Bifurcation of Shock Reflection Patterns

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joint work with Sunčica Čanić and Eun Heui Kim

simulations by Alexander Kurganov

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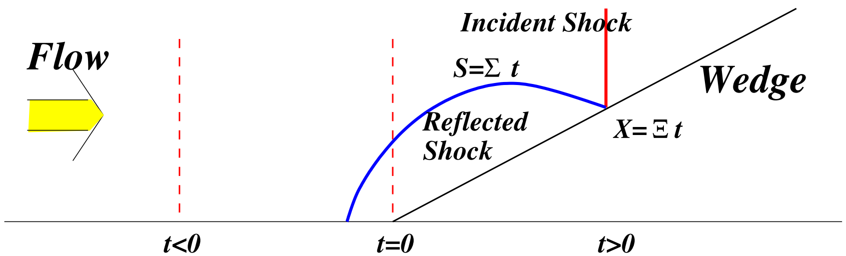
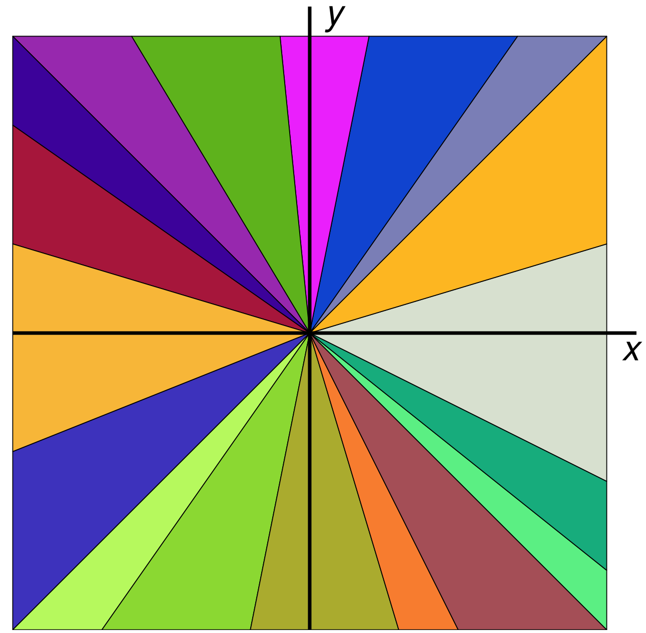
and the National Science Foundation, grant 03-06307.

Shock reflection by a wedge





Sectorial Data

Data: $U(x,y,0) = f\left(\frac{x}{y}\right)$

Similarity Variables:

$\xi = \frac{x}{t}, \eta = \frac{y}{t} \qquad U = U(\xi, \eta)$

Reduced System in Two V'bles

$$\partial_\xi(F - \xi U) + \partial_\eta(G - \eta U) = -2U$$

- Resolve 1-D far-field discont.: data for 2-D (IV)-BVP

Self-Similar Problems for $U_t + F_x + G_y = 0$

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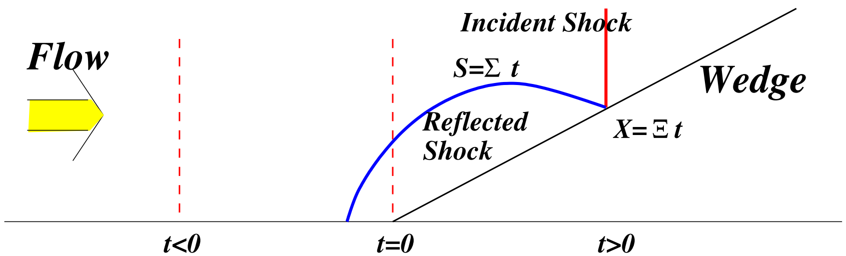

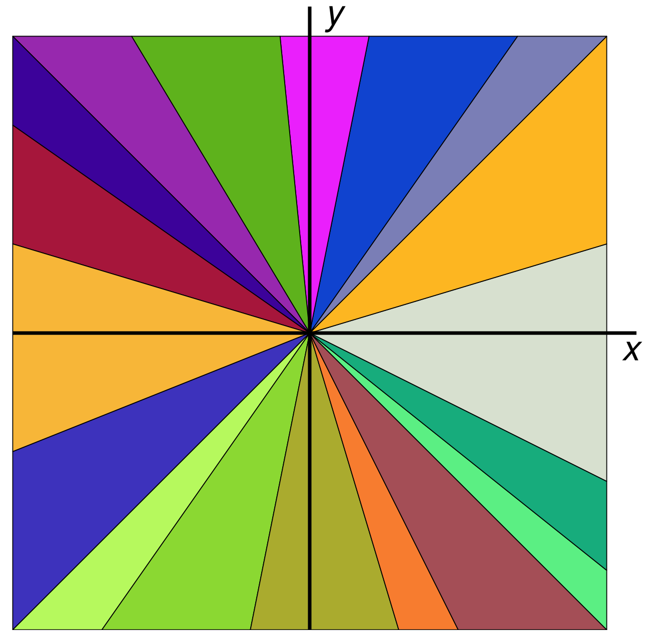

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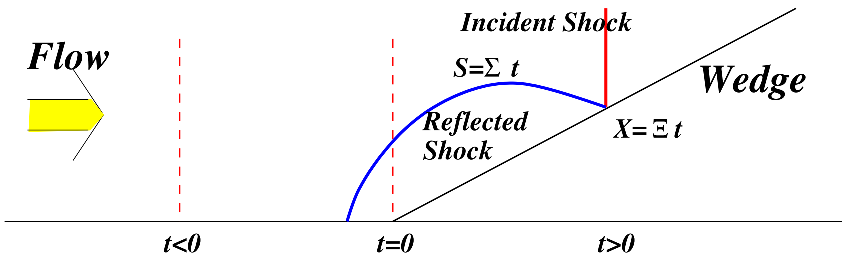

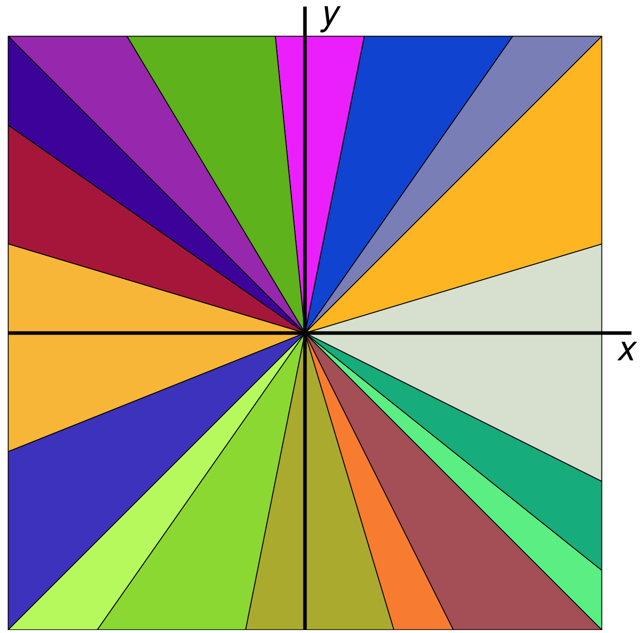

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- Type Changes: 'subsonic' region near origin
- Difficulties: Q-1-D probs w/o sol'n; subsonic FBP

Program with S. Čanić, E. H. Kim, G. Lieberman

- Two-dimensional cons law system $U_t + F_x + G_y = 0$
  Acoustic type structure (like linear W.E.)

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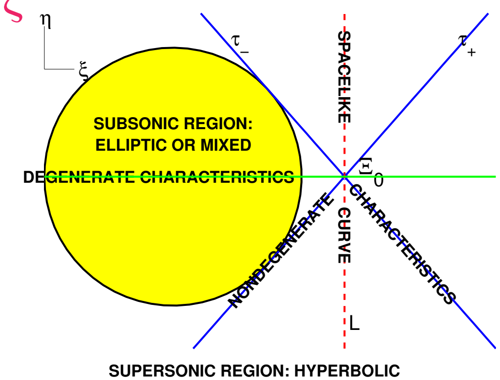
  Causality, determinacy (rel. 0)
  Change of type like steady TS:



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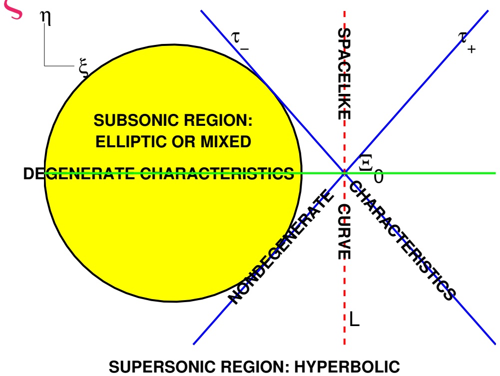
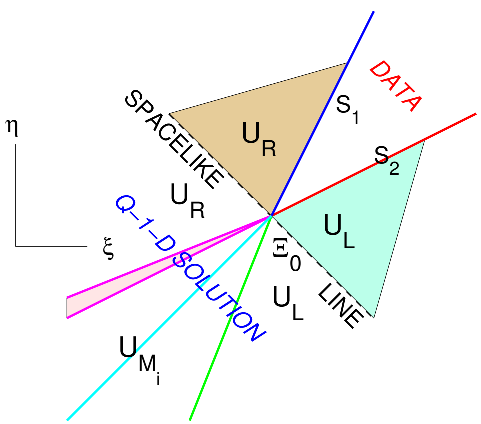
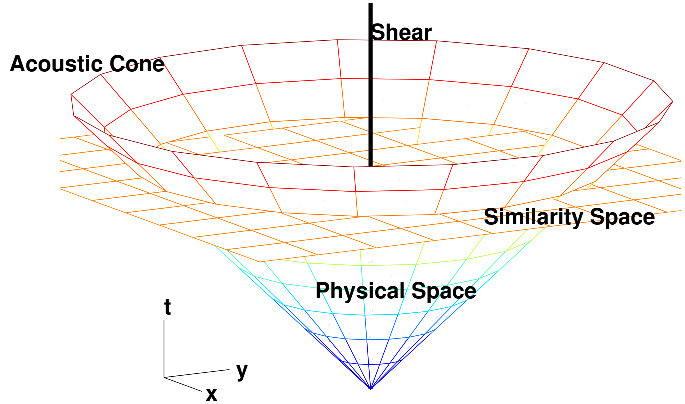




