Shocks, Singular Shocks and a Model for Two-Phase Flows

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April 8, 2003

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`http://www.math.uh.edu/~blk/blkp.html`

[a] Research supported by the Department of Energy, by the National Science Foundation, and by the Texas Advanced Research Program.

What this talk is about:

-conservation law approach to PDE

-shock solutions of CL, tests for admissibility

-structure of some model eqns for 2-phase or 2-comp't flows

What this talk is not about:

-how to model two-fluid flows

-how to compute two-fluid flows

-how to optimize, experiment on, or measure two-fluid flows

- Cons. Laws Model Dynamics of Fluid Flow on Acoustic Scale
  - well-posed equations are hyperbolic (real char speeds)
  - distinctive feature is discontinuous solutions (shocks)
- Loss of Hyperbolicity in Two-Fluid Equations
  - characteristics, which should be real, are instead complex
- Shocks, Singular Shocks and Nonhyperbolic Waves
  - generalizations of shocks in hyperbolic and nonhyperbolic probs
- Consequences in the Model Equations

Conservation Laws, Hyperbolicity and Well-Posedness

• Example: Isentropic Compressible Ideal Gas Flow:

Cons. of Mass: $\rho_t + (\rho u)_x = 0$

Cons. of Mom.: $(\rho u)_t + (\rho u^2 + p)_x = 0$

Closure Relation: $p = p(\rho)$

rewrite as

$$\boxed{\begin{aligned} \rho_t + m_x &= 0 \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x &= 0 \end{aligned}}$$

QL form $w_t + A(w)w_x = 0$

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + p'(\rho) & \frac{2m}{\rho} \end{pmatrix}$$

Characteristics: $\lambda = u \pm \sqrt{p'(\rho)}$

real and distinct if $p'(\rho) > 0$.

Hyperbolicity: acoustic waves from

dynamics and modeling

Focus of this research: new theory for new kinds of char. structure

Hyperbolicity closely related to well-posedness of IVP:

Hadamard example (linear): $u_{tt} + u_{xx} = 0$, $u(x,t) = \frac{1}{n^k} \sin nx \sinh nt$.

• Cons. Law System $w_t + q(w)_x = \ldots$ {lower and higher order terms}

Point of departure: 1^{st} order nonlinear diff. operator, ignoring

— balance terms (reaction, body forces, drag)

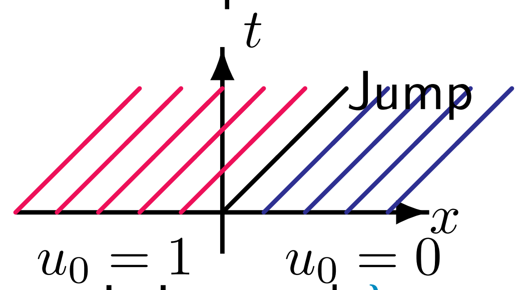
— dissipation (viscosity), surface tension, higher-dim. effects

Basic 1-D Hyperbolicity: Linear and Nonlinear

Linear Equation: waves move with characteristic speed λ

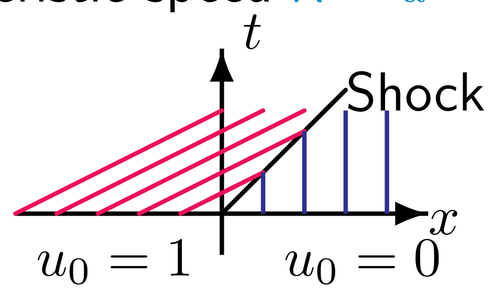
$$u_t + \lambda u_x = 0; \qquad u = f(x - \lambda t)$$

Jump Data: $u(x,t) = \begin{cases} u_L, & x < \lambda t \\ u_R, & x > \lambda t \end{cases}$



Nonlinear Equation: $u_t + uu_x = 0$, characteristic speed $\lambda = u$

Riemann Data: $u(x,t) = \begin{cases} u_L, & x < st \\ u_R, & x > st \end{cases}$



$$s = \frac{u_R^2/2 - u_L^2/2}{u_R - u_L} \approx u \text{ (char. speed) for small shocks only}$$

Rankine-Hugoniot relation for $w_t + q(w)_x = 0$: $s[w] = [q(w)]$

- Cons. form allows def'n of weak solution $\int w\varphi_t + q(w)\varphi_x = 0$
- No smooth well-posedness: Even C^∞ data $\Rightarrow$ discontinuous solution

Two Surprises

1. Some conservation laws fail to have classical shocks
2. Some physical models lead to nonhyperbolic equations

Motivating example: a 'two-fluid' model for two-component flow.

Work of our group: mathematical analysis to show that this equation and others like it, even though nonhyperbolic, have solutions with wavelike behavior.

