A Free Boundary Problem for a Quasi-linear Degenerate Elliptic Equation: Regular Reflection of Weak Shocks

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Abstract

We prove the existence of a solution to the weak regular reflection problem for the unsteady transonic small disturbance (UTSD) model for shock reflection by a wedge. In weak regular reflection, the state immediately behind the reflected shock is supersonic and constant. The flow becomes subsonic further downstream; the equation in self-similar coordinates is degenerate at the sonic line. The reflected shock becomes transonic and begins to curve there; its position is the solution to a free boundary problem for the degenerate equation. Using the Rankine-Hugoniot conditions along the reflected shock, we derive an evolution equation for the transonic shock, and an oblique derivative boundary condition at the unknown shock position. By regularizing the degenerate problem, we construct uniform bounds; we apply local compactness arguments to extract a limit that solves the problem. The solution is smooth in the interior and continuous up to the degenerate boundary.

This work completes a stage in our program to construct self-similar solutions of two-dimensional Riemann problems. In a series of papers, we developed techniques for solving the degenerate elliptic equations that arise in self-similar reductions of hyperbolic conservation laws. In other papers, especially in joint work with Gary Lieberman, we developed techniques for solving free boundary problems of the type that arise from Rankine-Hugoniot relations. For the first time, in this paper, we combine these approaches and show that they are compatible. Although our construction is limited to a finite part of the unbounded subsonic region, it suggests that this approach has the potential to solve a variety of problems in weak shock reflection, including Mach and von Neumann reflection in the UTSD equation, and the analogous problems for the unsteady full potential equation. © 2002 John Wiley & Sons, Inc.

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1 Introduction

With this paper, we complete a step in our study of self-similar solutions for the unsteady transonic small disturbance (UTSD) equation,

$$u_t + uu_x + v_y = 0\,, \quad v_x - u_y = 0\,. \tag{1.1}$$

This system, for a flow vector $U = (u, v)$, was established by Brio and Hunter [2] and Morawetz [20] to model shock reflection by a wedge in the case of weak shocks and small wedge angles. The two-dimensional UTSD equation, which can be derived by asymptotic reduction from the unsteady compressible gas dynamics equations [2] or the transonic full potential equation [20], is valid in the shock interaction region when an incident shock hits a semi-infinite wedge with corner at the origin. At the moment of impact, the position of the incident shock is $x = ay$, where a, the sole dimensionless parameter, measures the wedge angle scaled by the Mach number. The problem is self-similar and the solution is a function of $(\xi, \eta) = (x/t, y/t)$. The piecewise constant two-dimensional Riemann data consist of the upstream and downstream states, U_0 and U_1, separated by the incident shock, in the sector outside the wedge. In addition, the asymptotic reduction that produces equation (1.1) flattens the wedge and sends its vertex to $x = -\infty$ for positive t, so the initial value problem is posed in a half-plane with data (in the normalization derived by Brio and Hunter [2])

$$U(x, y, 0) = \begin{cases} U_0 = (0, 0)\,, & x > ay\,, \\ U_1 = (1, -a)\,, & x < ay\,, \end{cases}$$

and the boundary condition $v(x, 0) = 0$ on the negative x-axis.

Alternatively, one can enforce the boundary condition by giving Riemann data in the full plane, with a symmetry:

$$U(x, y, 0) = \begin{cases} U_0 = (0, 0)\,, & x > a|y|\,, \\ U_1 = (1, -a)\,, & y > 0,\ x < ay\,, \\ U_1^* = (1, a)\,, & y < 0,\ x < -ay\,. \end{cases} \tag{1.2}$$

For each $a > \sqrt{2}$ there are two possible reflected shocks, and, except for a narrow range $\sqrt{2} < a < a^* \equiv (1 + \sqrt{5}/2)^{1/2}$ where both reflected shocks are transonic, the stronger shock is transonic and the weaker, which is the one thought to occur physically, is supersonic, in a sense we will define below. (See [5] for the calculation of these values, and [1] and [20] for insight into the physical background.)

In our program of solving Riemann problems by using the reduced, self-similar equations, we replace initial data for a $(2 + 1)$–dimensional problem in space and time by boundary data on a curve $\xi^2 + \eta^2 = C$, and the Riemann problem then becomes a boundary value problem for the self-similar version of (1.1),

$$(u - \xi)u_\xi - \eta u_\eta + v_\eta = 0\,, \quad v_\xi - u_\eta = 0\,. \tag{1.3}$$

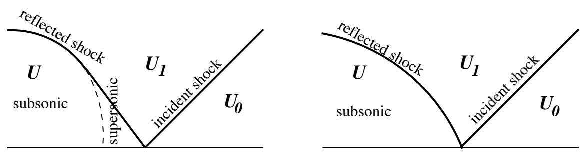

FIGURE 1.1. Supersonic and transonic regular reflection.

The data yield a uniform self-similar incident shock

$$S_1 : \xi = a\eta + \frac{1}{2} + a^2 \tag{1.4}$$

for $t > 0$, separating U_0 and U_1. The weaker of the two reflected shocks in regular reflection, with equation

$$\xi = -\frac{\eta}{a - \sqrt{a^2 - 2}} + \frac{1}{2} + a^2 , \tag{1.5}$$

meets S_1 at the wall at the point

$$\Xi_a = \left(\frac{1}{2} + a^2, 0\right) = (\xi_a, 0) , \tag{1.6}$$

and the state behind it [5] is

$$U_R = (u_R, 0) = (1 + a^2 - a\sqrt{a^2 - 2}, 0) . \tag{1.7}$$

However, a uniform state behind the reflected shock is not consistent with the Riemann data at infinity. Noting that equation (1.1) is hyperbolic, with the x, y-plane a characteristic surface, one sees that the reflected shock cannot remain rectilinear without violating causality. Hence, the state behind the reflected shock is nonuniform. Now, the self-similar equation changes type: It is hyperbolic (supersonic) far from the negative ξ-axis but elliptic (subsonic) when $\xi + \eta^2/4 - u < 0$. At parameter values $a > a^*$, the state U_R defined by equation (1.7) is supersonic at the reflection point Ξ_a and, by determinacy, the reflected shock remains straight and the downstream flow constant throughout the supersonic region. The reflected shock becomes curved where the flow becomes subsonic (see the left sketch in Figure 1.1), and it is the existence of this subsonic flow that we prove here. Our main result is the following theorem:

THEOREM 1.1 *For the UTSD equation with shock reflection Riemann data* (1.2) *in the case $a > a^*$, there exists a solution corresponding to* weak regular reflection. *The reflected shock is supersonic at $(\frac{1}{2} + a^2, 0)$, and the state behind it becomes subsonic further downstream, where it is a nonconstant solution to the UTSD equation satisfying the Rankine-Hugoniot conditions across the transonic shock and the symmetry condition at the wall. The nonconstant subsonic flow exists in at least a bounded neighborhood of the sonic line.*

In earlier papers [5, 7], we demonstrated how the study of self-similar solutions for some shock interaction and shock reflection problems reduces to a free boundary problem for an elliptic or degenerate elliptic equation. The main result in our previous paper [6, theorem 1.1] is the analogue of Theorem 1.1 for regular reflection in the case that the state immediately behind the reflected shock is strictly subsonic. That case, which we called transonic regular reflection, the right sketch in Figure 1.1, is simpler, because the subsonic problem is uniformly elliptic.

This paper complements [6]. Here we solve a free boundary problem for the UTSD equation in the subsonic region where the equation is degenerate on a part of the boundary adjacent to the free boundary. The difficulty that we have overcome is solving the free boundary problem when we do not have uniform ellipticity of the equation. At the fixed curve where the degeneracy occurs, the flow is continuous, and, as we show, the equation is strictly elliptic inside the subsonic region. Furthermore, the structure of the degeneracy is not that associated with the Tricomi equation. Although the Tricomi equation is connected with transonic flow, it appears as a linear equation representing steady transonic flow in hodograph variables. By contrast, when the quasi-steady equations representing self-similar flow (e.g., the system (1.3)) are transformed to the hodograph plane, they remain quasilinear. The structure of these equations (as measured either by the behavior of the characteristics on the hyperbolic side of the degeneracy or by an indicator such as the Fichera function on the elliptic side) is not that of the Tricomi equation but rather is a nonlinear version of an equation first studied by Keldysh [15]; to the best of our knowledge, the nonlinear equation had not been previously studied.