- Far field: Q-one-D R P:
  'Shock polars' at $\Xi_0 = (\xi_0, \eta_0)$
  'Self-similar' solution $U\left(\frac{\xi-\xi_0}{\eta-\eta_0}\right)$

Features of Subsonic Region

Degen. elliptic or mixed type
Keldysh type degen. (const U)
$x\phi_{xx} + \phi_{yy}$ (cf. $\phi_{xx} + x\phi_{yy}$)
Linear solution $\sqrt{x}w(x,y)$
Fichera: data on deg. bdry



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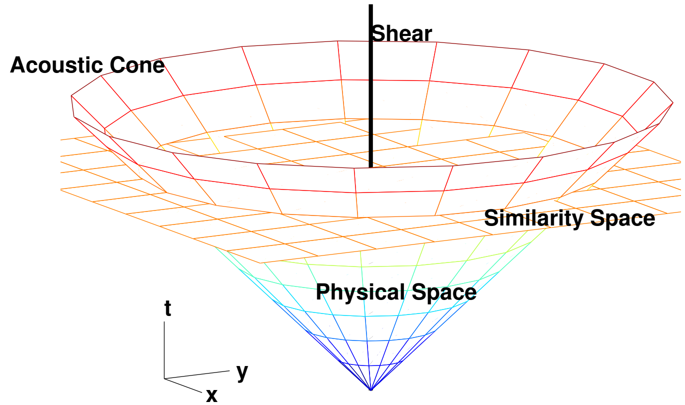

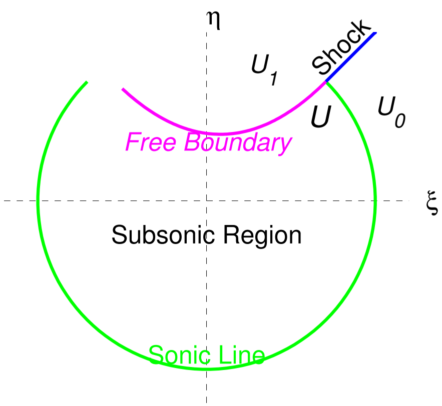

TS Shock Free bdry prob; $\xi = \kappa\eta + \chi$
RH rel. $\chi[U] = [F] - \kappa[G]$; $[U] = U - U_1$
$\kappa = \frac{d\xi}{d\eta} =$ slope $\chi = \xi - \kappa\eta =$ position
Overdetermined BC for elliptic eqn

Related work: Morawetz, Brio-Hunter, Rosales-Tabak, Y. Zheng, K. Song, Chen-Feldman, Serre, Zhang-Zheng, S.-X. Chen et al.

Prototypes: Gas Dynamics & NLWS

Comparison of Euler Equations & NLWS

$p = \rho^\gamma/\gamma$

Isentropic Gas Dynamics:

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0$$
$$(\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0$$

Nonlin. Wave Syst.:

$$\rho_t + m_x + n_y = 0$$
$$m_t + p_x = 0 \qquad m = \rho u$$
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Self-sim 2nd-order equation for ‘nonlinear’ variable ρ:

$$\big((c^2(\rho) - U^2)\rho_\xi - UV\rho_\eta\big)_\xi + \big((c^2(\rho) - V^2)\rho_\eta - UV\rho_\xi\big)_\eta \ldots = 0$$

$U = u - \xi, \quad V = v - \eta$ (‘ψ-vel.’)

$$\big((c^2 - \xi^2)\rho_\xi - \xi\eta\rho_\eta\big)_\xi + \big((c^2 - \eta^2)\rho_\eta - \xi\eta\rho_\xi\big)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0$$

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$$\left((c^2(\rho) - V^2)\rho_\eta - UV\rho_\xi\right)_\eta \dots = 0$$
$$U = u - \xi, \quad V = v - \eta \text{ (‘}\psi\text{-vel.’)}$$

$$\left((c^2 - \xi^2)\rho_\xi - \xi\eta\rho_\eta\right)_\xi$$
$$+\left((c^2 - \eta^2)\rho_\eta - \xi\eta\rho_\xi\right)_\eta$$
$$+\xi\rho_\xi + \eta\rho_\eta = 0$$

Transport equation for ‘linear’ variable:

$W = V_\xi - U_\eta = v_\xi - u_\eta =$ vorticity

$$UW_\xi + VW_\eta + (U_\xi + V_\eta + 1)W = 0$$

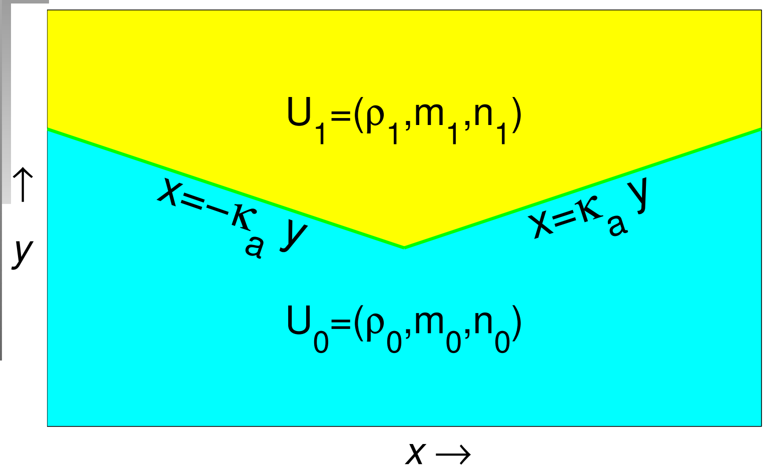
Nonlinear evolution equation

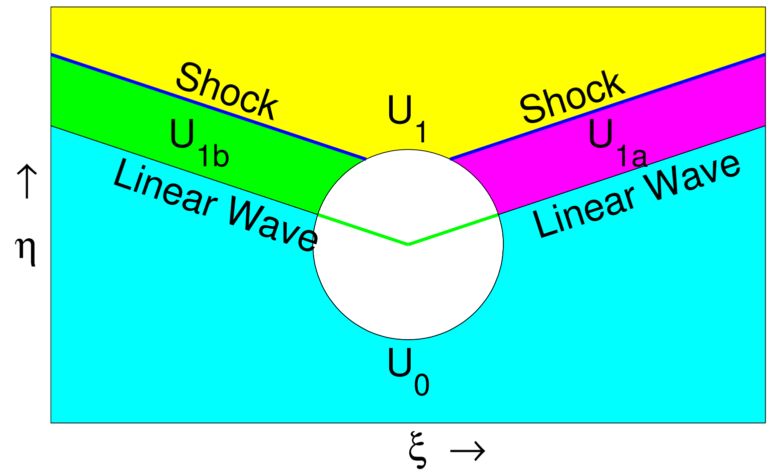
$w = n_\xi - m_\eta \qquad \overline{w}_t = 0$

Lin: $(\xi, \eta)\cdot\nabla w + w = 0$

or: $rm_r = p_\xi \quad rn_r = p_\eta$

Interacting Shocks: A Bifurcation Problem for NLWS





- Converging sector boundaries

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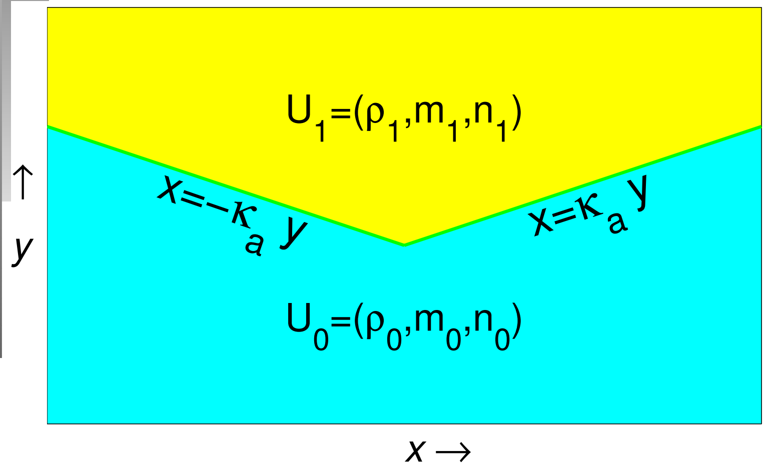

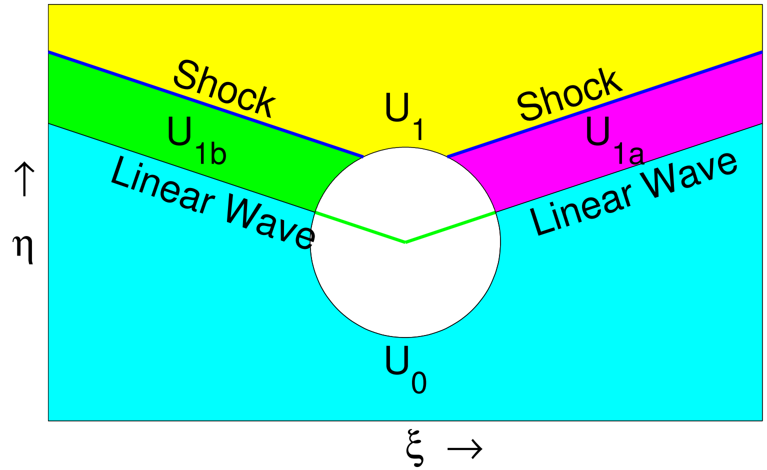

- Converging sector boundaries
- ‘Weak shock reflection’, von Neumann paradox

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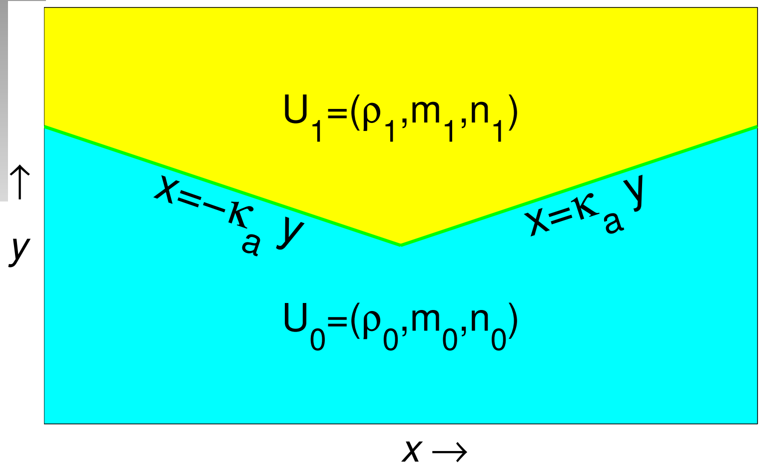

