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LACK OF HYPERBOLICITY IN THE TWO-FLUID MODEL FOR TWO-PHASE INCOMPRESSIBLE FLOW

Abstract. The two-fluid equations for two-phase flow, a model derived by averaging, analogy and experimental observation, have the property (in at least some commonly-occurring derivations) of losing hyperbolicity in their principal parts, those representing the chief entries in modeling conservation of mass and transfer of momentum and energy.

Much attention has centered on reformulating details of the model to avoid this awkwardness. This paper takes a different approach: a study of the nonhyperbolic operator itself. The objective is to understand the nature of ill-posedness in nonlinear, as distinct from linearized, models.

We present our initial study of the nonlinear operator that occurs in the two-fluid equations for incompressible two-phase flow. Our research indicates that one can solve Riemann problems for these nonlinear, nonhyperbolic equations. The solutions involve singular shocks, very low regularity solutions of conservation laws (solutions with singular shocks, however, are not restricted to nonhyperbolic equations). We present evidence, based on asymptotic treatment and numerical solution of regularized equations, that these singular solutions occur in the two-fluid model for incompressible two-phase flow. The Riemann solutions found using singular shocks have a reasonable physical interpretation.

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1. **Introduction: Nonhyperbolic Conservation Laws.** Systems of conservation laws which are not of classical, strictly hyperbolic type have been studied recently by mathematicians and other scientists. The original motivation was the

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appearance of such systems in models for the dynamics of complicated flow systems. Two examples are a model for three-phase convection-driven flow in porous media, described by Bell, Trangenstein and Shubin in [3]; and a model for two-phase elastodynamics, given by James in [12].

In the remainder of this section, we describe these models and give contexts in which nonlinear nonhyperbolic equations can be constructively discussed. Beginning with Section 2, we focus on a specific model system, arising from the two-fluid model for two-phase incompressible flow. We find an equivalent reduced system. In Section 3, we show that the system does not have classical Riemann solutions, and give an asymptotic argument that singular shocks appear; then singular shocks are used to solve the Riemann problem. Interpretation of the Riemann solution is given in Section 4, a discussion of balance terms in Section 5, and results on numerical approximation of singular shocks in Section 6.

We begin by noting that the use of conservation laws to model *steady* flows is classical, constituting a central topic in the celebrated monograph of Courant and Friedrichs, [6], published over fifty years ago; these equations are hyperbolic in type for supersonic flows. It is well known that conservation law models for steady flows which contain both supersonic and subsonic regimes (transonic flows) change type according to the flow speed, displaying both hyperbolic and nonhyperbolic regions in phase space. A mathematical theory for equations arising from this phenomenon was developed even earlier, [36]; however, as far as we know, serious engineering use was not made of these models until transonic flight became a reality in the 1970's. The first practical algorithm to simulate steady transonic flows by finite difference approximation was designed by Murman and Cole, [26]. Mathematically, there are fundamental differences between steady and unsteady change of type; not in the equations themselves, which are identical in form, but in the boundary conditions and entropy conditions for shocks, which are necessary for the formulation of well-posed problems. These have their genesis in the different notions of causality in the two types of systems: by embedding a steady transonic flow in a larger unsteady system one immediately recovers hyperbolicity. Some of this is discussed in [14]. In the remainder of this article we discuss only the more mysterious and controversial problem of unsteady change of type.

The three-phase porous medium flow example appeared when the authors of [3] stripped a set of equations commonly used in reservoir simulation down to their essentials. In a single space dimension, assuming a homogeneous medium, using Darcy's law to eliminate the pressure, ignoring gravity and dispersive effects, and choosing some well-known interpolations for the three-phase relative permeabilities ('Stone's model') results in a system of two conservation laws for two of the relative saturations. For most values of the empirically determined parameters in Stone's model, one finds a small region in the saturation triangle in which the characteristic speeds are complex. The authors found that this change of type caused no difficulties in computer simulations of the flow; this was probably because the finite-difference scheme used was first-order and moderately dissipative. The results did not seem to be highly dependent on the amount of dissipation; the finite-difference solutions appeared to be reliable approximations to straightforward shock and rarefaction structures, such as one would expect in a system of conservation laws. The authors of [3] rejected the non-hyperbolic model on the grounds that the predicted and computed solutions avoided states in the non-hyperbolic region — an avoidance which did not appear to have any physical basis.

A second kind of situation leading to change of type is exemplified by James's model in two-phase elasticity. The one-dimensional conservation law system arising from a stress-strain law $p(u)$ is

$$\begin{aligned} u_t - v_x &= 0 \\ v_t - p(u)_x &= 0. \end{aligned} \tag{1}$$

Here u is the displacement gradient (strain) and v the velocity of a point on a rod with reference coordinate x. An accepted model for two-phase elastostatics postulates a stress function $p = W'$ where W is a double-well potential. James used the dynamic model given by (1) with the stress function p of the two-phase static model. The characteristics of the system (1) are $\pm\sqrt{p'}$, and they are purely imaginary for the range of u in which $p' = W''$ is negative. Now, in this example there is some reason to suppose that states u for which W is convex down will be unstable; one certainly does not expect them to appear in steady-state configurations, and it would not be surprising for them to be absent from dynamic flows as well. To some extent, it was the expectation of seeing interesting consequences of this instability that prompted James to examine the model (1) in [12]. And, indeed, he found the usual shock and rarefaction structures of conservation laws, including propagating phase boundaries: shocks that divided material in one phase from a state in the other.

These are but two of many examples. At around the same time as James, Slemrod looked at a similar system as a model for the dynamics of a van der Waals gas near the critical temperature, [35]. Within the last ten years, a model that changes type has been used to describe some instabilities associated with chemotaxis, by Levine and Sleeman, [24], and also by other authors cited by them. Carpio, Chapman and Velázquez, [4], have introduced a change-of-type model for dislocation interactions in crystals, mainly as a mechanism to produce instability.

Previous work of Keyfitz, [13, 16], has looked at change of type models which appear in three-phase porous medium flow, and at a simple two-phase compressible flow model. Keyfitz and co-authors have also studied mathematical properties of systems that change type, [2, 14, 15, 22]; see the review articles, [17, 18, 19].

Can there be a theory of equations that change type? The Hadamard instability suggests that the initial-value problem for a linear elliptic problem is meaningless. Nonlinear problems may nonetheless have features which mitigate this conclusion.

1. Perturbations which are localized near a shock may be absorbed in the shock before they have grown very large. This formed the basis of results on shock stability in some systems which change type, by Ames and Keyfitz, [2], and by Keyfitz and Lopes, [22].
2. In a nonlinear equation, the instability saturates: once the solution takes values in the hyperbolic region, then it stops growing. This was observed by Sever, [34], as described below.
3. Finally, shock formation may cause decay even in the absence of an explicit dissipative mechanism. This feature arises even in a scalar convex conservation law, as is evidenced by L_∞ decay of periodic solutions, by contrast to L_∞ invariance in a linear transport equation. A suitable admissibility condition, which may be based on a vanishing viscosity limit, is required; however, decay is present even when viscosity is not, [23, Theorem 6.3].

We illustrate the second point by a simple example of nonlinear stabilization.

Consider a system of conservation laws of the form

$$w_t + q(w)_x = 0, \quad x \in \mathbb{R}, \;\; t > 0, \quad w(x,t) \in D = H \cup N \subset \mathbb{R}^n$$

equipped with an entropy density U which is strictly convex for w in the hyperbolic region H, and which satisfies

$$U|_{\partial H} = U_w|_{\partial H} = 0,$$

on ∂H, the boundary of H. For simplicity, we assume here that the nonhyperbolic region N is bounded, noting that this can be relaxed with additional assumptions on U. Denote by w_ϵ, $\epsilon > 0$, the solution of

$$w_{\epsilon,t} + q(w_\epsilon)_x = \epsilon w_{\epsilon,xx} \quad x \in \mathbb{R}, \;\; t > 0, \tag{2}$$

with $w_\epsilon(.,0)$ given, of bounded variation and independent of ϵ.

Proposition 1.1. *Assume that the w_ϵ satisfy* (2) *and*

$$w_\epsilon(x,t) \to w_\pm = w(\pm\infty, 0) \in H, \quad w_{\epsilon,x}(x,t) \to 0 \text{ as } x \to \pm\infty,$$

and that there exist constants c and L independent of ϵ such that

$$\int_{|x|>L(1+t)} |U(w_\epsilon(x,t) - U(w(x,0))|\, dx \le c;$$

then for any $t > 0$, $|w_\epsilon|$ and $U(w_\epsilon)$ are bounded in $L_{1,\mathrm{loc}}$ uniformly with respect to ϵ.

A proof is given in [34]; it is an easy application of the entropy inequality for such systems. For systems with these properties, bounded and weakly convergent sequences of approximate solutions (in the space of measures on $\mathbb{R}$, pointwise in t) may realistically be sought, despite the failure of hyperbolicity.