By now, a number of results have been obtained for degenerate elliptic equations of this type. In particular, Čanić and Keyfitz [3, 4] showed existence of both regular and singular solutions of the UTSD equation in weighted Sobolev spaces, using monotone operator methods. Those papers considered Dirichlet problems related to shock interaction data. Choi, Lazer, and McKenna [12] and later Choi and McKenna [13] used regularization and upper-lower solution methods to solve (in Hölder spaces) Dirichlet problems for anisotropic singular quasi-linear elliptic equations on convex, smooth domains. In a related example, Zheng [21] studied degenerate elliptic problems arising from the transonic pressure-gradient equations in a smooth and convex domain and found weak solutions in a Sobolev space. Later Choi and Kim [11] and Kim [16] extended the equation of [12] and [13] to include a fast-growing source term and established monotone iteration methods for anisotropic equations. Recently Čanić and Kim [9] provided a general approach to proving existence of solutions in Hölder spaces for a class of nonlinear Keldysh equations with structure conditions and found a structure condition sufficient to obtain interior ellipticity. That paper also described the boundary behavior of the solution.

The works cited above all concern domains with a fixed boundary, either partly or wholly degenerate, with Dirichlet boundary conditions. However, in the context of self-similar flows with shocks, part of the boundary of the subsonic region consists of a transonic shock. This boundary component is not known a priori but is determined by an interaction between the flow on the two sides of the shock via the Rankine-Hugoniot relations. To find the transonic shock, one can use the Rankine-Hugoniot conditions to derive an evolution equation for the shock and a boundary condition of oblique derivative type at the undetermined boundary. In work closely related to this paper, Čanić, Keyfitz, and Lieberman [8] studied perturbed steady transonic shocks via free boundary problems and developed the backbone of the method for solving free boundary problems arising in transonic shock configurations. We adapted the method of [8] in [6] to prove an existence theorem for the free boundary problem arising in transonic regular reflection for the UTSD equation. In both [8] and [6], the subsonic region is uniformly elliptic and does not exhibit degeneracy at any part of the boundary.

Thus the papers related to the present work consider either Dirichlet problems for degenerate elliptic equations or free boundary problems for strictly elliptic equations separately. As yet no study has considered the two complications at the same time. We do this for the first time here. Our method here combines that of [6, 8] in solving free boundary problems via a Schauder fixed-point theorem with that of [9, 12, 13, 16] in using regularized approximations to solve degenerate elliptic equations of Keldysh type.

Many interesting problems for the UTSD model for shock reflection remain unsolved. Specifically, if $a < \sqrt{2}$, regular reflection cannot occur at all, and some prototype for Mach or von Neumann reflection should appear as a solution to the Riemann problem. In [7] we suggest some ways solutions of these types could be realized in this model and define the subsonic problems, all involving free boundaries and degeneracies, which would have to be solved. The results of the present paper suggest that these problems may be tractable.

The outline of this paper is as follows. We first give the background necessary to state Theorem 1.2, a technical version of Theorem 1.1. To overcome the main difficulty, the lack of uniform ellipticity, we solve the free boundary problem for a regularized equation in Section 2. The key to removing the regularization is finding a local lower barrier that gives strict ellipticity in the interior of the subsonic region, independent of the regularization; this is accomplished in Section 3. In Section 4, we present compactness arguments that allow us to complete the proof of Theorem 1.2 and of Theorem 1.1. We conclude, in Section 5, by comparing this solution with some numerical simulations performed using the technique in [10].

1.1 Background on the UTSD Equation

We begin by recalling some facts about the UTSD equation; see [6]. In the coordinate system (ρ, η) with $\rho = \xi + \eta^2/4$, which we will use throughout this paper, equation (1.3) becomes

$$(u-\rho)u_\rho - \frac{\eta}{2}u_\eta + v_\eta = 0\,, \quad \frac{\eta}{2}u_\rho - v_\rho + u_\eta = 0\,. \tag{1.8}$$

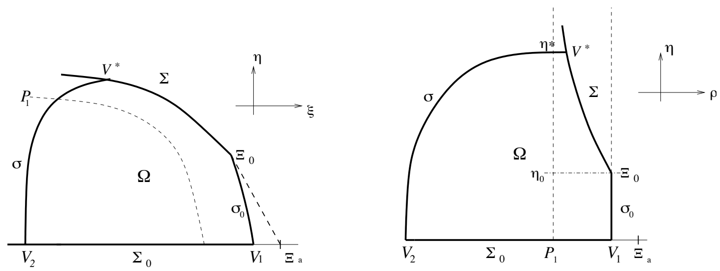

FIGURE 1.2. Domains in the two coordinate systems.

It is convenient to replace the system with the second-order equation obtained by eliminating v,

$$Qu \equiv \left((u-\rho)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = 0\,. \tag{1.9}$$

In regular reflection, the incident shock S_1 (1.4) and reflected shock S_2 meet at the point Ξ_a (1.6) on the wall. The state immediately behind the reflected shock is U_R (1.7). Near the reflection point Ξ_a the constant solution U_R is supersonic, but it becomes sonic at the parabola P_R defined by $\{\rho = \xi + \eta^2/4 = u_R\}$; $u_R = \rho_0 < \xi_a$ in our problem. The solution is nonconstant and the reflected shock is transonic and curved beyond P_R.

The governing equation (1.9) is degenerate at the sonic curve P_R. We solve the problem in a domain Ω bounded by the transonic reflected shock S_2, by the degenerate parabola P_R, by the wall $\eta = 0$, and by a cutoff boundary σ, introduced because of the unboundedness of the subsonic region. In [6] we showed that the reflected shock S_2 approaches the curve $\{\rho = 1\}$ asymptotically as $\eta \to \infty$; if $\rho(\eta)$ is the equation of S_2, then

$$\rho(\eta) = 1 + \mathcal{O}\left(\frac{1}{\eta}\right) \quad \text{as } \eta \to \infty\,. \tag{1.10}$$

We fix a suitable large $\eta^* > 0$ and define σ to be a smooth curve, shown in Figure 1.2, extending from the wall to S_2 and containing the segment $\eta = \eta^*$ for $\rho \geq 1$. We impose a Dirichlet boundary condition $u = f$ on σ, where f may be any function consistent with the asymptotic bound of [6]. In this paper, all we require is $\rho < f < u_R$ on σ and

$$m \equiv \min_\sigma f > 1\,.$$

As long as the state behind the reflected shock is supersonic, the reflected shock S_2 coincides with the rectilinear shock of equation (1.5). The sonic parabola P_R and the reflected shock S_2 meet at the point

$$\Xi_0 \equiv (\rho_0, \eta_0) = \left(u_R, a + \sqrt{a^2-2} - \sqrt{2a}\sqrt{a - \sqrt{a^2-2}}\right). \tag{1.11}$$

Thus the degenerate part of the boundary of Ω is the segment $\{(\rho_0, \eta) : \eta \in [0, \eta_0]\}$ of the parabola P_R, and the boundary condition there is $u = u_R = \rho_0$.

The boundary condition at the wall, $\eta = 0$, is $u_\eta(\xi, 0) = 0$.

We complete the formulation of the problem by giving the conditions that hold at the free boundary, the transonic portion of the shock S_2.

The Rankine-Hugoniot equations are

$$\frac{d\rho}{d\eta} = \frac{[\frac{1}{2}u^2 - \rho u]}{[v - \frac{\eta}{2}u]} = \frac{[\frac{\eta}{2}u - v]}{[u]}, \tag{1.12}$$

where the square brackets, $[\,\cdot\,]$, denote jumps across the shock $\rho(\eta)$. We replace the pair of equations (1.12) with an evolution equation for the shock,

$$\rho' = -\sqrt{\rho - (u+1)/2} = -\sqrt{\rho - \overline{u}}, \tag{1.13}$$

and an oblique derivative boundary condition for u at the unknown shock boundary,

$$\beta \cdot \nabla u = 0 \quad \text{where } \beta = (\beta^1, \beta^2) = \left(\rho'\left[\frac{7u+1}{8} - \rho\right], \frac{5u+3}{8} - \rho\right). \tag{1.14}$$

(Here $\overline{u} = (u+1)/2$ is the average of u on the two sides of the shock.) This equation was derived from (1.12) in [6].

Using $\nu = (-1, \rho')/\sqrt{1 + (\rho')^2}$, the inner unit normal to the subsonic domain along the shock $\rho(\eta)$, we get

$$\beta \cdot \nu = \frac{\beta \cdot (-1, \rho')}{\sqrt{1 + (\rho')^2}} = -\frac{\rho'(u-1)}{4\sqrt{1 + (\rho')^2}},$$

which is positive when $u > 1$ and $\rho' < 0$. In addition, $|\beta|$ is bounded if u, ρ, and ρ' are bounded. We establish uniform obliqueness in Section 2.