Potential applications of the analysis:

- understand what computer simulations produce
- determine whether model has predictive power
- allow systematic study of multiscale effects

Study of full system begins with CL operator
Begin with a hyperbolic equation with no classical sol'n

Riemann Solution with Singular Shocks to Hyperbolic Model System

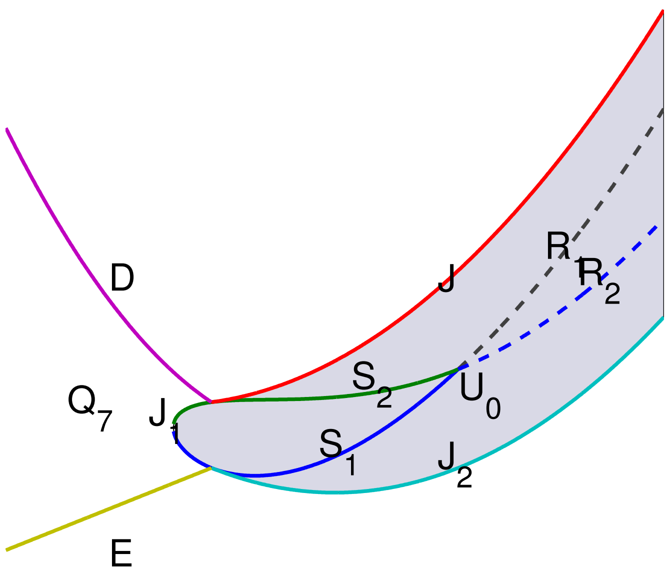
Isentropic isothermal gas dynamics equations ($\gamma = 1$) —

(smoothly) equivalent to strictly hyperbolic GNL system:

$$u_t + (u^2 - v)_x = 0 \quad \text{(velocity)}$$
$$v_t + (\tfrac{1}{3}u^3 - u)_x = 0 \quad \text{(enthalpy)}$$

where $\begin{cases} \rho = \rho(q) = e^q \\ v = \frac{1}{2}u^2 - q \end{cases}$

Riemann Problem



Theorem[K. & Kranzer]: $\exists$! sol. w. shks, raref. and sshks.

$\exists$ limits of approximate solutions:

- $w(x,t) = H + a^2\, t\, \mathbf{e}_2\, \delta(x - st)$

  $H = \begin{cases} w_-, & x < st \\ w_+, & x > st \end{cases}, \quad w_+ \in Q_7$

- $s[u] = [u^2 - v]$
- $a^2 \equiv s[v] - [\tfrac{1}{3}u^3 - u]$ RH deficit
- $u_0 \pm 1 > s > u_1 \pm 1$
- $w^\epsilon_t + q(w^\epsilon)_x \approx \epsilon w^\epsilon_{xx}$

Approximations to Singular Shocks in Hyperbolic Model Problem

"Self-similar viscosity": $w_t + q(w)_x = \epsilon t w_{xx}$, $w = w(\xi) = w(\frac{x}{t})$.

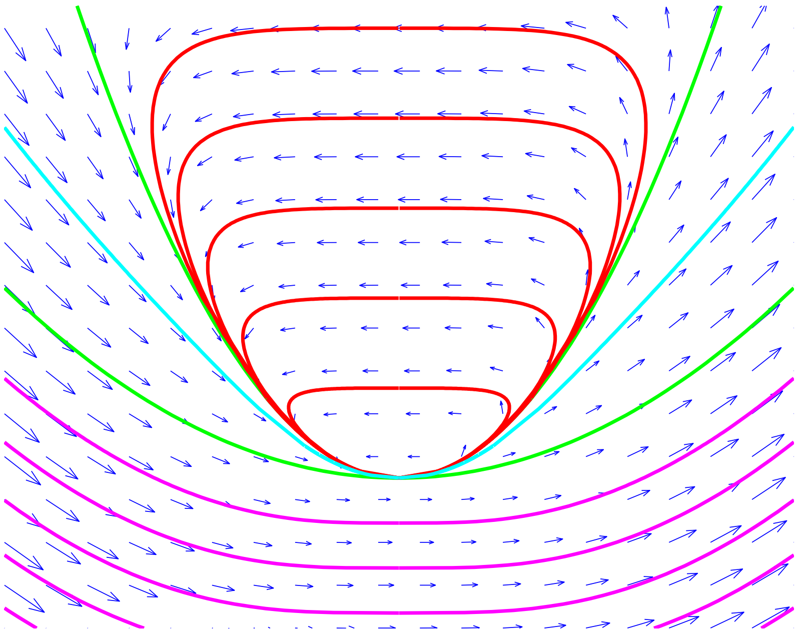
Singular shocks, near $\xi = s$ (inner expansion, width ϵ^2):

$$\widetilde{w} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\epsilon^p}\tilde{u}\left(\frac{\xi - s}{\epsilon^q}\right) \\ \frac{1}{\epsilon^r}\tilde{v}\left(\frac{\xi - s}{\epsilon^q}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\epsilon}\tilde{u}\left(\frac{\xi - s}{\epsilon^2}\right) \\ \frac{1}{\epsilon^2}\tilde{v}\left(\frac{\xi - s}{\epsilon^2}\right) \end{pmatrix}$$

Degenerate Vector Field

(Homoclinic Connection):

$$\begin{aligned} \tilde{u}' &= \tilde{u}^2 - \tilde{v} \\ \tilde{v}' &= \frac{\tilde{u}^3}{3} \end{aligned}$$



Match to outer expansion

$$\begin{aligned} u^\epsilon(\xi) &= h_1^\epsilon(\xi) + a\rho^\epsilon(\xi) \\ v^\epsilon(\xi) &= h_2^\epsilon(\xi) + a^2\delta^\epsilon(\xi) \end{aligned}$$

A Mathematical Theory for Singular Shocks (Sever)

