Pattern and Paradox: Shock Interactions in the Nonlinear Wave System

July 10, 2003

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joint work with Sunčica Čanić and Eun Heui Kim

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[a]Research supported by the Department of Energy, grant DE-FG-03-94-ER25222, and the National Science Foundation, grant 03-06307.

Outline of Program with S. Čanić, E. H. Kim, G. Lieberman

Similarity analysis of 2-D CL ($\xi = \frac{x}{t}$, $\eta = \frac{y}{t}$); piecewise const. data

Features:

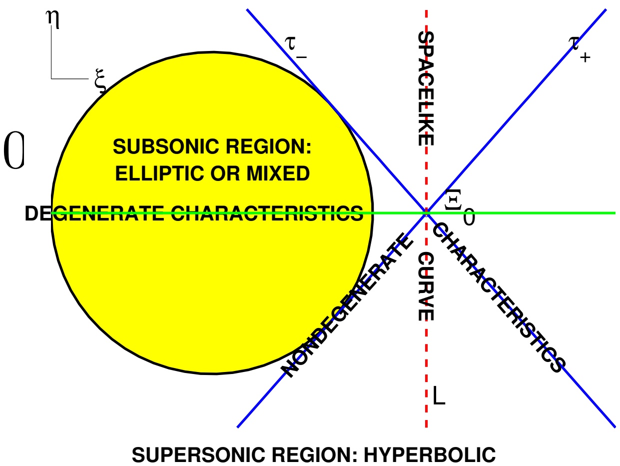
- 1-D waves in far field
- change of type in (ξ, η), like steady TS:

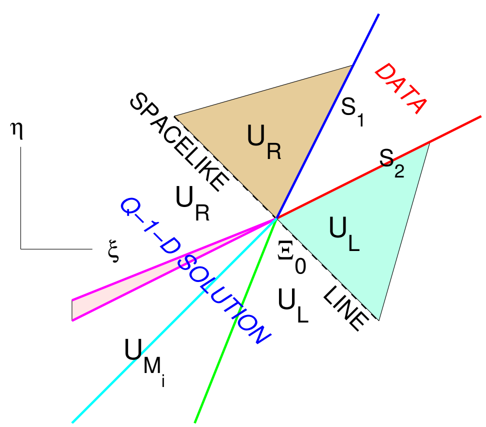
$U_t + F_x + G_y = 0$ to $(A-\xi)U_\xi + (B-\eta)U_\eta = 0$

Behavior of characteristics

Causality, determinacy

Acoustic type structure





- Quasi-one-D Riemann Problems:

'Shock polars' at $\Xi_0 = (\xi_0, \eta_0)$

'Self-similar' solution $U\left(\frac{\xi - \xi_0}{\eta - \eta_0}\right)$

cf. One-dimensional Riemann problem

Basic Features, continued

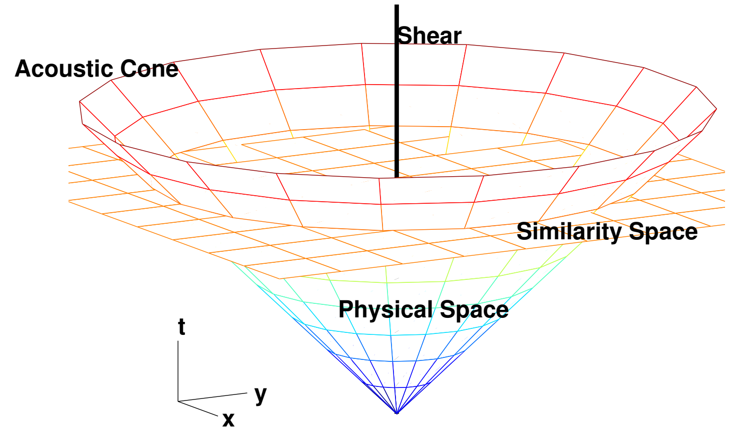
- Degenerate elliptic or mixed type

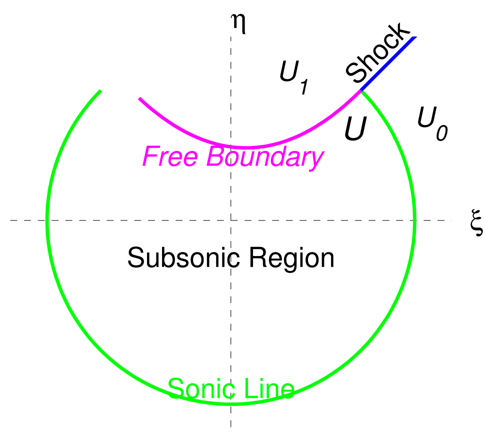
Degen. const U of Keldysh type:

$x\phi_{xx} + \phi_{yy}$ (cf. $\phi_{xx} + x\phi_{yy}$)

Linear solution $\sqrt{x}w(x,y)$

Fichera condition: data on deg. bdry





- Free boundary problems

RH relation $\chi[U] = [F] - \kappa[G]$; $[U] = U - U_1$

$\kappa = \frac{d\xi}{d\eta} =$ slope; $\chi = \xi - \eta\kappa =$ position

Overdetermined BC for elliptic equation

Related work: Morawetz, Brio-Hunter, Rosales-Tabak, Y. Zheng, K. Song, Chen-Feldman, Serre, Zhang-Zheng, S.-X. Chen et al.

Similarity Analysis of Two-Dimensional Systems: General Data

$U_t + F(U)_x + G(U)_y = 0,\ U \in \mathbb{R}^n$

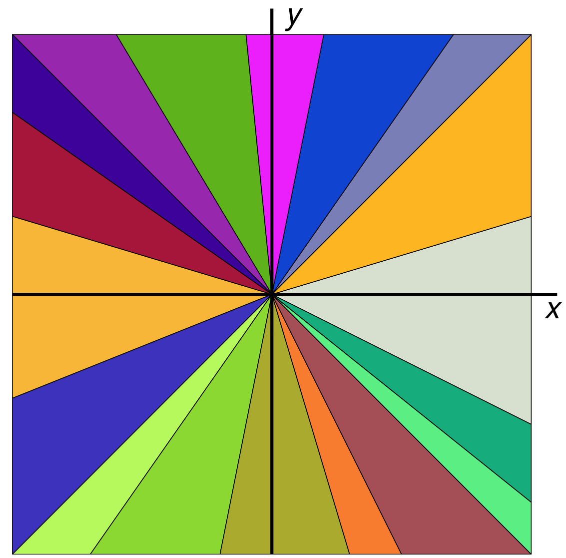
Data: $U(x, y, 0) = f\left(\frac{x}{y}\right)$

Similarity Variables:

$\xi = \frac{x}{t}, \eta = \frac{y}{t} \qquad U = U(\xi, \eta)$

Reduced System in Two Variables

$$\partial_\xi(F - \xi U) + \partial_\eta(G - \eta U)$$
$$\equiv \widetilde{F}_\xi + \widetilde{G}_\eta = -2U$$



Sectorially Constant Data

Method: resolve 1-D far-field discont.; give data for (IV)-BVP in 2-D

Type Changes: hyperbolic in far field; ‘subsonic’ region near origin

Difficulties: hyperbolic Q-1-D problems w/o solution; subsonic FBP

The Search for Prototype Systems: UTSD & NLWS

Comparison of Isentropic Gas Dynamics & NLWS

Isentropic Gas Dynamics: $p = \rho^\gamma/\gamma$

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0$$
$$(\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0$$

Nonlinear Wave System:

$$\rho_t + m_x + n_y = 0$$
$$m_t + p_x = 0 \qquad m = \rho u$$
$$n_t + p_y = 0 \qquad n = \rho v$$

Self-sim 2nd-order equation for nonlinear characteristic variable (ρ):

$$\big((c^2(\rho) - U^2)\rho_\xi - UV\rho_\eta\big)_\xi +$$
$$\big((c^2(\rho) - V^2)\rho_\eta - UV\rho_\xi\big)_\eta + \ldots = 0$$

$U = u - \xi, \quad V = v - \eta$ (‘pseudo-vel.’)

$$\big((c^2(\rho) - \xi^2)\rho_\xi - \xi\eta\rho_\eta\big)_\xi$$
$$+\big((c^2(\rho) - \eta^2)\rho_\eta - \xi\eta\rho_\xi\big)_\eta$$
$$+\xi\rho_\xi + \eta\rho_\eta = 0$$

