A Geometric Theory of Conservation Laws which Change Type*

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Abstract. The study of systems of conservation laws has both geometric and analytic aspects. The structure of shocks and their relation to characteristics of the partial differential equation are geometric in nature; these geometric pictures lead to admissibility criteria for shocks and the description of the propagation of a discontinuity as a Riemann solution.

On the other hand, justification of geometric admissibility criteria, the finding of appropriate spaces for existence and uniqueness of solutions, asymptotic behavior of systems and the convergence of approximations all involve substantial analysis and, ultimately, a consideration of the relation between the differential equations and the physical principles or the modelling assumptions which lead to a particular system.

In this survey, we illustrate the interplay between geometry and analysis in old and new kinds of conservation laws which have appeared in models recently. We present some models which change type in the sense that the characteristics of the linearized system are not everywhere real; such models occur in complicated flows, and may indicate instability in the system, either explicit or implicit. We formulate geometric admissibility criteria for shocks, and show how they can be validated analytically.

1. HYPERBOLIC CONSERVATION LAWS IN MODELLING AND ANALYSIS

Systems of hyperbolic conservation laws model many physical systems, including the unsteady motion of gases, liquids and elastic solids. Ideal gases, free surface flows, and mixtures (multiphase flows) are all studied using systems of partial differential equations that fall in this class. In such systems, a dominant feature which models important physical behavior is the manner in which nonlinearities lead to the phenomena of shock formation and shock propagation.

Hyperbolicity is a mathematical attribute: its significance for applications is the existence of wave speeds – characteristic speeds which govern the propagation of infinitesimal disturbances. Characteristics are a linear phenomenon. Shocks are macroscopic – finite – disturbances: they exhibit discontinuities in variables which are not characteristic, and they propagate at speeds which are noncharacteristic. The existence of both linear and

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nonlinear modes of wave propagation gives richness to the description of phenomena using nonlinear hyperbolic equations. The monograph of Courant and Friedrichs, [9], established the relevance of conservation laws to important aspects of compressible flow, while Lax, in a series of papers, [27, 28, 29, 30], expounded the basic geometry of shocks and characteristics. This stimulated further analytical work to establish existence and well-posedness theorems, such as Glimm's celebrated result, [10], which formed the focus of the field through the 1970's. The monographs of Majda, [36], and Smoller, [44], contribute to and describe these mathematical advances.

2. THE PROTOTYPE EQUATION AND ITS EXTENSIONS

We can recapitulate the development of the subject by describing a prototype equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) \equiv u_t + u\, u_x = 0, \tag{2.1}$$

(sometimes called the inviscid Burgers equation, or the Hopf equation) which represents a disturbance, u, advecting in a single space dimension with a local velocity (characteristic speed) equal to the amplitude of the disturbance. If one examines the Cauchy problem for (2.1) with data $u(x,0) = u_0(x)$, then simple characteristic tracing shows that, unless u_0 is a monotonically increasing function of x, a classical solution exists for only a finite time: any downward-sloping part of the data will steepen, and the solution, given implicitly by the formula

$$u(x_0 + u_0(x_0)t, t) = u_0(x_0)$$

develops an infinite slope at

$$T = \inf_{x_0} \frac{-1}{u_0'(x_0)},$$

where the infimum is taken over points where u_0' is negative.

Classical theory defines weak solutions of (2.1) as follows. Integrate against an arbitrary smooth test function in (2.1) and then replace the resulting equation by an inequality. A bounded measurable function $u(x,t)$ is said to be a weak solution of (2.1) if

$$\iint \left\{ |u(x,t) - k|\varphi_t + \operatorname{sign}(u(x,t) - k)\left(\frac{u^2}{2} - \frac{k^2}{2}\right)\varphi_x \right\} dx\, dt \geq 0 \tag{2.2}$$

for all constants k and nonnegative test functions φ. A unique weak solution of (2.1) which assumes Cauchy data $u_0(x) \in L^\infty$ exists for all $t > 0$. This theorem was proved by Kružkov, [26], (for any scalar equation in any number of space dimensions).

Now, inequality (2.2) incorporates a so-called *entropy condition*; such a condition is necessary to obtain well-posedness (uniqueness and continuous dependence on data). We can explain this as follows. Equation (2.2) implies the equation

$$\iint \left(u\varphi_t + \frac{u^2}{2}\varphi_x \right) dx\, dt = 0, \tag{2.3}$$

the integrated (or *weak*) form of (2.1), and this in turn implies the *Rankine-Hugoniot relation*

$$s = \frac{dx}{dt} = \frac{[u^2/2]}{[u]} = \frac{u_L + u_R}{2} \tag{2.4}$$

along a discontinuity $x(t)$ separating two regions of smooth flow, $u_L(x,t)$ on the left and $u_R(x,t)$ on the right. Conversely, any piecewise smooth solution of (2.1) which satisfies (2.4) also satisfies (2.3), but there may be extraneous solutions (corresponding to 'rarefaction shocks', for example). Equation (2.2) is satisfied only if an additional condition is met at discontinuities, namely

$$u_L > u_R\,; \tag{2.5}$$

and this rules out extraneous solutions. Inequality (2.5) is the *admissibility condition* for shocks for (2.1). In fact, since then

$$u_L > s > u_R, \tag{2.6}$$

one sees that the shock speed is intermediate between the characteristic speeds of the states on either side, and the geometry is such that the forward characteristics intersect the shock from both sides. Inequality (2.6) called the *geometric entropy condition*, GEC, or Lax entropy condition for shocks.

For the prototype equation, several other admissibility criteria are known to be equivalent to the GEC or to (2.2). All have some physical and modelling significance. The first is the *entropy/entropy-flux* criterion: if η (the 'entropy') is any *convex* function of u and q (the 'entropy flux') is another function of u such that

$$\partial_t \eta(u) + \partial_x q(u) = 0$$

for smooth solutions, then this criterion requires that

$$\iint \Big(\eta \varphi_t + q \varphi_x \Big)\, dx\, dt \geq 0 \text{ for } \varphi \geq 0 \in \mathcal{D}$$

for admissible weak solutions. (Here, $\mathcal{D}$ is the space of test functions.)

A second condition is the *viscous profile* criterion: consider

$$u_t + \left(\frac{u^2}{2}\right)_x = \epsilon u_{xx} \tag{2.7}$$

which is parabolic and hence irreversible if $\epsilon > 0$. Admissible shocks in (2.1), between two constant states u_R and u_L satisfying (2.4), are those which can be approximated by self-similar solutions

$$u = u\left(\frac{x - st}{\epsilon}\right)$$

of (2.7). For weak solutions which are not piecewise smooth, one can also consider a more general viscosity criterion which admits as a weak solution any bounded measurable solution of (2.1) which is a limit of solutions of (2.7).