The boundary of Ω is a curvilinear quadrilateral whose sides are $\sigma_0 = \{(\rho_0, \eta) : \eta \in [0, \eta_0]\}$, the part of P_R forming the degenerate boundary; $\Sigma = \{(\rho(\eta), \eta) : \eta \in (\eta_0, \eta^*)\}$, the part of S_2 that forms the free boundary; σ, the cutoff boundary; and Σ_0, the wall.

The corners are $\Xi_0 = (\rho_0, \eta_0)$, where the degenerate boundary σ_0 and the free boundary Σ meet; $V^* = (\rho(\eta^*), \eta^*) = (\rho^*, \eta^*)$, where Σ and the cutoff boundary σ meet; V_2, where σ and Σ_0 meet; and V_1, where σ_0 meets the wall Σ_0.

We let $\mathbf{V} = \{\Xi_0, V_1, V_2, V^*\}$ denote the set of corners. See Figure 1.2.

1.2 The Free Boundary Problem

We now restate the main result of the paper in the precise form we will use to construct the proof. Besides using the second-order operator Q, introducing the cutoff boundary σ, and manipulating the Rankine-Hugoniot conditions, we make one other important change: We modify the shock evolution equation, (1.13), so that the square root remains well defined for all approximations and so that ρ' remains strictly negative all the way up to η^*. For this we define the function

$$g(x) = \begin{cases} x\,, & x \geq \delta^*\,, \\ \delta^*\,, & x \leq \delta^*\,, \end{cases}$$

where $\delta^* > 0$ is chosen in equation (2.6) and replace (1.13) by

$$\rho' = -\sqrt{g(\rho - \overline{u})}\,. \tag{1.15}$$

We may further modify g in a small neighborhood of δ^* so that g' is continuous and $0 \leq g' \leq 1$. We turn to the question of removing the cutoff at the end of Section 4.

Incorporating the reductions we have introduced, we state the main technical result.

THEOREM 1.2 *For any $a > a^*$ and for any data $f \in C^{2+\alpha_0}$ with $\rho < f < u_R$ and $f > 1$, there exists a $\delta_1 > 0$ such that for any $0 < \delta^* < \delta_1$, the following problem has a solution in Ω:*

Equation:

$$Qu = \left((u-\rho)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = 0 \in \Omega\,. \tag{1.16}$$

Free boundary conditions:

$$\left.\begin{aligned} &Nu = \beta(u,\rho)\cdot\nabla u = 0 \\ &\frac{d\rho}{d\eta} = -\sqrt{g\left(\rho - \frac{u+1}{2}\right)}\,, \quad \rho(\eta_0) = \rho_0 \end{aligned}\right\} \quad on\ \Sigma \equiv \{\rho = \rho(\eta)\}$$

Fixed boundary conditions:[1]

$$u = f \ \ on\ \sigma\,, \quad u = \rho_0 \ \ on\ \sigma_0\,, \quad u_\eta = 0 \ \ on\ \Sigma_0\,.$$

The solution (u,ρ) is in $C^{2+\alpha'}(\overline{\Omega}\setminus(\sigma_0\cup\mathbf{V}))\cap C(\overline{\Omega})\times C^{2+\alpha'}(0,\eta_0)$ for some $\alpha' \in (0,\alpha_0)$.

The foremost difficulty in this problem is that the operator Q loses ellipticity on σ_0. To handle this, we approximate Q by a sequence of regularized operators:

$$Q^\varepsilon u = \left((u-\rho+\varepsilon)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = 0\,. \tag{1.17}$$

In the solution of the free boundary problems for Q^ε, the transonic shock position now depends on ε as does the cutoff boundary σ, and we write $\Sigma^\varepsilon = \{\rho^\varepsilon(\eta) : \eta \in (\eta_0,\eta^*)\}$ for the free boundary, σ^ε for the cutoff boundary, and Ω^ε for the domain.

The proof of Theorem 1.2 proceeds in the following steps:

[1] The boundary condition on σ is a standard Dirichlet condition. Note, however, that σ also depends on the free boundary since the curve terminates at $\rho(\eta^*)$.

STEP 1: Show existence for the regularized free boundary problems and obtain a priori bounds on the solutions u^ε and $\rho^\varepsilon(\eta)$ uniformly in ε. Existence follows directly by the method of [6].

STEP 2: Construct a local lower barrier for u^ε that is independent of ε to give local ellipticity uniformly in ε. This is the crucial step that allows us to use a local compactness argument.

STEP 3: Obtain a convergent subsequence and show that the limit solves the problem. Using the local barrier we constructed in the previous step, and applying regularity, compactness and a diagonalization argument, we prove this claim.

In Section 2, we carry out step 1, in Section 3 we do the construction in step 2, and in Section 4 we establish step 3 and complete the proof of the main theorem.

2 Existence of a Solution to the Regularized Problem

In this section, we obtain a solution to the regularized free boundary problem, using the method in [6]. We also obtain a priori bounds, independent of ε, on solutions of the sequence of regularized problems. The theorem for the regularized equation is as follows:

THEOREM 2.1 *For each* $0 < \varepsilon < 1$, *there exists a solution* $(u^\varepsilon, \rho^\varepsilon) \in C^{2+\alpha}(\overline{\Omega^\varepsilon} \setminus \mathbf{V}) \cap C^\alpha(\overline{\Omega^\varepsilon}) \times C^{2+\alpha}((\eta_0, \eta^*))$ *of*

$$Q^\varepsilon u = \Big((u - \rho + \varepsilon)u_\rho + \frac{u}{2}\Big)_\rho + u_{\eta\eta} = 0 \ \text{ in } \Omega^\varepsilon ,$$

$$Nu = \beta(u, \rho) \cdot \nabla u = 0 , \quad \rho' = -\sqrt{g(\rho - \overline{u})} , \quad \rho(\eta_0) = \rho_0 \ \text{ on } \Sigma^\varepsilon , \tag{2.1}$$

$$u|_{\sigma^\varepsilon} = f , \quad u|_{\sigma_0} = \rho_0 , \quad u_\eta|_{\Sigma_0} = 0 ,$$

where $0 < \alpha = \alpha(\varepsilon) < 1$. *Moreover, for every* $0 < \varepsilon < 1$, *the following bounds hold uniformly in* ε: $\rho < u^\varepsilon(\rho, \eta) \le \rho_0$ *for all* $(\rho, \eta) \in \Omega^\varepsilon$, *and* $\rho_L^* \le \rho^\varepsilon(\eta) \le \rho_0$ *in* $[\eta_0, \eta^*]$ *where* $\rho_L^* > 1$ *is a constant independent of* ε.

PROOF: The proof is similar to [6, theorem 3.2]; there are three parts. In the first part, we fix a function $\rho(\eta)$ that belongs to a closed, convex set $\mathcal{K}$, defining a fixed boundary component Σ that approximates the free boundary, Σ^ε. Since we cannot assume that $u - \rho$ is nonnegative, we introduce a cutoff function h,

$$h(x) = \begin{cases} x , & x \ge 0 , \\ 0 , & x \le 0 , \end{cases}$$

(appropriately smoothed near zero so that h' is continuous and bounded). We replace Q^ε by the modified regularized operator

$$\widetilde{Q}^\varepsilon u = \Big((h(u - \rho) + \varepsilon)u_\rho + \frac{u}{2}\Big)_\rho + u_{\eta\eta} = 0 \tag{2.2}$$

and later show that the cutoff can be removed. Equation (2.2) is strictly elliptic, with ellipticity ratio depending on ε. We establish existence of solutions of (2.2) with fixed boundary conditions.

In the second part, we derive L^∞ bounds $\rho < u(\rho,\eta) < \rho_0$ in Ω on the solution u of (2.2) with fixed boundary conditions. This allows us to remove the cutoff h and gives L^∞ bounds uniformly in ε.

Finally, in the last part, we complete the proof by defining a map J on the set $\mathcal{K}$ and showing J has a fixed point.

