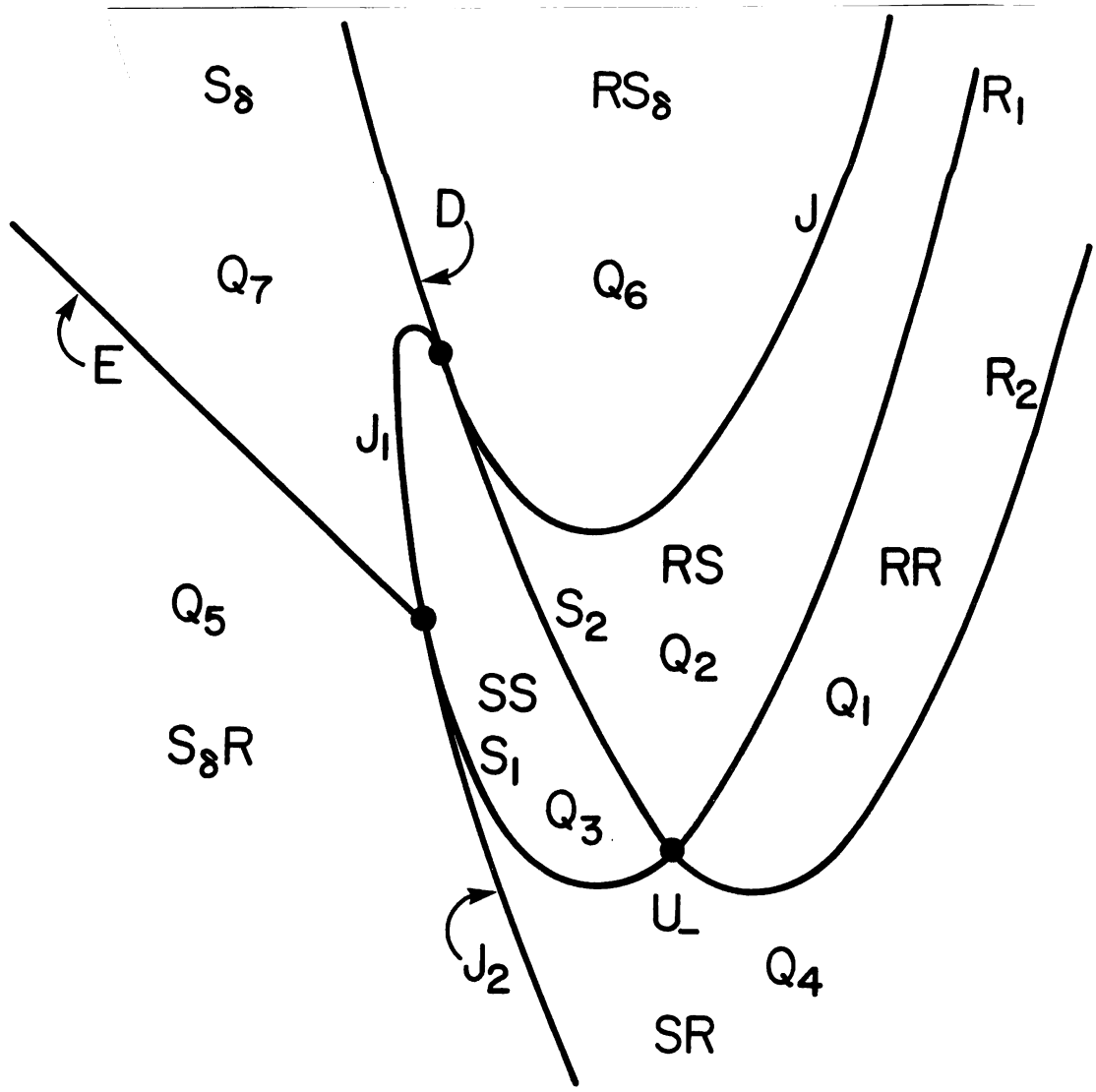

Figure 4

$$S_\delta(U_-) = \{U \notin Q_3 \mid u < u_-;\ u(u_- - 1) - u_-(u_- - 1) \le v - v_- \le u^2 + (1 - u_-)u - u_-\}\ . \tag{19}$$

On the lower boundary, E, of S_δ, $s = \lambda_2(u)$, and thus a singular shock to E can be continued via a 2-rarefaction to any point in a sector Q_5 below E. On the upper boundary, $D(U_-)$, $s = \lambda_1(u_-)$, and so in the region Q_6 above D there is a solution to the Riemann problem consisting of 1-rarefactions followed by singular shocks joining U to U_m, where $U_m \in R_1(U_-)$, and $U \in D(U_m)$. The regions Q_5 and Q_6 adjoin the sectors Q_2 and Q_4, in which there are classical solutions. Thus, letting Q_7 denote the sector S_δ in which the solution to the Riemann problem consists of a singular shock alone, we have described a "generalized Riemann solution" of our original

problem in the entire plane. This is illustrated in Figure 4.

An analytical justification of the asymptotic derivation presented here, consisting of an existence theorem for solutions of (4) and (5) for all $\varepsilon > 0$ and a demonstration of the qualitative behavior of the solutions in sectors Q_5, Q_6 and Q_7 will be the subject of [5].

IV. CONCLUSIONS

We have presented, in (1), an example of a system of conservation laws for which the "large data" Riemann problem may not have a solution. The obstruction to solving the Riemann problem for (1) can be described as a consequence of the (easily verified) compactness of the Hugoniot locus in the u-v plane; this is related (although we have not established how general is the connection) to the fact that the two families of characteristic speeds are not globally distinct: the <u>global</u> character of this system is not that of a strictly hyperbolic problem. In fact, the system

$$\begin{cases} u_t + (u^2 - v)_x = 0 \\ v_t + (\frac{1}{3}u^3 - k^2 u)_x = 0 \end{cases} \tag{20}$$

(which happens to correspond to the example considered in [1]), for which the characteristic speeds are $u \pm k$, can be rescaled ($u \mapsto ku$, $v \mapsto k^2 v$, $x \mapsto kx$) to the form (1), with a similar correspondence of solutions to the Riemann problem. In the limiting case, $k = 0$, (which is "unphysical" in [1], since k is a Hooke's constant), (20) becomes a nonlinear system with globally coincident characteristics. In this case the quadrant, Q, of classical Riemann solutions degenerates to a single, semi-infinite curve.

It is instructive to note that the linearization of this degenerate system about a constant state $u = a$, say, leads to the equation

$$w_{tt} + 2aw_{tx} + a^2 w_{xx} = 0 \tag{21}$$

for $w = u - a$; this equation is of a degenerate type (it might be called weakly hyperbolic at best). The solution of (21) with Cauchy data

$$w(x, 0) = w_0(x), \qquad w_t(x, 0) = v_0(x),$$

is

$$w(x, t) = w_0(x + at) + t[v_0(x + at) - aw_0'(x + at)]. \tag{22}$$

The solution in (22) of the linear problem (21) exhibits both growth in t and loss of a derivative; one can read both these features in the approximate solution constructed asymptotically in this paper: although the limit solution defined by (16) - (19) is described by finite-valued states, the limiting process itself involves a sequence which is unbounded. It does not appear possible to speak of a solution to (1) and (2), even in the weakest sense, without invoking functions which are more singular than the data. Thus, although it is possible that the behavior of the approximate solutions constructed in this paper is strongly affected by the approximation we have chosen, there is also the possibility that this behavior is characteristic of a type of global failure of strict hyperbolicity which should be further investigated.

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