The Cauchy problem for nonhyperbolic systems is obviously not well-posed in the same sense as for hyperbolic systems. We propose to avoid this difficulty by restricting the initial data to a class for which the Cauchy problem is well-posed. For example, we consider Riemann problems for a nonhyperbolic pair of conservation laws below, hoping ultimately to extend the possible form of the initial data and thus obtain a model with predictive value.

In this context, computational methods based on Riemann solvers, such as Glimm's and Godunov's schemes and front-tracking algorithms, are regularly used for theoretical analysis and do not explicitly require hyperbolicity. To be precise, such methods require solution of Riemann problems with suitable estimates. At states where a system is not hyperbolic, the Hugoniot locus cannot be of classical form locally, and small data Riemann problems are not classically solvable. However, it may be possible to extend the solution class, as we do in this paper for a model problem. In this context, locally bounded measures such as delta-shocks and singular shocks may be introduced; these may be compatible with some front-tracking algorithms, even if not with random choice or averaging schemes such as Glimm's or Godunov's.

The following system, discussed by Sever, [34], serves as an example. It also introduces singular shock solutions, which will be discussed in detail later in this paper. The system

$$\begin{aligned} u_t + \left(\frac{u^2}{2} - e\right)_x &= 0 \\ \left(\frac{u^2}{2} + e\right)_t + \left(\frac{u^3}{3}\right)_x &= 0, \end{aligned} \tag{3}$$

although it does not have complex characteristics, is not hyperbolic, as it has a single characteristic speed $\lambda = u$ of multiplicity two with a single, nondegenerate characteristic family.

The system (3) does not admit classical shocks; however, Riemann problems can be uniquely solved by introducing singular shocks, [31, 34]. Thanks to the exceptionally rich symmetry group possessed by the system, a convergent sequence of front-tracking approximations is obtained, provided the initial data satisfies

$$u(\cdot, 0) \in BV(\mathbb{R})$$

and $e(\cdot, 0)$ is piecewise constant, with

$$\sum_i |e(x_i + 0, 0) - e(x_i - 0, 0)|^{1/2} < \infty,$$

where $\{x_i\}$ are the points of discontinuity of $e(\cdot, 0)$. Given such data, for all $t > 0$ we find $u(\cdot, t) \in BV(\mathbb{R})$ and $e(\cdot, t)$ a locally bounded measure on $\mathbb{R}$. At each t, the isolated singularities appearing in the solution $u(., t)$, $e(., t)$, are weak limits, in the space of measures on $\mathbb{R}$, of approximations with viscous structure. The limits satisfy equation (3) weakly in the dual of the space of functions $\theta \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^+ \to \mathbb{R})$ such that $\theta_x = 0$ in a neighborhood containing the singular support of e [34].

2. **Model Equations for Incompressible Flow.** In the remainder of this paper, we focus on a model for incompressible flow. The details of the solution illustrate three points raised in the introduction: by focusing on Riemann problems, we localize the perturbation of a nonhyperbolic state; the oscillations we find are finite in amplitude; and the oscillations have a fixed frequency.

Consider the simplest model for two-phase, one-dimensional incompressible flow, exhibited under this name in Drew and Passman's book, [10, page 248]. The authors distinguish between continuous and dispersed phases, but this distinction is unimportant here. We shall examine two applications of this model in Sections 4 and 5 of this paper. The first is stratified flow in which two continuous phases are separated by an interface. The second is dilute bubbly flow in which a vapor is dispersed in a liquid. We call the phases 1 and 2, and their densities (assumed constant here) ρ_1 and ρ_2. Let α_i be the volume fraction of the i-th phase, u_i the velocity and p_i the pressure of the phase. Then conservation of mass and balance of momentum yield the four equations:

$$\partial_t(\alpha_i) + \partial_x(\alpha_i u_i) = 0 \tag{4}$$

$$\partial_t(\alpha_i \rho_i u_i) + \partial_x(\alpha_i \rho_i u_i^2) + \alpha_i \partial_x p_i = F_i, \tag{5}$$

where F_1 and F_2 are balance terms involving the interfacial force density, gravity and other forces. In many common models, this term does not contain derivatives of the variables, and hence does not influence the type of the equation. Initially, we shall proceed as though the F_i were both zero. They represent important physical effects, but our first goal is to make mathematical sense of the nonlinear operator in (4), (5).

The six unknowns are reduced to four by the relations

$$\alpha_1 + \alpha_2 = 1 \tag{6}$$

and

$$p_1 = p_2 \equiv p. \tag{7}$$

The first says merely that the two fluids fill the physical space (a channel, say); from equation (4), each is separately conserved. The second gives rise to a "single pressure model", and is often altered to alter the type of the equation. However, in at least one simple case, one-dimensional averages of stratified flows, where α_1 and α_2 represent the fraction of a channel filled by each fluid, then the pressures in each phase are equal, except for the effects of surface tension. In the present context, surface tension is considered a higher-order effect which acts only to regularize the flow, and is ignored.

Questions arise from the outset about the momentum equations, (5), which are not in conservation form. This is a consequence of approximations which are made during the averaging process in order to obtain a closed system, and which tacitly assume smooth solutions. Dal Maso, LeFloch and Murat, [8], showed that a consistent mathematical theory can be developed to cover equations like (5) which are not in conservation form, but it does not provide a basis for choosing one form of the equations over another. In fact, the momentum equations, (5), are often replaced by a pair, easily put in conservation form, which is equivalent only for smooth solutions, [10, p 248]:

$$\alpha_i \rho_i \left(\partial_t u_i + u_i \partial_x u_i\right) + \alpha_i \partial_x p_i = F_i. \tag{8}$$

A closed system of conservation laws, equivalent to (4), (8), and equivalent to (4), (5) for smooth solutions, can be obtained using (6) and (7). For this procedure, the forces F_i are unimportant and are neglected in the interest of notational simplicity.

The two equations in (4) both govern the time evolution of a single volume fraction, while there is no term at all for the time evolution of the pressure. Drew and Passman describe the quasilinear system (4), (5) as having two infinite-speed characteristics. Technically, this is so. (The incompressible Euler equations, governing the evolution of a single ideal fluid, have the same feature of an infinite speed.) An interpretation of this feature is that two of the four variables adjust instantaneously to changes in the other two, and therefore one can reduce (4), (5) to a system of two equations for two variables, and solve by integration for the other two. The following procedure, devised by Dafermos [7], gives a tidy reduction.

Adding the two equations in (4) and using (6) gives $\partial_x(\alpha_1 u_1 + \alpha_2 u_2) = 0$, and so

$$\alpha_1 u_1 + \alpha_2 u_2 = f(t). \tag{9}$$

If we assume that the flow conditions are constant at one point, an inflow boundary for example, then we can replace (9) by

$$\alpha_1 u_1 + \alpha_2 u_2 = K. \tag{10}$$

Now, this observation eliminates one variable, and we can eliminate the pressure by subtracting multiples of the two equations in (8); the pressure can later be found by integrating one of those equations. Hence, we can write (4), (5) as a system of two equations in two conserved quantities, from which we can recover the others. Define two affine functions of α_1, α_2, u_1 and u_2:

$$\beta = \rho_2 \alpha_1 + \rho_1 \alpha_2 \tag{11}$$

$$v = \rho_1 u_1 - \rho_2 u_2 - (\rho_1 - \rho_2) K. \tag{12}$$

Then, from (6) and (11) we obtain

$$\alpha_1 = \frac{\rho_1 - \beta}{\rho_1 - \rho_2}, \qquad \alpha_2 = \frac{\beta - \rho_2}{\rho_1 - \rho_2}, \tag{13}$$

and from (10) and (12), using (13), we obtain

$$u_1 = \left(\frac{\beta - \rho_2}{\rho_1 - \rho_2}\right)\frac{v}{\beta} + K, \qquad u_2 = \left(\frac{\beta - \rho_1}{\rho_1 - \rho_2}\right)\frac{v}{\beta} + K. \tag{14}$$

We simplify the expressions by assuming $\rho_1 - \rho_2 = 1$; this can be achieved by rescaling the space or time variable. Thus, α_1, α_2, u_1 and u_2 are expressed in terms of β and v.

Using (6), (7), (10), (11) and (12) in (4) and (8), with $F_i = 0$, one obtains a system in conservation form for β and v:

$$\beta_t + \left(vB_1(\beta) + K\beta\right)_x = 0 \tag{15}$$

$$v_t + \left(v^2 B_2(\beta) + Kv\right)_x = 0, \tag{16}$$

where

$$B_1(\beta) = \frac{(\beta - \rho_1)(\beta - \rho_2)}{\beta}, \qquad B_2(\beta) = \frac{\beta^2 - \rho_1\rho_2}{2\beta^2}. \tag{17}$$

We write the system as

$$U_t + F_x = 0$$

with state variable $U = (\beta, v)^t$ and flux function $F = (vB_1(\beta), v^2B_2(\beta))^t + KU$. The physical range for β is $\rho_2 \le \beta \le \rho_1$. The system (15), (16) contains the same information as (4), (5) or (4), (8), always ignoring the balance terms, as the α_i and u_i can be recovered from equations (13) and (14), and the pressure found by integrating (5).