A third condition is that of *linearized stability* under inviscid perturbation of a uniform shock

$$u_0(x,t) = \begin{cases} u_L, & x \leq st \\ u_R, & x > st \end{cases}.$$

Perturbation data are taken to be $u_L + \epsilon \tilde{u}_L(x)$ for $x < 0$ and $u_R + \epsilon \tilde{u}_R(x)$ for $x > 0$ and a perturbed solution of the form

$$\begin{aligned} u(x,t) &= u_0(x,t) + \epsilon v(x,t) + \mathcal{O}(\epsilon^2) \\ s(t) &= st + \epsilon \phi(t) + \mathcal{O}(\epsilon^2) \end{aligned}$$

is sought. Linearizing leads to two quarter-plane problems:

$$\begin{cases} v_t + u_L v_x &= 0, & x < st \\ v(x,0) &= \tilde{u}_L(x), & x < 0 \end{cases}$$

$$\begin{cases} v_t + u_R v_x &= 0, & x > st \\ v(x,0) &= \tilde{u}_R(x), & x > 0 \end{cases}$$

and an evolution equation for the shock:

$$\phi'(t) = \frac{1}{u_R - u_L}\Big\{(u_R - s)v(st+0,t) - (u_L - s)v(st-0,t)\Big\}\,.$$

Well-posedness of this linear system implies admissibility of the shock by this criterion. One can also formulate a nonlinear stability question, by asking whether the perturbed data lead to a unique classical solution to the quasilinear problem on each side of the perturbed shock (one expects such a solution to exist for only a finite time); this is a nonlinear existence problem, and its solution involves a free-boundary problem; see [31], where such problems are studied.

Finally, we mention the criterion of *evolutionarity* of a uniform shock: here one replaces the initial data of a uniform shock by a smooth function asymptotic to the shock values u_R and u_L as $x \to \infty$ and $x \to -\infty$ and asks whether the shock discontinuity will evolve from smooth data in forward time. This question is related to the asymptotic stability of shocks and has been much studied by Liu and co-workers; see, for example, [8], [34] and [35].

Relationships among these criteria are explored in the references.

2.1 EXTENSIONS OF THE PROTOTYPE

All the above criteria have been applied to the Hopf equation and all confirm the geometric entropy condition. Of greater interest in physics and engineering are systems of conservation laws of the form

$$u_t + f(u)_x \equiv \partial_t \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \partial_x \begin{pmatrix} f_1(u_1,\ldots,u_n) \\ f_2(u_1,\ldots,u_n) \\ \vdots \\ f_n(u_1,\ldots,u_n) \end{pmatrix} = 0 \tag{2.8}$$

where the state components u_i represent densities of mass, momentum, energy, ions and so on and the f_i are fluxes of these variables. Write the system in quasilinear form,

$$u_t + \frac{\partial f}{\partial u} u_x \equiv u_t + A(u)u_x = 0;$$

and introduce characteristic quantities, as follows. Let λ_i be an eigenvalue of A and $\vec{r}_i$ and $\vec{\ell}_i$ the corresponding right and left eigenvectors, so

$$\Big(A(u) - \lambda_i(u)I\Big)\vec{r}_i(u) = 0; \qquad \vec{\ell}_i A(u) = \lambda_i \vec{\ell}_i.$$

The system is written in *characteristic form* as

$$\vec{\ell}_i\, u_t + \lambda_i \vec{\ell}_i\, u_x = 0$$

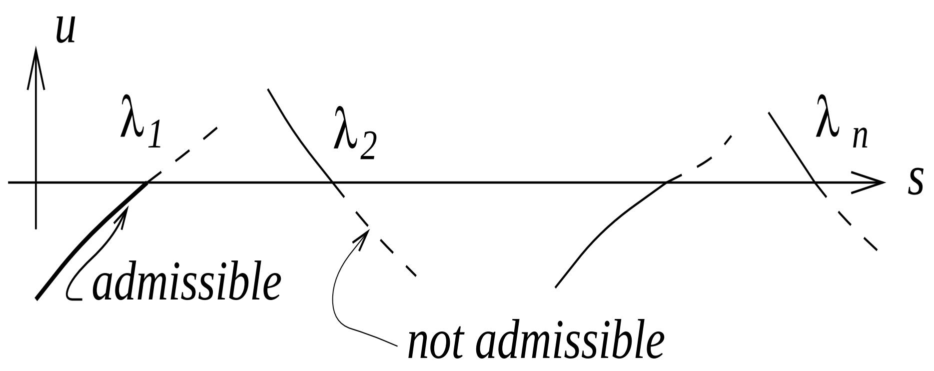

Figure 1: The Hugoniot Locus Viewed as a Bifurcation Problem. Each real, simple eigenvalue spawns a transcritical curve of nontrivial solutions. Each subcritical branch is admissible near λ_i.

for $i = 1, 2, \ldots, n$. Within each equation, differentiation occurs in one direction only. The corresponding *characteristic variable* is $w_i = \vec{\ell}_i u$. The Hopf equation is a useful model for how the system behaves only if each λ_i is *real* (hyperbolic) and *monotonic* in w_i, which means $\vec{r}_i \cdot \nabla\lambda_i \neq 0$ (this condition is called *genuine nonlinearity*).

In particular, as with formula (2.3), the weak form of (2.8) implies a Rankine-Hugoniot relation $s[u] = [f(u)]$ for shocks; it is useful to write this in the form

$$V(u, s; u_L) \equiv s(u - u_L) - (f(u) - f(u_L)) = 0 \tag{2.9}$$

where $V : \mathbf{R}^{n+1} \to \mathbf{R}^n$ is a mapping of u, parameterized by s. Seeking nontrivial solutions of (2.9) leads to a bifurcation problem in s: a transcritical bifurcation occurs at $s = \lambda_i(u_L)$ if $\lambda_i(u_L)$ is real, simple and genuinely nonlinear; see Figure 1. By analogy with the scalar equation, one obtains the geometric entropy condition for systems,

$$\lambda_i(u_L) > s > \lambda_i(u_R), \tag{2.10}$$

for a shock of the i-th family, bifurcating from $\lambda_i(u_L)$. One can justify this condition by means of the principles mentioned already — entropy, viscosity, linearized stability, nonlinear stability and evolutionarity. The References include studies of this sort, and citations of many more. Interaction between different characteristic families complicates the extension of scalar results to systems. For example, to validate the viscous profile criterion one must establish the existence of heteroclinic connections in the corresponding dynamical system. Figure 2 sketches the dynamics of a typical orbit in phase space.