Transport equation for linear characteristic variable:

$W = V_\xi - U_\eta = v_\xi - u_\eta =$ vorticity

$$UW_\xi + VW_\eta + (U_\xi + V_\eta + 1)W = 0$$

Nonlinear evolution equation

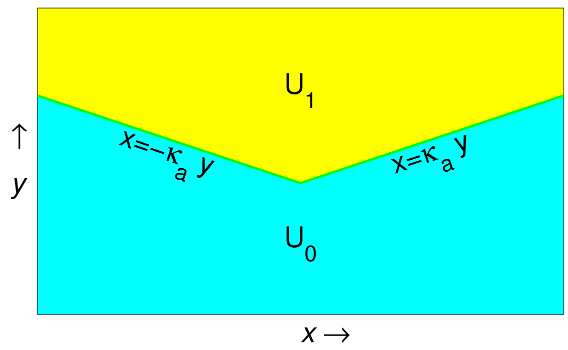
$w = n_\xi - m_\eta \qquad w_t = 0$

$(\xi, \eta) \cdot \nabla w + w = 0$ Linear

or: $rm_r = p_\xi \qquad rn_r = p_\eta$

The Search for Prototype Data

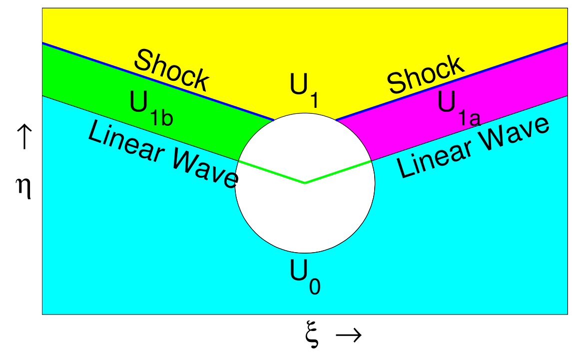
Interacting Shocks: A Bifurcation Problem for NLWS



2-state data: U_0, U_1

Data give 2 shocks

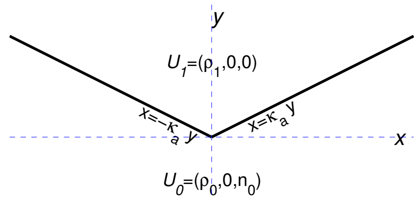
Far field soln: 4 waves



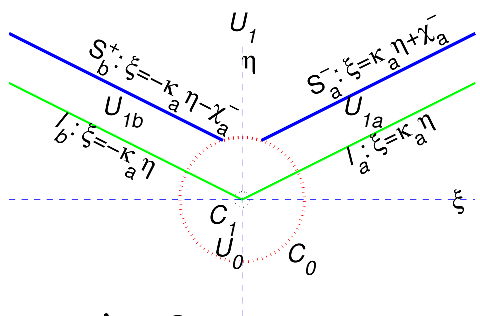
- Symmetric prototype for converging sector boundaries
- 'Weak shock reflection', von Neumann paradox
- Features

1. 2 parameters: $\rho_0/\rho_1 > 1$ and κ_a (Mach # and wedge angle)
2. Incident shocks: $\xi = \kappa_a \eta - \chi$, $\xi = -\kappa_a \eta + \chi$
3. Small κ: two local solutions –'weak' and 'strong' regular reflection
4. Large κ: curved shock, weak reflected wave (Čanić talk)
5. Intermediate values of κ: no sol'n from shock polars (Q1D RP)

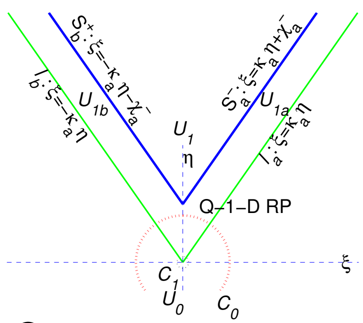
Interacting Shock Problem for the Nonlinear Wave System