The system for the conserved quantities $U = (\beta, v)^t$ can conveniently be written in symmetric or gradient form [25],

$$\Phi_{z,t} + \Psi_{z,x} = 0, \tag{18}$$

with

$$z = (v, \beta)^t, \quad \Phi(z) = \beta v, \quad \text{and} \quad \Psi(z) = \tfrac{1}{2}v^2 B_1(\beta) + K\beta v.$$

The mapping $z \mapsto U = \Phi_z$ is obviously globally invertible, although Φ cannot be strictly convex in z as the system (18), which is equivalent to (15), (16), turns out not to be hyperbolic, [11, 25].

3. **Mathematical Analysis of the System.** To simplify the analysis of the Riemann problem, we take $K = 0$ (this can be effected by a Galilean change of variables) and consider the system

$$\beta_t + (vB_1(\beta))_x = 0, \tag{19}$$

$$v_t + (v^2 B_2(\beta))_x = 0, \tag{20}$$

on the strip $\rho_2 \le \beta \le \rho_1$. The Jacobian matrix is

$$A = \frac{\partial F}{\partial(\beta, v)} = \begin{bmatrix} vB_1' & B_1 \\ v^2 B_2' & 2vB_2 \end{bmatrix}.$$

A calculation gives

$$B_1' = 2B_2 \quad \text{and} \quad B_2' = \frac{\rho_1\rho_2}{\beta^3}.$$

The eigenvalues of A are

$$\lambda = 2vB_2(\beta) \pm v\sqrt{B_1 B_2'} \tag{21}$$

and since $B_1 \le 0$ and $B_2' > 0$ on the physical range of β, the system is never strictly hyperbolic. In fact, the eigenvalues have nonzero imaginary part except

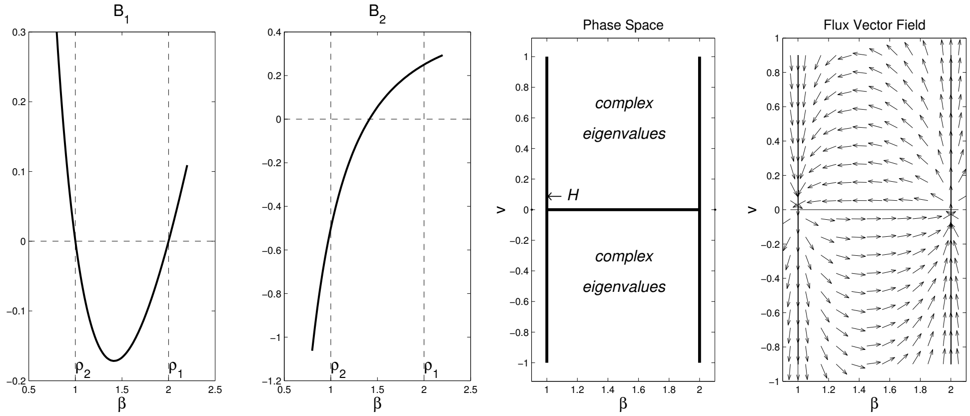

FIGURE 1. B_1, B_2, Physical Space and the Flux Vector

when $\beta = \rho_i$ or $v = 0$. This corresponds to the calculation in [10] of the finite characteristic speeds.

There is a subset of phase space, shaped like the letter 'H', in which there is a single, real characteristic speed. We will call this set H. Since B_2 is positive when $\beta = \rho_1$ and negative when $\beta = \rho_2$ (see Figure 1), the characteristic speed increases in the positive v-direction on the right leg of the 'H' and decreases on the left leg. Each leg of the 'H' is an *invariant set* for the system (19), (20); that is, if initial data $(\beta_0(x), v_0(x))$ is given in the set, then the solution remains in the set for all $t > 0$. This is a simple consequence of the structure of the system: if, initially, $\beta = \rho_i$, then $B_1(\beta) = 0$, so $\beta_t = 0$ and β remains constant. On these invariant sets, the system reduces to the scalar equation

$$v_t + B_2(\rho_i)(v^2)_x = 0$$

with quadratic flux function, convex up at ρ_1 and convex down at ρ_2. The β-axis is also an invariant set, on which the system reduces to a linear equation $\beta_t = 0$, with zero characteristic speed. On the vertical lines in H, solutions in the form of shock and rarefaction waves can be found, while the horizontal line admits contact discontinuities. One can pose Riemann problems with data in H, and if the local wavespeeds are ordered so that they increase from left to right, then a stable solution, completely contained in H, exists. For example, if $U_L = (\rho_2, v_L)$ with $v_L > 0$ and $U_R = (\rho_1, v_R)$ with $v_R > 0$ are the left and right states, respectively, then the solution consists of a rarefaction from U_L to the state $(\rho_2, 0)$, followed by contact discontinuity to the state $(\rho_1, 0)$ and a rarefaction to U_R. The right front of the left rarefaction and the left front of the right rarefaction both travel with speed zero, the speed of the contact, so the solution appears to be a single wave. These are the only cases in which classical Riemann solutions exist.

We now examine other Riemann data in the physical region. Since the region is bounded by invariant lines, we conjecture that it is also invariant; that is, if the initial data are in the region, the solution stays there for all $t > 0$. (A variant of the theory of Chueh, Conley and Smoller, [5], could perhaps be applied to find invariant regions.)

3.1. **The Hugoniot Locus and the Hyperbolic Region.** The part of phase space with $\beta < \rho_2$ or $\beta > \rho_1$ does not correspond to physically realizable states.

Nevertheless, it is useful to know something about the system there. Restricting attention to $\beta > 0$, since the flux functions are discontinuous at zero, we see that the system is strictly hyperbolic in this part of space. Since it is possible that shocks might join states in this region to states in the nonhyperbolic physical region, we calculate the Hugoniot locus. The Rankine-Hugoniot equations are

$$s[\beta] = [vB_1], \quad s[v] = [v^2 B_2], \tag{22}$$

where $[\cdot]$ represents the jump in a quantity across a shock of speed s. From these, we obtain the equation for the Hugoniot locus, by eliminating s:

$$[v][vB_1] - [\beta][v^2 B_2] = 0. \tag{23}$$

Solving, we find that given a state $U_0 = (\beta_0, v_0)$, the set of states (β, v) that can be joined to U_0 by a shock is

$$v = v_0 \left(\frac{B_1(\beta) + B_1(\beta_0) \pm \sqrt{Disc}}{2(B_1(\beta) - [\beta]B_2(\beta))} \right), \quad s = \frac{[vB_1]}{[\beta]} \tag{24}$$

where

$$Disc = \frac{[\beta]^2 \rho_1 \rho_2}{\beta^2 \beta_0^2} (\beta + \beta_0 - 2\rho_1)(\beta + \beta_0 - 2\rho_2).$$

Notice that when both U and U_0 are in the physical region, or if one is on its boundary, H, then the average of β and β_0 lies strictly between ρ_1 and ρ_2, so $Disc$ is negative and there are no real solutions to (23).

The hyperbolic region consists only of physically unrealizable states. However, when performing numerical simulations on the model equations, it is of course important to ask whether states outside the physical region may be connected to those inside by stable shocks, which might appear in computations. Thus we look briefly at the Hugoniot locus for hyperbolic points. When U_0 is in the hyperbolic region, the locus consists of four semi-infinite branches; two of the branches enter the physical region and are asymptotic to the line $\beta = \beta_*(\beta_0)$, where β_* is the value which makes the denominator zero in (24). It can be checked that as β_0 decreases to ρ_1, the asymptote β_* increases to ρ_1. Thus, the behavior of this system near the change of type locus is different from that of other systems which have been studied, in which the Hugoniot locus forms a "shock polar" which crosses the locus of change of type, [15]. This is presumably related to the fact that the right eigenvectors of the Jacobian matrix A are almost tangent to the change of type locus when U is near the locus; at the locus itself, the unique eigenvector is tangent to H, the change of type locus. This differs from the normal form in [15].

3.2. **Singular Shocks in the Model System.** Owing to the lack of shock solutions, we are motivated to seek singular shock solutions, as studied in [20], [21], [30], [33] and [34]. One way to look for such functions is to consider the self-similar viscosity approximation,

$$U_{\epsilon,t} + F(U_\epsilon)_x = \epsilon t U_{\epsilon,xx}, \tag{25}$$

which for fixed ϵ admits solutions depending only on $\xi = x/t$. If we seek self-similar solutions which are concentrated near a particular speed, $\xi = s$, then introducing $\tau = (\xi - s)/\epsilon$ we get the reduced form of (25):

$$-(s + \epsilon\tau)\dot{U}_\epsilon + A(U_\epsilon)\dot{U}_\epsilon = \ddot{U}_\epsilon \tag{26}$$

with $\dot{} = d/d\tau$. The following development uses formal asymptotics; a proof of existence of approximate solutions to (26) and their convergence properties can be found in [29].