Although there are slight differences in the application of the different criteria to moderate sized shocks, all lead to (2.10) for sufficiently small shocks if λ_i is a real, simple, genuinely nonlinear characteristic value at u_L.

2.2 LESSONS OF THE PROTOTYPE

Before studying the adaptations which must be made when the classical theory no longer suffices, we summarize the experience with the shock geometry of nonlinear conservation laws as follows.

1. When wave speeds depend on the amplitude of signals, solutions do not remain smooth in general, but instead shocks form in finite time.

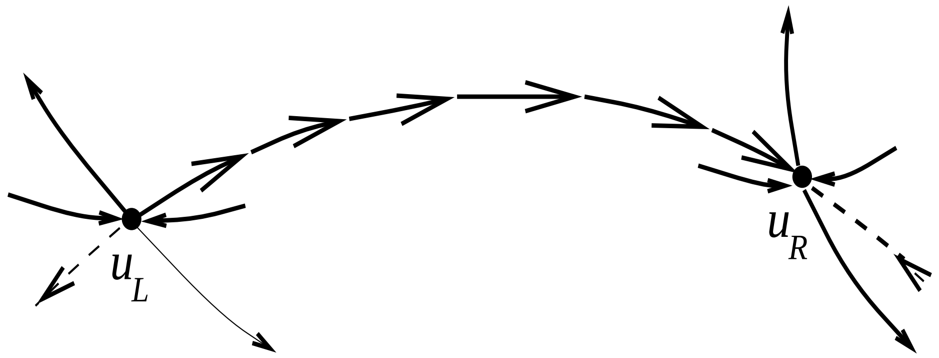

Figure 2: The Dynamics of the Viscous Profile Criterion

2. Sharp signals, in the form of shocks, propagate at noncharacteristic speeds. (By contrast, a linear hyperbolic system obeys a weak Huygens' principle: discontinuities are confined to the characteristic variety. The reader is referred to [12] or [39] for properties of linear systems.)

3. Even though a system may be strictly hyperbolic, this condition is no longer sufficient for well-posedness of the initial-value problem, as it is for linear hyperbolic equations. The ill-posedness can be described as a lack of uniqueness and a lack of continuous dependence in the weak formulation; the weak formulation is incomplete; the underlying flow is reversible for smooth solutions but irreversible for discontinuous ones. Completing the formulation requires further constraints: specifying a direction for the time variable, or imposing continuity under taking limits of explicitly irreversible processes involving entropy or viscosity. The constraints may be formulated as physical principles (entropy or viscosity) or as self-contained analysis (linear or nonlinear perturbation, evolutionarity), or via the simple formal geometric condition. These are all, for the main part, equivalent.

3. FAILURE OF THE PROTOTYPE AND CLASSIFICATION OF NEW PHENOMENA

As the previous section indicated, classical shock admissibility theory for a scalar equation is based on a characteristic speed, $\lambda(u)$, which depends monotonically on u, and this generalizes well to a system for which all the characteristic speeds, λ_i, are real and distinct and in which each depends monotonically on the corresponding characteristic variable. The remainder of this paper concerns problems where these nondegeneracy conditions fail. Systems with degeneracies arise in applications, and have been much discussed. A partial list of papers which explore interesting systems from different areas of science is included in the Reference section; see [1], [2], [4], [5], [13], [15], [19], [20], [21], [42], [43] and [46] in particular. In this paper, we describe the mathematical framework underlying extensions of the classical model.

3.1 THE FAILURE OF GENUINE NONLINEARITY

Genuine nonlinearity fails to hold in a scalar equation, $u_t + f(u)_x = 0$, if f is not convex. If we ignore fields which are linearly degenerate everywhere, that is, for which $\vec{r}_i \cdot \nabla\lambda_i \equiv 0$, then there are two ways to regard a linear degeneracy: either from a local viewpoint,

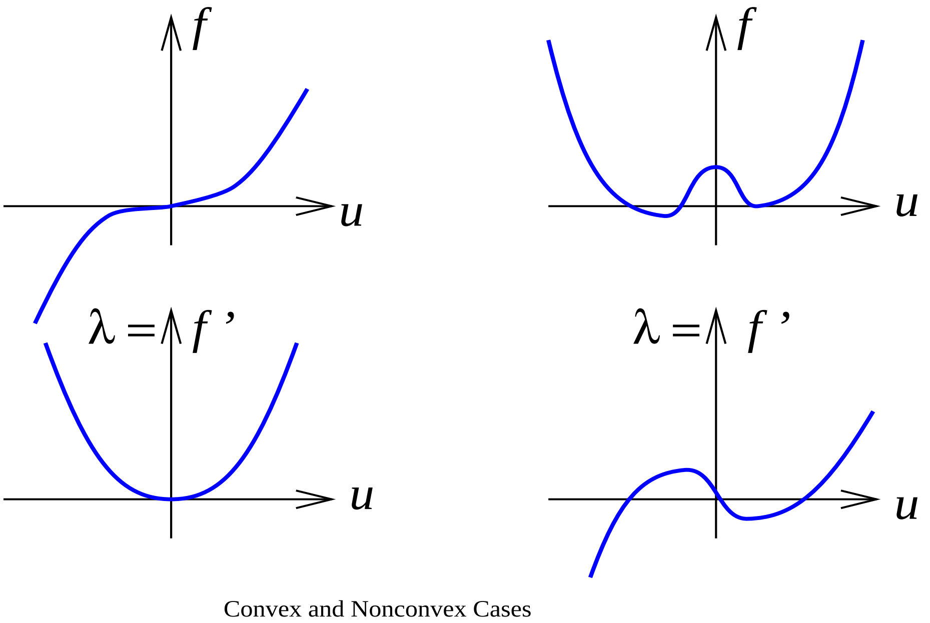

Figure 3: Local and Global Failure of Genuine Nonlinearity

near a point where $\vec{r}_i \cdot \nabla\lambda_i = 0$ (a scalar model is $f = u^3$), or from a global viewpoint, as a perturbation of a system which is convex (a model is $f = u^4 - u^2$). Figure 3 sketches local and global models: flux functions are shown in the first row with their derivatives, the characteristics speeds, below them.