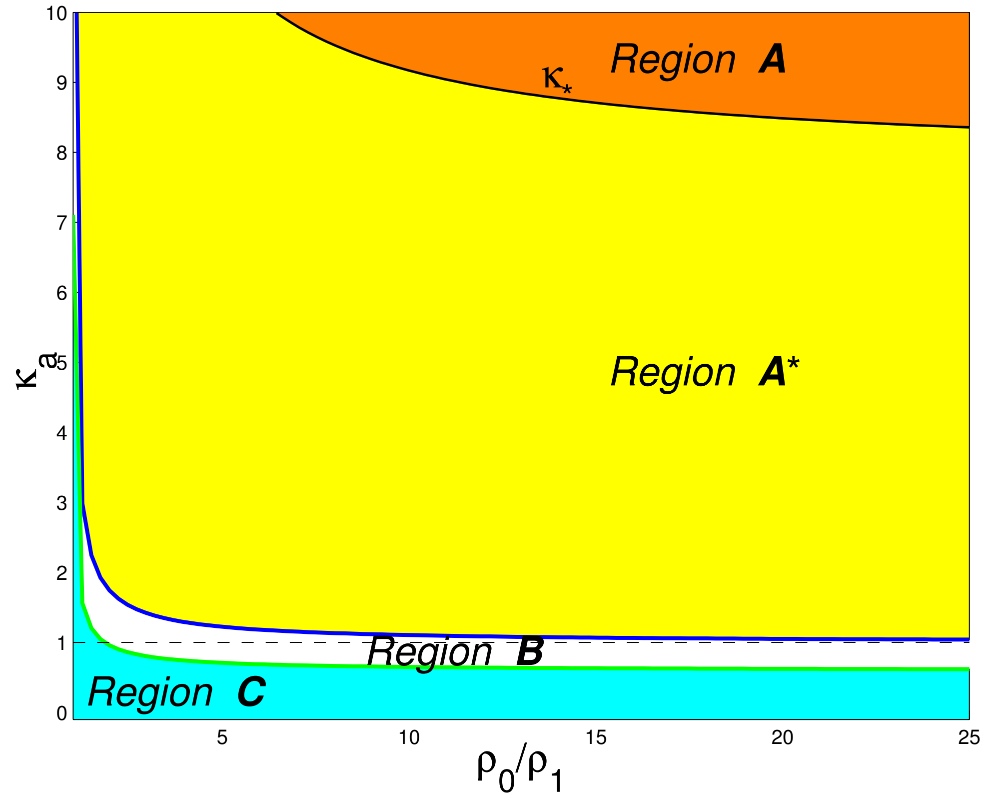
Data



A+A*: Shock meets C_0



C: Q-1-D RP solvable



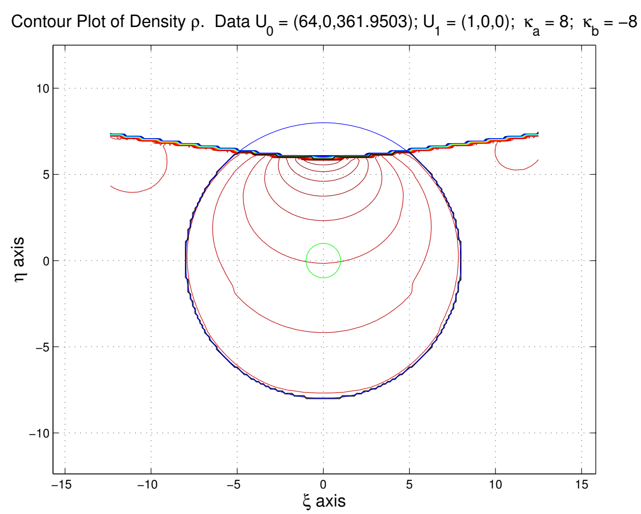
3 regions: **A+A*** Weak MR possible

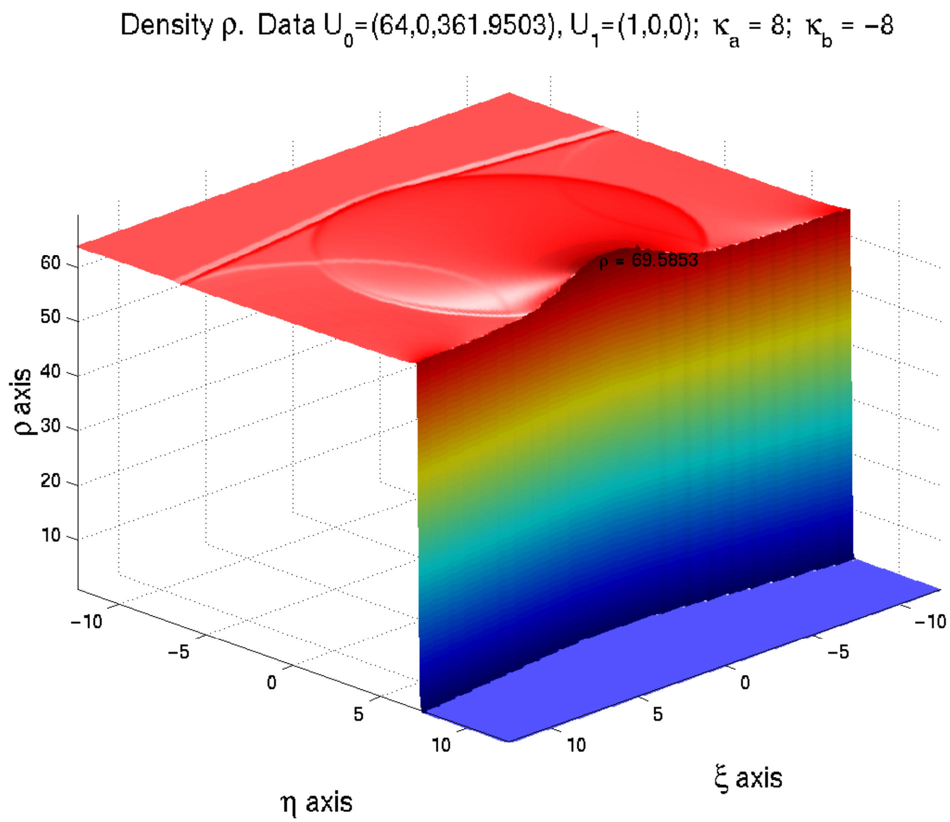
C RR possible

B neither possible

Theory and Numerical Simulations of the Solution

Region A (Weak Mach Reflection)





Sonic circle

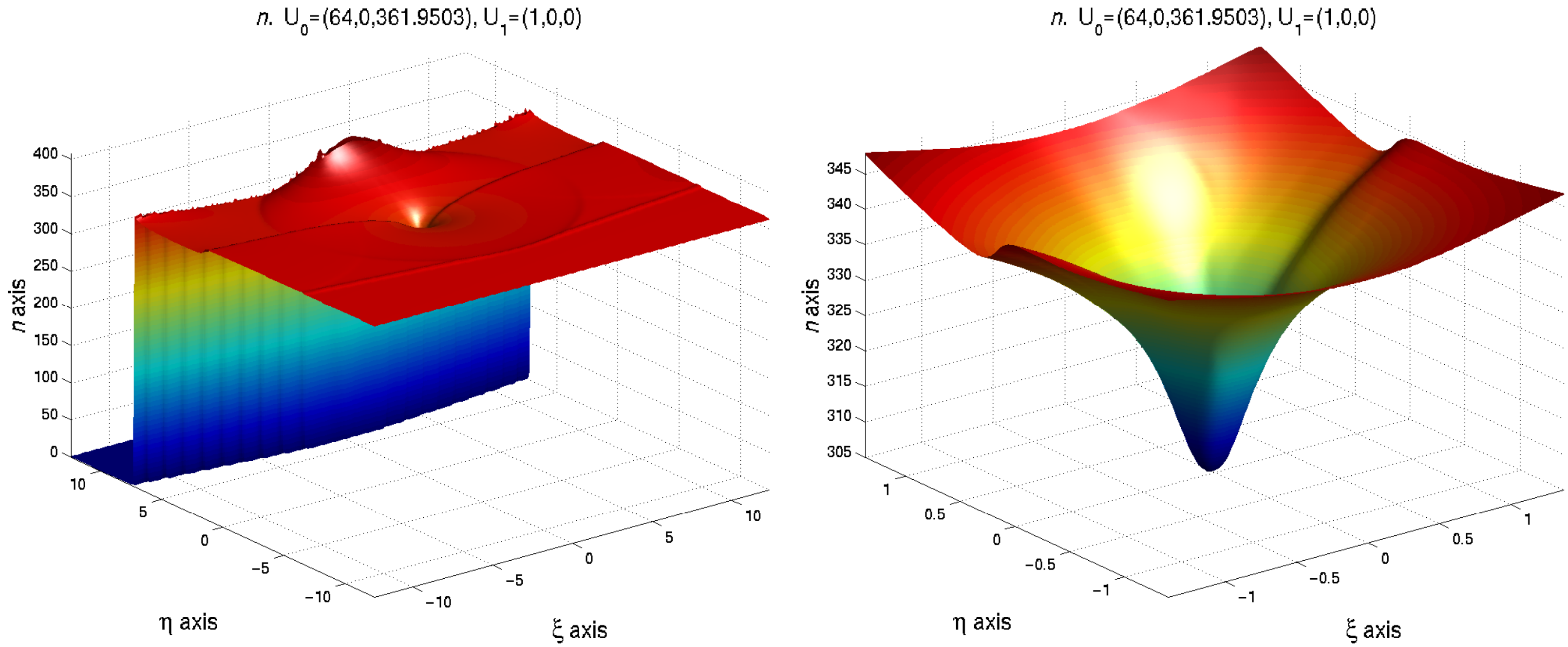
$$C_0 = \{\xi^2 + \eta^2 = c^2(\rho_0)\}$$

Supersonic soln known

U continuous at C_0

$\partial U/\partial r$ singular

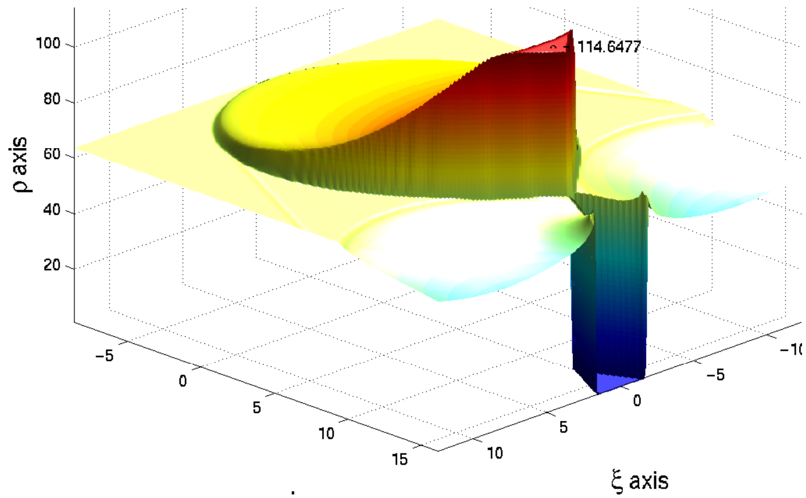
Same Case: $\kappa_a = 8$, $\rho_0 = 64$, Momentum Component n

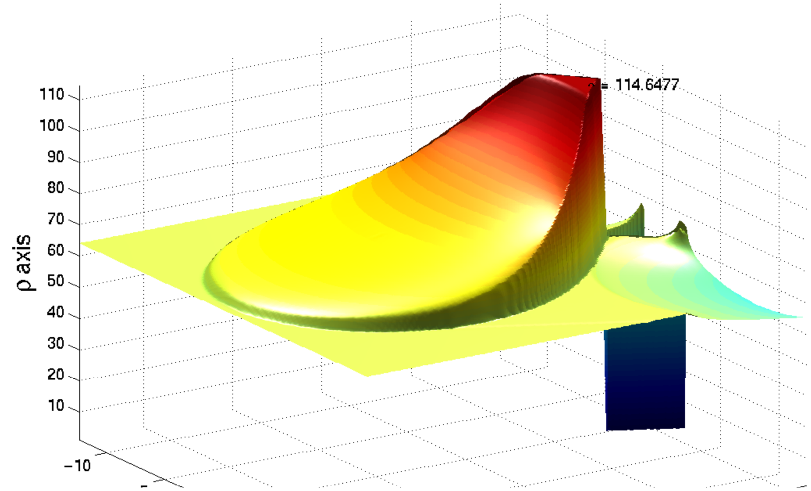


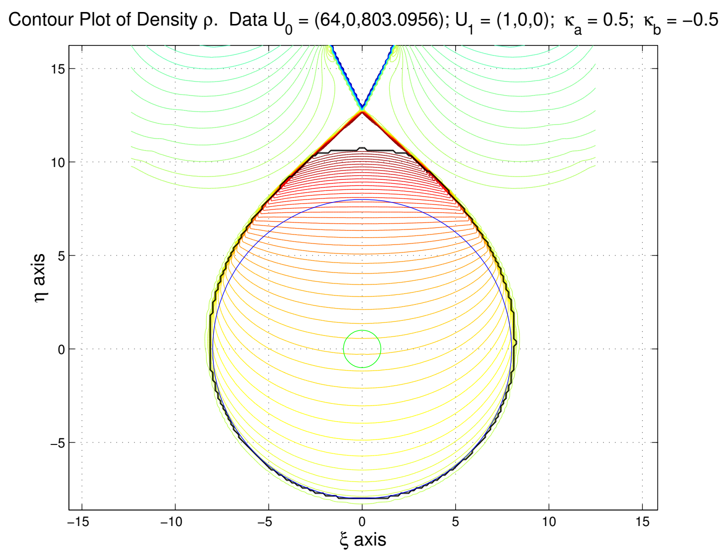
Simulation of full field and close-up near $(0, 0)$