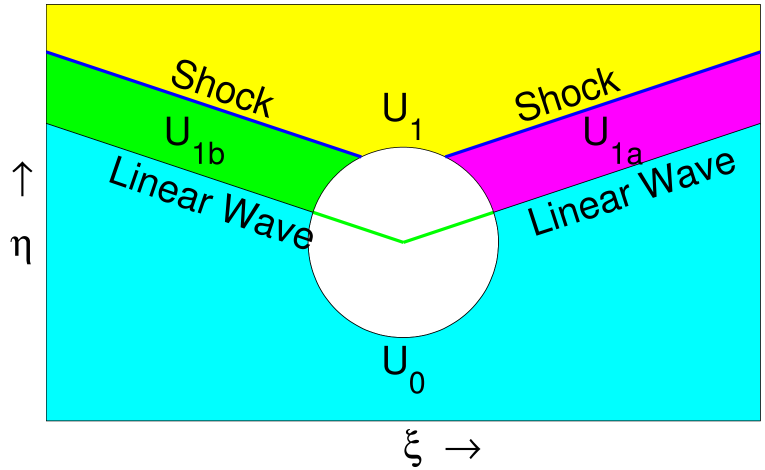

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- 2 parameters: $\rho_0/\rho_1 > 1$ and κ_a (Mach #, angle)

Prototype Data

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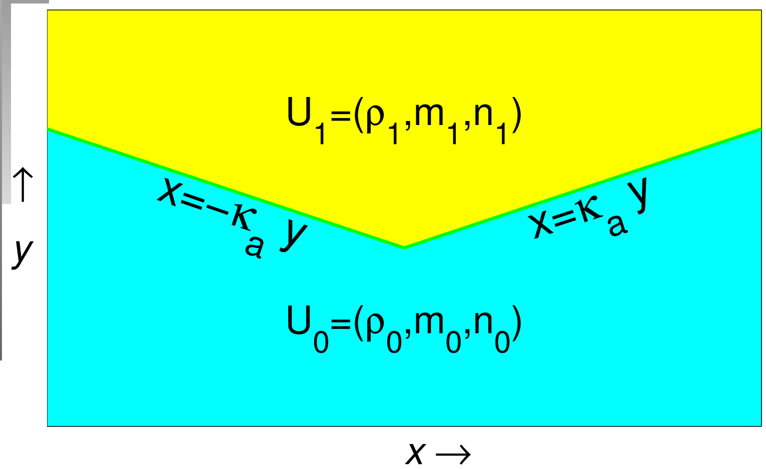

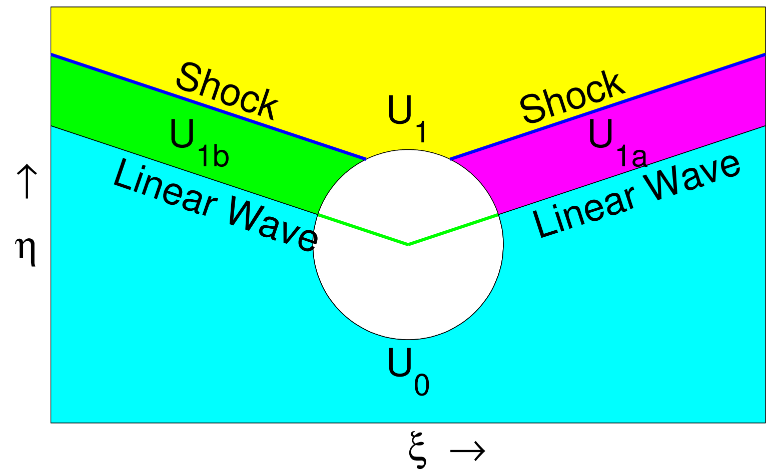

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- Small κ_a: ‘weak’ and ‘strong’ reg. reflection

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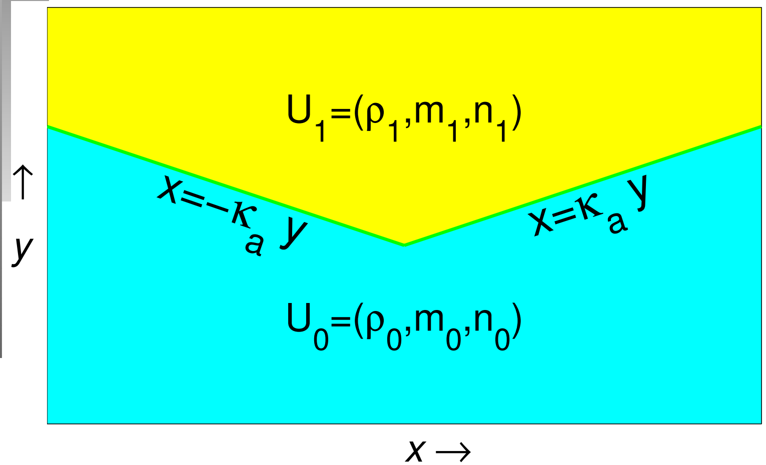

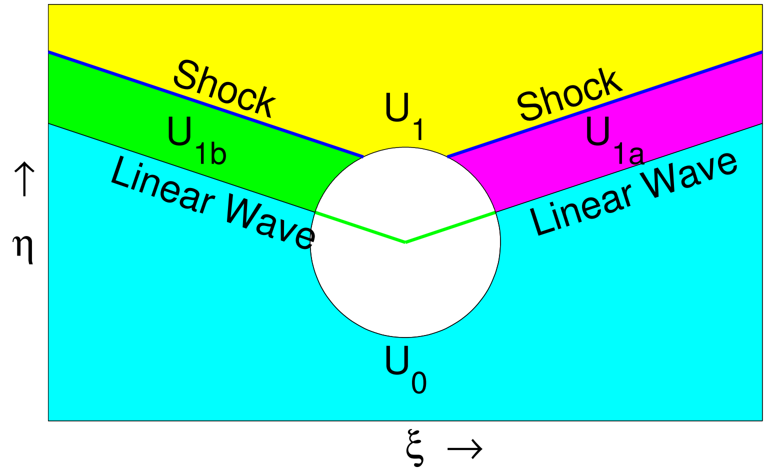

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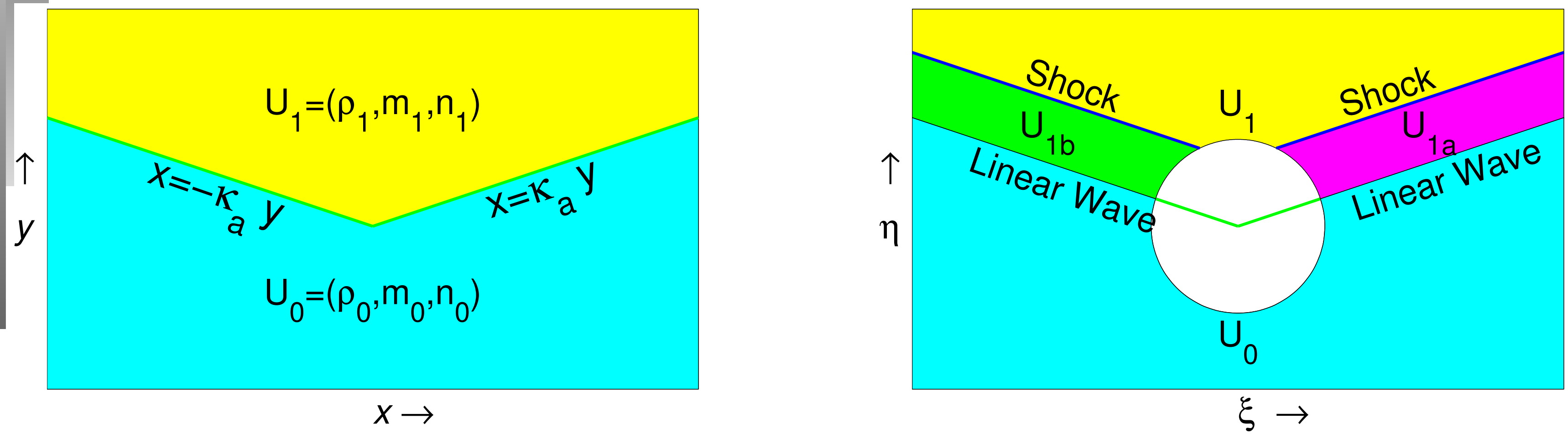