- Approximate solutions $\{w^\epsilon\}$: $w^\epsilon_t + q(w^\epsilon)_x \rightharpoonup 0$
- Classical weak solutions: $w^\epsilon \to w$ and $q(w^\epsilon) \to q(w)$
- Singular weak solutions: $w^\epsilon \rightharpoonup w$ and $q(w^\epsilon) \rightharpoonup Q \neq q(w)$ (at sing)
- Or $w^\epsilon \to w$ in measure space where $q(w)$ not defined

Comparison of different kinds of singular weak solutions:

1. Measure-valued solutions [Azevedo et al, Chen-Frid]: oscillatory — found in model quadratic change-of-type systems
2. Delta-shocks [Tan, Zhang, Zheng; Bouchut, James; E, Rykov & Sinai]: $|q(w)| \leq c(1+|w|)$, $w \in D$, so $w^\epsilon \rightharpoonup w$ in space of measures $\Rightarrow q(w^\epsilon) \rightharpoonup Q$ in space of measures
3. Singular shocks [KK; Schaeffer, Schecter, Shearer; Sever]: $q(w^\epsilon)$ not locally bdd meas. unif. w. r. to ϵ (cf. u^3 term in KK ex.)

$$u_t + (u^2 - v)_x = 0 \qquad\qquad v_t + (\tfrac{1}{3}u^3 - u)_x = 0$$

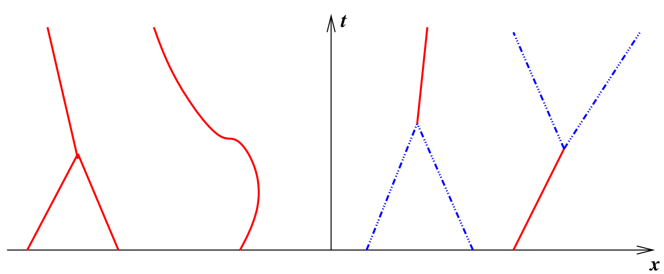
Singular Shock Weak Solutions in Hyperbolic Problems

1. "something worse can happen" [pace Glimm]
2. *Theorem [Sever]: A pair of strictly hyperbolic, genuinely nonlinear CL, with a convex entropy and satisfying a set of asymptotic conditions, admits singular shock solution limits of approximate solutions to $w_t + f(w)_x = \epsilon w_{xx}$ (or $\epsilon t w_{xx}$).*
3. Singular shock solutions:

$$w(x,t) = \widetilde{w}(x,t) + \sum_i M_i(t)\chi_{I_i}(t)\delta\big(x - x_i(t)\big) = \widetilde{w} + w_s$$

$\widetilde{w}$: nonsingular weak solution on open set $\mathbb{R}^2_+ \backslash \big\{ \cup_i \{x = x_i(t)\}\big\}$

$I_i = [I_i^-, I_i^+) \subset \mathbb{R}_+$; $M_i \in L^\infty : I_i \to \mathbb{R}^2$; singular mass



$$\dot{M}_i = a^2 = \text{R-H deficit}$$

$$w_t + Q_x = 0$$

$$Q = f(\widetilde{w}) + \sum_i A_i(t)\chi_{I_i}(t)\delta\big(x - x_i(t)\big)$$

Finite # of singular shocks at each t.

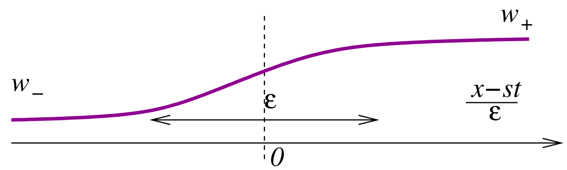
Sing Shock: Red Shock: Blue

(Existence for Cauchy problem, KLS)

A Little Viscous Profile Analysis

Admissibility criterion: $\exists\{w^\epsilon\}$, $w^\epsilon \rightharpoonup w$, $w^\epsilon(\pm\infty) = w_\pm$

$$w^\epsilon_t + q^\epsilon_x - \epsilon w^\epsilon_{xx} \rightharpoonup 0 \quad \text{or} \quad w^\epsilon_t + q^\epsilon_x - \epsilon t w^\epsilon_{xx} \rightharpoonup 0, \quad \text{as } \epsilon \to 0.$$

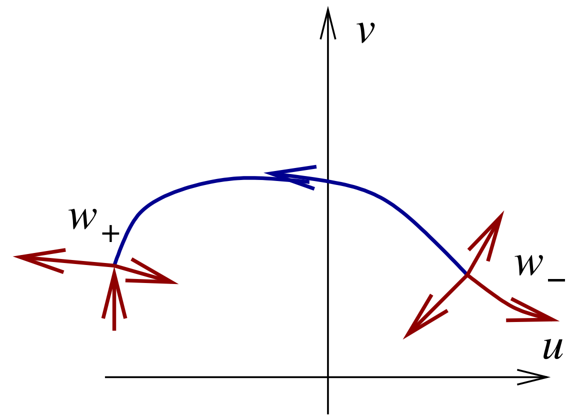


Classical: $\frac{x-st}{\epsilon} = \eta$, $w'' = q(w)' - sw'$;

Integrate: $w' = q(w) - sw + z$

$z = s(w_+) - q(w_+) = s(w_-) - q(w_-)$

$\Rightarrow s[w] = [q]$, $z = 0$, w. l. o. g.



