Pattern and Paradox: Shock Interactions in the Nonlinear Wave System

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Outline of Program with S. Čanić, E. H. Kim, G. Lieberman

Similarity analysis of 2-D CL ($\xi = \frac{x}{t}$, $\eta = \frac{y}{t}$); piecewise const. data

Features:
• 1-D waves in far field
• change of type in $(\xi, \eta)$, like steady TS:

\[ U_t + F_x + G_y = 0 \text{ to } (A-\xi)U_\xi + (B-\eta)U_\eta = 0 \]

Behavior of characteristics
Causality, determinacy
Acoustic type structure

• Quasi-one-D Riemann Problems:
  ‘Shock polars’ at $\Xi_0 = (\xi_0, \eta_0)$
  ‘Self-similar’ solution $U \left( \frac{\xi-\xi_0}{\eta-\eta_0} \right)$
  cf. One-dimensional Riemann problem
Basic Features, continued

- Degenerate elliptic or mixed type
  Degen. const $U$ of Keldysh type:
  \[ x\phi_{xx} + \phi_{yy} \text{ (cf. } \phi_{xx} + x\phi_{yy} \text{)} \]
  Linear solution $\sqrt{x}w(x, y)$
  Fichera condition: data on deg. bdry

- Free boundary problems
  RH relation $\chi[U] = [F] - \kappa[G]; \ [U] = U - U_1$
  $\kappa = \frac{d\xi}{d\eta} = \text{slope}; \ \chi = \xi - \eta\kappa = \text{position}$
  Overdetermined BC for elliptic equation

Similarity Analysis of Two-Dimensional Systems: General Data

\[ U_t + F(U)x + G(U)y = 0, \quad U \in \mathbb{R}^n \]

Data: \( U(x, y, 0) = f \left( \frac{x}{y} \right) \)

Similarity Variables:
\[ \xi = \frac{x}{t}, \quad \eta = \frac{y}{t} \quad \Rightarrow \quad U = U(\xi, \eta) \]

Reduced System in Two Variables
\[ \partial_\xi (F - \xi U) + \partial_\eta (G - \eta U) \equiv \tilde{F}_\xi + \tilde{G}_\eta = -2U \]

Method: resolve 1-D far-field discont.; give data for (IV)-BVP in 2-D

Type Changes: hyperbolic in far field; ‘subsonic’ region near origin

Difficulties: hyperbolic Q-1-D problems w/o solution; subsonic FBP
The Search for Prototype Systems: UTSD & NLWS

Comparison of Isentropic Gas Dynamics & NLWS

Isentropic Gas Dynamics: \( p = \rho^\gamma / \gamma \)

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0 \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0 \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0
\end{align*}
\]

Nonlinear Wave System:

\[
\begin{align*}
\rho_t + m_x + n_y &= 0 \\
m_t + p_x &= 0 \\
n_t + p_y &= 0 \\
m &= \rho u \\
n &= \rho v
\end{align*}
\]

Self-sim 2nd-order equation for nonlinear characteristic variable \( (\rho) \):

\[
\begin{align*}
((c^2(\rho) - U^2)\rho_\xi - UV \rho_\eta)_\xi + ((c^2(\rho) - \xi^2)\rho_\xi - \xi \eta \rho_\eta)_\xi \\
((c^2(\rho) - V^2)\rho_\eta - UV \rho_\xi)_\eta + \ldots &= 0 \\
U &= u - \xi, \quad V = v - \eta \quad ('pseudo-vel.') \\
+ (c^2(\rho) - \eta^2)\rho_\eta - \xi \eta \rho_\xi)_\eta + \xi \rho_\xi + \eta \rho_\eta &= 0
\end{align*}
\]

Transport equation for linear characteristic variable:

\[
\begin{align*}
W &= V_\xi - U_\eta = v_\xi - u_\eta = \text{vorticity} \\
UW_\xi + VW_\eta + (U_\xi + V_\eta + 1)W &= 0 \\
W_t &= 0 \\
(w = n_\xi - m_\eta) \\
(\xi, \eta) \cdot \nabla w + w &= 0
\end{align*}
\]

Linear or:

\[
\begin{align*}
rm_r = p_\xi \\n r_r &= p_\eta
\end{align*}
\]
The Search for Prototype Data
Interacting Shocks: A Bifurcation Problem for NLWS

2-state data: \( U_0, U_1 \)
Data give 2 shocks
Far field soln: 4 waves

- Symmetric prototype for converging sector boundaries
- ‘Weak shock reflection’, von Neumann paradox
- Features
  1. 2 parameters: \( \rho_0/\rho_1 > 1 \) and \( \kappa_a \) (Mach # and wedge angle)
  2. Incident shocks: \( \xi = \kappa_a \eta - \chi \), \( \xi = -\kappa_a \eta + \chi \)
  3. Small \( \kappa \): two local solutions –‘weak’ and ‘strong’ regular reflection
  4. Large \( \kappa \): curved shock, weak reflected wave (Čanić talk)
  5. Intermediate values of \( \kappa \): no sol’n from shock polars (Q1D RP)
Interacting Shock Problem for the Nonlinear Wave System