Part 1: *Existence for the Fixed Boundary Problem.* We begin with a function $\rho = \rho(\eta)$, defining an approximate boundary Σ. Assume $\rho \in \mathcal{K}$, where $\mathcal{K} = \{\rho(\eta) : \eta \in [\eta_0, \eta^*]\}$ is a closed, convex subset of the Banach space $C^{1+\gamma_1}([\eta_0, \eta^*])$ for some $\gamma_1 \in (0,1)$. The functions in $\mathcal{K}$ satisfy

(1) $\rho(\eta_0) = \rho_0$; $\rho_0 = u_R$ is given in (1.7); and we know that $1 < \rho_0 < \xi_a$.
(2) $-\sqrt{\rho_0 - 1} \le \rho' \le -\sqrt{\delta^*} < 0$ where δ^* is specified in Part 3.
(3) $\rho_L(\eta) \le \rho(\eta) \le \rho_R(\eta)$, where ρ_L and ρ_R are also defined in Part 3.

The boundary value problem is

$$\begin{gathered}
\widetilde{Q}^\varepsilon u = \left((h(u-\rho)+\varepsilon)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = 0 \quad \text{in } \Omega\,, \\
Nu = \beta(u,\rho)\cdot\nabla u = 0 \quad \text{on } \Sigma\,, \\
u|_\sigma = f\,, \quad u|_{\sigma_0} = \rho_0\,, \quad \text{and} \quad u_\eta|_{\Sigma_0} = 0\,.
\end{gathered} \tag{2.3}$$

Because of the cutoff h and the regularization of the coefficients, $\widetilde{Q}^\varepsilon$ is strictly elliptic, with ratio depending on ε. Since the domain has corners, the weak solutions lie in a weighted Hölder space $H_{1+\alpha}^{(-\gamma)} = H_{1+\alpha;\Omega\cup(\partial\Omega\setminus\mathbf{V})}^{(-\gamma)}$. (Weighted Hölder spaces are defined in [14, p. 90]; the spaces we use here, with weights at the corners, were also used in [6] and [8].) Elements of this space have some regularity except at the set of corner points; that is, they are in $C^{1+\alpha}(\overline{\Omega}\setminus\mathbf{V}) \cap C^\gamma(\overline{\Omega})$. The exponent $\gamma \in (0,1)$ depends on the corner angles, and $\alpha \in (0,\gamma)$. From theorem 4.1 in [6], for a given $\rho(\eta) \in \mathcal{K}$, there exists a solution $u \in H_{1+\alpha}^{(-\gamma)}$ of the boundary value problem (2.3), where $\gamma = \gamma(\varepsilon)$ and $\alpha = \alpha(\varepsilon)$.

Part 2: *A Priori Bounds*. Applying propositions 3.3, 3.4, and 3.5 of [6] to any solution of $\widetilde{Q}^\varepsilon u = 0$, we deduce strict ellipticity and a priori bounds; that is, $\rho < u(\rho,\eta) < \rho_0$ and $u > m$ in Ω, where $m = \min f$. A Hopf-lemma type of argument (see proposition 3.5 in [6]) shows that $u - \rho > 0$ on Σ. Hence, the cutoff h is the identity in $\overline{\Omega}$, and we can replace the operator $\widetilde{Q}^\varepsilon$ by Q^ε. Thus, the solution u of (2.3) satisfies

$$Q^\varepsilon u = 0 \text{ in } \Omega\,, \quad Nu|_\Sigma = 0\,, \quad u|_\sigma = f\,, \quad u|_{\sigma_0} = \rho_0\,, \quad u_\eta|_{\Sigma_0} = 0\,. \tag{2.4}$$

A key point is that the a priori bounds, ρ_0 and m, are independent of ε.

Part 3: *Existence for the Free Boundary Problem*. For each $\rho \in \mathcal{K}$, using the corresponding solution u of (2.4), we define the map J on $\mathcal{K}$ by $\tilde{\rho} = J\rho$ where

$$\tilde{\rho}'(\eta) = -\sqrt{g\left(\tilde{\rho}(\eta) - \frac{u(\rho(\eta), \eta) + 1}{2}\right)} \quad \text{and} \quad \tilde{\rho}(\eta_0) = \rho_0 \,. \tag{2.5}$$

By proposition 4.6 in [6], we find that

$$\tilde{\rho}(\eta) > 1 + \frac{m-1}{4} > 1 \quad \forall \eta \in [\eta_0, \eta^*] \quad \text{if } \delta^* < \frac{1}{4}\left(\frac{m-1}{\eta^* - \eta_0}\right)^2 . \tag{2.6}$$

We let $\delta_1 \equiv \big((m-1)/4(\eta^* - \eta_0)\big)^2$ and $\rho_L^* \equiv 1 + (m-1)/4$. Then $\tilde{\rho} > \rho_L^* > 1$ independent of ε. Moreover, we obtain an L^∞ bound for ρ this way; for instance, we have $\rho_L \le \rho^\varepsilon \le \rho_R$ in $[\eta_0, \eta^*]$, where we can choose

$$\rho_L(\eta) = \max\left\{\rho_0 - (\eta - \eta_0)\sqrt{\rho_0 - 1},\, \rho_L^* + (\eta^* - \eta)\sqrt{\delta *}\right\}, \tag{2.7}$$

$$\rho_R(\eta) = \rho_0 - (\eta - \eta_0)\sqrt{\delta *}\,. \tag{2.8}$$

These bounds are independent of ε.

The argument above shows that for every $0 < \delta^* < \delta_1$, J maps $\mathcal{K}$ into itself. Proposition 4.2 in [8] shows that the map J is compact, when γ_1 is chosen sufficiently small, and is continuous. By the Schauder fixed-point theorem, there is a fixed point $\rho^\varepsilon \in H_{1+\gamma}[\eta_0, \eta^*]$. Using the fact that equation (2.3) has a solution for this Σ^ε and the corresponding σ^ε, we establish the existence of a solution $(u^\varepsilon, \rho^\varepsilon) \in H_{1+\alpha}^{(-\gamma)} \times H_{1+\gamma}$ of the free boundary problem (2.1) for sufficiently small $\gamma = \gamma(\varepsilon)$ and $\alpha = \alpha(\varepsilon)$.

Now, the regularity of ρ^ε can be improved from $C^{1+\alpha}$ to $C^{2+\alpha}$, since $\rho^{\varepsilon\prime}$ satisfies equation (1.15), in which the entries on the right side have $\rho^\varepsilon \in C^{1+\alpha}$ and $u^\varepsilon \in H_{1+\alpha}^{(-\gamma)}$, so $u^\varepsilon \in C^{1+\alpha}$ on Σ^ε. In addition, we have assumed $f \in C^{2+\alpha_0}$, and so regularity arguments such as theorem 6.2 and theorem 6.30 in [14] ensure that the solution $u^\varepsilon \in H_{1+\alpha}^{(-\gamma)}$ is in fact in $C^{2+\alpha}(\overline{\Omega} \setminus \mathbf{V})$.

This completes the proof. ∎

We also have, from the second property of $\mathcal{K}$, a priori bounds for ρ^ε and $\rho^{\varepsilon\prime}$ uniformly in ε. A priori bounds for ρ^ε define the bounds for the segment $\eta = \eta^*$ of σ^ε where $\rho \ge 1$; the maximum interval is $[1, \rho_R(\eta^*)]$.

We note again that α is small and depends on ε.

3 The Local Lower Barrier

This section is devoted to establishing a local lower barrier, given in Lemma 3.2. This uniform barrier will allow us to obtain strict ellipticity uniformly in ε. In [9], Čanić and Kim found a structure condition that was sufficient to yield a nontrivial global lower barrier for degenerate elliptic problems of this type. Because we consider here mixed and free boundary problems rather than Dirichlet problems, we are able to construct local barriers only; however, this is sufficient. Our method is close to that of [12].

To state the lemma, we define a collection of balls $\mathcal{B}$ that provides a cover for Ω^ε for all sufficiently small ε; we prove the lemma in each ball from $\mathcal{B}$.

DEFINITION 3.1 The collection $\mathcal{B}$ consists of balls $B = B_R(X_1)$ of radius $R \in (0, 1)$ centered at $X_1 = (\rho_1, \eta_1)$ that satisfy $B \cap \sigma_0 = \varnothing$ and one of the following four conditions:

(i) $B_R(X_1) \subset \Omega^\varepsilon$ for all ε,
(ii) $B_R(X_1) \cap \sigma^\varepsilon \neq \varnothing$ for some ε,
(iii) $X_1 \in \Sigma_0$, or
(iv) $B_R(X_1) \cap \Sigma^\varepsilon \neq \varnothing$ for some ε.