For the scalar equation, complete geometric admissibility criteria are known, [37, 38]. (There is also a theory for systems, [32, 33], and most criteria have been verified here also.) The original geometric entropy condition, (2.10), is no longer correct, but it can be replaced by an inequality based on convex hulls of f. The best way to describe shock geometry here is to formulate a *Riemann problem*: an initial value problem with piecewise constant data

$$u_0(x) = \begin{cases} u_L, & x \le 0 \\ u_R, & x > 0 \end{cases} .$$

If the data constitute an admissible shock, it propagates along $x = st$; otherwise the solution to this problem consists of a combination of admissible shocks, constant states, and smooth functions of x/t, called centered rarefaction waves, which satisfy the equation. The Riemann solution for nonconvex f is as follows. (See the illustration in Figure 4.) An admissible shock can be constructed between u_L and u_R if $u_L > u_R$ and if the flux function f, for values of u between u_R and u_L, lies below the chord joining $(u_L, f(u_L))$ to $(u_R, f(u_R))$. If f does not have this property, then one constructs the *concave hull* of f between those two points; the solution joining states u_L and u_R consists of rarefactions (in the intervals where f coincides with its concave hull) interpolating shocks (which now satisfy the chord condition). If $u_L < u_R$, one constructs the convex hull and proceeds the same way. The shocks in this construction are admissible by the entropy/entropy-flux and viscosity criteria. The linearized stability condition, like the GEC, fails here because it is based on the characteristic speeds only at the two end states. Similarly, nonlinear inviscid perturbation would give the correct result only if the perturbations are large enough. For example, if the shock is replaced by a smooth transition, then one recovers the evolutionarity criterion, which correctly predicts the shock structure.

New phenomena appear here. The generic local picture is given by $f = u^3$, with $u_L > 0 > u_R$, and contains a *shock-fan* configuration, consisting of a shock attached to a rarefaction whose head or tail speed is exactly that of the shock, as shown in Figure 4.

Another new feature is the existence of *avoided states*: for a given pair of states u_L and u_R, whether we take u_L to be the state on the right or the state on the left, the shock-fan configuration which arises will omit a range of values of u in phase space. An interpretation of this property is that probes of the system, based on fixing those two states as the upstream or downstream states, will not show a uniform distribution of intermediate values of u, such as one would see in a genuinely nonlinear system. One imagines a sort of wrinkle, caused by lack of uniform convexity, in the flux. The intermediate values are not unstable in any technical sense, but as the system evolves in time, they will disappear into the shock even if they are present initially.

Another phenomenon, which occurs in the global picture, is *splitting of shocks*: if the end states u_L and u_R are not constant but vary in time, then a single large shock may split into a shock-fan configuration. One manifestation of this in the numerical computation of shocks using shock-capturing methods which introduce numerical viscosity is that shock profiles may show intermediate layers. An example leading to shock splitting is illustrated in Figure 5.

3.2 DEGENERACIES INVOLVING INTERACTION OF FAMILIES

A variety of phenomena occur when we consider degeneracies in systems of conservation laws. An elementary classification begins with the bifurcation equation (2.9). When genuine nonlinearity fails, the bifurcation may not be transcritical (the theory of the last subsection can be formulated from this viewpoint); higher-order degeneracies at the bifurcation point result in more complicated shock-fan solutions.

Another nondegeneracy condition fails when the eigenvalues of A are not everywhere distinct; that is, the characteristic polynomial, $p(u, \lambda)$, of $A(u)$ and its derivative p_λ have a common root. This may happen at a point, or along a curve, or a higher-dimensional submanifold of the state space $\Omega \subset \mathbf{R}^n$ or in all of Ω. A complete classification does not yet exist, but interesting cases have been identified. If two eigenvalues coincide on a codimension one submanifold Σ of Ω, and all eigenvalues are real and distinct on $\Omega \sim \Sigma$, then the system is *nonstrictly hyperbolic*, NSH. Examples of such systems have been extensively studied, for example in [13], [22] and [23]. For any NSH flux function, f, there exist nearby fluxes, $\tilde{f}$ (nearby in the sense that $\|f(u) - \tilde{f}(u)\|$ is small in a neighborhood of Σ) which correspond to strictly hyperbolic systems which fail to be genuinely nonlinear somewhere near Σ, and other $\tilde{f}$ for which the system has some non-real eigenvalues in a neighborhood of Σ. Eigenvalue coincidence occurs in numerous models of physical phenomena, so it is not surprising that change of type does also. We explore this in the next section. Figure 6 illustrates some cases schematically.

We comment briefly on the degeneracies pictured in the bottom row of Figure 6. The first has been described as a lack of global separation of eigenvalues. A branch of solutions of (2.9) emerging from one eigenvalue is globally coincident with a branch from another eigenvalue. When this occurs, a new and stronger type of singularity may appear in shock solutions, [24], [40]. Admissibility for singular shocks is very much an open question at the moment.

Another interesting possibility is an *umbilic point*: this term describes eigenvalue coincidence at a point, u_0, in Ω; eigenvalues are real and distinct in a neighborhood of u_0. Classification of the shock structure and solution of Riemann problems for this