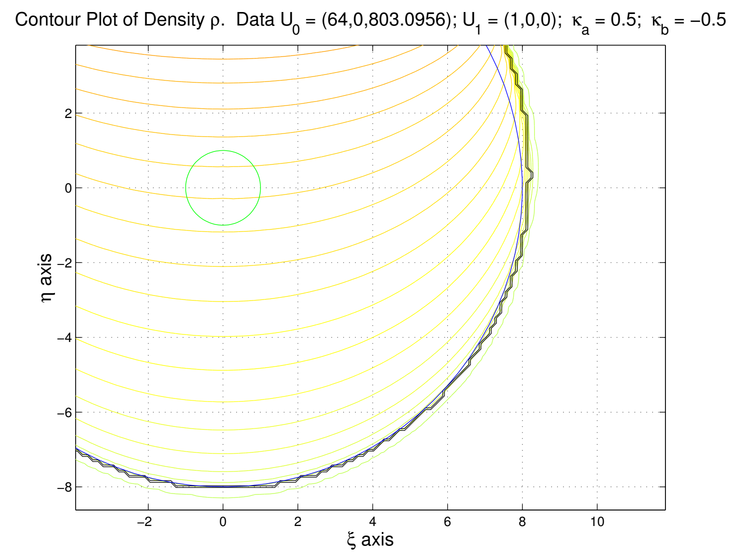
Logarithmic singularity in n at $(0, 0)$

Regular Reflection, $\kappa_a = 0.5$: Simulation of WRR





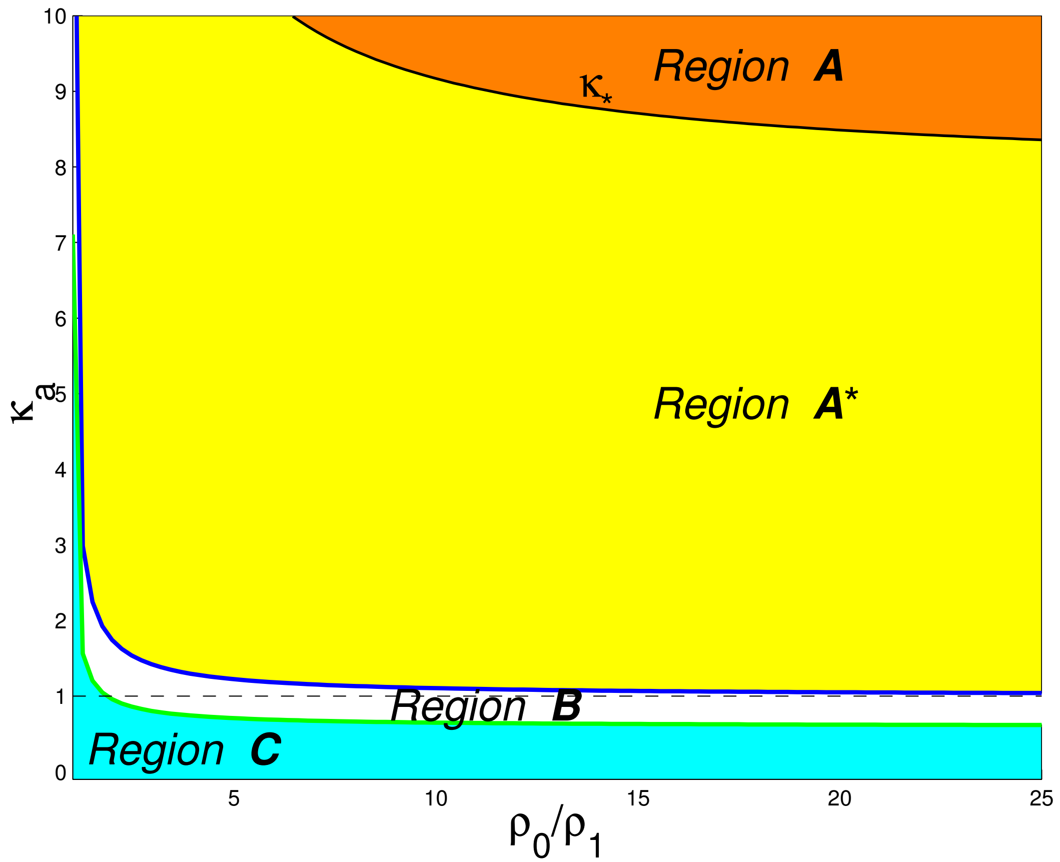




Bifurcation Diagram

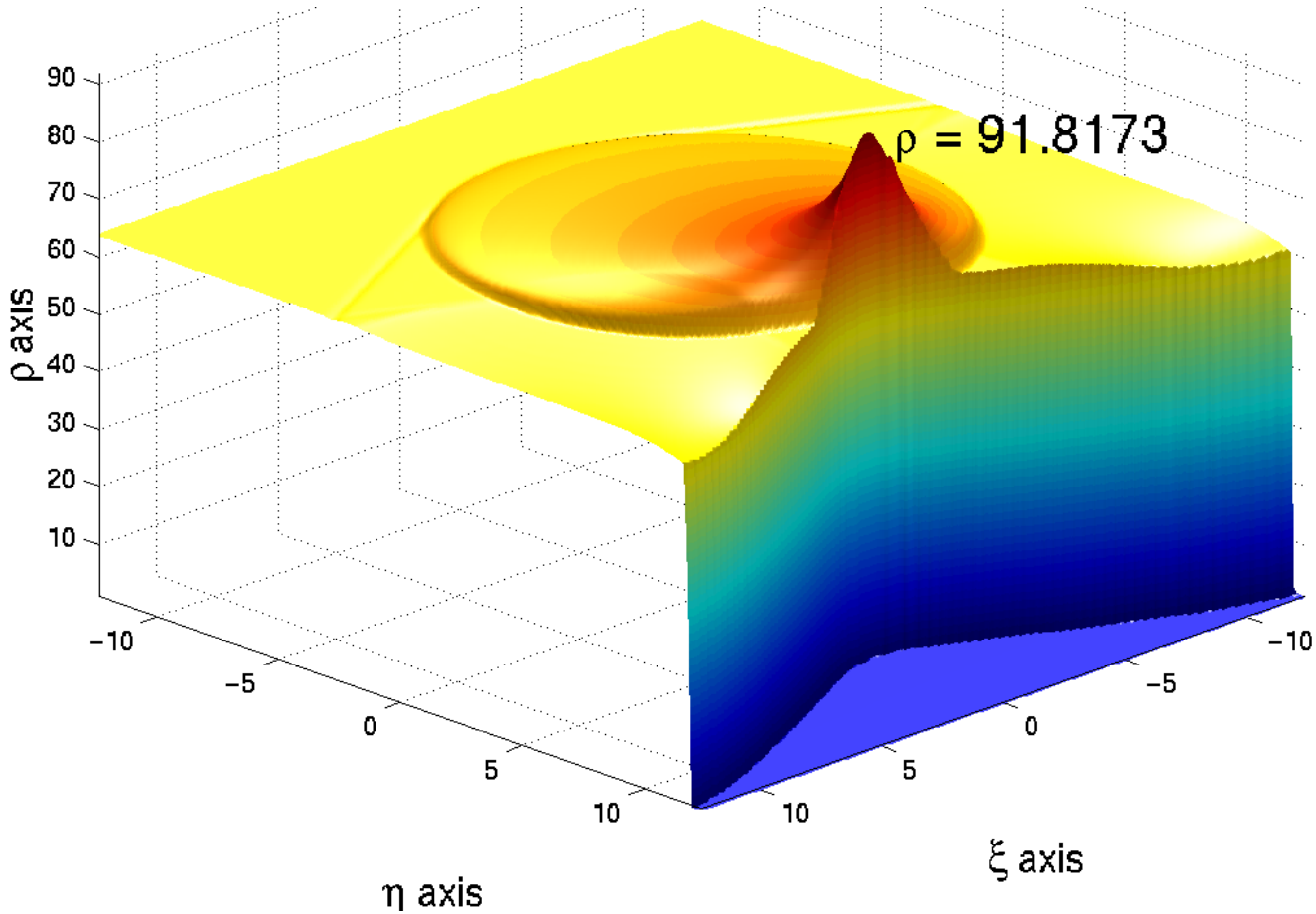
Region **A**: Analytic solution for $\kappa > \kappa*$
('technical' condition in gradient estimate)