In any ball in $\mathcal{B}$, we construct a lower barrier independent of ε by means of the following lemma:

LEMMA 3.2 *For any $B = B_R(X_1) \in \mathcal{B}$, there exists a $\delta > 0$ independent of ε such that*

$$u^\varepsilon - \rho \geq \phi \equiv \delta(R^2 - |X - X_1|^2) \quad \textit{in } B \cap \overline{\Omega^\varepsilon}\,. \tag{3.1}$$

PROOF: The function $w^\varepsilon \equiv u^\varepsilon - \rho$ satisfies the quasi-linear equation

$$F^\varepsilon w \equiv (w + \varepsilon) w_{\rho\rho} + w_{\eta\eta} + w_\rho^2 + \frac{3}{2} w_\rho + \frac{1}{2} = 0\,. \tag{3.2}$$

The operator F^ε exhibits a positive constant source term, which means there is a positive δ that gives $F^\varepsilon > 0$. For, by a calculation,

$$F^\varepsilon \phi \geq -2\delta(\delta + 1) - 2\delta + (-2\delta(\rho - \rho_1))^2 + \frac{3}{2}(-2\delta(\rho - \rho_1)) + \frac{1}{2} > 0 \tag{3.3}$$

when δ is sufficiently small. Now, since $w^\varepsilon = u^\varepsilon - \rho$ is positive in Ω^ε and satisfies $F^\varepsilon w^\varepsilon = 0$, we have

$$0 > F^\varepsilon w^\varepsilon - F^\varepsilon \phi = G^\varepsilon (w^\varepsilon - \phi) \tag{3.4}$$

where $G^\varepsilon \equiv (w^\varepsilon + \varepsilon)\partial_\rho^2 + \partial_\eta^2 + (w_\rho^\varepsilon + \phi_\rho + \frac{3}{2})\partial_\rho - 2\delta$ can be thought of as a linear elliptic operator acting on $w^\varepsilon - \phi$. Thus, we have $G^\varepsilon(w^\varepsilon - \phi) < 0$ in B_R.

Now, we prove the lemma for balls of each type.

CASE 1: $B_R(X_1) \subset \Omega^\varepsilon$ for all ε.

Since δ is a positive constant, w_ρ^ε is bounded inside Ω, and $w^\varepsilon > 0 = \phi$ on ∂B_R, we can apply the strong maximum principle [14, theorem 3.5] to $G^\varepsilon(w^\varepsilon - \phi)$ on B_R to get (3.1).

CASE 2: $B_R(X_1) \cap \sigma^\varepsilon \neq \varnothing$ for some ε.

On σ^ε, $u^\varepsilon - \rho = f - \rho > 0$. Now, by the choice of f, η^*, and $\delta*$, we can ensure that $w^\varepsilon \geq \delta > 0$ on σ^ε for some $\delta > 0$, uniformly in ε. Hence, the argument of case 1 applies for any ε for which $B \cap \sigma^\varepsilon \neq \varnothing$, and δ is again independent of ε.

CASE 3: $X_1 \in \Sigma_0$.

From the definition of ϕ, when $X_1 \in \Sigma_0$ we have $\phi_\eta = 0$ on Σ_0. Using $G^\varepsilon(u^\varepsilon - \phi) < 0$, we apply the strong maximum principle [14, theorem 3.5] on $B_R(X_1) \cap \Omega^\varepsilon$. Since $w^\varepsilon > 0 = \phi$ on $\partial B_R(X_1) \cap \Omega^\varepsilon$, a nonpositive minimum of $w^\varepsilon - \phi$ could occur only on $B_R(X_1) \cap \Sigma_0$. However, since $w^\varepsilon_\eta = \phi_\eta = 0$ on Σ_0, if we assume such a minimum exists, then the Hopf lemma [14, lemma 3.4] gives a contradiction. Hence we get (3.1) in this case.

CASE 4: $B_R(X_1) \cap \Sigma^\varepsilon \neq \varnothing$ for some ε.

Using the inequality $G^\varepsilon(w^\varepsilon - \phi) < 0$, we again apply the strong maximum principle to conclude that the minimum must occur at the boundary of $B_R(X_1) \cap \Omega^\varepsilon$. Since we have $u^\varepsilon - \rho > 0$ and $\phi = 0$ on $\partial B_R(X_1)$, if we assume that $u^\varepsilon - \rho - \phi \le 0$ for some $X \in B_R(X_1) \cap (\Omega^\varepsilon \cup \Sigma^\varepsilon)$, then the absolute minimum must occur on $B_R(X_1) \cap \Sigma^\varepsilon$.

Now, suppose that there is a nonpositive minimum at $X_{\min} \in \Sigma^\varepsilon \cap B_R(X_1)$ and that $u^\varepsilon - \rho^\varepsilon - \phi = -\mu$ with $\delta \ge \mu > 0$ there. (Since $u^\varepsilon - \rho^\varepsilon > 0$, μ cannot be bigger than δ.) Again, $G^\varepsilon(w^\varepsilon - \phi) < 0$, from (3.4), and the Hopf lemma [14, lemma 3.4] ensures that the derivative of $w^\varepsilon - \phi$ in the outward normal direction, $\nu = (1, -\rho^{\varepsilon\prime})$, at $X_{\min}$ is negative:

$$0 > \frac{\partial (w^\varepsilon - \phi)(X_{\min})}{\partial \nu} = u^\varepsilon_\rho - 1 - \phi_\rho - \rho^{\varepsilon\prime}(u^\varepsilon_\eta - \phi_\eta)\,, \tag{3.5}$$

and, since $X_{\min}$ is a minimum point on Σ^ε,

$$0 = (w^\varepsilon - \phi)'(X_{\min}) = \rho^{\varepsilon\prime}(u^\varepsilon_\rho - 1 - \phi_\rho) + (u^\varepsilon_\eta - \phi_\eta)\,. \tag{3.6}$$

Using (3.5) and (3.6), we get $0 > (1 + (\rho^{\varepsilon\prime})^2)(u^\varepsilon_\rho - 1 - \phi_\rho)$, which implies that

$$u^\varepsilon_\rho < 1 + \phi_\rho\,. \tag{3.7}$$

Now we combine the oblique derivative boundary condition (1.14) with (3.6) to get

$$0 = \beta \cdot \nabla u^\varepsilon = u^\varepsilon_\rho \rho^{\varepsilon\prime}\left[\frac{2u^\varepsilon - 2}{8}\right] + \big(\phi_\eta + \rho^{\varepsilon\prime}(1+\phi_\rho)\big)\left[\frac{5u^\varepsilon + 3}{8} - \rho^\varepsilon\right].$$

Using $\rho^{\varepsilon\prime} < 0$, $u^\varepsilon > 1$, and (3.7) in the last equation, we get

$$0 > (1+\phi_\rho)\rho^{\varepsilon\prime}\left[\frac{7u^\varepsilon + 1}{8} - \rho^\varepsilon\right] + \phi_\eta\left[\frac{5u^\varepsilon + 3}{8} - \rho^\varepsilon\right].$$

Now, using $u^\varepsilon - \rho^\varepsilon - \phi = -\mu$ and $|\phi - \mu| \le \delta$ at $X_{\min}$, the last inequality becomes

$$\begin{aligned} 0 &> \rho^{\varepsilon\prime}\left[\frac{1-\rho^\varepsilon}{8}\right] + \rho^{\varepsilon\prime}\left[\frac{7\phi - 7\mu}{8}\right] \\ &\quad - 2\delta\Bigg((\rho^\varepsilon - \rho_1)\rho^{\varepsilon\prime}\left[\frac{1-\rho^\varepsilon + 7\phi - 7\mu}{8}\right] \\ &\qquad + (\eta - \eta_1)\left[\frac{3 - 3\rho^\varepsilon + 5\phi - 5\mu}{8}\right]\Bigg) \\ &\ge \rho^{\varepsilon\prime}\left[\frac{1-\rho^\varepsilon}{8}\right] \\ &\quad + \delta\left(-\frac{7|\rho^{\varepsilon\prime}|}{8} - 2|\rho^{\varepsilon\prime}|\left[\frac{|1-\rho^\varepsilon| + 7\delta}{8}\right] - 2\left[\frac{3|1-\rho^\varepsilon| + 5\delta}{8}\right]\right). \end{aligned}$$

Now, $\rho^\varepsilon \ge \rho_L^* > 1$, where $\rho_L^* = 1 + (m-1)/4$ is independent of ε (see the proof of Theorem 2.1); $\rho^{\varepsilon\prime} \le -\sqrt{\delta *} < 0$ and $|\rho^{\varepsilon\prime}|$ is bounded independently of ε by the properties of the set $\mathcal{K}$. That is, the coefficient of δ on the last line is bounded independently of ε; thus we can choose δ independently of ε so that the last line becomes strictly positive. This contradiction completes the proof. $\square$

Lemma 3.2 implies that in each $B_R(X_1)$, we have $u^\varepsilon - \rho \ge \phi > 0$. Thus in $B_{3R/4}(X_1) \cap \overline{\Omega^\varepsilon}$ we get uniform ellipticity independent of ε. In fact, we have

$$\begin{aligned} 0 < \frac{7}{16}\delta R^2 \le \phi \le \lambda_{\min} \le \lambda_{\max} \le \\ \max\{u - \rho + \varepsilon, 1\} \le \rho_0 + \max|\rho| + 1 < \infty\,. \end{aligned} \tag{3.8}$$

We will use this result in the next section to provide a bootstrap argument to improve the regularity and also to prove local compactness.