- Converging sector boundaries
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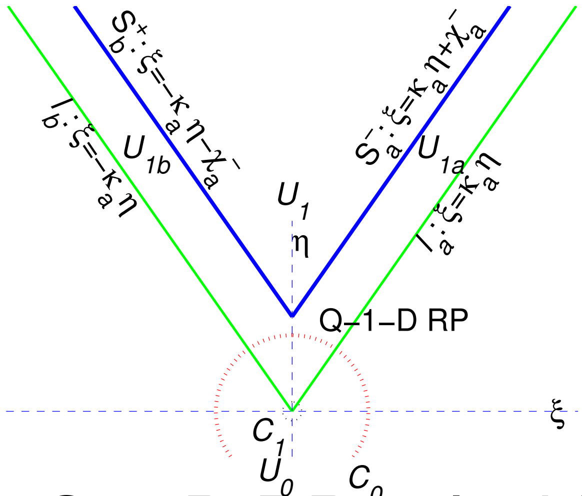
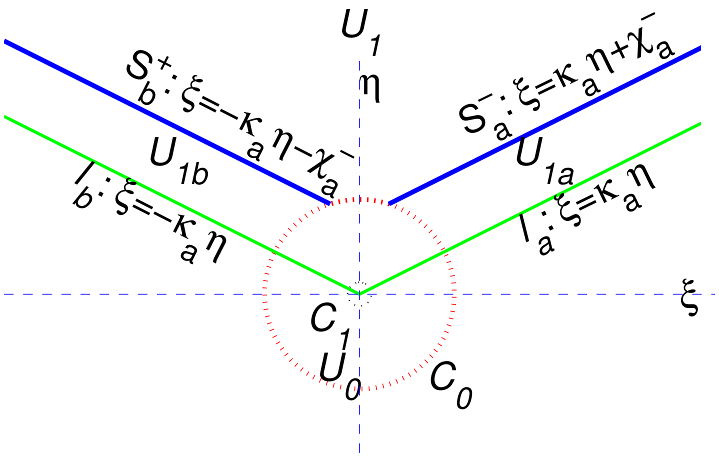
- 2 parameters: $\rho_0/\rho_1 > 1$ and κ_a (Mach #, angle)
- Small κ_a: ‘weak’ and ‘strong’ reg. reflection
- Large κ_a: curved shock, weak reflected wave
- Intermediate κ_a: no sol’n from Q-1-D RP

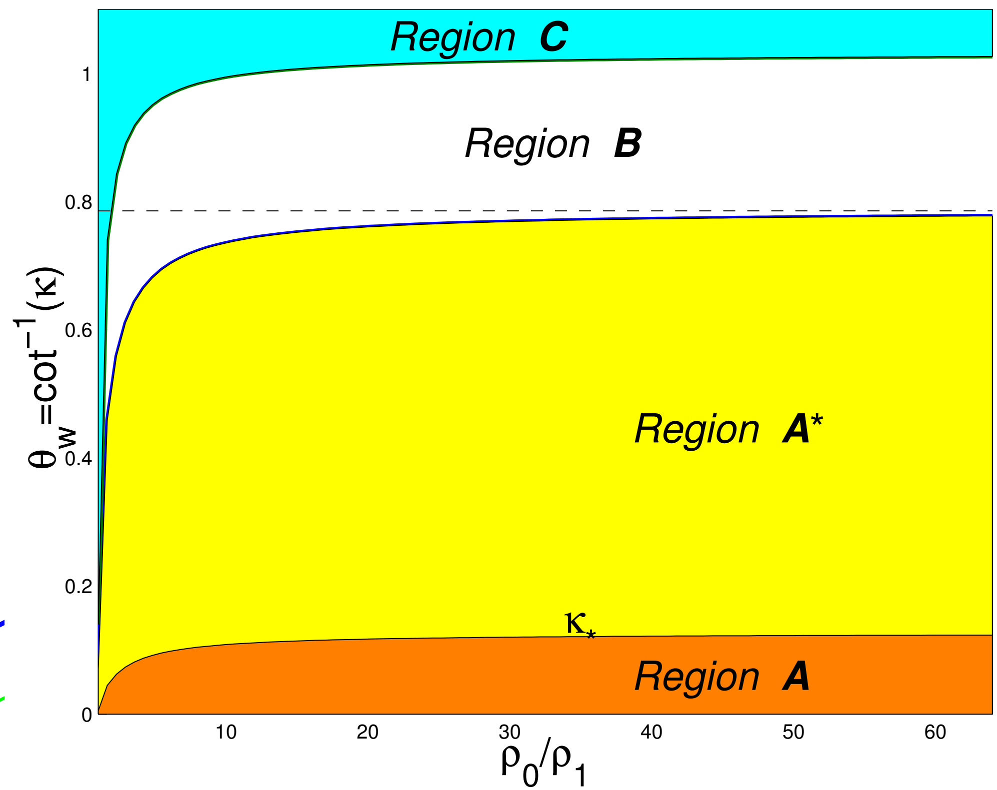
Interacting Shocks for NLWS



C: Q-1-D RP solvable



A+A*: Shk meets C_0



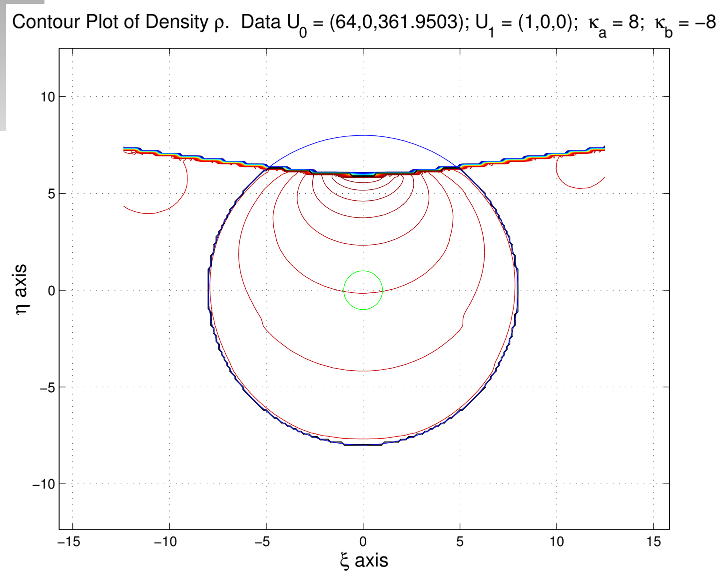
3 reg: **A+A*** Weak MR possible

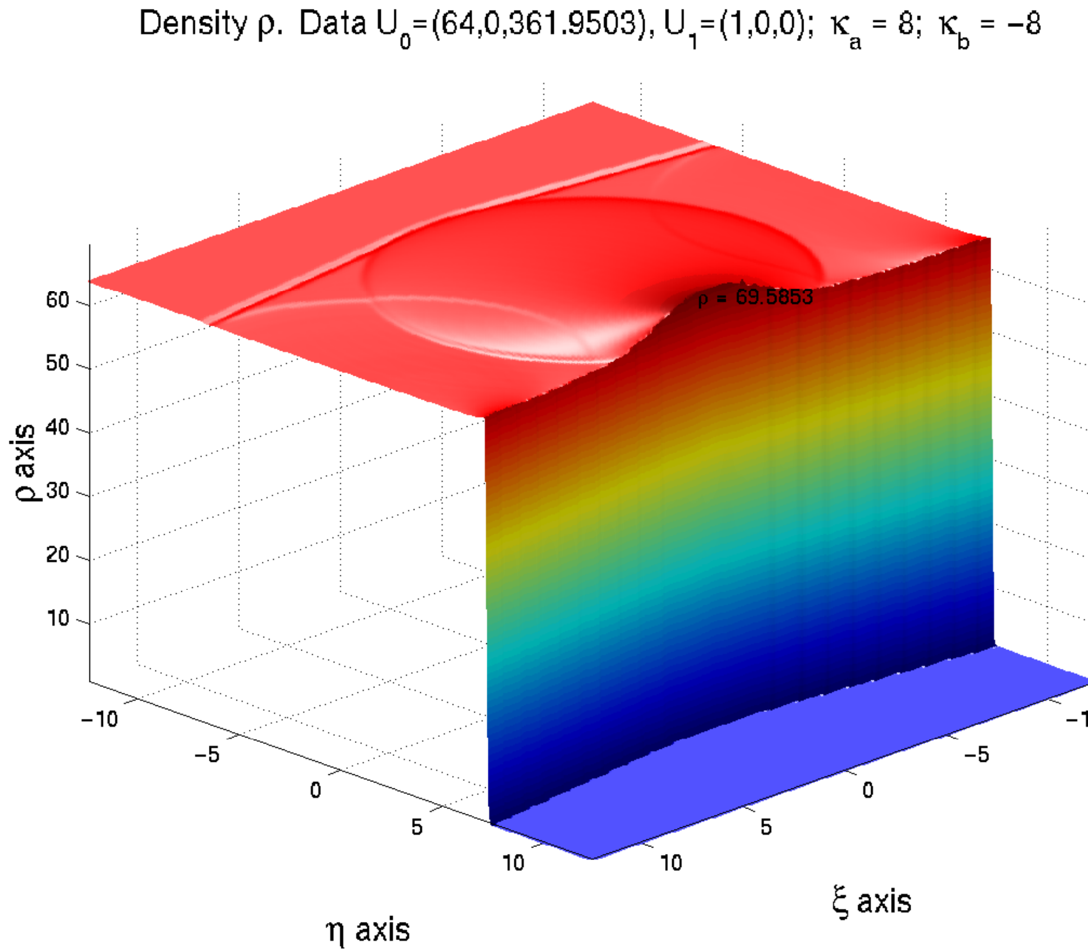
C RR possible

B neither possible

Numerical Simulations

Region A (Weak Mach Reflection)





Sonic circle

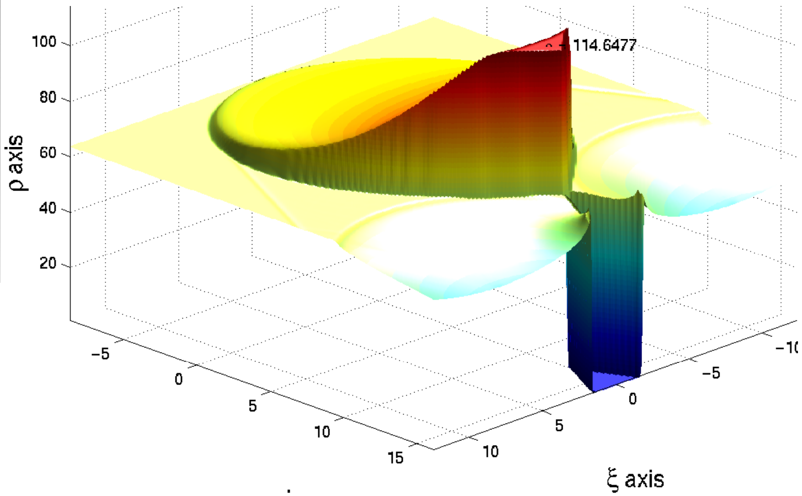
$$C_0 = \{\xi^2 + \eta^2 = c^2(\rho_0)\}$$