Data

$U_1 = (\rho_1, 0, 0)$

$U_0 = (\rho_0, 0, n_0)$

$\eta_{a}^l: \xi = \kappa_{a} \eta$

$\eta_{b}^l: \xi = -\kappa_{a} \eta$

$\eta_{a}^b: \xi = \kappa_{a} \eta + \chi_{a}$

$\eta_{b}^b: \xi = -\kappa_{a} \eta - \chi_{a}$

$A + A^*: \text{Shock meets } C_0$

$C: \text{Q-1-D RP solvable}$

$3 \text{ regions: } A + A^* \text{ Weak MR possible}$

$C: \text{RR possible}$

$B: \text{neither possible}$
Theory and Numerical Simulations of the Solution
Region A (Weak Mach Reflection)

Sonic circle

\[ C_0 = \{ \xi^2 + \eta^2 = c^2(\rho_0) \} \]

Supersonic soln known

\( U \) continuous at \( C_0 \)

\( \partial U/\partial r \) singular
Same Case: $\kappa_a = 8$, $\rho_0 = 64$, Momentum Component $n$

Simulation of full field and close-up near $(0, 0)$
Logarithmic singularity in $n$ at $(0, 0)$
Regular Reflection, $\kappa_a = 0.5$: Simulation of WRR
Bifurcation Diagram

Region A: Analytic solution for $\kappa > \kappa^*$

('technical' condition in gradient estimate)

Region C: Local solution for weak or strong RR (CKK, UTSD)
Maybe Not Just Technical

Angle \( \kappa_a = 2 \), Region \( \mathbf{A}^* \): Apparent Reflected Shock

\[ \rho = 91.8173 \]
Same Case, $\kappa_\alpha = 2$, $\rho_0 = 64$, Close-up of ‘triple point’

Reflected shock with zero strength at ‘triple point’?
Simulations in Region $\mathbf{B}$: $\kappa_a = 1$, $\rho_0 = 64$: von Neumann Paradox

Contour Plot of Density $\rho$. Data $U_0 = (64, 0, 507.9222)$, $U_1 = (1, 0, 0)$; $\kappa_a = 1$; $\kappa_b = -1$

Contour Plot of Density $\rho$. Data $U_0 = (64, 0, 507.9222)$, $U_1 = (1, 0, 0)$; $\kappa_a = 1$; $\kappa_b = -1$
Same Case, $\kappa_a = 1$, $\rho_0 = 64$, Different Orientation
Same Case, $\kappa_\alpha = 1$, $\rho_0 = 64$, Momentum Components

Close-up showing apparent triple point

$m, U_0 = (64,359.1553,359.1553), U_1 = (1,0,0)$

$n, U_0 = (64,359.1553,359.1553), U_1 = (1,0,0)$
Bifurcation Diagram

Region $A^*$: Weak reflected shock or reflected shock with zero strength at triple point (conjecture)

Region $B$: Neither $A$, $A^*$ nor $C$ type solution exists
Possible Behavior at ‘Triple Point’, Region $\mathbf{B}$

**Proposition:** NO nontrivial sol’ns to R-H eq’ns for constant states \{\(u_0, u_1, u_2\}\) separated by shock lines $S_a, S_b, S_c$.

**Proposition:** $\exists$ nontrivial sol’ns to R-H eq’ns for states \(\{u_0, u_1, u_2, u_3\}\) separated by shock lines $S_a, S_b, S_c + \text{linear wave}$.

- States $u_2$ and $u_3$ must be subsonic (causality)
- Only discont. supp. in sub. reg. is lin. wave
- Only lin. waves are those in data
Supersonic bubble

- Numerical results of Tesdall and Hunter on UTSD eqn
- SIAP, 2003
- Quasi-steady simulation
- Cascade of embedded supersonic regions
Scenario for a Triple Point in NLWS: Embedded Supersonic Region

- Construction of states $U_M$ (sonic), $U_m$ (supersonic)
- One parameter family, param. by $\xi_M$ (det. by far field)
- Supersonic bubble not a domain of determinacy (analysis needed)
- Cascade due to singular hyperbolic nature of UTSD?
- Numerical evidence lacking for NLWS
Simulations on NLWS by Kurganov

Angle $\kappa_a = 1$, $\rho_0 = 4$: Region B: Apparent Triple Point

Contour Plot of Density $\rho$. Data $U_0 = (4,4.7434,4.7434)$; $U_1 = (1,0,0)$: $\kappa_a = \text{Inf}; \kappa_b = 0$
Same Case, $\kappa_a = 1$, $\rho_0 = 4$, Close-up of ‘triple point’