Heteroclinic connection:

$\lambda_1(w_-) > s > \lambda_1(w_+)$, $s < \lambda_2(w_\pm)$:

$\Rightarrow$ Lax geometric entropy condition

Dafermos-DiPerna viscosity: $\xi = \frac{1}{\epsilon}\left(\frac{x}{t} - s\right) = \frac{\eta}{t}$

$w'' - q' + sw' = -\epsilon\xi w' \equiv z'$, $\quad w(\pm\infty,\epsilon) = w_\pm$, $z(\pm\infty,\epsilon) = 0$

LGEC $\Rightarrow \exists$ connecting orbit w. $z \sim 0$, w bdd., unif. in ϵ

Singular: $s[w] - [q(w)] = C$ (RH deficit); $\lambda_i(w_-) > s > \lambda_i(w_+)$,

$i = 1, 2$ ('overcomp') $\Rightarrow \exists$ conn. orbit w. z bdd, unif. in ϵ (Sever)

A New Phenomenon: Evanescent Singular Shocks in KK

$$w(x,t) = \widetilde{w}(x,t) + M(t)\chi_I(t)\delta\big(x - X(t)\big)$$

$\dot{M} = C = s[w] - [q(w)] =$ R-H deficit

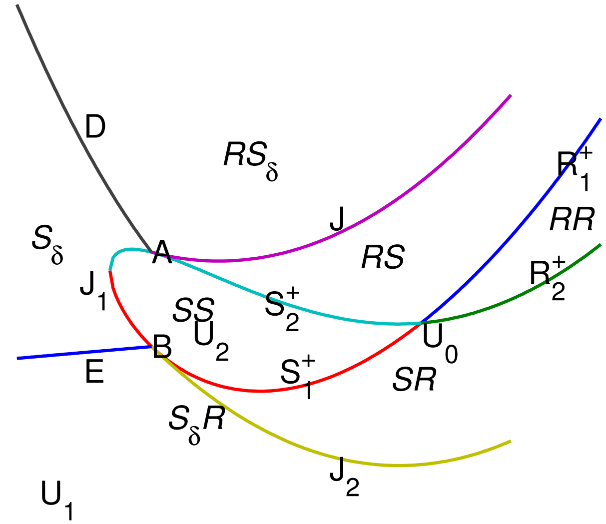
IVP: $w(x,0) = \begin{cases} w_-, & x < 0 \\ w_+, & x > 0 \end{cases} + A\delta(x),$

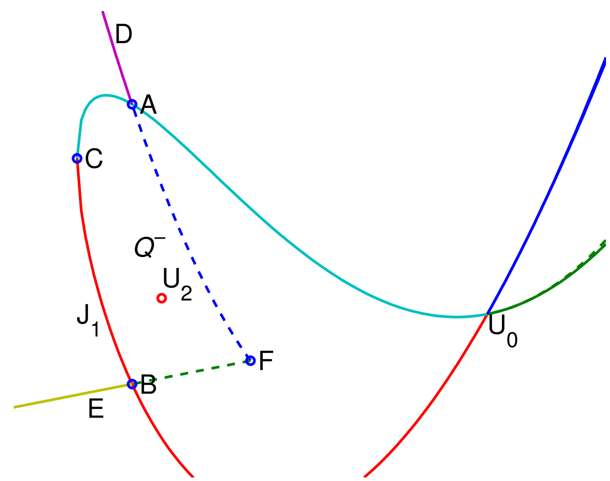
$A > 0$ but $w_+ \in Q_-(w_-)$:

Singular mass decays to 0 in finite time and ‘fission’ occurs.

IC in movie: 3 states U_0, U_1, U_2

Simulation: evolution of viscous solution





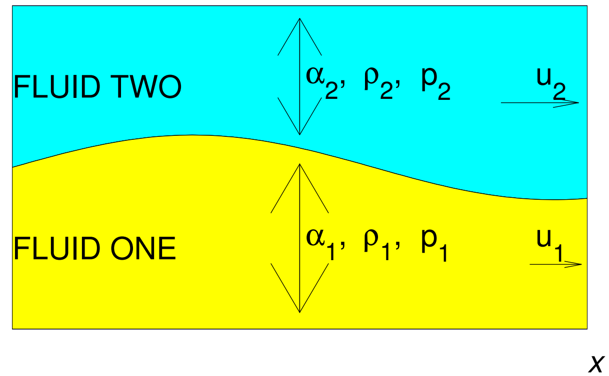
Two-Phase Flow: An Incompressible One-Dimensional Model

Ex: stratified pipe flow – 1-D avg.

Conservation of Mass & Momentum:

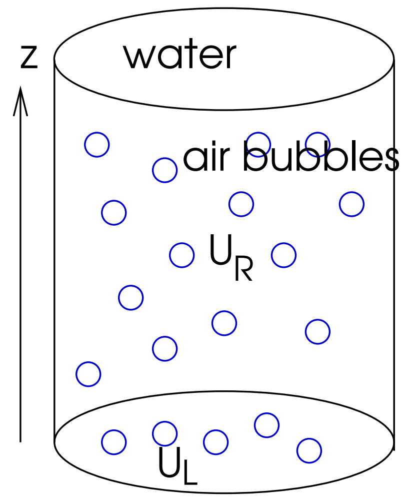
$$\partial_t(\alpha_i\rho_i) + \partial_x(\alpha_i\rho_i u_i) = 0$$

$$\partial_t(\alpha_i\rho_i u_i) + \partial_x(\alpha_i\rho_i u_i^2) + \alpha_i\partial_x p_i = F_i$$