4 Proof of the Main Theorem

In this section we discuss how to obtain a convergent subsequence from the regularized solutions of equation (2.1), given by Theorem 2.1. Since the a priori bounds for the sequence of solutions and the local lower barrier in Lemma 3.2 are independent of ε, we can use regularity and local compactness to show that the sequence of approximate solutions contains a convergent subsequence. Furthermore, the limit satisfies the equations. This method was introduced by Choi, Lazer, and McKenna [12, 13] and later used in Kim [16] and in Čanić and Kim [9] for a more general class of equations that has a degeneracy at the boundary, similar to our problem, and a Dirichlet condition there. While those papers dealt only with Dirichlet boundary conditions, our paper is complicated by the mixed boundary conditions and the necessity of solving for the free boundary, and thus it requires a more careful approach than the results cited above.

We handle the existence of the limit locally, using the local lower barrier constructed in Lemma 3.2. We first find a limit of ρ^ε to fix the domain, Ω. Then we show that a limit u of u^ε exists in $\Omega \cup (\Sigma_0 \cup \sigma)$. We can do this because of the uniform C^1 bounds on ρ^ε independent of u^ε. Next, we show that the limiting function u satisfies the oblique derivative boundary condition and that (u, ρ) satisfies the free boundary condition. Finally, in Lemma 4.4, we show that the limit satisfies $u = \rho_0$ on σ_0 and is continuous up to the degenerate boundary σ_0.

LEMMA 4.1 *The sequence ρ^ε has a convergent subsequence whose limit ρ is in $C^\gamma([\eta_0, \eta^*])$ for all $\gamma \in (0, 1)$.*

PROOF: In Theorem 2.1, we obtained a sequence ρ^ε of solutions to the equation $\rho^{\varepsilon\prime} = -\sqrt{g(\rho^\varepsilon - \overline{u}^\varepsilon)}$ in the set $\mathcal{K}$. The second property of the set $\mathcal{K}$,

$$-\sqrt{\rho_0 - 1} \le \rho^{\varepsilon\prime} \le -\sqrt{\delta^*}\,, \tag{4.1}$$

gives C^1 bounds on ρ^ε, uniformly in ε. Thus by the Arzela-Ascoli theorem, ρ^ε has a convergent subsequence, and the limit $\rho \in C^\gamma([\eta_0, \eta^*])$ for all $\gamma \in (0, 1)$. □

Since the limit value $\rho(\eta^*) = \lim \rho^\varepsilon(\eta^*)$ is also established, the curves σ^ε tend to a limit, σ. (Notice that the curves σ^ε differ only in the endpoint of the segment on $\eta = \eta^*$, $1 \le \rho \le \rho^\varepsilon(\eta^*)$.) As a consequence, the corresponding subsequence Ω^ε also has a limit, Ω.

In the remaining lemmas, without further comment, we carry out the limiting argument using the convergent subsequence of ρ^ε, which we again call ρ^ε.

LEMMA 4.2 *The sequence u^ε has a limit $u \in C^{2+\alpha'}(\Omega \cup \Sigma_0 \cup \sigma \setminus \{V^*, V_2\})$ for some $\alpha' > 0$, and u satisfies the equation*

$$Qu = \left((u-\rho)u_\rho + \frac{1}{2}u\right)_\rho + u_{\eta\eta} = 0 \quad \text{in } \Omega\,,$$

and the boundary conditions $u_\eta = 0$ on Σ_0 and $u = f$ on σ. In addition, $\rho < u \le \rho_0$ in Ω.

PROOF: For the proof, we use local compactness arguments and uniform L^∞ bounds for u^ε: $m < u^\varepsilon \le \rho_0$. Our arguments are similar to those used in [13, theorem 1].

Fix Ω', a compact subset of $\Omega \cup \Sigma_0 \cup \sigma$. There exists an ε' (depending on Ω') such that for all $\varepsilon \le \varepsilon'$, $\Omega' \subset \Omega^\varepsilon \cup \Sigma_0^\varepsilon \cup \sigma^\varepsilon$. Since Ω' is compact, it has a finite cover by balls from $\mathcal{B}$; on each ball Lemma 3.2 holds for all sufficiently small ε so ϕ is a lower barrier for $u^\varepsilon - \rho$. We now use the uniform L^∞ bounds and treat the problem as a linear equation. Since we have uniform ellipticity by equation (3.8) in $B_{3R/4} \cap \overline{\Omega^\varepsilon}$, we first apply local Hölder estimates from [14]: theorem 8.22 for case 1 of Lemma 3.2, and theorem 8.27 for cases 2 and 3 of the lemma. We find that $\|u^\varepsilon\|_{C^\alpha(\overline{B_{R/2} \cap \Omega'})} \le C$, where $\alpha \in (0, 1)$ and C are independent of ε. With this estimate of the coefficients of Q^ε in (1.17) and with the boundedness of u^ε, we apply the standard Schauder estimates of [14]: Theorems 8.32 and 6.2 for the interior, and theorem 8.33 and lemma 6.5 for the boundary of

$\overline{B_{R/4} \cap \Omega'}$. We get $\|u^\varepsilon\|_{C^{2+\alpha}(\overline{B_{R/4}\cap\Omega'})} \leq C$. By the Arzela-Ascoli theorem, there exists a $C^{2+\alpha'}(\overline{B_{R/4} \cap \Omega'})$-convergent subsequence for any $\alpha' < \alpha$. A covering argument immediately gives a $C^{2+\alpha'}(\Omega')$-convergent subsequence.

Now we let Ω' vary and use a diagonalization argument to obtain a subsequence of u_ε that converges in $C^{2+\alpha'}_{\text{loc}}(\Omega \cup \Sigma_0 \cup \sigma \setminus \{V^*, V_2\})$ to a limit $u \in C^{2+\alpha'}(\Omega \cup \Sigma_0 \cup \sigma \setminus \{V^*, V_2\})$, which satisfies $Qu = 0$ in Ω. Also, since the limiting function is in $C^{2+\alpha'}(\Omega \cup \Sigma_0 \cup \sigma \setminus \{V^*, V_2\})$, it clearly satisfies $u_\eta = 0$ on Σ_0 and $u = f$ on σ.

Moreover, since $u^\varepsilon - \rho \geq 7\delta R^2/16$ in $B_{3R/4}$ for every ε, we get $u - \rho \geq 7\delta R^2/16$ in $B_{3R/4}$; thus $u > \rho$. Finally, $u \leq \rho_0$ in Ω. □

In the next lemma, we show that the limiting functions u and ρ satisfy both the shock evolution equation (1.15) and the oblique derivative boundary condition $Nu = 0$ on Σ.

LEMMA 4.3 *The limits ρ and u satisfy*

$$\rho' = -\sqrt{g(\rho - \overline{u})} \quad \textit{and} \quad Nu = \beta(u, \rho) \cdot \nabla u = 0 \ \textit{ on } \Sigma\,, \tag{4.2}$$

and $\rho \in C^{2+\alpha'}((\eta_0, \eta^*))$ and $u \in C^{2+\alpha'}(\Omega \cup \Sigma)$ for some $\alpha' > 0$.

PROOF: Recall that in case 4 of Lemma 3.2, we showed that

$$u^\varepsilon - \rho \geq \phi > 0 \quad \text{in } B_R(X) \cap (\Sigma^\varepsilon \cup \Omega^\varepsilon)\,. \tag{4.3}$$

This inequality holds if $B_R(X) \cap \Sigma^\varepsilon \neq \varnothing$.