We look for solutions of the form $U(x/t) = U(\xi)$ with growth in the second component:

$$U_\epsilon = \begin{pmatrix} \beta_\epsilon \\ v_\epsilon \end{pmatrix} = \begin{pmatrix} \tilde\beta\left(\frac{\xi-s}{\epsilon^q}\right) \\ \frac{1}{\epsilon^r}\tilde v\left(\frac{\xi-s}{\epsilon^q}\right) \end{pmatrix}. \tag{27}$$

Substituting into the equation, we find that nontrivial solutions can be found only if $q = 1 + r$, and they satisfy

$$\tilde\beta'' = (\tilde v B_1(\tilde\beta))' - \epsilon(\epsilon^{r+1}\eta + s)\tilde\beta'$$
$$\tilde v'' = (\tilde v^2 B_2(\tilde\beta))' - \epsilon(\epsilon^{r+1}\eta + s)\tilde v';$$

here the variable is $\eta = (\xi - s)/\epsilon^q$; we can set $r = 1$, $q = 2$ to simplify the development, though this is not the only choice possible. Setting $\epsilon = 0$ and integrating once, we obtain a dynamical system with nonhyperbolic (in the dynamical systems sense) equilibria:

$$\tilde\beta' = \tilde v B_1(\tilde\beta) \tag{28}$$
$$\tilde v' = \tilde v^2 B_2(\tilde\beta). \tag{29}$$

This has the dynamics of the flux vector field sketched in Figure 1. We want orbits asymptotic to zeros of F (this is how we integrated once and eliminated the constant), and so the solutions of interest correspond to heteroclinic orbits. There are two kinds: orbits from $(\rho_1, 0)$ to $(\rho_2, 0)$ in the upper half-plane, and orbits from $(\rho_2, 0)$ to $(\rho_1, 0)$ in the lower half-plane.

We construct the singular shock by expanding in powers of ϵ. The singular part of the shock can be developed as the inner part of a matched asymptotic expansion with an outer part in the form of a (possibly) regular shock, $\overline{U}(\tau) = \overline{U}((\xi - s)/\epsilon)$, with end states $U^\pm$ outside the shock layer. Here $\tau = \epsilon^r\eta = (\xi - s)/\epsilon$. Now we derive a Rankine-Hugoniot type condition on the end states. In the standard procedure, one writes

$$\frac{d}{d\tau}\left(\frac{dU_\epsilon}{d\tau} - F(U_\epsilon) + sU_\epsilon\right) = -\epsilon\tau\frac{dU_\epsilon}{d\tau} \tag{30}$$

and then integrates between $\tau = -A$ and $\tau = B$, where A and B are positive numbers of order $1/\epsilon$ – that is, outside the shock layer. Thus,

$$\begin{aligned} \frac{dU_\epsilon}{d\tau} - F(U_\epsilon) + sU_\epsilon\Big]_{-A}^{B} &= -\epsilon\int_{-A}^{B}\tau\frac{dU_\epsilon}{d\tau}\,d\tau \\ &= \epsilon\int_{-A}^{B}\big(U_\epsilon(\tau) - U_H(A,B)\big)\,d\tau, \end{aligned} \tag{31}$$

where

$$U_H(A,B) = \begin{cases} U_\epsilon(-A), & \tau < 0 \\ U_\epsilon(B), & \tau > 0 \end{cases}$$

and we have integrated by parts. By hypothesis, $dU_\epsilon/d\tau \to 0$ as A, $B \to \infty$, so we obtain

$$s[U_\epsilon] - [F(U_\epsilon)] = \epsilon\int_{-\infty}^{\infty}\big(U_\epsilon(\tau) - U_H(\infty,\infty)\big)\,d\tau. \tag{32}$$

In a standard shock profile, $U_\epsilon(\tau)$ is a bounded trajectory which approaches the critical points U_L and U_R exponentially fast, so the improper integral is bounded and we recover the Rankine-Hugoniot conditions. In fact, even without hyperbolicity the condition

$$\mathrm{Re}(\lambda(U^-)) > s > \mathrm{Re}(\lambda(U^+)) \tag{33}$$

guarantees exponential convergence, and we now find a generalized Rankine-Hugoniot condition for this case. (If $s = \mathrm{Re}(\lambda)$ at one or both of the states connected this way, the improper integral in (32) may not exist, and this approach would have to be modified. This remains an open problem.) We analyse the right side of (32) for the system (19), (20). Note first that with U_ϵ given by (27), β_ϵ is bounded, as can be seen from (28), noting that (30) controls the growth of β_ϵ near $\tau = 0$ and recalling the exponential decay at infinity. Hence, the first component of the right side of (32) is of order ϵ, so the first Rankine-Hugoniot condition holds, and the generalized Rankine-Hugoniot condition is of the form

$$s(\beta^+ - \beta^-) = v^+ B_1(\beta^+) - v^- B_1(\beta^-) \tag{34}$$

$$s(v^+ - v^-) = (v^+)^2 B_2(\beta^+) - (v^-)^2 B_2(\beta^-) + C \tag{35}$$

where C may have any finite value (corresponding to different trajectories in the heteroclinic connection) but is positive for trajectories in the upper-half-plane, negative in the lower.

It will be observed that for a given (β^-, v^-), the set of (β^+, v^+) satisfying (34), (35), forms an open region of phase space. This is typical for a pair of conservation laws. More generally, for a system of dimension n, the limiting values and speed of singular shocks satisfy m of the Rankine-Hugoniot relations. Thus the set of w^+ which can be connected to a given state w^- by a singular shock is a manifold of dimension $n + 1 - m$. For a singular shock (as distinct from a delta-shock) $1 \leq m \leq n - 1$. So for a pair, $n = 2$, we necessarily have $m = 1$ and $n + 1 - m = 2$.

We can use equations (28) and (29) to describe the blow-up in v_ϵ as ϵ approaches zero. For definiteness, take $C > 0$ in (35); then the functions v_ϵ have a very sharp, positive peak in τ at $\tau = 0$. In a neighborhood of $\tau = 0$, β_ϵ decreases rapidly from $\rho_1 - 0$ to $\rho_2 + 0$, (this follows from (28)), motivating the approximation

$$\beta_\epsilon(\tau) \approx \begin{cases} \rho_1, & \tau < 0 \\ \rho_2, & \tau > 0 \end{cases}. \tag{36}$$

The approximation (36) is remarkably well justified in computations. From the formula (17) for $B_2(\beta)$ and (36),

$$B_2(\beta_\epsilon(\tau)) \approx \begin{cases} 1/2\rho_1, & \tau < 0 \\ -1/2\rho_2, & \tau > 0 \end{cases}. \tag{37}$$

Let A and B be such that $v_\epsilon(-A)$, $v_\epsilon(B)$ are bounded uniformly in ϵ, and such that (29) approximates the behavior of v_ϵ in the interval $(-A, B)$. Then using (29), (35) and (37), we find that the second component of the right side of (32), the Rankine-Hugoniot deficit, is

$$\begin{aligned} C &= \epsilon \int_{-A}^{B} v_\epsilon \, d\tau + \mathcal{O}(\epsilon) = \epsilon \left(\int_{-A}^{0} v_\epsilon \, d\tau + \int_{0}^{B} v_\epsilon \, d\tau \right) + \mathcal{O}(\epsilon) \\ &= \epsilon \left(\int_{v_\epsilon(-A)}^{v_\epsilon(0)} \frac{v_\epsilon}{v_{\epsilon,\tau}} \, dv_\epsilon + \int_{v_\epsilon(B)}^{v_\epsilon(0)} \frac{v_\epsilon}{|v_{\epsilon,\tau}|} \, dv_\epsilon \right) + \mathcal{O}(\epsilon) \\ &= \epsilon \left(2\rho_1 \int_{v_\epsilon(-A)}^{v_\epsilon(0)} \frac{dv_\epsilon}{v_\epsilon} + 2\rho_2 \int_{v_\epsilon(B)}^{v_\epsilon(0)} \frac{dv_\epsilon}{v_\epsilon} \right) + \mathcal{O}(\epsilon) \\ &= 2\epsilon \left(\rho_1 + \rho_2 \right) \log v_\epsilon(0) + \mathcal{O}(\epsilon). \end{aligned}$$

Thus the peak value of v_ϵ is approximately

$$v_\epsilon(0) \approx \exp\left(\frac{C}{2\epsilon(\rho_1+\rho_2)}\right), \tag{38}$$

again remarkably well verified in computations; see Figure 5 in Section 6. We comment in passing that a more lengthy analysis, avoiding the approximation (36), leads to the same asymptotic estimate (38).

Sever showed in [34] that a suitable admissibility condition for singular shocks in similar systems is given by (33). As we shall see, we also need to allow singular shocks in which one of the inequalities is weak, in order to solve Riemann problems.