Assumptions:

- incompressible, isentropic
- ρ_i const., $\rho_1 - \rho_2 = 1$
- $\alpha_1 + \alpha_2 = 1$ (saturated)
- $u_2 \neq u_1$ (2-velocity)
- $\boxed{p_2 \equiv p_1}$ (1-pressure)
- relaxes to other models



Notes:

- $F_i = \alpha_i\rho_i g + M_i$ (drag)
- Conservation form ambiguous

Simplified Coordinates

$$\partial_t \alpha_i + \partial_x(\alpha_i u_i) = 0$$

$$\rho_i \partial_t u_i + \rho_i u_i \partial_x u_i + \partial_x p = F_i/\alpha_i$$

Reduction to 2 eqns:

- $\beta = \rho_2\alpha_1 + \rho_1\alpha_2$: mass
- $v = \rho_1 u_1 - \rho_2 u_2 - K$: mom
- $K(t) = \alpha_1 u_1 + \alpha_2 u_2$: inlet

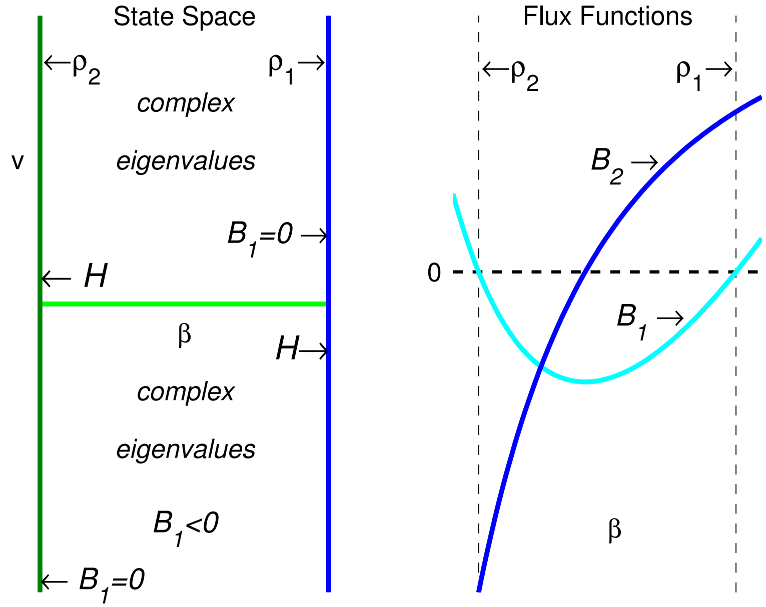
Equiv. System for Mass and Mom:

$$\boxed{\begin{aligned} &\beta_t + (vB_1(\beta) - K\beta)_x = 0 \\ &v_t + (v^2 B_2(\beta) - Kv)_x = G \end{aligned}}$$

$$B_1(\beta) = \frac{(\beta-\rho_1)(\beta-\rho_2)}{\beta}$$

$$B_2(\beta) = \frac{\beta^2 - \rho_1\rho_2}{2\beta^2}$$

$$\rho_2 \le \beta \le \rho_1$$



Characteristics:

$$\lambda = 2vB_2(\beta) \pm v\sqrt{B_1 B_2'}$$

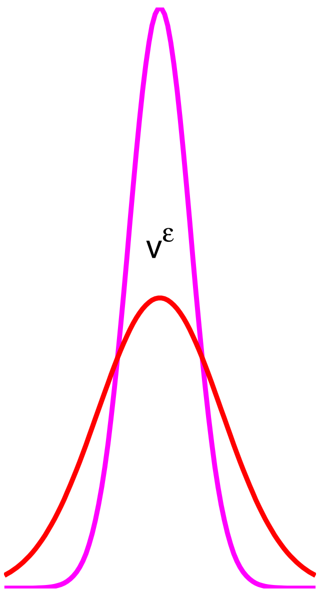
Complex: $B_1 \le 0$, $B_2' > 0$.

- $H = \{\beta = \rho_i\} \cup \{v = 0\}$
- *Linearly unstable*, $\rho_2 < \beta < \rho_1$

Riemann Problem, Incompressible System, Nonhyperbolic Operator

Appearance of Singular or "Phase Boundary" Shocks

R P $\quad w_t + q(w)_x = 0, \quad w(x,0) = \begin{cases} w_-, & x < 0 \\ w_+, & x > 0 \end{cases}$



Approx: $w_t + q(w)_x = \epsilon t w_{xx}$, $w = w(\xi) = w(x/t)$.