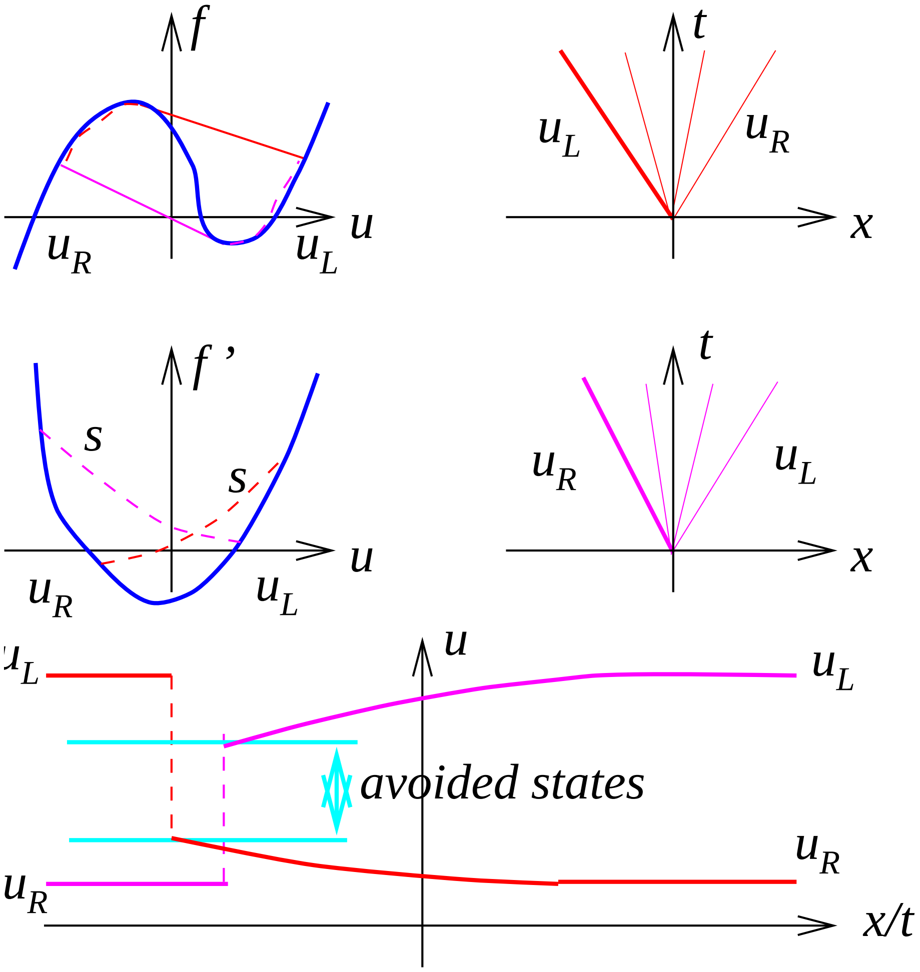

Figure 4: Riemann Solution for Nonconvex f and the Phenomenon of Avoided States. For f and f' as sketched, the solution in physical space is given in the top right diagram. The corresponding solution when the state labelled u_L is on the right and u_R is on the left is given in the center right diagram. The bottom diagram sketches the solution profiles in both cases.

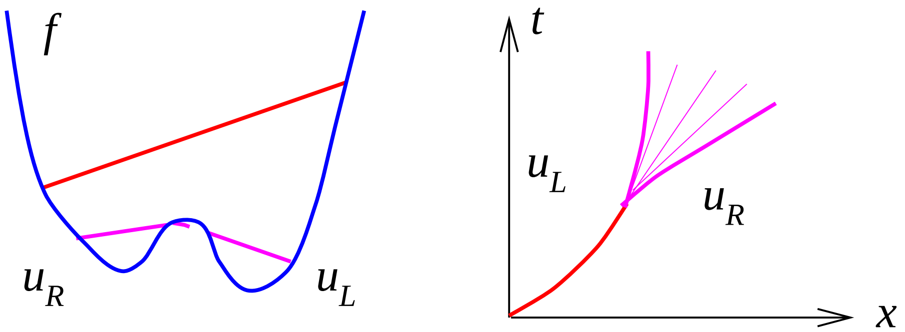

Figure 5: Splitting of Shocks. In this example, initial data are given so that the value of u left of the shock, u_L, decreases in time, and u_R increases.

problem has been the subject of many papers, beginning with [41]; new kinds of shock geometry appear, and shocks may be admissible under modifications of the conditions described in the previous section; change of type occurs in perturbations of these systems. Overall, there are many open questions relating to umbilic points.

4. THE THEORY OF UNSTEADY CHANGE OF TYPE

Examples of systems that change type occur in modelling of two-phase mixtures, [46], three-phase porous-media flow, [5], [42], and simplified models of fluids or solids undergoing phase transitions, [1], [15], [43], [45]. Initially, many such models were rejected, as it was believed that they would lead to catastrophic ill-posedness of the system; hence, it was felt, they had appeared only because of some significant omission or error in the modelling assumptions. However, a mathematical theory of systems which change type is beginning to emerge, as we shall outline in this section. This research effort continues, and many open questions remain. Models which change type appear to be incomplete rather than catastrophically ill-posed. To some extent, quasilinear hyperbolic systems are also incomplete, as the earlier sections have made clear. Change of type systems share this property but are more problematic. Efforts to improve the models through better understanding of the physics behind them continue to be the focus of much engineering and computational work in problems of multiphase flow and elastoplasticity. Here, we outline a basis for a mathematical theory.

A point to emphasize at the start is one which has its analogue in the linear degeneracy dichotomy of Section 3: one can regard change of type either from a local or from a global perspective, and one must usually do both in order to solve a problem completely. The admissibility of a single shock is typically a local problem, and this is discussed in Section 4.1. On the other hand, resolution of a discontinuity (Riemann problem) is typically global. By a global perspective we mean that we can regard a flux function in a system that changes type as a perturbation of one which is well understood, for example, strictly hyperbolic. This will be discussed in Section 4.2.

4.1 SHOCKS NEAR THE SONIC LINE

A relatively complete picture of shock construction near a curve where a system changes type can be given in the least degenerate case. The prototype model is a system of two