We define a *singular shock* to be the limit of the $\mathcal{O}(\epsilon)$ layer, and identify it by the outer limits, U^+ and U^-, and the shock speed, s, satisfying (34) and (35); singular shocks exist only if v^+ and v^- have the same sign, and C also has that sign. The asymptotic structure we have developed above establishes the following Proposition.

Proposition 3.1. *If a singular shock connects U^- and U^+, then both states are in the same vertical half-plane. The end states satisfy the generalized Rankine-Hugoniot conditions,* (34) *and* (35)*, where C may have any finite value and is positive for end states in the upper-half-plane, negative in the lower. Singular shocks satisfy an admissibility condition,*

$$\mathrm{Re}(\lambda(U^-)) \geq s \geq \mathrm{Re}(\lambda(U^+)). \tag{39}$$

Strict inequalities in (39) yield *strictly overcompressive* singular shocks, which are locally isolated transitions. On the other hand, when equality holds in one of the conditions in (39), then a singular shock may form part of a complex wave pattern, as it may lie at the head or tail of a rarefaction. A calculation gives

Corollary 3.1. *For a strictly overcompressive singular shock with left state U^-, the right state U^+ lies in the interior of a cusped triangular region $Q(U^-)$ bounded by the curves*

$$v^+ = v^-\left(\frac{2B_2(\beta^-)(\beta^+-\beta^-)+B_1(\beta^-)}{B_1(\beta^+)}\right) \tag{40}$$

and

$$v^+ = v^-\left(\frac{B_1(\beta^-)}{B_1(\beta^+)-2B_2(\beta^+)(\beta^+-\beta^-)}\right). \tag{41}$$

On the boundary segment (40)*, $s = \mathrm{Re}(\lambda^-)$, and on* (41)*, $s = \mathrm{Re}(\lambda^+)$.*

The curve (41) meets H at a point U_0 defined by

$$U_0(U^-) = \left(\rho_2, -\frac{v^-B_1(\beta^-)}{2B_2(\rho_2)(\rho_2-\beta^-)}\right) \quad \text{or} \quad \left(\rho_1, -\frac{v^-B_1(\beta^-)}{2B_2(\rho_1)(\rho_1-\beta^-)}\right)$$

according as v is positive or negative. The significance of this point is that the singular shock from U^- to U_0 may form part of a composite wave, with a rarefaction on the right. We can give an analogous description of overcompressive shocks from the viewpoint of a fixed state U^+ on the right; in this case, there is a unique point $U_1(U^+)$ to which U^+ can be joined by a singular shock preceded by a rarefaction. The curves (40) and (41) and the region Q where overcompressive shock solutions exist are illustrated in Figure 2.

Some shocks which are not strictly overcompressive can form part of a wave pattern. For example, if the singular shock connects states U and U_0 with $U_0 \in H$ the point defined above, then the wave may continue with a rarefaction. Take U

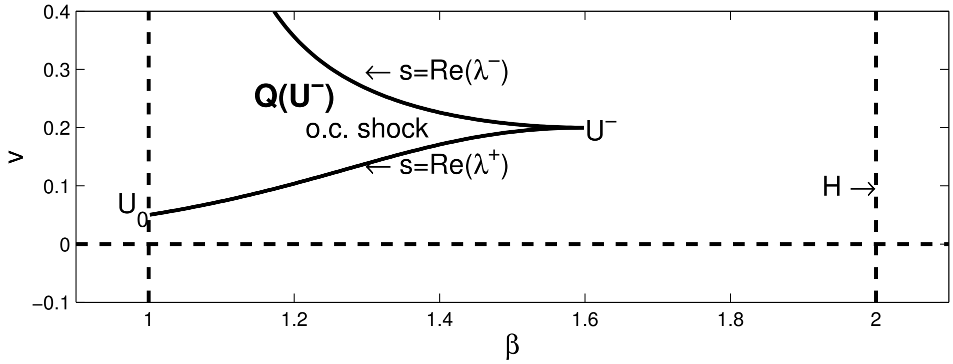

FIGURE 2. The Singular Shock Region for $\rho_1 = 2$, $\rho_2 = 1$

to be a given state in the interior of the physical region. Then the first Rankine-Hugoniot equation, (34), gives (since $B_1(\beta_0) = 0$),

$$s = \frac{vB_1(\beta)}{\beta - \beta_0}, \tag{42}$$

with $\beta_0 = \rho_i$. In particular, s does not depend on the value of v_0. Now, we have either $\mathrm{Re}(\lambda(U)) > s = \lambda(U_0)$, for the state U_0 illustrated in Figure 2, or $\lambda(U_1) = s > \mathrm{Re}(\lambda(U))$, for the corresponding state U_1, with U now the state on the right. We can compare $s = vB_1(\beta)/(\beta - \rho_i)$ with $\mathrm{Re}(\lambda(U)) = 2vB_2(\beta)$; to fix ideas suppose that $v > 0$. Then we find

$$s > \mathrm{Re}(\lambda(U)) \Leftrightarrow \beta < \rho_i;$$

that is, U is the state on the right if and only if $\beta_0 = \rho_1 > \rho_2$. When U is the state on the left of the singular shock, then U_0 must lie on the left leg of H ($\beta_0 = \rho_2$). When $v < 0$, the situation is reversed: a state U in the interior of the physical region will jump to a state on the right side of the region, with $\beta_0 = \rho_1$, via a singular shock with U on the left; in the other case, U will be the right state of a singular shock connecting it to U_1 with $\beta = \rho_2$. From (42), a singular shock with U on the left has negative speed, while a singular shock with U on the right travels with positive speed.

If $s = \lambda(U_0)$ (or $s = \lambda(U_1)$), then the composite wave consists of a singular shock followed (or preceded) by a rarefaction between U_0 (or U_1) and a state U_M on the same leg of H. The condition $s = \lambda(U_0)$ fixes v_0 to have the value

$$v_0 = v\frac{B_1(\beta)}{2(\beta - \rho_i)B_2(\rho_i)} = v\frac{(\beta - \rho_j)\rho_i}{\beta(\rho_i - \rho_j)}, \tag{43}$$

as calculated above, where $\beta_0 = \rho_i$ and j is the other index ($j = 3 - i$). For either sign of v and either leg of H, we have $|v_M| < |v_0|$; that is, v_M is closer to the origin than v_0.

3.3. **Solution of Riemann Problems.** The construction detailed in Section 3.2 provides a self-similar solution to any Riemann problem,

$$U(x, 0) = \begin{cases} U_L, & x < 0 \\ U_R, & x > 0 \end{cases}, \tag{44}$$

by giving a recipe for solving Riemann problems with any states U_L and U_R. Several representations of a typical Riemann solution are illustrated in Figure 3.

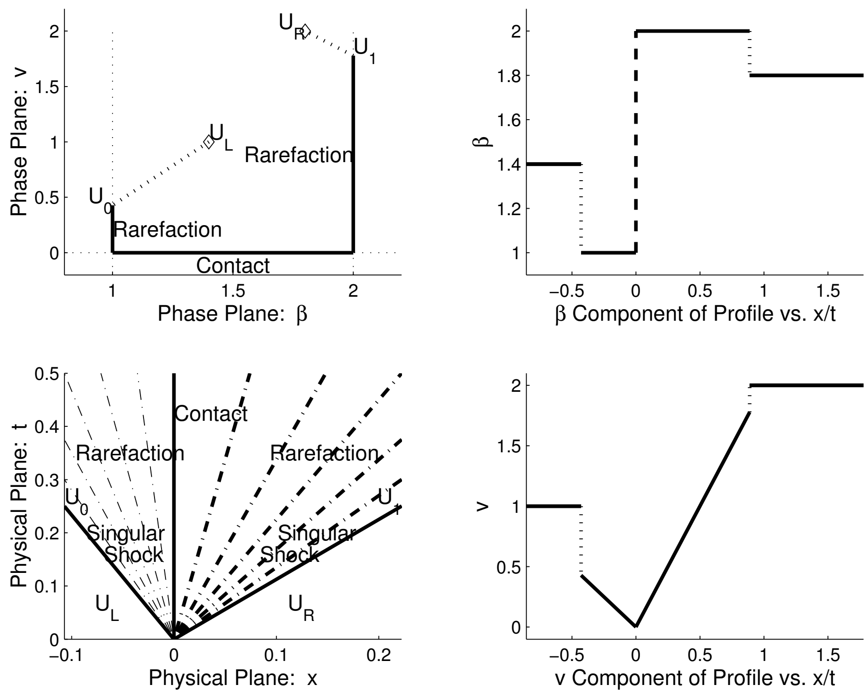

FIGURE 3. Four Views of the Riemann Solution for $\rho_1 = 2$, $\rho_2 = 1$

As is common, the solution of a given nonconstant ($U_L \neq U_R$) Riemann problem is determined by specifying distinct intermediate states