Region **C**: Local solution for weak or strong RR (CKK, UTSD)

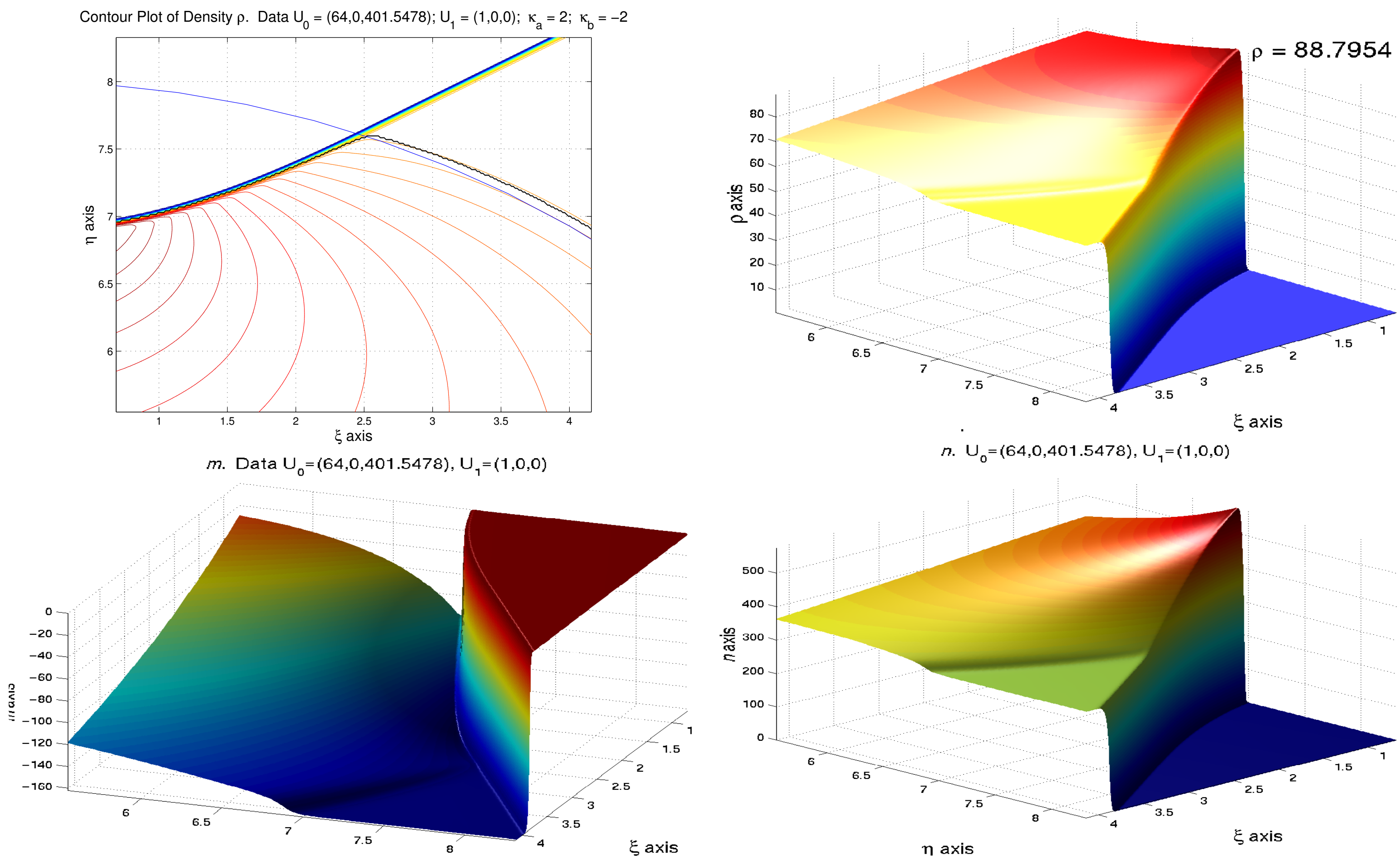


Maybe Not Just Technical

Angle $\kappa_a = 2$, Region $\mathbf{A}^*$: Apparent Reflected Shock

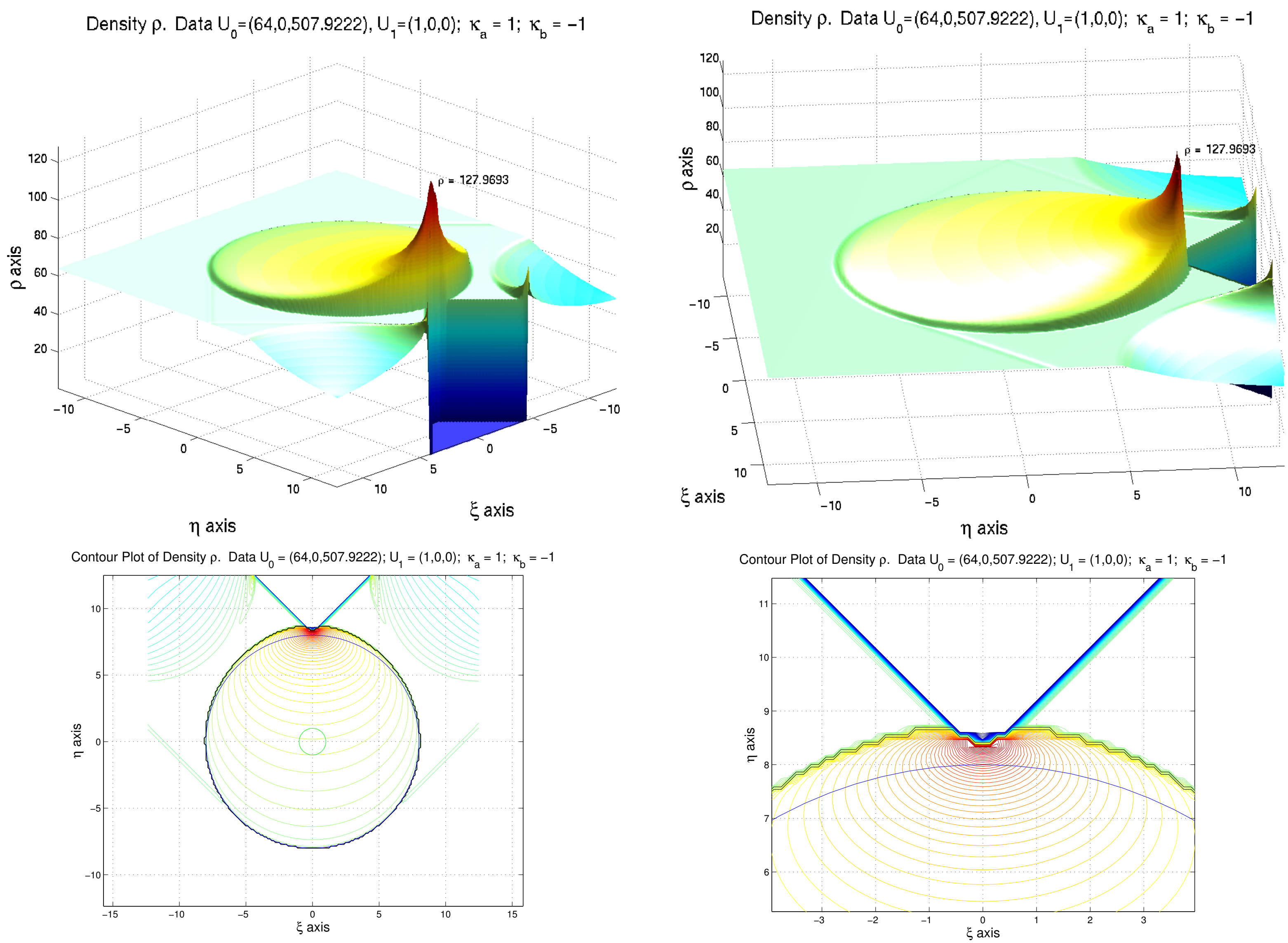


Same Case, $\kappa_a = 2$, $\rho_0 = 64$, Close-up of 'triple point'