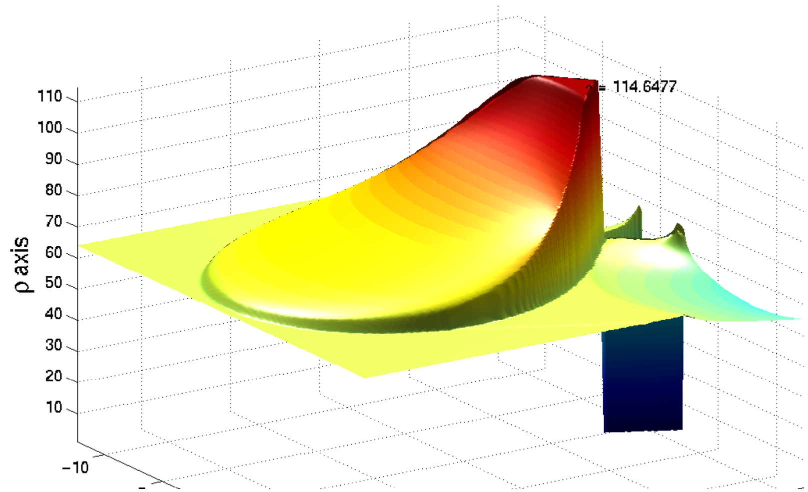
Supersonic soln known

U continuous at C_0

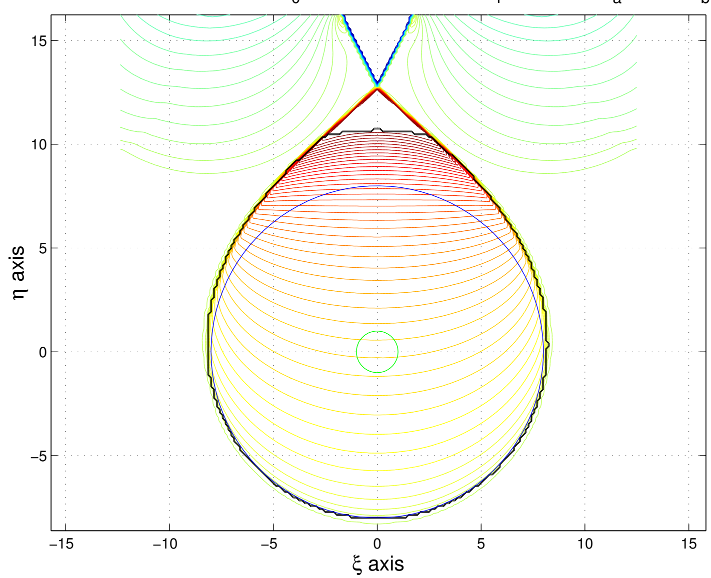
$\partial U / \partial r$ singular

Regular Reflection, $\kappa_a = 0.5$: Density





Contour Plot of Density ρ. Data U_0 = (64,0,803.0956); U_1 = (1,0,0); κ_a = 0.5; κ_b = −0.5



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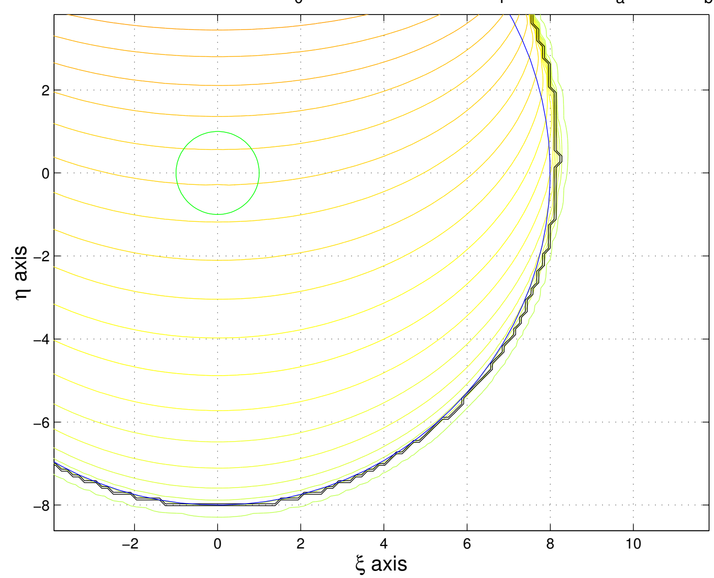
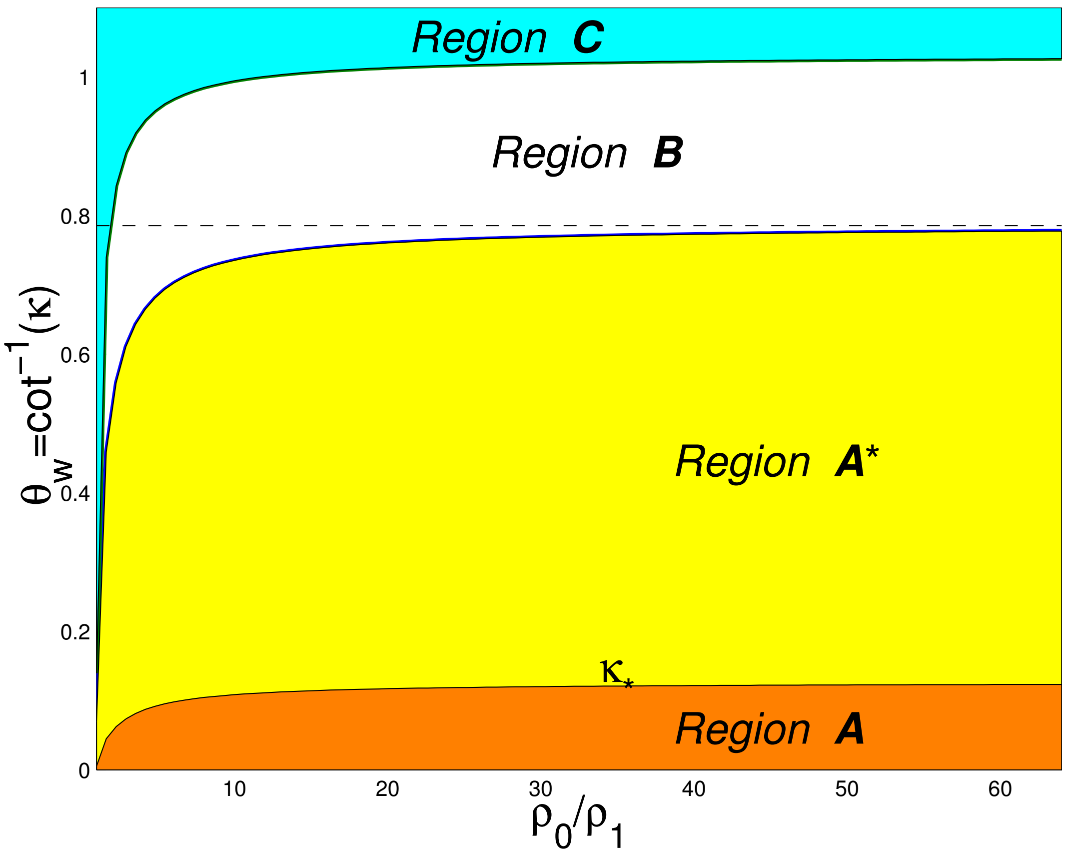


Current State of Analysis

Region **A**: Analytic solution for $\kappa > \kappa*$
('technical' condition in gradient estimate)
Region **C**: Local $\exists$ for weak or strong RR (CKK, UTSD)

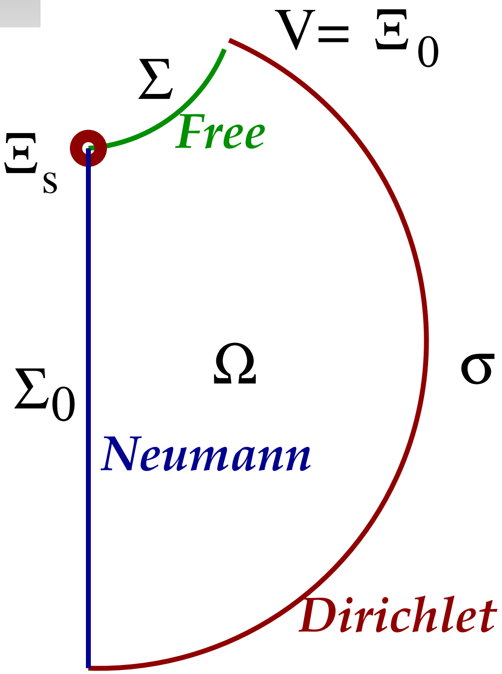


Weak Mach Stem (Large κ)

Degenerate Elliptic Free Boundary Problem

Theorem: Solution exists for global problem for NLWS.

$$Q \equiv \left((c^2-\xi^2)\rho_\xi - \xi\eta\rho_\eta\right)_\xi + \left((c^2-\eta^2)\rho_\eta - \xi\eta\rho_\xi\right)_\eta + \xi\rho_\xi + \eta\rho_\eta$$



$Q(\rho) = 0$ (degenerate elliptic) in Ω

Deg. BC $c^2(\rho) = \xi^2 + \eta^2 = c^2(\rho_0)$ on σ

Neumann BC $\rho_\xi = 0$ (symm) on Σ_0

Free boundary from RH equations:

$N(\rho) \equiv \beta \cdot \nabla\rho = 0$ (oblique deriv) on Σ

Shock $\frac{d\eta}{d\xi} = \frac{\eta^2 - s^2}{\xi\eta + \sqrt{s^2(\xi^2+\eta^2-s^2)}}$ $\quad s^2 = \frac{[p]}{[\rho]}$