$$U_L = U^0, U^1, \dots, U^M = U_R \tag{45}$$

such that for $j = 1, \dots, M$, U^{j-1} is connected on the left to U^j on the right by an admissible wave, with the wave speeds increasing with j. We adopt the standard convention that the speed of a discontinuity connecting U^{j-1} to U^j must be strictly less than that of a discontinuity conecting U^j to U^{j+1}. Otherwise, U^j is omitted, and U^{j-1} is considered connected directly to U^{j+1}. For this system, the set of admissible waves is limited to the following: *entropy shocks*, satisfying (22) and $\lambda(U^-) > s > \lambda(U^+)$, specifically with

$$U^{j-1} = \begin{pmatrix} \rho_2 \\ v^- \end{pmatrix}, \quad U^j = \begin{pmatrix} \rho_2 \\ v^+ \end{pmatrix}, \quad v^- < v^+,$$

or

$$U^{j-1} = \begin{pmatrix} \rho_1 \\ v^- \end{pmatrix}, \quad U^j = \begin{pmatrix} \rho_1 \\ v^+ \end{pmatrix}, \quad v^- > v^+;$$

rarefaction waves, with

$$U^{j-1} = \begin{pmatrix} \rho_2 \\ v^- \end{pmatrix}, \quad U^j = \begin{pmatrix} \rho_2 \\ v^+ \end{pmatrix}, \quad v^- > v^+,$$

or

$$U^{j-1} = \begin{pmatrix} \rho_1 \\ v^- \end{pmatrix}, \quad U^j = \begin{pmatrix} \rho_1 \\ v^+ \end{pmatrix}, \quad v^- < v^+;$$

contact discontinuities, satisfying (22) with

$$U^{j-1} = \begin{pmatrix} \beta^- \\ 0 \end{pmatrix}, \quad U^j = \begin{pmatrix} \beta^+ \\ 0 \end{pmatrix}, \quad \rho_2 \le \beta^-, \beta^+ \le \rho_1; ,$$

and *singular shocks*, satisfying (34), (35), and (39) with

$$U^j \in Q(U^{j-1}) \tag{46}$$

as shown in Figure 2 and described in Proposition 3.1 and Corollary 3.1.

Denoting a Riemann solution with intermediate states U^j by $\{U_L, U_R\}$, we adopt a metric based on the intermediate states. We define

$$|\{U_L, U_R\} - \{\widetilde{U}_L, \widetilde{U}_R\}| = \max_j \min_k |U^j - \widetilde{U}^k| + \max_k \min_j |U^j - \widetilde{U}^k| \tag{47}$$

(For a nonhyperbolic system in which the classical partition of waves into families does not apply, such a metric is more useful than the standard metric based on wave strengths.)

In studying the continuous dependence of Riemann solutions on the Riemann data, we use a notion of *continuous connection* based on the metric (47).

Definition 3.1. *A Riemann solution $\{U_L, U_R\}$ is said to be* continuously connected *to $\{\widetilde{U}_L, \widetilde{U}_R\}$ if for any $\varepsilon > 0$ there exists a sequence*

$$\{U_L, U_R\} = \{W_L^0, W_R^0\}, \{W_L^1, W_R^1\}, \dots, \{W_L^N, W_R^N\} = \{\widetilde{U}_L, \widetilde{U}_R\} \tag{48}$$

such that

$$|\{W_L^j, W_R^j\} - \{W_L^{j-1}, W_R^{j-1}\}| \le \varepsilon, \quad j = 1, \dots, N. \tag{49}$$

We note that this concept is different from the notion of structural stability of Riemann solutions introduced by Schecter, Marchesin and Plohr, [32], which requires that the entire wave structure, not merely the set of states achieved, be similar.

Theorem 3.1. *The Riemann problem is solvable in the class of self-similar solutions containing only admissible waves for any U_L, U_R in the physical region.*

Remark. For fixed U_L, the number M in (45) depends on U_R. As M is integer-valued and not constant, it cannot depend continuously on U_R.

Proof. The largest possible value of M in (45) is 5, which occurs when both U_L and U_R are in the interior of the physical region and v_L and v_R are nonzero and of the same sign. The case of positive v_L and v_R is shown in Figure 3. In this case

$$U^1 = U_0(U_L) \in \partial Q(U_L) \cap \{\beta = \rho_2\},$$

and U^1 is connected to U_L by a singular shock of speed $s = \lambda(U^1)$; $U^2 = (\rho_2, 0)^t$ is connected to U^1 by a rarefaction; $U^3 = (\rho_1, 0)^t$ is connected to U^2 by a contact discontinuity; $U^4 = U_1(U_R)$, as shown in Figure 3, is connected to U^3 by a rarefaction, and is such that U_R is connected to U^4 by a singular shock of speed $s = \lambda(U^4)$.

For the case $v_L > 0$, $v_R < 0$, $M = 3$ suffices, with the same point U^1; now $U^2 = (\rho_2, v_2)^t$ is connected to U^1 by a rarefaction, and $v_2 < 0$ is such that U_R is connected to U^2 by a singular shock of speed $s = \lambda(U^2)$.

The case of negative v_L is entirely similar.

These solutions can be continuously connected to the cases of v_L or v_R equal to zero, and to the cases that U_L or U_R are on the boundary of the physical region, (typically with reduced values of M). ☐

There is no possibility of uniqueness of the solution obtained in Theorem 3.1 for all U_L, U_R. Recalling H as shown in Figures 1 and 2, for $U_L = U_R \notin H$, there is a constant solution and also a solution with $M = 5$ as obtained in Theorem 3.1. However the following holds.

THEOREM 3.2. *The solution obtained in Theorem 3.1 is unique unless*

$$U_R \in Q(U_L), \quad U_L, U_R \notin H. \tag{50}$$

REMARK. The region $Q(U_L)$ is closed, so the case $U_L = U_R$ is included in (50).

Proof. First we consider the case $M > 1$ in (45). We claim that necessarily $U^j \in H$, $j = 1, \ldots, M-1$. Otherwise, the connections U^{j-1} to U^j and U^j to U^{j+1} are both singular shocks, which is incompatible with the overcompressibility condition (39). Now if $U_L \notin H$, then necessarily $U^1 = U_0(U_L)$ the point illustrated in Figure 2 and Figure 3, and correspondingly if $U_R \notin H$, then $U^{M-1} = U_1(U_R)$ as shown in Figure 3.

Thus we consider the uniqueness of solutions with both U_L and U_R in H. The uniqueness of weak solutions (with no singular shocks), including trivial solutions, follows from the entropy condition for ordinary shocks. Thus suppose that the solution contains a singular shock connecting U^{j-1} to U^j,

$$U^j \in Q(U^{j-1}), \quad U^{j-1}, U^j \in H. \tag{51}$$

Observing that $H = \{vB_1(\beta) = 0\}$, from (34) it follows that the speed of such a singular shock is zero, and thus from the ovcercompressibility condition (39) that $\lambda(U^j) \leq 0 \leq \lambda(U^{j-1})$. Thus no connections to such a singular shock are possible, and the only solutions of the form (51) correspond to $M = 1$ in (45).

Thus there can be no more than one solution with $M > 1$, and the only way uniqueness can fail is that there exists a solution with $M > 1$ and a solution with $M = 1$, necessarily a singular shock. This is the case when (50) holds, but we claim that only the $M = 1$ solution exists if either U_L or U_R is in H. It suffices to show that there is no solution with $M > 1$ in the case

$$U_L \notin H, \quad U_R \in Q(U_L) \cap H. \tag{52}$$

Suppose otherwise. Then from (52), $U^1 = U_0(U_L)$ must be connected to U_R by an entropy shock. But the speed of the singular shock connecting U_L to U^1 is $\lambda(U^1)$, which exceeds the speed of an entropy shock connecting U^1 on the left to U_R on the right. $\square$

Next we consider the continuous connection of solutions, say as U_R is varied with U_L fixed.

THEOREM 3.3. *All of the solutions constructed in Theorem 3.1 are continuously connected in the sense of Definition 3.1.*

Proof. Fix $\underline{U} = (\rho_2, 0)^t$; from Theorem 3.2, it follows that $\{\underline{U}, \underline{U}\}$ is the trivial solution. For any U_L and U_R in the physical region, it will suffice to show that $\{U_L, \underline{U}\}$ is continuously connected to $\{\underline{U}, \underline{U}\}$, and that $\{\underline{U}, \underline{U}\}$ is continuously connected to $\{\underline{U}, U_R\}$. We prove the second statement, the first being entirely similar. The Riemann solution $\{\underline{U}, U_R\}$ contains at most three waves, each of which corresponds to a finite segment within H or to a finite segment of the boundary of $Q(U_1)$ for some point U_1 as shown in Figure 3. Connecting these segments, we obtain a continuous, piecewise smooth trajectory $U(\Lambda)$, $0 \le \Lambda \le 1$, with $U(0) = U$ and $U(1) = U_R$. Given $\varepsilon > 0$, choose N sufficiently large and in (48) set

$$W_L^j = \underline{U}, \quad W_R^j = U(j/N), \quad j = 0, 1, \ldots, N.$$

Then, as the intermediate states in the solution $\{W_L^{j-1}, W_R^{j-1}\}$ coincide with those of $\{W_L^j, W_R^j\}$, it follows from (47) that

$$|\{W_L^j, W_R^j\} - \{W_L^{j-1}, W_R^{j-1}\}| \le 2 \left| U\left(\frac{j}{N}\right) - U\left(\frac{j-1}{N}\right) \right|,$$

so (49) holds. □

A stronger condition, of the form

$$|\{U_L, U_R\} - \{U_L, U_R'\}| \le c|U_R - U_R'|, \tag{53}$$

does not hold in general. Counterexamples are easily constructed by taking $U_R \in H$, or by taking v_R and v_R' of opposite sign, or by taking U_R close to the boundary of $Q(U_L)$ in the case that (50) holds and choosing the $M = 1$ solution for $\{U_L, U_R\}$. However, we have the following stability result.