Since by Lemma 4.1 we have a convergent subsequence of ρ^ε with limit ρ, there exists $0 < \varepsilon_0 < 1$ such that if $B_R(X_1)$ is centered at $X_1 = (\rho(\eta_1), \eta_1) \in \Sigma$ with radius $R < 1$, then for each $\varepsilon \leq \varepsilon_0$ we have by Lemma 3.2

$$\phi \leq u^\varepsilon - \rho \quad \text{in } B_R(X_1) \cap (\Omega^\varepsilon \cup \Sigma^\varepsilon)\,.$$

Using this ϕ as a local lower solution in $B_R(X_1)\cap(\Omega^\varepsilon\cup\Sigma^\varepsilon)$, we obtain an ellipticity ratio from equation (3.8), independent of ε.

Thus in $\overline{B_{3R/4}} \cap (\Omega^\varepsilon \cup \Sigma^\varepsilon)$, we have uniform ellipticity, L^∞ bounds on ρ^ε and u^ε, and uniform obliqueness (which depends only on the C^1 bound of ρ^ε (4.1) and on $\sup u^\varepsilon$), where all these bounds are uniform in ε, and we can apply local Hölder estimates [17, lemma 2.1] to get

$$\|u^\varepsilon\|_{C^\alpha(\overline{B_{R/2}(X_1)}\cap(\Sigma^\varepsilon\cup\Omega^\varepsilon))} \leq C\,, \tag{4.4}$$

where C and α depend on $\sup u^\varepsilon$, $\sup \rho^\varepsilon$, and (4.1) and are independent of ε. Since ρ^ε is a solution of (1.15) and $\sup \rho^\varepsilon$ is uniform in ε, the Hölder estimate (4.4) immediately gives us

$$\|\rho^\varepsilon\|_{C^{1+\alpha}(B_{R/2}(\eta_1))} \leq C\,, \tag{4.5}$$

where $B_{R/2}(\eta_1)$ is a closed interval on (η_0, η^*) with half-length $R/2$, centered at η_1 (it is the projection of $B_{R/2}$ on the η-axis). Now, taking account of the uniform $C^{1+\alpha}$ bound on ρ^ε (4.5) and the uniform C^α bound on u^ε in $\overline{B_{R/2}} \cap (\Sigma^\varepsilon \cup \Omega^\varepsilon)$

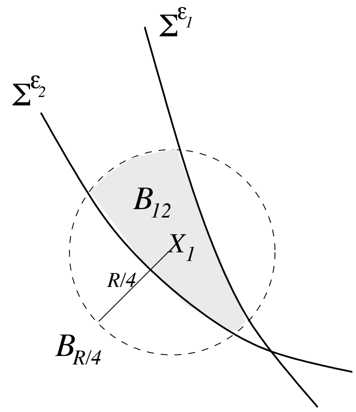

FIGURE 4.1. The domain B_{12}.

(4.4), we apply Schauder estimates (as in theorem 8.33 and lemma 6.29 of [14] or theorem 4.21 in [19]) to get

$$\|u^\varepsilon\|_{C^{1+\alpha}(\overline{B_{R/3}}\cap(\Sigma^\varepsilon\cup\Omega^\varepsilon))} \le C\,. \tag{4.6}$$

Since ρ^ε is a solution of the differential equation (1.15), and the bounds (4.6) and (4.5) are uniform in ε, we repeat the argument to get another derivative

$$\|\rho^\varepsilon\|_{C^{2+\alpha}(B_{R/3}(\eta_1))} \le C\,. \tag{4.7}$$

Then by Schauder estimates (lemmas 6.5 and 6.29 in [14] or theorem 4.21 in [19]) once again, we have

$$\|u^\varepsilon\|_{C^{2+\alpha}(\overline{B_{R/4}}\cap(\Sigma^\varepsilon\cup\Omega^\varepsilon))} \le C \tag{4.8}$$

uniformly in ε.

From (4.7), the uniform $C^{2+\alpha}$ bound on ρ^ε, there exists a subsequence of ρ^ε that converges in $C^{2+\alpha'}(B_{R/3}(\eta_1))$ for any $\alpha' < \alpha$. Thus, by covering arguments and diagonalization, we obtain a subsequence of ρ^ε that converges in $C^{2+\alpha'}_{\text{loc}}$ to a limiting function $\rho \in C^{2+\alpha'}((\eta_0,\eta^*))$.

It remains to prove that ρ and u satisfy equations (4.2) on Σ. For each $B_{R/4}(X_1)$ centered at $X_1 \in \Sigma$ and $\varepsilon_i \le \varepsilon_0$ where $i = 1, 2$, consider the domain, say $B_{12} \subset B_{R/4}(X_1)$, created by the boundaries Σ^{ε_1} and Σ^{ε_2} in $B_{R/4}(X_1)$, as in Figure 4.1.

Now, either u^{ε_1} or u^{ε_2} is undefined in B_{12}; however, we use (4.7) and (4.8) to define smooth extensions such that $u^{\varepsilon_i} \equiv u^{\varepsilon_i}|_{\Sigma^{\varepsilon_i}\cap B_{R/4}}$ in B_{12} for the corresponding $i = 1, 2$, and (4.8) holds in B_{12} as well for the other function. Suppose that u^{ε_1} is defined in B_{12}. Since $u^{\varepsilon_1} - \rho \ge \phi > 0$, letting $z = u^{\varepsilon_1} - u^{\varepsilon_2}$, we get

$$L^{\varepsilon_1}z \equiv ((u^{\varepsilon_1} - \rho + \varepsilon_1)z_\rho)_\rho + z_{\eta\eta} + \frac{z_\rho}{2} = f_1(u^{\varepsilon_i}, Du^{\varepsilon_i}, D^2u^{\varepsilon_i})\,,$$

in B_{12}, and

$$\beta(\rho^{\varepsilon_1}, u^{\varepsilon_1}) \cdot \nabla z = g_1(\rho^{\varepsilon_i}, u^{\varepsilon_i}, Du^{\varepsilon_i}) \quad \text{on } \Sigma^{\varepsilon_1} \cap B_{R/4}\,, \tag{4.9}$$

$$\beta(\rho^{\varepsilon_2}, u^{\varepsilon_2}) \cdot \nabla z = g_2(\rho^{\varepsilon_i}, u^{\varepsilon_i}, Du^{\varepsilon_i}) \quad \text{on } \Sigma^{\varepsilon_2} \cap B_{R/4}\,; \tag{4.10}$$

here $i = 1, 2$, $\|z\|_{L^\infty} \le 2\rho_0$ is bounded, and we denote by f_1, g_1, and g_2 the remaining terms in the corresponding equations. Using the elliptic equation $L^{\varepsilon_1} z = f_1$ in B_{12} with oblique derivative boundary conditions on z at $\Sigma^{\varepsilon_1} \cap B_{R/4}$ (4.9) and $\Sigma^{\varepsilon_2} \cap B_{R/4}$ (4.10), and Dirichlet boundary conditions on the rest of the boundary $\partial B_{12} \subset \overline{B_{R/4}}$, we can apply $H^{(-\gamma)}_{1+\alpha}$ estimates [18, theorem 1] in B_{12} to get

$$\begin{aligned} |z|^{(-\gamma)}_{1+\alpha} &\le C|\rho^{\varepsilon_1} - \rho^{\varepsilon_2}|_{L^\infty}\Big(\sup_{B_{12}} d_\Sigma^{1+\alpha-\gamma} d^{1-\alpha}|f_1| + |g_1|^{(1-\gamma)}_\alpha + |g_2|^{(1-\gamma)}_\alpha + |z|_\gamma\Big) \\ &\le C|\rho^{\varepsilon_1} - \rho^{\varepsilon_2}|_{L^\infty}\,. \end{aligned} \tag{4.11}$$

Here C depends on the $C^{2+\alpha}$ bounds on u^{ε_i} and the $C^{1+\alpha}$ bound on ρ^{ε_i}, and all bounds are uniform in ε in $B_{R/4}$. Note here that d_Σ is the distance from the corners where the Dirichlet boundary and the oblique derivative boundary meet on $\partial B_{R/4}$. Thus, since ρ^ε converges, the limit of the right-hand side of (4.11) becomes zero as ε_1 and ε_2 go to zero. Applying a covering argument and diagonalization as before, we get $u^\varepsilon \to u$ in $C^{1+\alpha'}_{\text{loc}}$ as $\rho^\varepsilon \to \rho$.