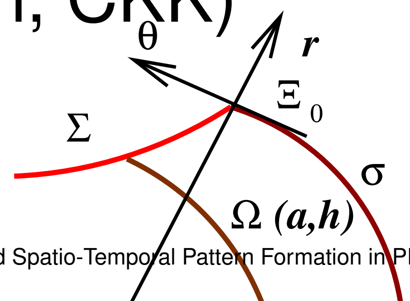
$\rho = \rho_{\text{max}}$ at $\Sigma \cap \Sigma_0$ (part of D. bdry)

Fixed Point Theorem (CK & Lieberman, CKK)

- New difficulties:

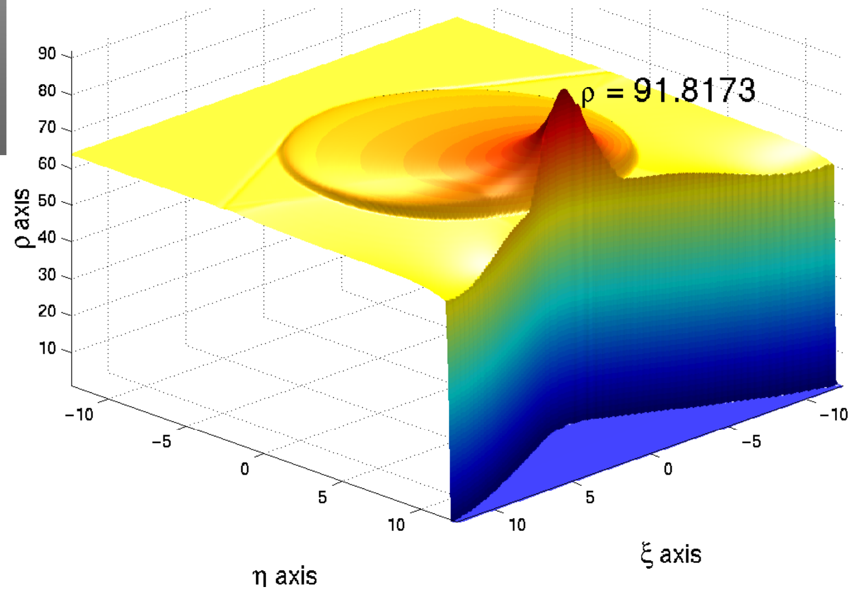
N not unif. oblique;

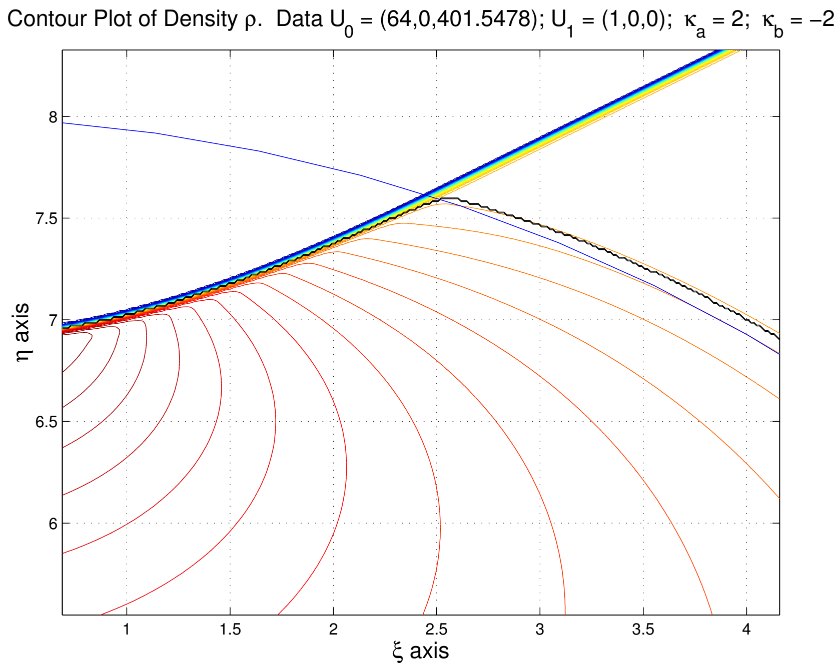
est. at degenerate corner



Maybe Not Just Technical

Angle $\kappa_a = 2$, Region **A***: Apparent Reflected Shock

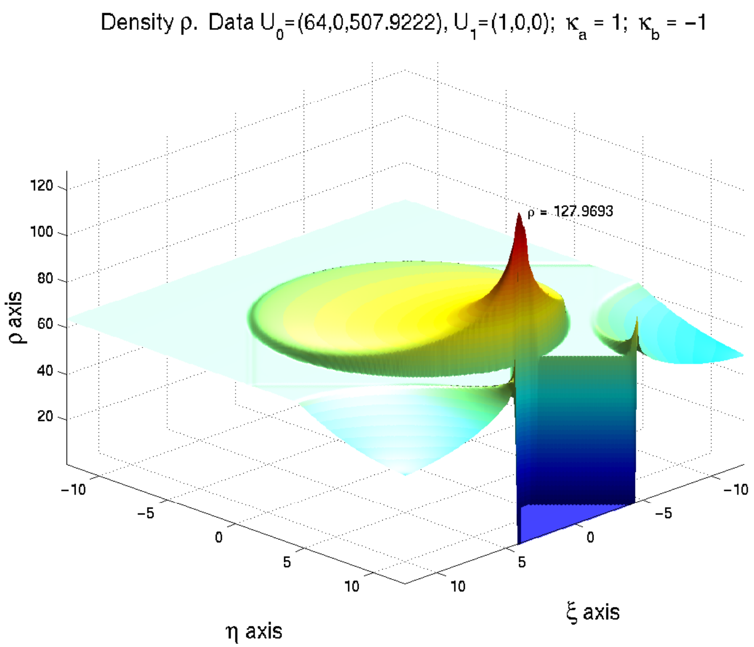


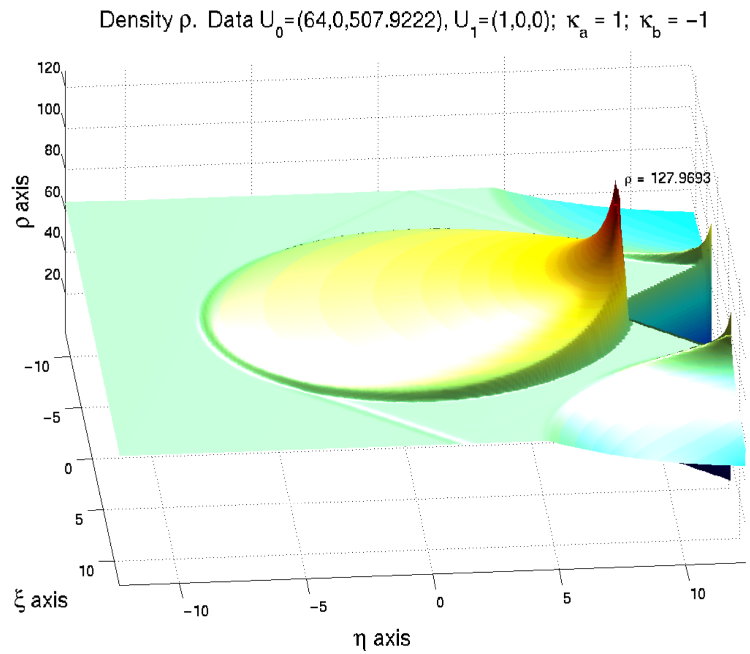


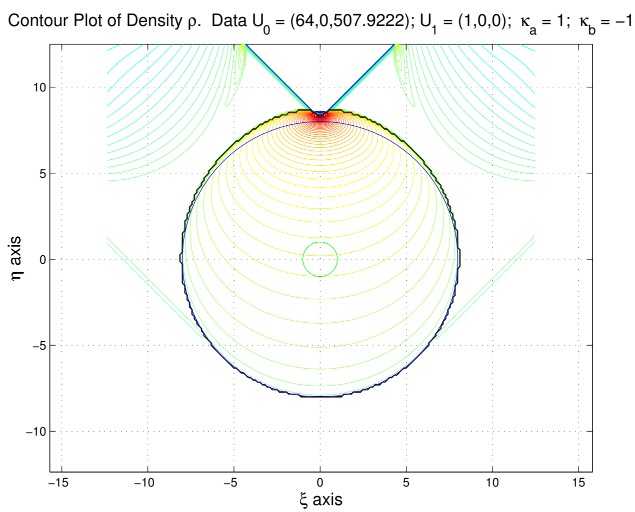
Conjecture: WR shock or reflected shock with zero strength at triple point

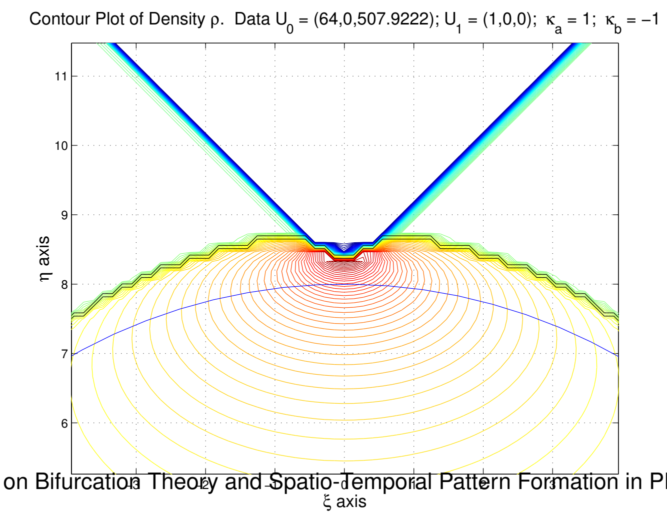
Simulations in Region B

$\kappa_a = 1$, $\rho_0 = 64$: 'von Neumann Paradox'

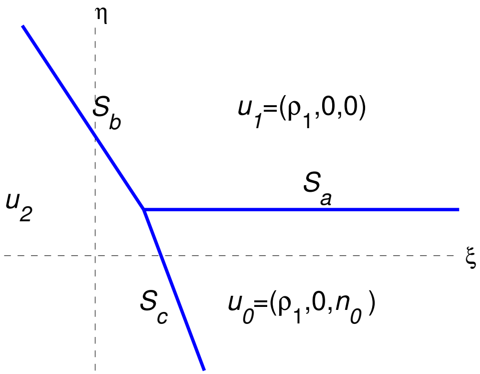






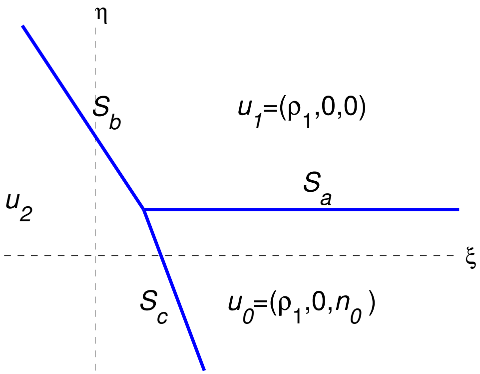


‘Triple Point’, Region B



- Proposition: NO nontrivial sol’ns to R-H eq’ns for constant states $\{u_0, u_1, u_2\}$ separated by shock lines S_a, S_b, S_c.