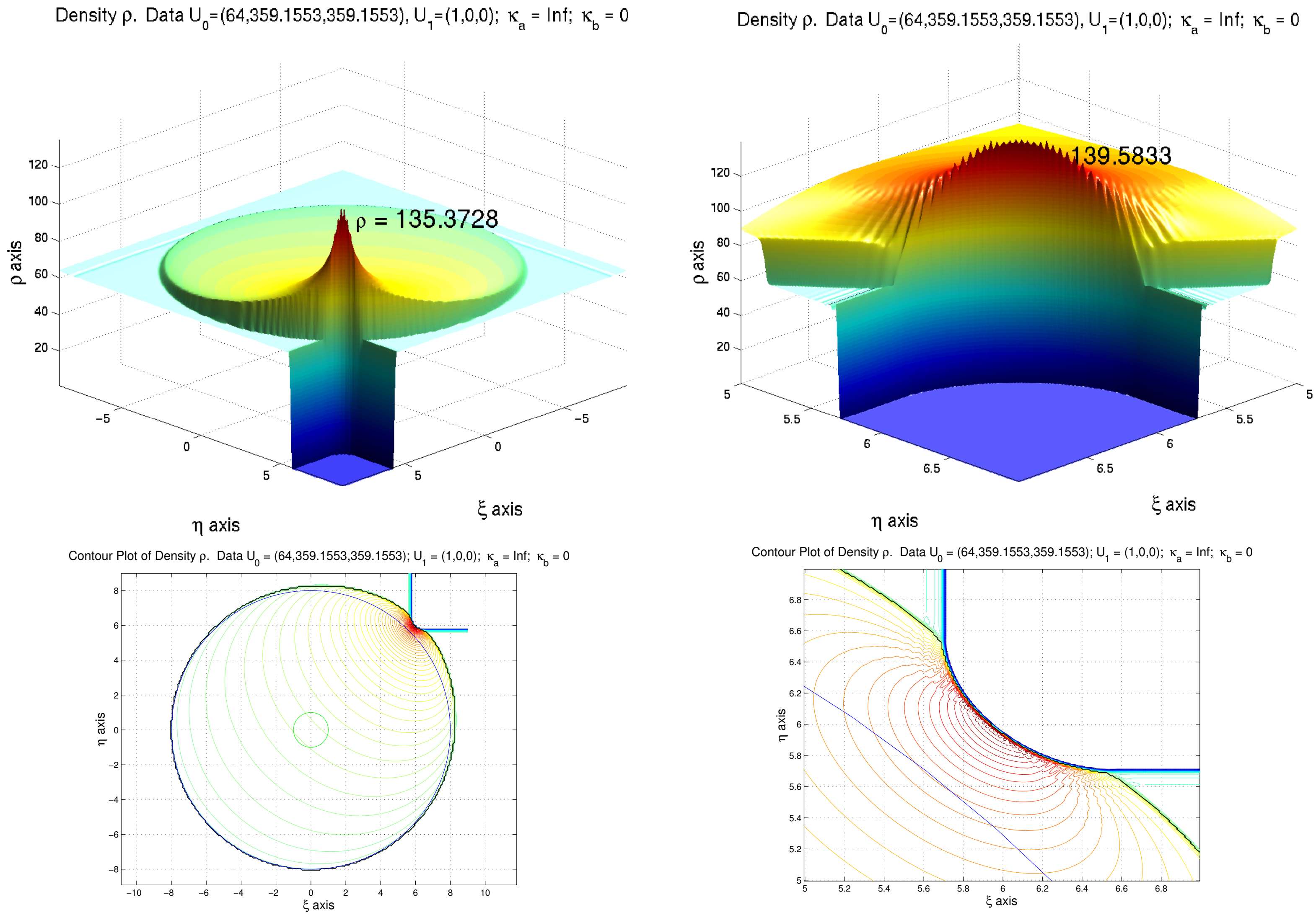


Reflected shock with zero strength at 'triple point'?

Simulations in Region $\mathbb{B}$: $\kappa_a = 1$, $\rho_0 = 64$: von Neumann Paradox