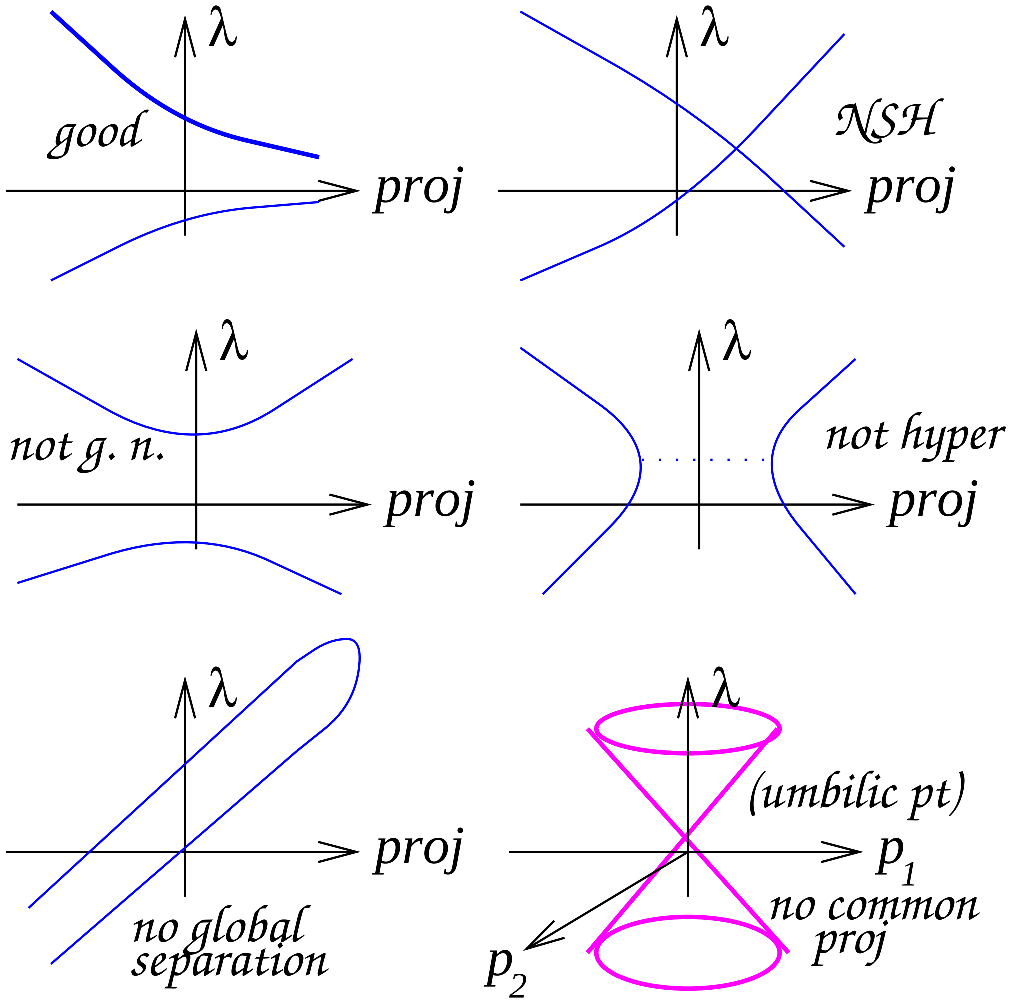

Figure 6: Types of Interactions. The horizontal axis represents some common projection of the eigenspace in the neighborhood of a pair of eigenvalues. The top left diagram shows the prototype, nondegenerate situation: characteristic and shock speeds vary monotonically along each bifurcating branch. Eigenvalue coincidence, top right, can be perturbed in two ways, as shown in the center row. One direction leads to a strictly hyperbolic system where genuine nonlinearity fails, left; and the other to a system which changes type locally, right. The bottom row illustrates two other degeneracies. In the picture on the left, the eigenvalues are distinct and the bifurcation branches are locally monotonic, but not globally separated. The situation on the right occurs when the coincidence locus is a point or higher-codimension submanifold: there is no common projection of eigenvectors; new phenomena appear in this case.

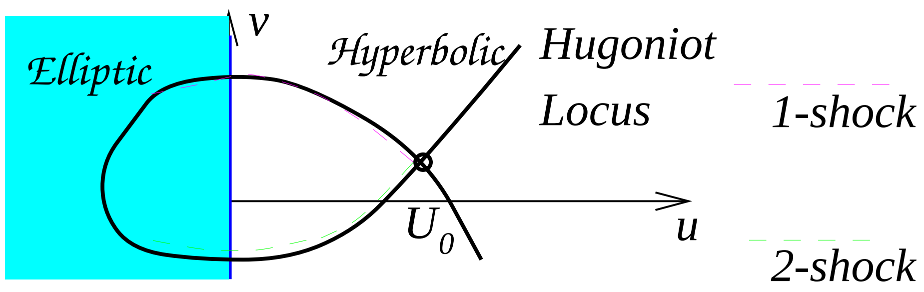

Figure 7: The Hugoniot Loop. The bifurcation diagram is projected into state space; the portions of the diagram corresponding to standard shocks of the first and second characteristic families are highlighted.

equations

$$\begin{aligned} u_t - v_x &= 0 \\ v_t - uu_x &= 0. \end{aligned} \tag{4.1}$$

This system is hyperbolic in the right half-plane, $\mathcal{H}$, nonhyperbolic in the left half-plane, $\mathcal{E}$. The v-axis forms the change of type locus. We shall refer to this as the 'sonic line', by analogy with steady transonic flow; some authors call it a 'coincidence locus' because the eigenvalues of A, in this case $\pm\sqrt{u}$, are equal there. We prefer to reserve this term for eigenvalue coincidence in a NSH problem. The following theorem, [17], describes the Hugoniot locus (solutions of the Rankine-Hugoniot relation for a fixed $U_0 \equiv (u_0, v_0)$) near the sonic line.

Theorem 4.1 *The system (4.1) gives a normal form for characteristics and for solutions of the Rankine-Hugoniot relation near the curve where a system $u_t + f(u)_x = 0$ changes type. If $U_0 \in \mathcal{H}$, the Hugoniot locus contains a loop; if $U_0 \in \mathcal{E}$, then the locus is detached from U_0, and consists entirely of points in $\mathcal{H}$.*

If a flux f satisfies a set of defining and nondegeneracy conditions at a point $u_0 \in \mathbf{R}^n$, then the characteristic and shock structure will be qualitatively the same as that of (4.1) in a neighborhood of U_0. The system (4.1) is a prototype for local behavior of systems which change type, analogous to the Hopf equation; see [6], where the prototype equation is derived by weakly nonlinear geometric optics techniques.

Figure 7 shows the Hugoniot loop. Since Theorem 4.1 shows that for any shock connection, the state on one side must be hyperbolic, we can take the base state to be in $\mathcal{H}$ in the discussion which follows.

We ask which shock configurations are admissible. The admissibility criteria which gave a mathematical theory for the prototype scalar equation and its hyperbolic generalizations can be used to answer this question. The results are not completely self-consistent; they do consistently indicate that at least some shocks joining points in $\mathcal{H}$ to points in $\mathcal{E}$ are admissible. Not all admissibility criteria can be applied to this problem. For example, there does not exist a convex entropy function for a system which changes type, since the existence of a convex entropy function actually implies that the system can be put in symmetric hyperbolic form and is therefore hyperbolic. The evolutionarity criterion would apply, but has not yet been tested. At the moment, there are three tests which have been made: linearized stability, viscous profiles and nonlinear stability. They have been applied to systems equivalent to (4.1) as well as to some more complex systems. The most delicate is the viscous profile criterion, and we state its conclusions first, [18].