‘Triple Point’, Region B

- 

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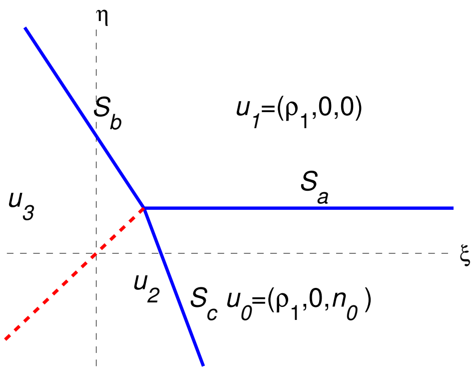
- 

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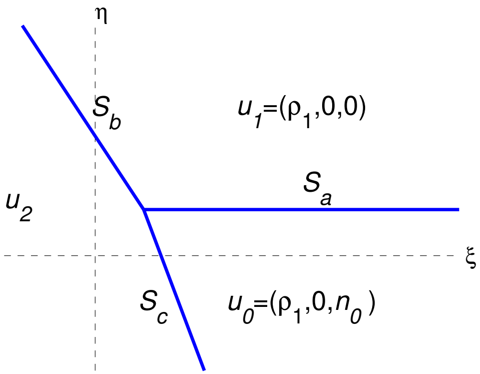
- 

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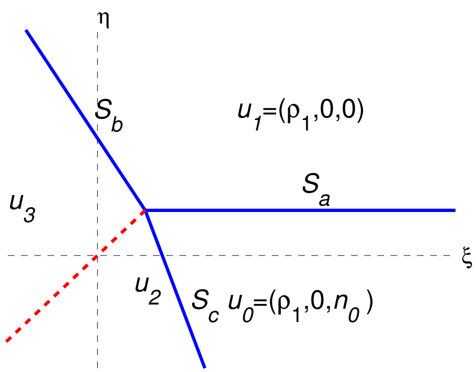
- 

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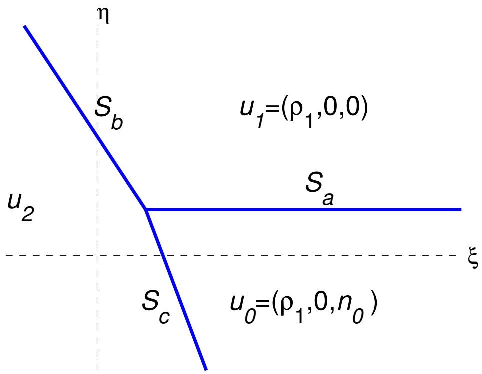
‘Triple Point’, Region B

- 

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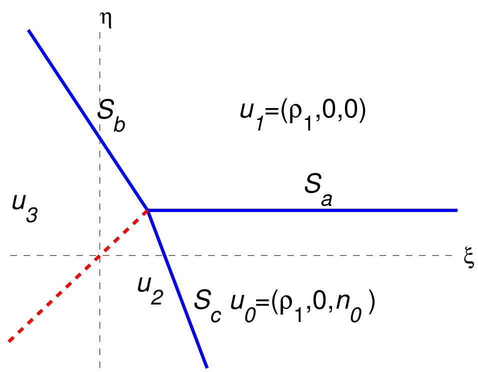
- 

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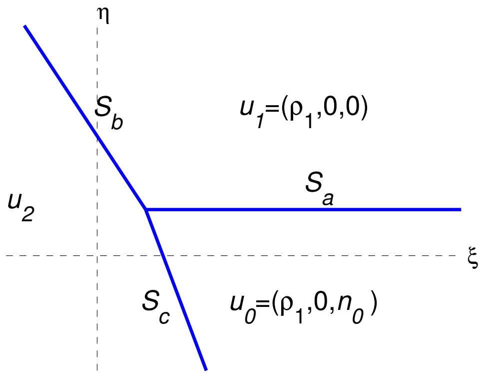
  - Subsonic wave must be lin.

'Triple Point', Region B

- 

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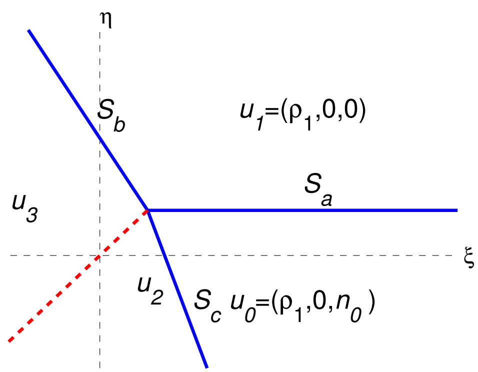
- 

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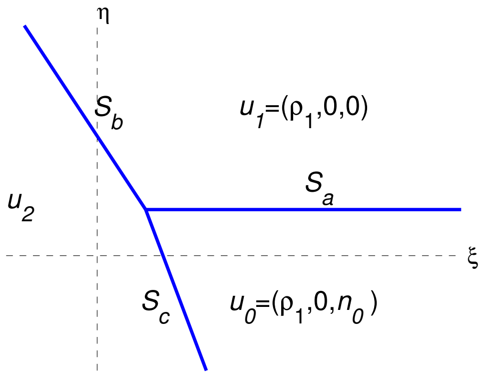
- 

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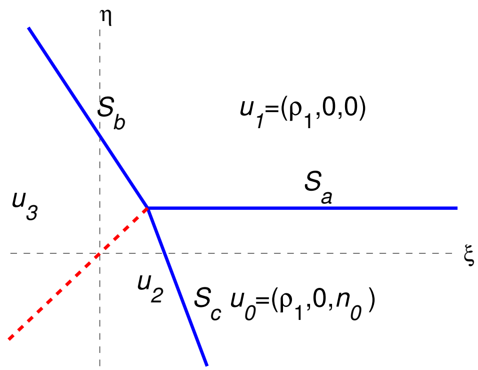
- 

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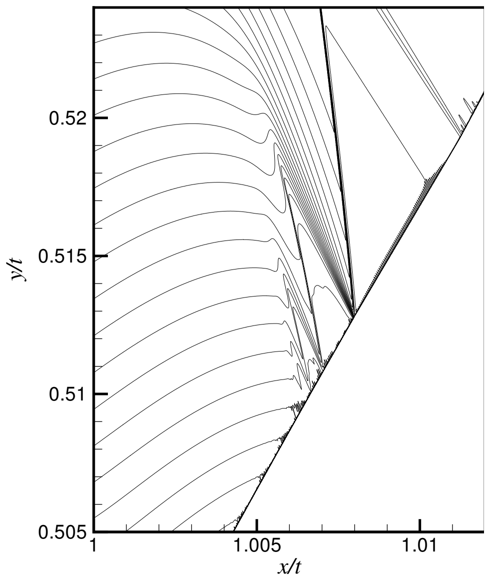
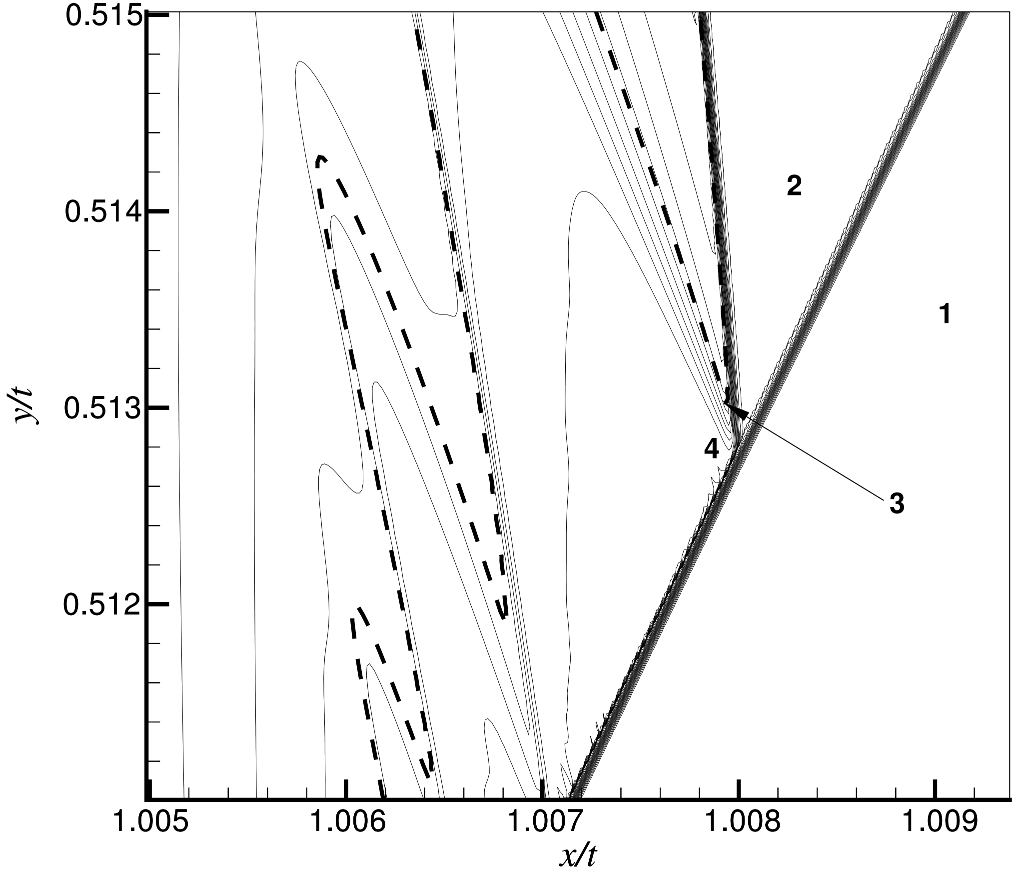
- Cannot occur in NLWS (cf. gas dynamics)