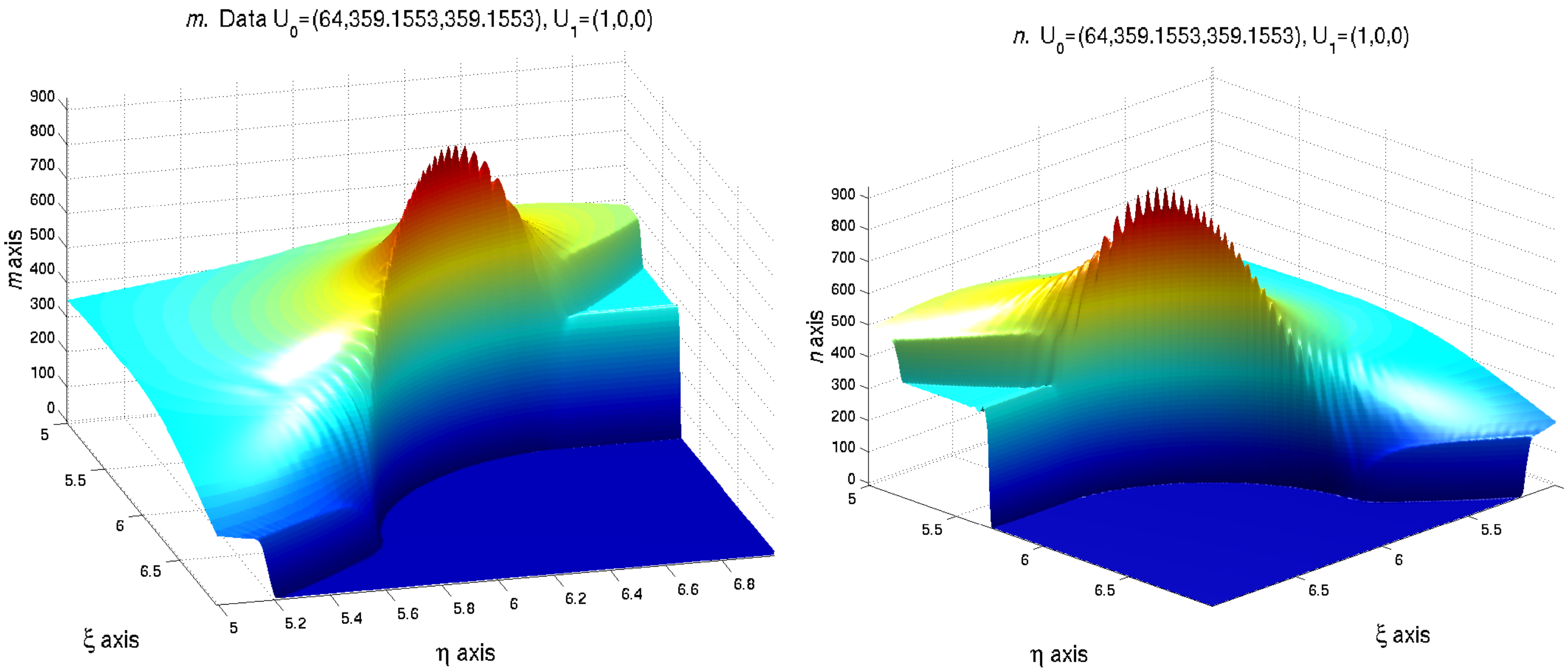


Same Case, $\kappa_a = 1$, $\rho_0 = 64$, Different Orientation



Same Case, $\kappa_a = 1$, $\rho_0 = 64$, Momentum Components

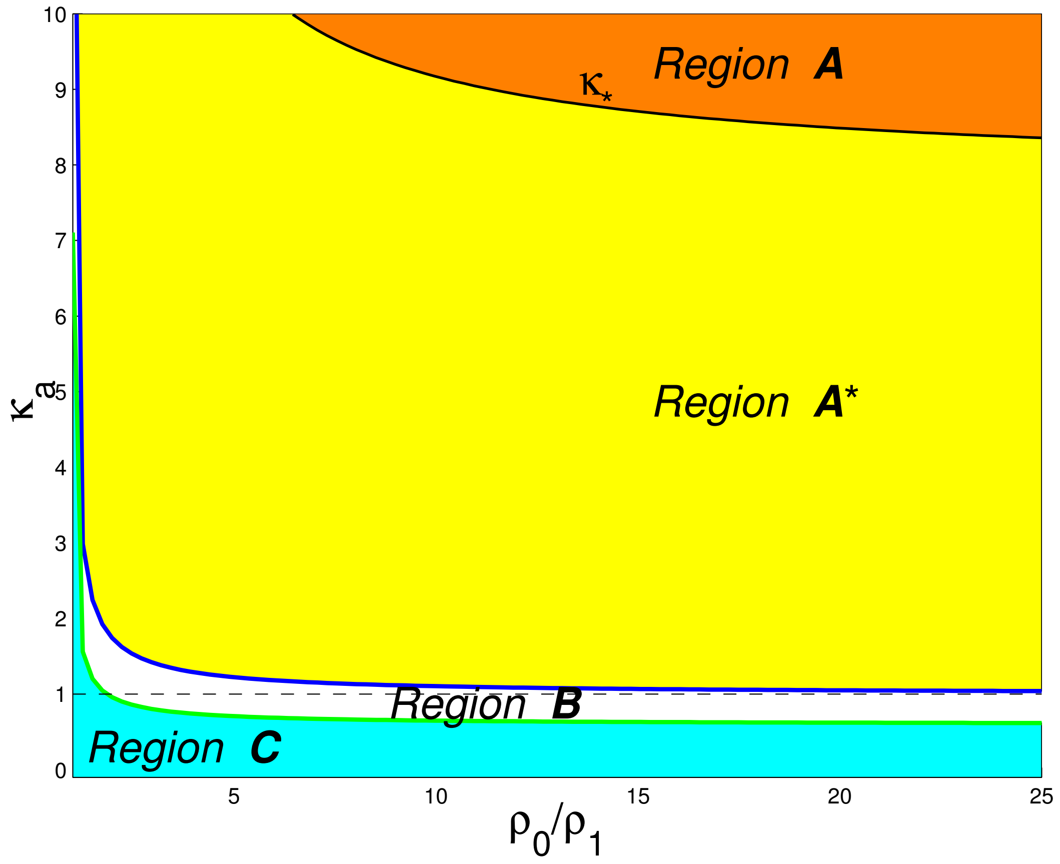
Close-up showing apparent triple point



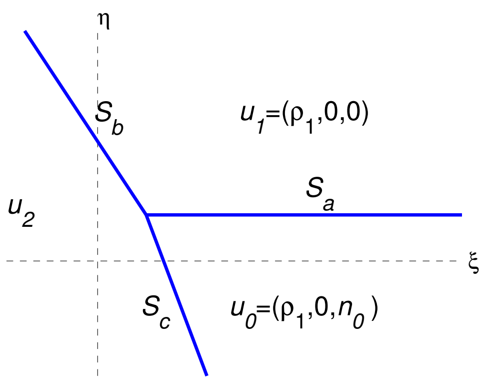
Bifurcation Diagram

Region **A***: Weak reflected shock or reflected shock
with zero strength at triple point (conjecture)

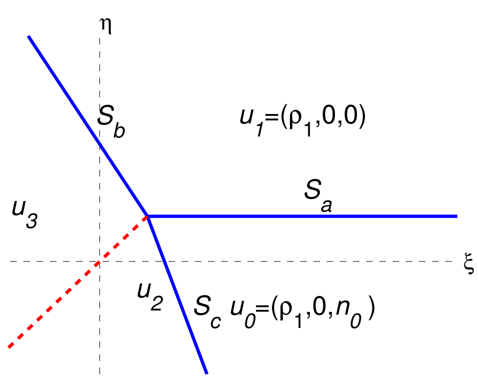
Region **B**: Neither **A**, **A*** nor **C** type solution exists



Possible Behavior at 'Triple Point', Region $\mathbf{B}$

- 

Proposition: NO nontrivial sol'ns to R-H eq'ns for constant states $\{u_0, u_1, u_2\}$ separated by shock lines S_a, S_b, S_c.

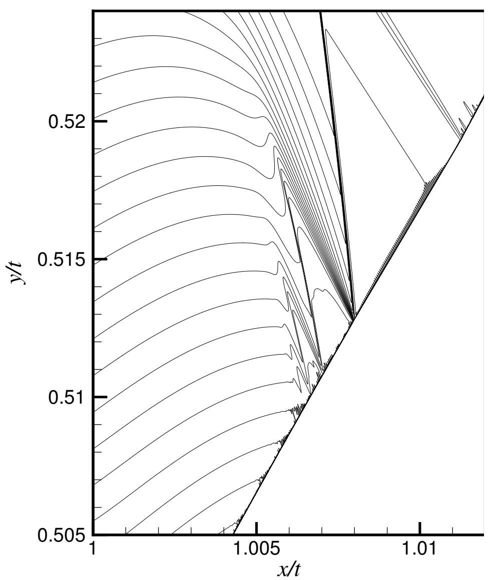
- 

Proposition: $\exists$ nontrivial sol'ns to R-H eq'ns for states $\{u_0, u_1, u_2, u_3\}$ separated by shock lines S_a, S_b, S_c + linear wave.

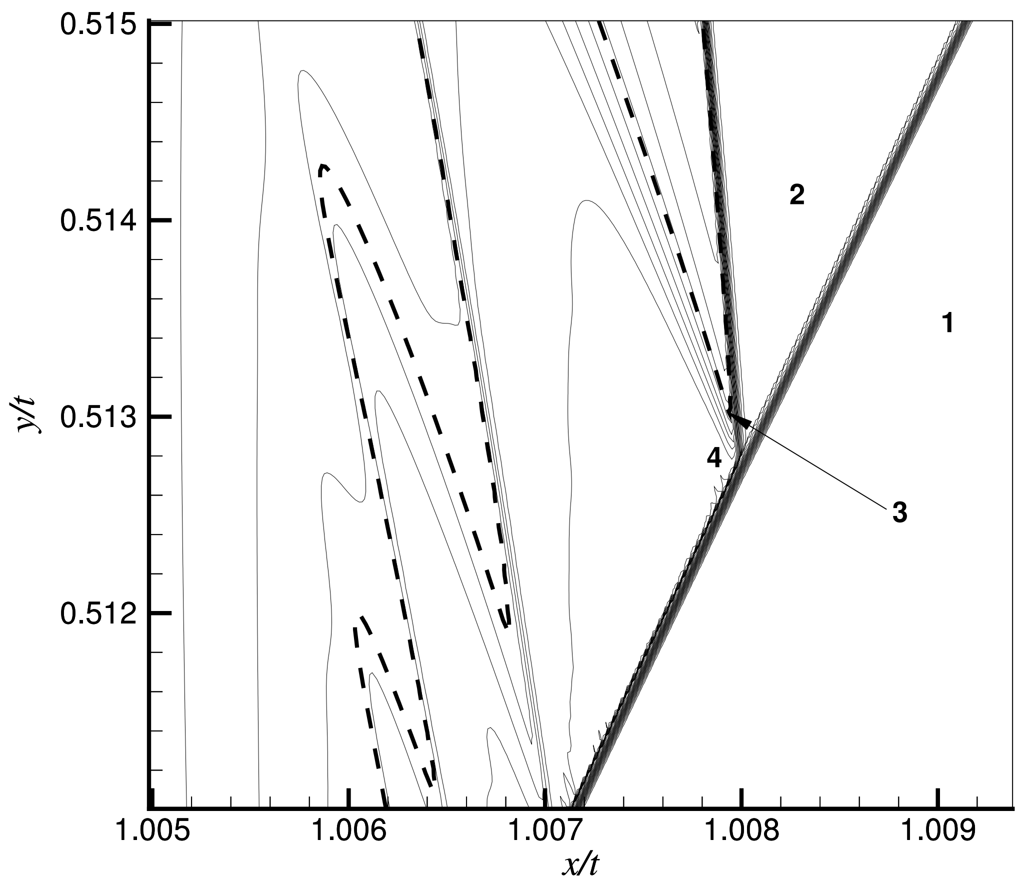
- States u_2 and u_3 must be subsonic (causality)
- Only discont. supp. in sub. reg. is lin. wave
- Only lin. waves are those in data