Theorem 3.4. *Assume that neither U_R nor U_R' is in H, that $v_R v_R' > 0$, and that if* (50) *holds, one chooses the $M = 5$ solution for both $\{U_L, U_R\}$ and $\{U_L, U_R'\}$. Then* (53) *holds, with the constant c depending only on an upper bound for $|v_R|$, $|v_R'|$.*

Proof. Under these assumptions, the value of M is either 3 or 5, depending only on the sign of $v_L v_R$. For definiteness, we consider the case that v_R and v_R' are positive. Then the intermediate states in the two solutions coincide except for the M–1[st] states, which are of the form (ρ_1, v_1), (ρ_1, v_1') (the states $U_1(U_R)$ and $U_1(U_R')$ respectively). Thus from the metric (47),

$$|\{U_L, U_R\} - \{U_L, U_R'\}| \le 2\big(|U_R - U_R'| + |v_1 - v_1'|\big). \tag{54}$$

Using (43), the values of v_1, v_1' are given explicitly in terms of U_R and U_R':

$$v_1 = v_R \frac{1 - \rho_2/\beta_R}{1 - \rho_2/\rho_1}, \qquad v_1' = v_R' \frac{1 - \rho_2/\beta_R'}{1 - \rho_2/\rho_1}, \tag{55}$$

and (53) follows easily from (54) and (55). □

3.4. **Alternative Forms of the System.**

The derivation of (15), (16) in Section 2 tacitly assumes smooth solutions. In particular, weak solutions are not preserved under the nonlinear operations carried out on the equations in that section. As our analysis suggests, however, distribution solutions with singular shocks are to be expected. In this context we ask about alternative formulations.

Equations (4), unlike (5), are in conservation form and unambiguous with regard to discontinuous solutions. Given the conditions (6), (7), the only independent affine combinations of equations (4), (5) in conservation or balance form are (4) and an equation of total momentum balance

$$\partial_t(\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_2\rho_2 u_2^2 + p) = F_1 + F_2. \tag{56}$$

Theorem 3.5. *There is no pair of conservation laws in the variables β, v, equivalent for weak solutions to* (4), (56) *(with $F_1 + F_2 = 0$).*

Remark. This implies that any adopted model in the variables β, v requires an exchange of the conserved quantities, with the implied lack of validity for weak or distribution solutions.

Proof. Any such conservation laws, of the form

$$\eta(\beta, v)_t + \zeta(\beta, v)_x = 0, \tag{57}$$

are necessarily satisfied by smooth solutions of (15), (16), provided that $\eta(\beta, v)$, $\zeta(\beta, v)$ satisfy

$$\zeta_\beta = K\eta_\beta + 2vB_2(\beta)\eta_\beta + v^2 B_{2,\beta}(\beta)\eta_v \tag{58}$$

$$\zeta_v = K\eta_v + B_1(\beta)\eta_\beta + 2vB_2(\beta)\eta_v. \tag{59}$$

The pair (58), (59) is solvable if and only if η satisfies a degenerate elliptic equation in phase space,

$$v^2\eta_{vv} = \frac{\beta^2(\beta-\rho_1)(\beta-\rho_2)}{\rho_1\rho_2}\eta_{\beta\beta}. \tag{60}$$

Therefore conservation laws equivalent to (4), (56) for weak solutions correspond to solutions of (60) with η an affine combination of α_1, α_2, and $\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2$. There is, however, only one independent such solution, given by

$$\eta(\beta, v) = \beta. \tag{61}$$

In particular, η given by

$$\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2 = \frac{v(\rho_1-\beta)(\beta-\rho_2)}{\beta(\rho_1+\rho_2)} - K\beta + K(\rho_1+\rho_2) \tag{62}$$

obviously does not satisfy (60), as it is affine in v but not in β. Since any pair of conservation laws in β, v must include a second, independent solution of (60), the proof is complete. □

Using (13), (34) we observe that (15) is equivalent to either component of (4) for weak solutions and for distribution solutions containing singular shocks as described above. This equation is an expression of conservation of mass of each phase separately, presumably a highly desirable feature in an such models.

4. **Interpretation of the Riemann Solution.** We have presented a solution to the Riemann problem for the system (19), (20), which is equivalent to (4), (5) except for three things: we have omitted the balance terms, we have made a convenient choice of the conserved quantities, and we have written the solution in scaled variables β and v, and with a spatial scale centered at the center of volume of the flow (as a result of setting $K = 0$, see equation (10)) rather than laboratory coordinates.

It is now straightforward to rewrite the solution in the original variables. We note first that reintroducing the factor of K replaces the system we have been analysing, (19) and (20), by (15) and (16), which differs by a scalar term, corresponding to $x \mapsto x - Kt$. In particular, the Riemann solutions are identical, except that all velocities (characteristic, shock and rarefaction speeds) are augmented by the constant amount K.

Next, the volume fractions and flow speeds, in laboratory coordinates, are given by (13) and (14). We note that at a point (x, t) in the solution where one volume fraction is zero, the flow speed corresponding to the material which is present is K, which is consistent with the physical assumptions, while the absent phase is assigned a mathematical value which seems to have no physical significance. In particular, in the rarefaction waves, which always correspond to pure single-phase states, the actual flow is uniform throughout the wave. Contact discontinuities still correspond to transitions in the flow where the mixture ratios change without a discontinuity in

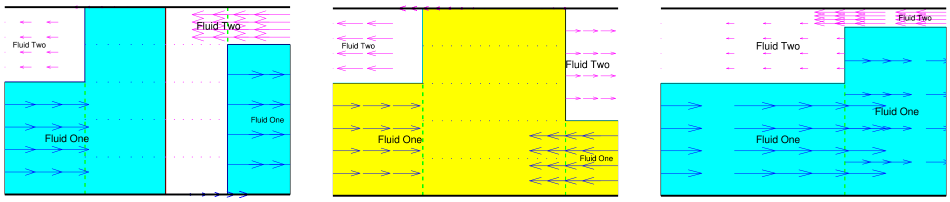

FIGURE 4. Three Views of Stratified Pipe Flow

velocity. Singular shocks, on the other hand, whether between states with different mixture ratios or between mixed and pure states, are transitions accompanied by a discontinuity in the velocity.

Three examples of Riemann solutions are illustrated in Figure 4 for the case of stratified flow in a pipe. The solution on the left contains a composite wave with two singular shocks, two rarefactions and a contact discontinuity; that in the middle, two singular shocks separated by a rarefaction; and that on the right a single overcompressive singular shock separating two regions of mixed flow.

5. **Inclusion of Applied Forces.** The applied forces F_1, F_2 in (5) or (8) are an essential feature of models of either dispersed or stratified flow. Inclusion of these terms results in a modification of (16) of the form

$$v_t + (v^2 B_2(\beta) + Kv)_x = \frac{F_1}{\alpha_1} - \frac{F_2}{\alpha_2} \equiv G(\beta, v). \tag{63}$$

When F_1 and F_2 depend only on ρ_i, α_i, u_i, $i = 1, 2$ and not on any derivatives, the system (15), (63) is a pair of balance laws. Results on the existence of solutions and well-posedness of the Cauchy problem for such systems have recently been obtained in [1], albeit under different structural conditions than those encountered here.

In the case of vertical flow one anticipates buoyancy forces of the form

$$F_2^{\text{buoy}} = -F_1^{\text{buoy}} = \alpha_1 \alpha_2 g(\rho_1 - \rho_2) \tag{64}$$

with g the gravitational constant. The resulting system (15), (63) with

$$G(\beta, v) = -g(\rho_1 - \rho_2)$$

obtained from (63), (64) and (6), was employed in [9] to obtain a solution of the Ransom faucet problem [28].

Making the approximations

$$\alpha_2 \ll \alpha_1, \quad \rho_2 \ll \rho_1, \quad u_1, u_2 \text{ uniformly bounded }, \tag{65}$$

models of drag forces for dispersed flow are obtained in [10] and [37], in both cases of the form

$$F_2^{\text{drag}} = -F_1^{\text{drag}} = c\alpha_2 |u_2 - u_1|(u_2 - u_1), \tag{66}$$

with c an effective drag coefficient.