Theorem 4.2 *Consider a similarity solution* $u = u\left(\frac{x-st}{\epsilon}\right)$ *of*

$$u_t + f(u)_x = \epsilon D u_{xx},$$

connecting a state u_0*, on the left or on the right, with another state* u *on the Hugoniot locus of* u_0*. Then, for* D *near* I*, and* u_0 *near the sonic line, there exist viscous profiles between* u_0 *and points on the Hugoniot loop near* u_0*. There are two types of admissible shocks: 1-shocks with* u_0 *on the right and 2-shocks with* u_0 *on the left. The interval where profiles exist extends into the elliptic region, but does not include the entire loop. The end points of the intervals depend on* D *and on* f*; when* $D = I$*, the two intervals for the flux in the normal form (4.1) cover the entire loop; however, this property is structurally unstable; for nearby matrices* D *and nearby fluxes, there is generically an open interval of the loop where no profiles exist.*

For the normal form (4.1) and $D = I$, the proof of Theorem 4.2 shows that the two intervals meet at the unique point on the Hugoniot loop where $s = \Re(\lambda(U))$. This point also distinguishes candidates for linearized stability in Theorem 4.3 below, but is not necessarily a criterion for the viscous profile condition. In fact, we have

Corollary 4.1 *For an open set of fluxes and viscous matrices,* D*, the condition* $s \geq \Re(\lambda(U))$ *is not sufficient to give viscous profiles for* 2*-shocks, nor* $s \leq \Re(\lambda(U))$ *for* 1*-shocks.*

As was the case for hyperbolic problems, the linearized stability criterion is less restrictive than viscous profiles. We have, [3],

Theorem 4.3 *For a system of two equations which changes type in a way characterized by the normal form (4.1), the point* $s = \Re(\lambda(U))$ *divides the Hugoniot loop into two intervals; on one interval, 1-shock connections with* U_0 *on the right are linearly stable; on the other, the 2-shocks with* U_0 *on the left satisfy linearized stability.*

Figure 8 shows the geometry of linearized stability. There is a nonlinear version of this theorem, [25].

Theorem 4.4 *Consider perturbed Cauchy data of a uniform shock solution of an equation with normal form (4.1), where the uniform shock is linearly stable and satisfies a strict inequality* ($s < \Re(\lambda)$ *or* $s > \Re(\lambda)$)*. If the perturbed data are in a class of uniformly analytic functions, and small in a suitable norm, then there is a time* T*, depending on geometric constants but uniform in the perturbations, for which there is a unique piecewise smooth solution which satisfies the Rankine-Hugoniot condition across a single curve of discontinuity. Furthermore, the perturbed solution is close to the uniform one. As the perturbations go to zero, the solution approaches the uniform shock, uniformly in a fixed time interval* T*.*

4.2 LARGE SCALE EFFECTS OF CHANGE OF TYPE

An analysis of change of type in the large begins by supposing that a nonhyperbolic region in state space Ω appears (or disappears) as the flux function is perturbed. When $\mathcal{E}$ is small, one might consider it as shrinking, under perturbation, and it could shrink to a point or to a line. Prototypes for the two limiting cases, assuming the simplest case of a system of two equations, are

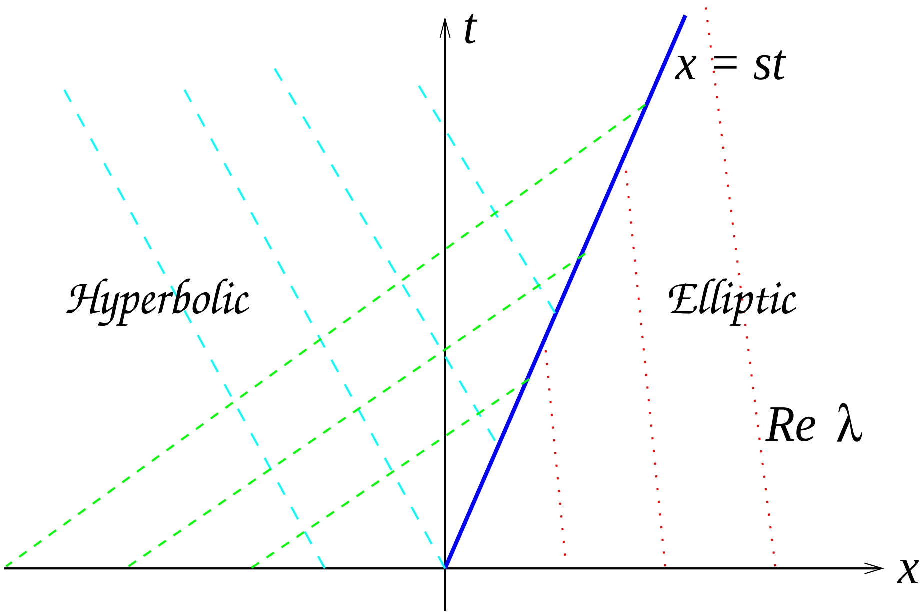

Figure 8: A Case Where Linearized Stability Holds

1. Umbilic Point: $\left\{ \begin{array}{l} u_t + (au^2 + buv + cv^2)_x = 0 \\ v_t + (du^2 + euv + fv^2)_x = 0 \end{array} \right.$

2. NSH Line: normal form $\left\{ \begin{array}{l} u_t + v_x = 0 \\ v_t + (u^3)_x = 0 \end{array} \right.$

Interestingly, the first gives rise to much more complicated behavior than the second. This is because a hyperbolic system with an umbilic point is itself very complicated, containing a number of normal forms for Riemann problems, as well as a number of subcases which require new kinds of shocks for their solution. A classification was begun in [41]; further developments are reported, for example, in [14]. Perturbations of these systems to include elliptic regions become yet more complicated, and a catalog, even of Riemann solutions, has not been completed. Examination of admissibility conditions and well-posedness of these systems is the subject of much current research; see [7] and [11] for example.

On the other hand, the normal form for a system with a NSH line in $\Omega \subset \mathbf{R}^2$ (this line is sometimes called the coincidence locus) is a simple system for which the Riemann problem and shock admissibility have been well studied, beginning with [23]. Although it is known from examples that some NSH systems may display a lack of well-posedness, [47], this system exhibits good behavior. A perturbation of the normal form which displays change of type is

$$\left\{ \begin{array}{l} u_t + v_x = 0 \\ v_t + (u^3 - \epsilon u)_x = 0 \end{array} \right. . \tag{4.2}$$

This system is hyperbolic outside a narrow band of elliptic states. The Riemann problem for this and related equations has been solved, in several ways, using several admissibility criteria. The solution includes the cases that one or both of u_L and u_R lie in $\mathcal{E}$. Figure 9 shows the projection of a typical Riemann solution in state space. Some properties