Singular shocks, near $\xi = s$ (inner exp, width $f(\epsilon)$):

$$\widetilde{w} = \begin{pmatrix} \beta \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\epsilon^p}\tilde{\beta}\left(\frac{\xi-s}{\epsilon^q}\right) \\ \frac{1}{\epsilon^r}\tilde{v}\left(\frac{\xi-s}{\epsilon^q}\right) \end{pmatrix} = \begin{pmatrix} \tilde{\beta}\left(\frac{\xi-s}{\epsilon^2}\right) \\ \frac{1}{\epsilon}\tilde{v}\left(\frac{\xi-s}{\epsilon^2}\right) \end{pmatrix}$$

Find heteroclinic connections

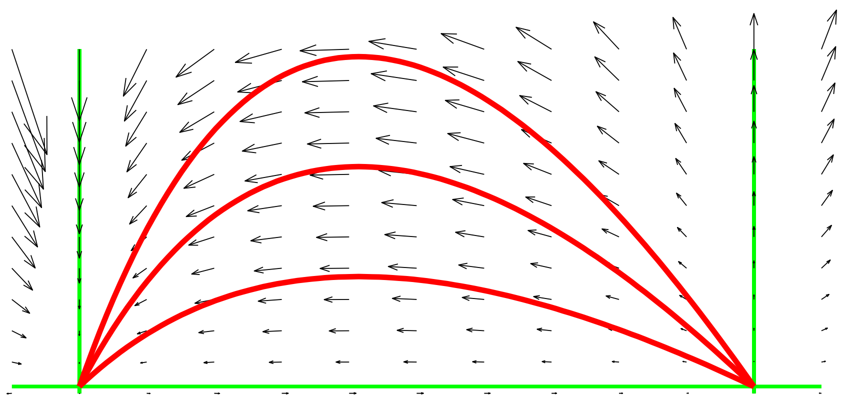
with amplitude $e^{1/\epsilon}$ in

$$\tilde{\beta}' = \tilde{v} B_1(\tilde{\beta})$$
$$\tilde{v}' = \tilde{v}^2 B_2(\tilde{\beta})$$

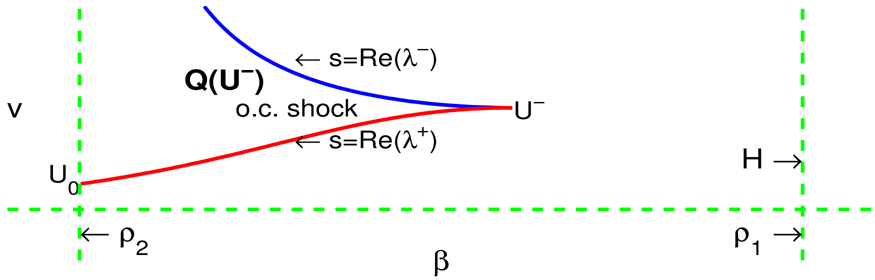
Complete with outer expansion

$\overline{w}((\xi - s)/\epsilon)$ to connect w_-, w_+

Flux Vector Field



Singular Shock Solutions to Riemann Problem



Singular shock: $w_+ \in Q(w_-)$

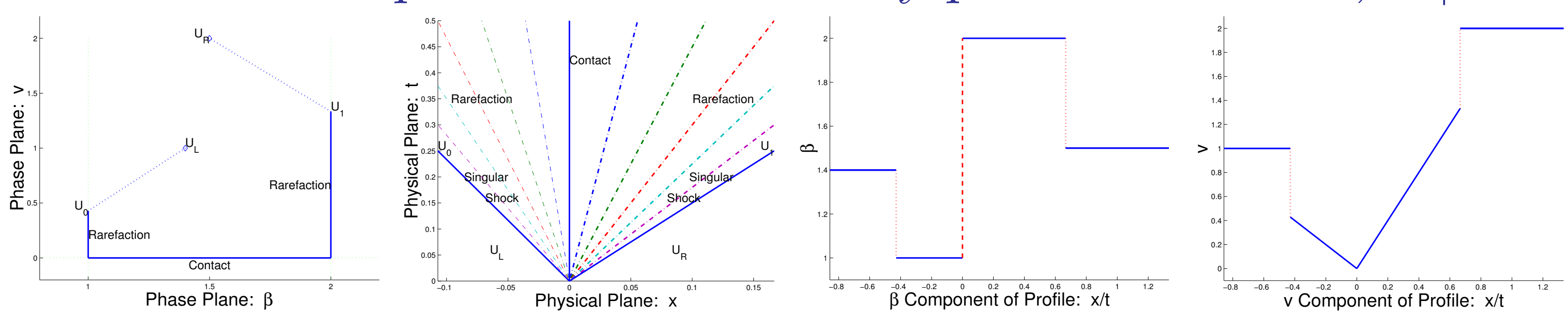
Generalized R-H cond:

$$s[\beta] = [vB_1(\beta)]$$

$$s[v] = [v^2 B_2(\beta)] + C$$

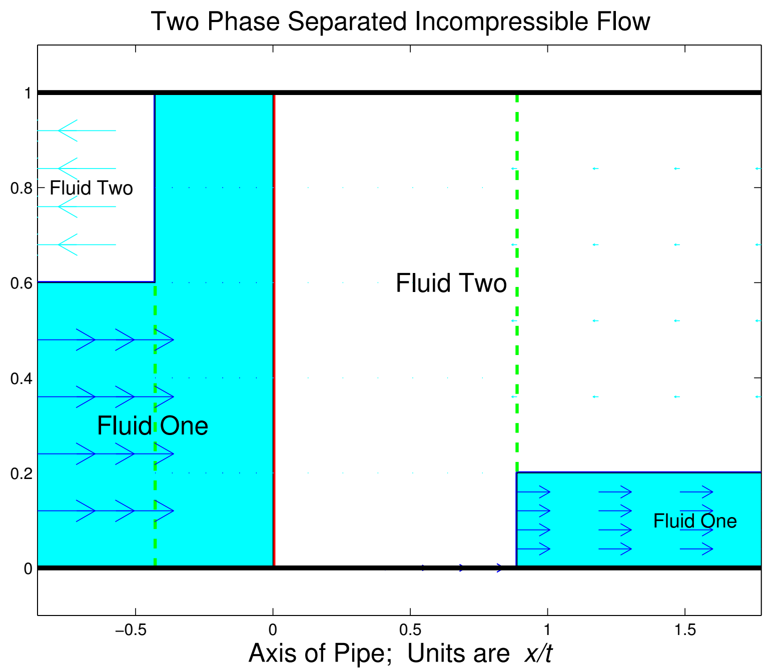
- Overcompressive admissibility condition, $\Re(\lambda^-) \geq s \geq \Re(\lambda^+)$