Since we have $\rho^\varepsilon(\eta) \to \rho(\eta)$ in $C^{2+\alpha'}_{\text{loc}}$, and since we now know that the subsequence u^ε converges to u in $C^{1+\alpha'}_{\text{loc}}$, we get

$$\rho^{\varepsilon\prime} = -\sqrt{g\left(\rho^\varepsilon - \frac{u^\varepsilon + 1}{2}\right)} \to -\sqrt{g\left(\rho - \frac{u+1}{2}\right)}\,;$$

thus $\rho' = -\sqrt{g(\rho - \overline{u})}$. Finally, we have

$$0 = \beta(u^\varepsilon, \rho^\varepsilon) \cdot \nabla u^\varepsilon(\rho^\varepsilon(\eta), \eta) \to \beta(u, \rho) \cdot \nabla u(\rho(\eta), \eta)\,,$$

and thus $\beta(u, \rho) \cdot \nabla u = 0$ on Σ.

We can apply a local argument to check ellipticity: Since $u^\varepsilon - \rho \ge 7\delta R^2/16$ in $\overline{B_{3R/4}} \cap (\Omega^\varepsilon \cup \Sigma^\varepsilon)$, then the limit u satisfies $u - \rho \ge 7\delta R^2/16$ in $\overline{B_{3R/4}} \cap (\Omega \cup \Sigma)$ as well, so we get strict ellipticity. From this, we infer higher regularity: The limiting function u is in $C^{2+\alpha'}(\Omega \cup \Sigma)$. □

The final task is to prove continuity of u up to the degenerate boundary σ_0.

LEMMA 4.4 *The limit u satisfies $u = \rho_0$ on σ_0 and $u \in C(\overline{\Omega})$.*

PROOF: In Lemma 4.2 we showed that $\rho < u^\varepsilon \le \rho_0$ and that the limiting function u satisfies $\rho < u \le \rho_0$. Then, taking a limit as an interior point X approaches a boundary point $X_0 \in \sigma_0$, we get

$$\rho_0 = \lim_{X \to X_0} \rho \le \lim_{X \to X_0} u \le \rho_0.$$

Therefore $u \in C(\overline{\Omega})$. □

Lemmas 4.1, 4.2, 4.3, and 4.4 show that there exists a solution pair $(u, \rho) \in C^{2+\alpha'}(\overline{\Omega} \setminus (\sigma_0 \cup \mathbf{V})) \cap C(\overline{\Omega}) \times C^{2+\alpha'}(0, \eta_0)$ satisfying (1.16) and the free and fixed boundary conditions. This completes the proof of Theorem 1.2.

Remark. At the corner points V^* and V_2, the solution u^ε is Hölder-continuous with an exponent γ that depends on the corner angles, the a priori L^∞ bounds on u^ε, and the ellipticity ratios. The a priori L^∞ bounds and ellipticity ratios are independent of ε, and the corner angle at V_2 is fixed, while the corner angle at V^* is uniformly bounded in ε (because of the uniform properties of the set $\mathcal{K}$ established in Theorem 2.1). Thus, γ is independent of ε; therefore, the limit function u is Hölder-continuous with exponent γ at V^* and V_2.

PROOF OF THEOREM 1.1: To complete the proof of Theorem 1.1, we note that the flow in the supersonic region was constructed in the introduction. Theorem 1.2 provides a candidate for the first velocity component u of the subsonic flow in an arbitrary bounded region near the sonic line; the second component is reconstructed by integration from the ρ-axis:

$$v(\rho, \eta) = \int_0^\eta \frac{y}{2} u_y - (u - \rho) u_\rho dy$$

and is in the same space as u.

Finally, we show that the cutoff function g is the identity in a part of the domain near Σ_0.

The cutoff g can be removed unless $\rho - \overline{u} \le \delta *$. At η_0,

$$\rho - \overline{u} = \rho(\eta_0) - \frac{u(\rho(\eta_0), \eta_0) + 1}{2} = \frac{\rho_0 - 1}{2} > 0\,.$$

By Theorem 2.1, we know that $u \le \rho_0$, so the cutoff g will not apply until $\rho(\eta)$ becomes less than $(\rho_0 + 1)/2 + \delta^*$, which is strictly less than ρ_0 since $\rho_0 > 1$ and $\delta *$ is small. Therefore there is a finite neighborhood of $\rho(\eta_0)$ along the shock in which the solutions u and ρ we have found solve $\rho' = -\sqrt{\rho - (u + 1)/2}$. A bound for the size of the neighborhood is $(\rho_0 + 1)/2 \le \rho \le \rho_0$.

Thus Theorem 1.2 implies Theorem 1.1. □

We note that if, on the interval $[\eta_0, \eta^*]$, $\rho - \overline{u}$ becomes smaller than $\delta *$ for any $\delta * > 0$, then in the limit $\delta * \to 0$ the solution we have found in this theorem tends to a pair (u, ρ) with $\rho' = 0$ at a point on the shock Σ, beyond which a solution to the problem as formulated here does not exist. In fact, beyond that point one would need to consider the more general free boundary condition $(\rho')^2 = \rho - \overline{u}$ and to seek a solution with ρ now increasing. Such a solution would be incompatible with our asymptotic formula for the shock position; in addition, it appears (by comparison with a linear shock) to violate causality.

Now, because the UTSD equation does not incorporate the correct physics as $x \to -\infty$, there is no real significance to solving the problem in the entire plane. However, to allow matching of the UTSD solution to a far-field flow, it would be

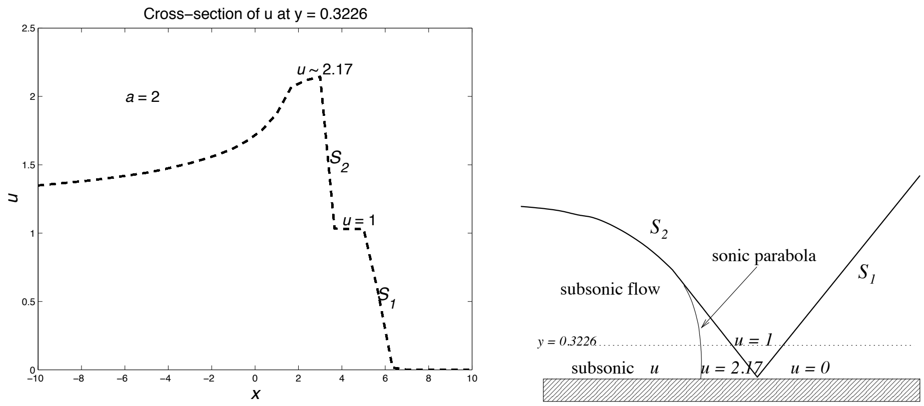

FIGURE 5.1. Results of a numerical simulation of the solution.

useful to be able to find a solution in an arbitrarily large bounded region, with a cutoff function like our f on the downstream cutoff boundary. If it is the case that g cannot be removed because a solution to $\rho' = -\sqrt{\rho - \overline{u}}$ does not exist for arbitrarily large intervals $[\eta_0, \eta^*]$, then the matching problem becomes much more delicate: One would need to develop a solution to the exterior problem (involving a solution with a smooth shock for the full set of Euler equations) all the way up to the finite neighborhood whose existence has been proved in Theorem 1.1.

5 Conclusions

We present a numerical simulation of the solution discussed in this paper, performed using the numerical solver developed in [10] for $a = 2$. At this value regular reflection occurs, and one of the two solutions is supersonic immediately behind the reflection point, as described in Section 1.1. The left picture in Figure 5.1 shows a cross section $u(x, 0.3226)$ of the first velocity component u at the value $y = 0.3226$ and $t = 1$. On the right is a sketch of the physical plane indicating the cross section. In the left picture one sees the constant state $u = 0$ ahead of the shock S_1, the state $u = 1$ between S_1 and S_2, and an approximation to the state behind the reflected shock S_2, which is near the predicted value $u_R \approx 2.17$ (see (1.7)) at S_2. The subsonic part of the solution is to the left of the point P.

The behavior of the solution to the UTSD equation at the degenerate boundary was studied in [3, 4, 9]; there it was shown that a solution taking its minimum at the degenerate boundary exhibits a square root singularity, while a solution that is increasing as one approaches the boundary is continuously differentiable with slope exactly $\frac{1}{2}$.

Here, the subsonic part of the solution is increasing towards the sonic point P. As predicted by the theory, the solution found in this simulation appears to be the differentiable solution, studied in [4].

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