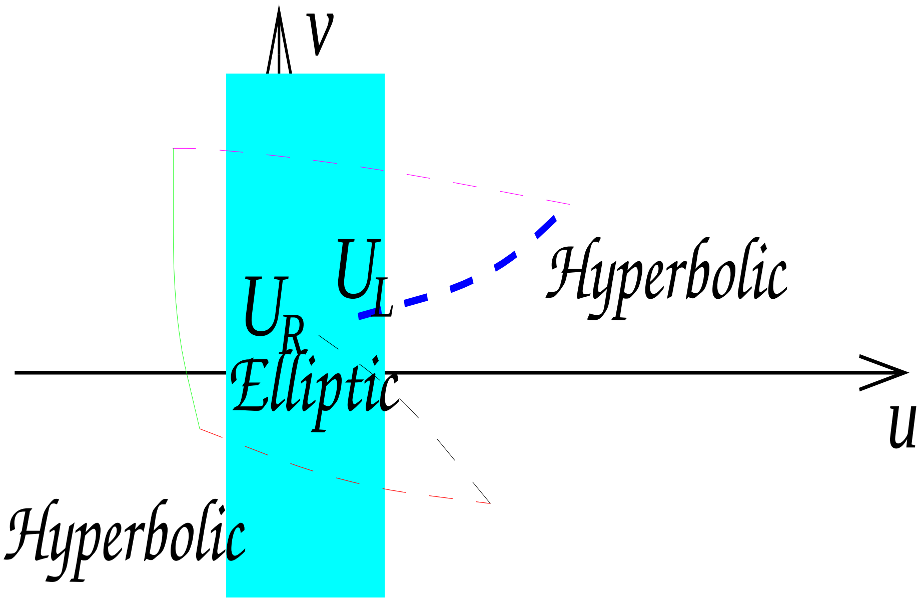

Figure 9: A Riemann Solution: Data in the Elliptic Region

of this solution are the following. Transitions between points in $\mathcal{E}$ and points in $\mathcal{H}$ are admissible shocks under at least one of the local criteria of the previous subsection. For most hyperbolic data, the Riemann problem has solution close to the NSH solution, and it lies outside $\mathcal{E}$. However, for data in a small band (proportional to $\sqrt{\epsilon}$ in (4.2)), the solution requires a new kind of $\mathcal{H}$-to-$\mathcal{H}$ shock, which does not satisfy the GEC. This kind of shock (sometimes called "undercompressive" or "transitional") occurs also in NSH problems of umbilic type which do not change type. A variant of this normal form has been devised, [16], whose Riemann solution uses only standard shocks which satisfy the GEC. Although the example in [16] seems rather contrived, it reinforces the point that the new kinds of shocks which appear are typically present in related hyperbolic systems and are not a direct consequence of change of type.

Another example of a change of type system which can be regarded as a perturbation of a coincidence line was analyzed by Vinod, [48]. In this example a quadratic system could be regarded as a perturbation either of a NSH coincidence line flux or of an umbilic, homogeneous quadratic, flux. For fluxes close to the coincidence line limit, Riemann solutions close to the NSH solution were found to exist. In this case, the appropriate Riemann data lie in the hyperbolic region.

In any case, all the Riemann problems for model equations which have been solved display an avoidance of elliptic states in their solutions. This was noted already on the basis of numerical calculation of Riemann solutions in [2] and [5]. However, it is not clear that this indicates instability any more than does the avoidance phenomenon already observed, see Figure 4, when genuine nonlinearity fails.

An interesting feature of the Riemann solution emerges when one sets $u_R = u_L \in \mathcal{E}$ in problem (4.2); an admissible nontrivial Riemann solution exists, and it is close to the admissible solution which is constructed when u_R is close to u_L. In order to obtain uniqueness, it would be necessary to reject the trivial solution as inadmissible. There are obvious difficulties with this approach. One heuristic resolution is the following.

One must begin with a careful statement of some physical basis for studying a Riemann problem, including a recognition that a theory centered on Riemann problems can apply only in a situation in which only self-similar solutions are to be expected. For example, this is the case if one is looking either at a very large time or a very small one.

4.3 LESSONS OF CHANGE OF TYPE MODELS

At the moment, there is no general theory, either based on quadratic fluxes or using perturbations, which asserts existence or uniqueness of Riemann solutions for systems which change type. As the numerous examples and counterexamples, those cited in the references and others, indicate, the closest approach to a general theory is likely to be one which involves perturbations of coincidence line NSH fluxes. Even here, examples have been devised to show that existence in the class of classical shock and rarefaction waves as well as uniqueness under any of the classical admissibility criteria can fail. However, most of these failures are present already in the NSH limit.

In none of the examples constructed so far is there evidence of the catastrophic failure of well-posedness, the famous Hadamard instability, [44], associated with the Cauchy problem for linear systems which are nonhyperbolic. The Riemann solution constructions for these equations, pictured for example in Figure 9, show that instead of the exponential growth in amplitude of disturbances predicted by linear theory, one finds a nonlinear saturation in amplitude which is essentially independent of the frequency of the disturbance. The following conjecture has not been rigorously confirmed, but numerical experiments suggest it is valid. One could take Cauchy data in the elliptic region, say for equation (4.2) or any system with reasonable Riemann solutions, and approximate by piecewise constant data, and then evolve an approximate solution using the Glimm approximation, [10]. There would be an instantaneous explosion of the total variation of the solution to an approximate size $\mathcal{O}(1/h)$ where h is the mesh size. This would be followed by a rapid collapse of the total variation as the high-frequency waves interact and merge. Asymptotically, one might expect to recover the Riemann solution of Figure 9. In fact, when the elliptic region is narrow, all wave speeds are close to the characteristic speed at the coincidence line, and the approach to an asymptotic state will be slower than for classical strictly hyperbolic systems (see [34] for the classical case). Nonetheless, this reasoning shows the way nonlinear interactions could potentially eliminate the high-frequency catastrophe in the Hadamard instability.

Another conclusion emerges from the construction of a Riemann solution for data in $\mathcal{E}$: disturbances are seen to propagate in space-time even though no real wave speeds exist. That is, the existence of shock waves allows one to define a substitute for the wave speeds for points in $\mathcal{E}$. In this way, a nonlinear theory complements linear hyperbolic theory.

We summarize the results reported in this section as follows.

1. Shock discontinuities exist in systems that change type and define a nonlinear mode of wave propagation even for states for which no real wave speeds exist.

2. Nonlinear saturation and nonlinear wave interactions (which are dissipative or irreversible in nature) remove the catastrophic high-frequency illposedness associated with linear change of type.

3. Details of the nonlinear solution structure depend on admissibility criteria. The solution is more sensitive to the choice of criterion than is the case for hyperbolic systems.

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