Proposition: *In the class of self-similar solutions, with the weakly overcompressive admissibility condition for singular shocks and the standard Lax admissibility condition for regular shocks, a solution of the Riemann problem exists for any pair of states* w_-, w_+.



- Singular shock + contact + rarefaction form composite wave.
- Cont. dependence on Riemann data except as w_+ crosses ∂Q.

Stratified Pipe Flow Model: Flow Separation

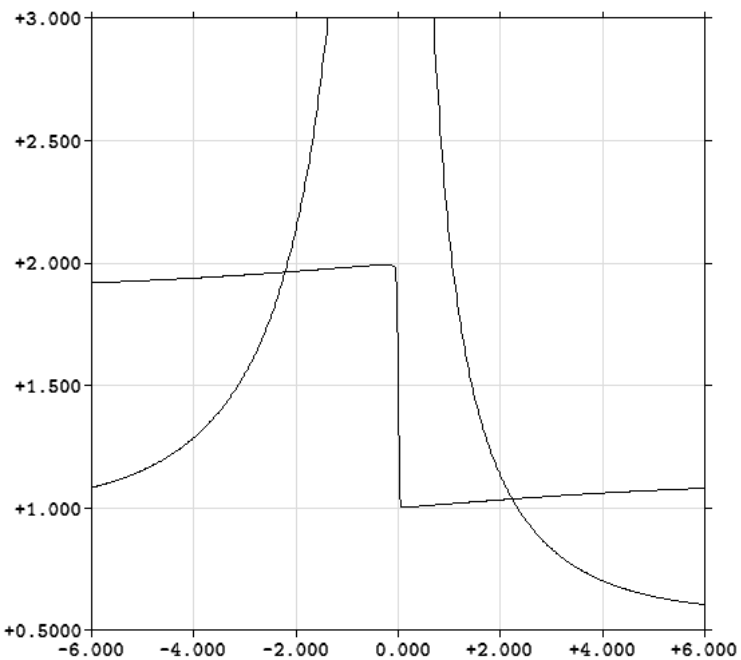


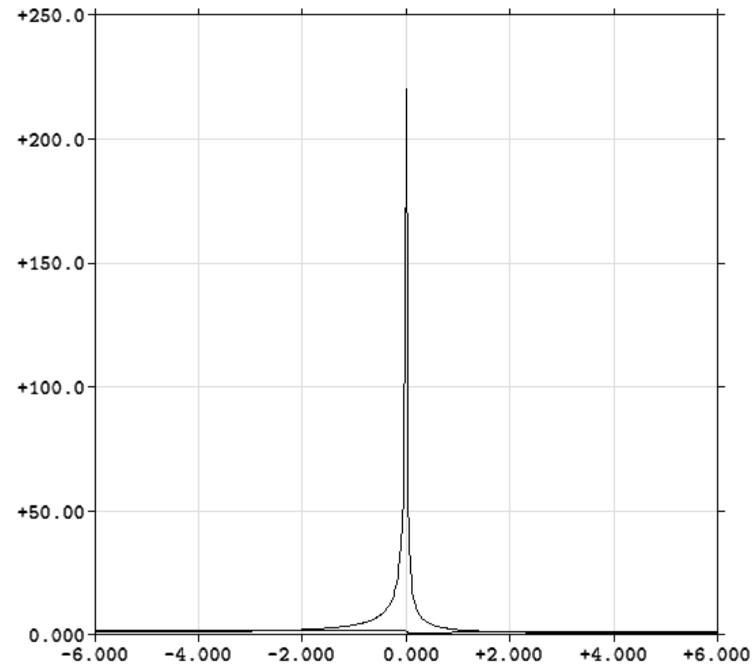
1. Spatial scale centered at contact discontinuity
2. Singular shock (singularity in velocity/mom) at moving phase boundary
3. Stable single-phase states appear
4. Left phase boundary moves upstream relative to contact discontinuity and downstream relative to left inflow boundary

- ‘Rarefaction’ occurs in absent phase (constant flow in other phase)
- Trivial Riemann data ($w_- = w_+$) lead to nonconstant solution, similar to above (ill-posedness!); expected for nonhyperbolic problem
- With no surface tension or drag, Bernoulli effect dominates
- Scale for origin for Riemann problem is implicit