The expression (66) is not in a form suitable for inclusion in the present formulation, as the resulting expression for G is not defined in the limit α_1 decreasing to zero, or in the presence of singular shocks, where $u_2(., t) - u_1(., t)$ is defined only as a measure. As both of these conditions contradict the assumptions (65), suitable modification of the expression (66) is not difficult.

Finally, we cite the study [27], of drag forces depending on the derivatives of u_1 and u_2, which concludes that such models exhibit mathematical structure similar to that of (15), (16).

6. **Numerical Approximations.** We consider here the numerical approximation of solutions to the self-similar viscosity equation (25) which was introduced and studied in Section 3.2. We seek a solution of the form $U(x,t) = U(\tau)$, with $\tau = (x/t - s)/\epsilon$, satisfying the second-order ordinary differential equation (26). We choose data $U^{\pm} = \lim_{\tau \to \pm\infty} U(\tau)$ which correspond to left and right states of a singular shock of speed s in the inviscid equation. Needless to say, a grid with infinite extent is impossible to fully accommodate numerically. Therefore, we restrict the problem to a finite domain chosen large enough to contain the solution's exponential transition. For $\epsilon \sim 10^{-2}$, $\tau^- = -40 < \tau < 40 = \tau^+$ has proven to be much more than adequate.

A variant of the shooting method is applied to solve the self-similar viscosity equation cast as a two point boundary value problem. In order to utilize a generic fourth-order accurate, adaptive-grid Runge-Kutta integration scheme, the differential equation (26) must be reduced to first order. We expect its solution to be poorly behaved near $\tau = 0$. Therefore, writing $U = (\beta, v)^t$ and $F(U) = (vB_1(\beta), v^2 B_2(\beta))^t$, see Section 3.2, we introduce the auxiliary variable $X = -\epsilon\tau\dot{U}$ to find a four-dimensional first-order system equivalent to (26):

$$\begin{aligned} \dot{U} &= F(U) - s\,U + X \\ \dot{X} &= -\epsilon\tau\,[\,F(U) - s\,U + X\,], \end{aligned} \tag{67}$$

where again $\dot{} = d/d\tau$. We solve (67) on the finite interval (τ^-, τ^+) and require its solution to satisfy the boundary condition $U(\tau^{\pm}) = U^{\pm}$. This particular form for the first order system is motivated by our result in Section 3.2 where we found that $X = \dot{U} - F(U) + s\,U$ is expected to be fairly well behaved through a singular shock profile; see equation (30).

Shooting from one boundary point to the other for (67) is necessarily poorly conditioned even for moderately small ϵ. This is due to the fact that $U(\tau)$ is essentially constant near the the boundary points but is wildly varying in the singular shock's inner layer around $\tau = 0$. For this reason, our numerical procedure shoots from $\tau = 0$. Let $W(W_0; \tau) = (U(\tau), X(\tau))^t$ with given initial data $W_0 = (U(0), X(0))^t$; integrate (67) to τ^+ to obtain $W(W_0; \tau^+)$, and also integrate backwards to τ^- to obtain $W(W_0; \tau^-)$. This yields $U(W_0; \tau^+)$ and $U(W_0; \tau^-)$. The idea then is to solve the implicitly defined four-dimensional system

$$\begin{aligned} U(W_0; \tau^-) &= U^- \\ U(W_0; \tau^+) &= U^+ \end{aligned} \tag{68}$$

for the unknown W_0. So instead of integrating over the *hump* near $\tau = 0$, which would be the case by shooting from one boundary point to the other, here we attempt to integrate *down hill* only.

We solve (68) by Newton's linearization. However even for our procedure of shooting from $\tau = 0$, the Newton domain of attraction can be very small when ϵ is small. For this reason, we apply ϵ-continuation. That is, suppose a sequence of converged solutions to (68), say $W_0(\epsilon_1), \ldots, W_0(\epsilon_n)$, is known for decreasing $\epsilon_1 > \cdots > \epsilon_n$. An approximation to $W_0(\epsilon_{n+1})$ is computed by using high order extrapolation on the previously determined $W_0(\epsilon_k)$, and used as a Newton first guess to the ϵ_{n+1} problem. Then, this first guess is updated by Newton's method until convergence, thus yielding $W_0(\epsilon_{n+1})$. In our results presented below, we use at most quadratic extrapolation and reduce ϵ by 2% at every continuation step.

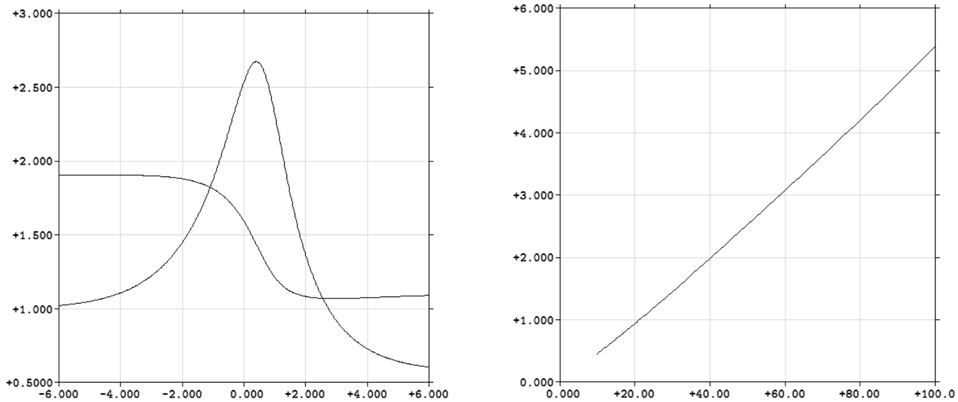

FIGURE 5. (a) β_ϵ and v_ϵ at $\epsilon = .05$; (b) $\log(v_\epsilon(0))$ vs. $1/\epsilon$

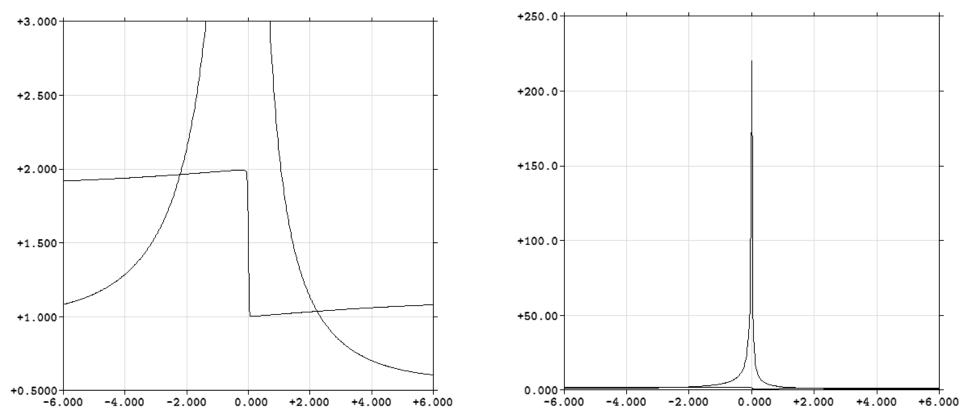

FIGURE 6. Two Views of β_ϵ and v_ϵ vs. $\tau = x/\epsilon t$; $\epsilon = .01$

The above algorithm is applied to the problem of resolving the solution to (67) with densities given by $\rho_1 = 2$ and $\rho_2 = 1$ in the functions $B_1(\beta)$ and $B_2(\beta)$. We considered far field data $U^- = (1.9, 1.0)^t$ and $U^+ = (1.1, 1.1/1.9)^t$, which, according to the theory presented earlier, correspond to single overcompressive singular shock with speed $s = 0$ in the inviscid problem. The adaptive grid ODE integrator is set to enforce a maximum local truncation error of 10^{-12}. Newton's method is said to be converged when its residual is less than 10^{-10}. Figure 5(a) depicts $U = (\beta, v)^t$ within the singular layer when $\epsilon = 0.05$. Note that the spatial domain is given in units $\tau = x/(t\epsilon)$. The singular component v is quite evident on inspection. The β component is well behaved and lies within the invariant region $\rho_2 \leq \beta \leq \rho_1$. Figure 6 depicts the solution when $\epsilon = 0.01$. The singular component v now has completely blown off the displayed scale in the left picture. Also note that β has almost sharpened into a saw-tooth – remarkable considering the spatial $x/(t\epsilon)$ scaling.

In Figure 5(b) we plot $\log(v(0))$ as a function of $1/\epsilon$ where ϵ ranges from 0.10 to 0.01; ($1/\epsilon$ ranges from 10 to 100). The asymptotic analysis given in Section 3.2

(equation (38)) implies this plot should approximate a straight line when ϵ is sufficiently small. Clearly the analysis is vindicated by the numerical result. Moreover, the asymptotic analysis tells us the slope of the line should be 1/6 of the v component's Rankine-Hugoniot deficit; 0.05540166 for this example problem. Using the computed data at $1/\epsilon = 10$ and $1/\epsilon = 100$, we find the slope of the plot's secant line is 0.05489607.

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