Supersonic bubble

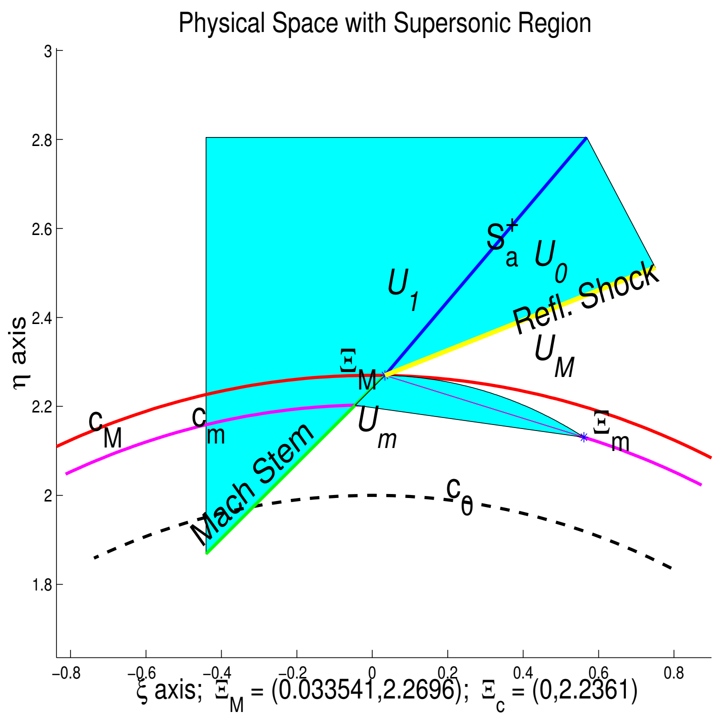
- Numerical results of Tesdall and Hunter on UTSD eqn
- SIAP, 2003
- Quasi-steady simulation
- Cascade of embedded supersonic regions

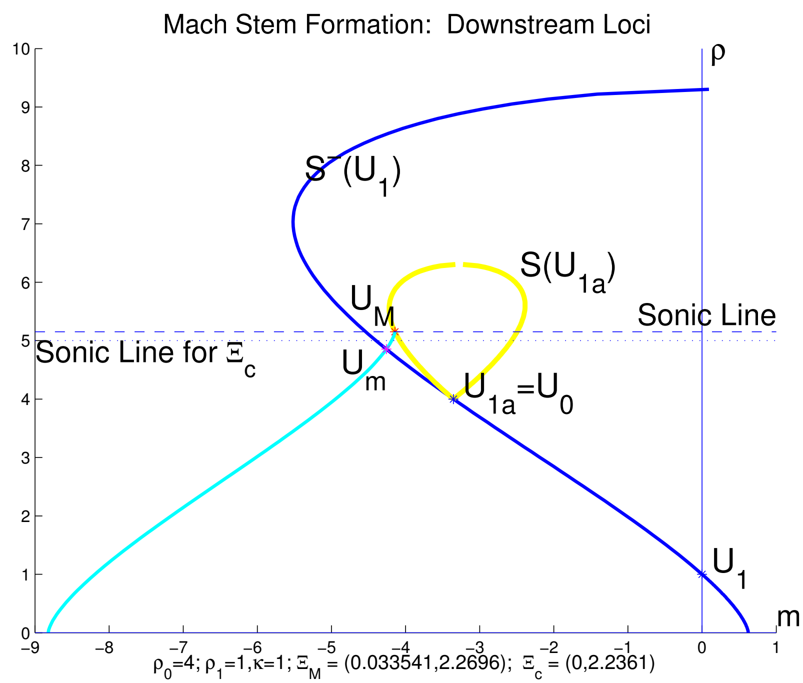


ALLEN M. TESDALL AND JOHN K. HUNTER



Scenario for a Triple Point in NLWS: Embedded Supersonic Region

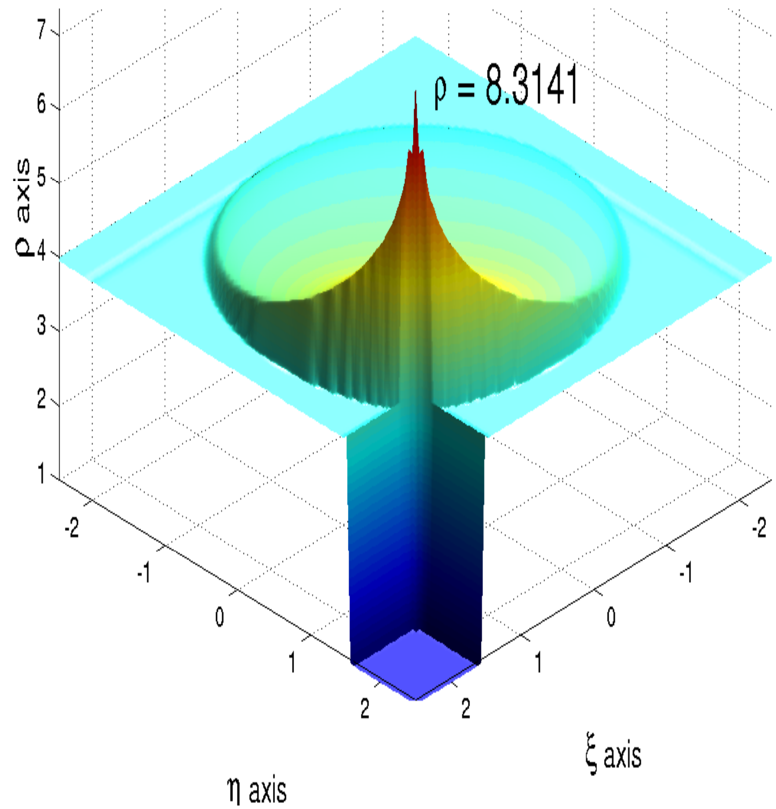


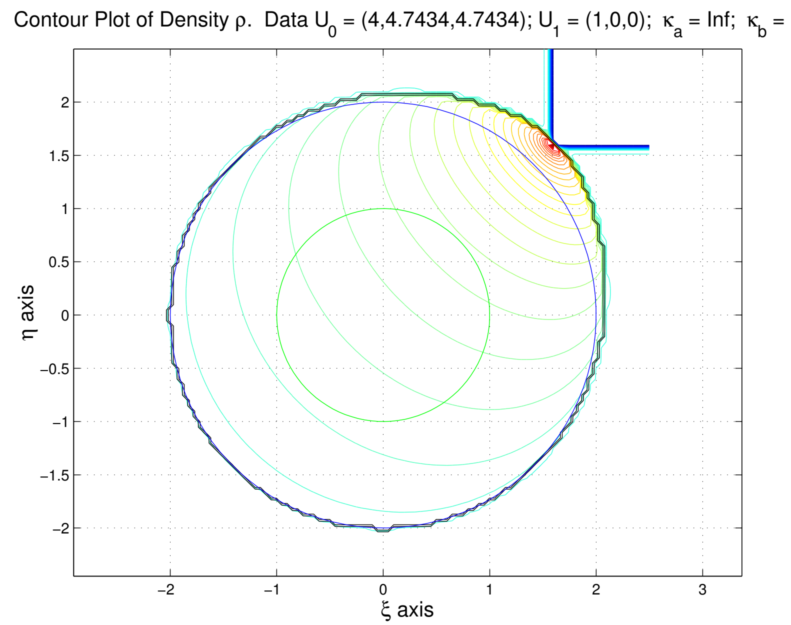


- Construction of states U_M (sonic), U_m (supersonic)
- One parameter family, param. by ξ_M (det. by far field)
- Supersonic bubble not a domain of determinacy (analysis needed)
- Cascade due to singular hyperbolic nature of UTSD?
- Numerical evidence lacking for NLWS

Simulations on NLWS by Kurganov

Angle $\kappa_a = 1$, $\rho_0 = 4$: Region $\mathbf{B}$: Apparent Triple Point





Same Case, $\kappa_a = 1$, $\rho_0 = 4$, Close-up of 'triple point'