Numerical Simulation of Singular Shock: Self-Similar Viscosity

$$w_t + q_x = \epsilon t w_{xx}$$

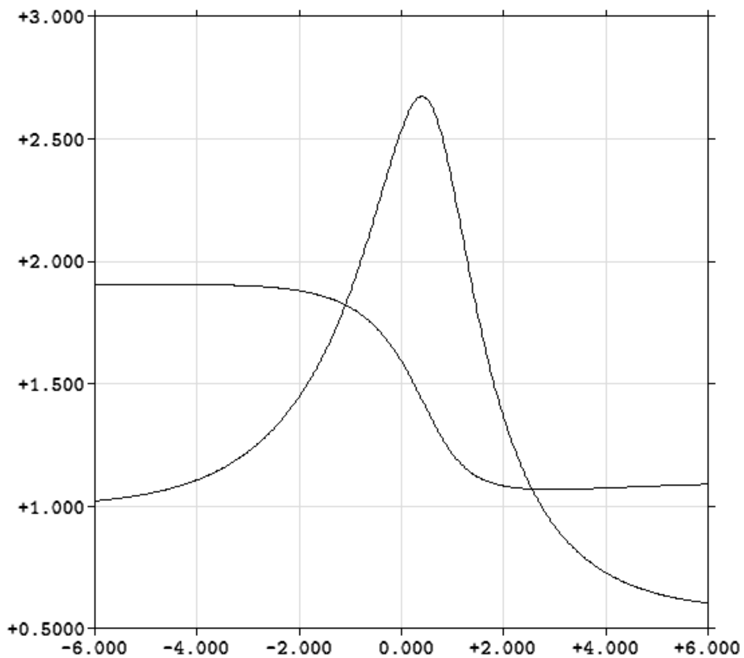


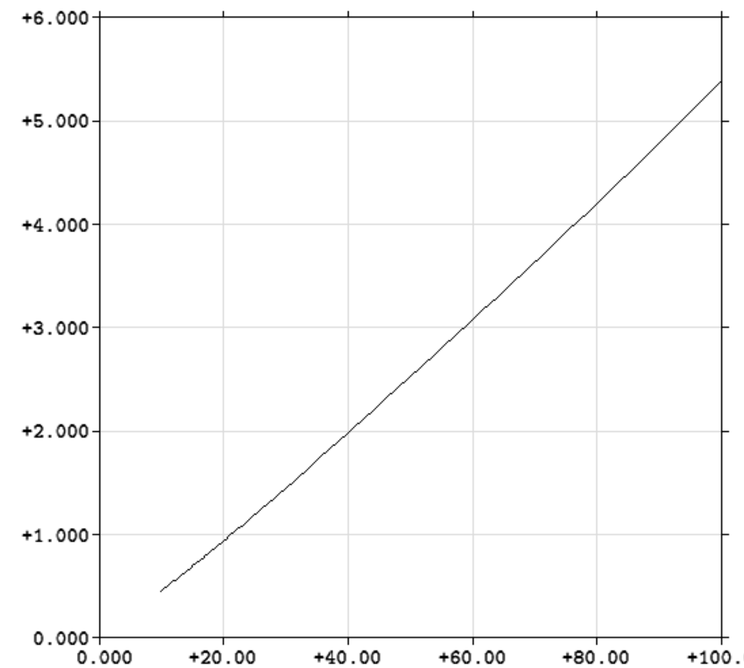


Profiles of β and v vs. $x/\epsilon t$ for $\epsilon = .01$

Exponential Growth Rate of Peak Height

Analysis: $v^{\epsilon}(0) \approx \exp\left(\frac{C}{2\epsilon(\rho_1+\rho_2)}\right)$





Profiles of β and v, $\epsilon = .05$ $\qquad$ $\log(v^{\epsilon}(0))$ vs. ϵ

Balance Terms: Bubble Column with Gravity and Interfacial Drag

Preliminary Results

Two-fluid, single-pressure, two-velocity, incompressible, 1-D model:

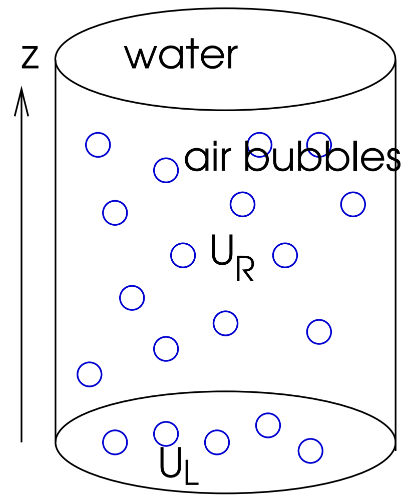
— same operator as stratified pipe flow

Typical balance term in β-v equations:

$$\beta_t + (vB_1(\beta) - K\beta)_z = 0$$

$$v_t + (v^2 B_2(\beta) - Kv)_z = G$$

$$G = -\left(g + \frac{A|v|v}{\beta^2}\right)$$



Solution tends to curve of stable equilibria.

Shock formation vs. trend to equilibrium.

Relaxation to single-velocity model?

Summary

- Two-fluid models of multicomponent systems are nonhyperbolic.
- Stratified flow: interfaces are unstable.
- Analysis of nonhyperbolic equations is possible using CL theory.
- Results are physically plausible.
- Nonlinear effects mitigate catastrophic instability.
- Viscosity/drag damp high-/low-frequency oscillations.
- Novel shocks appear (but may be damped by drag).
- Mathematical analysis may be useful: shows what is being computed in simulations using two-fluid model.
- Two-pressure model relaxes to equal pressure.
- Two-velocity model studied here may approach single-velocity limit if drag terms present.