Supersonic Patch

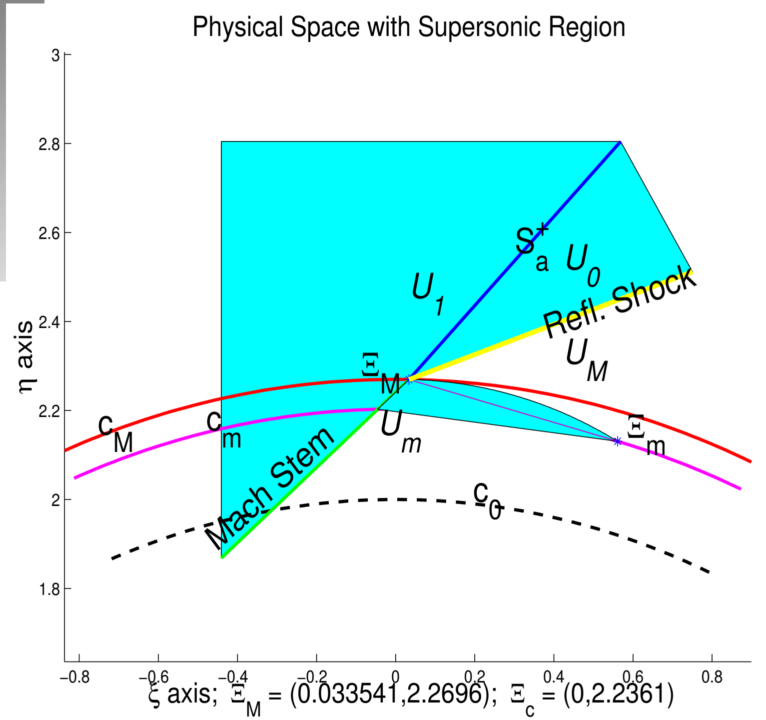
- Tesdall & Hunter, UTSDE
- SIAP, 2003
- Quasi-steady
- Embedded supersonic regions

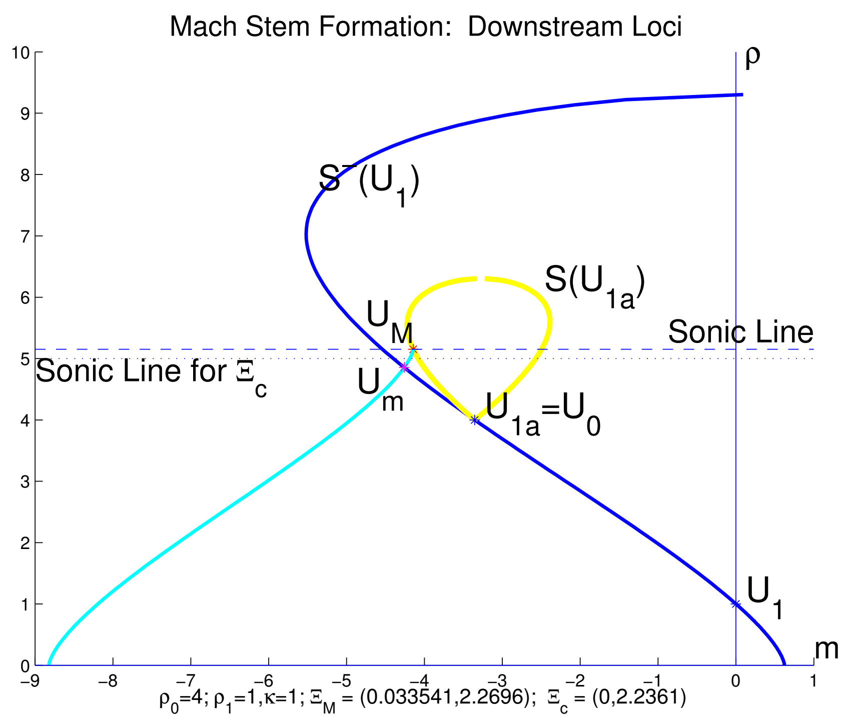


ALLEN M. TESDALL AND JOHN K. HUNTER



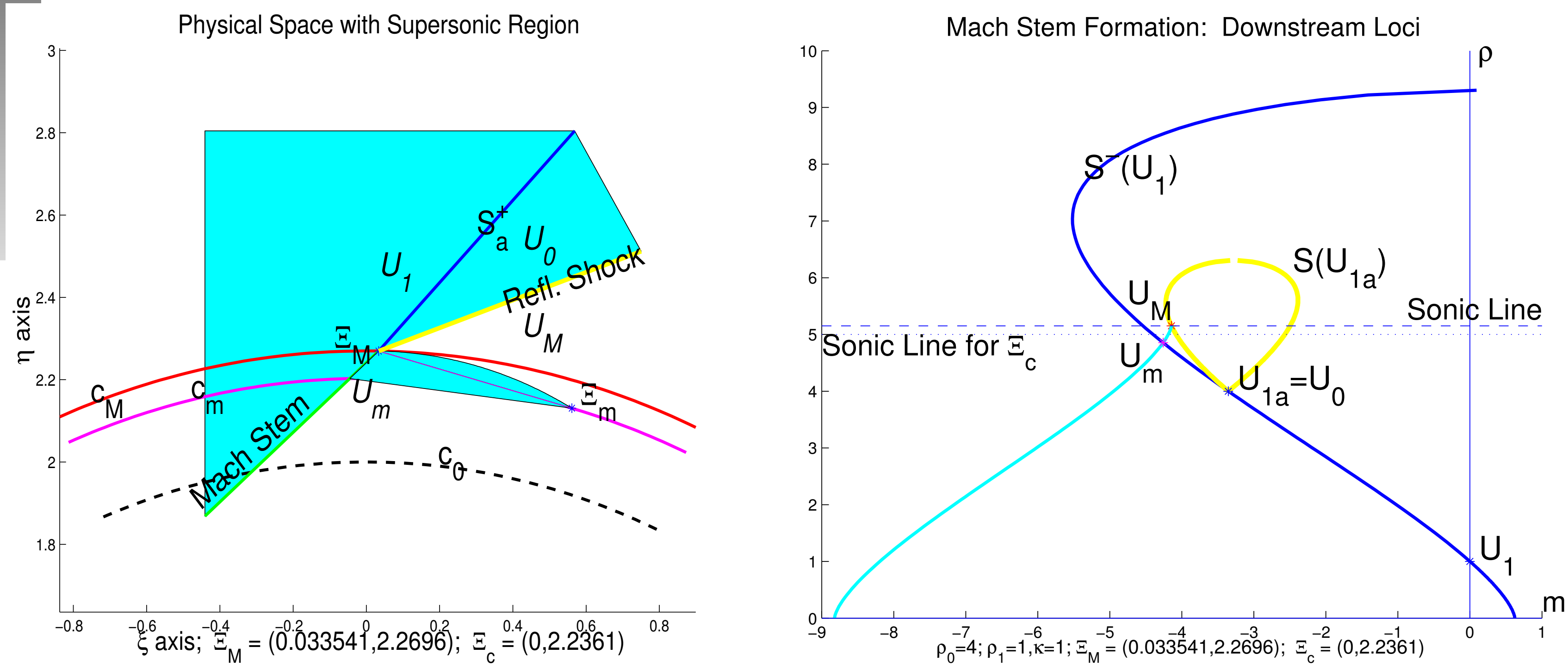
Embedded Supersonic Region in NLWS





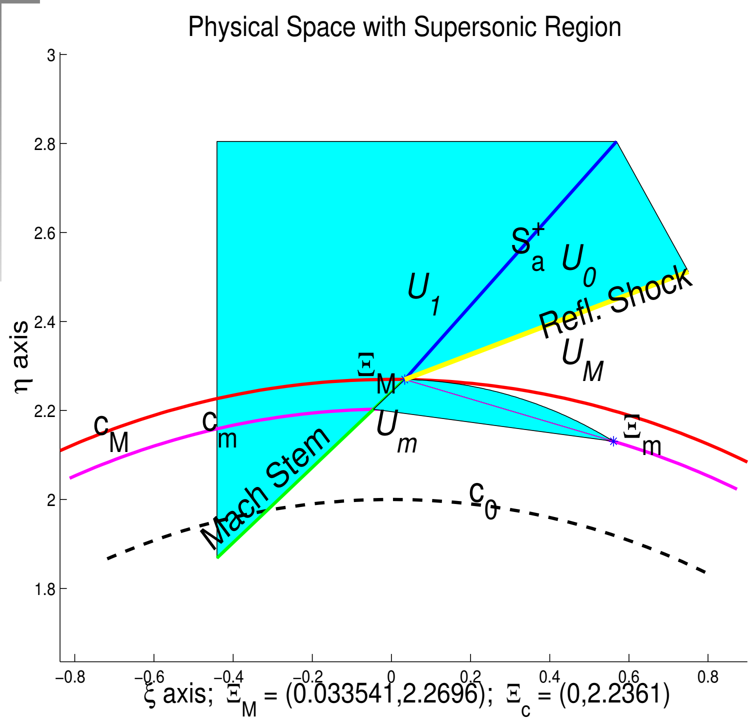
- Construction of states U_M (sonic), U_m (supersonic)

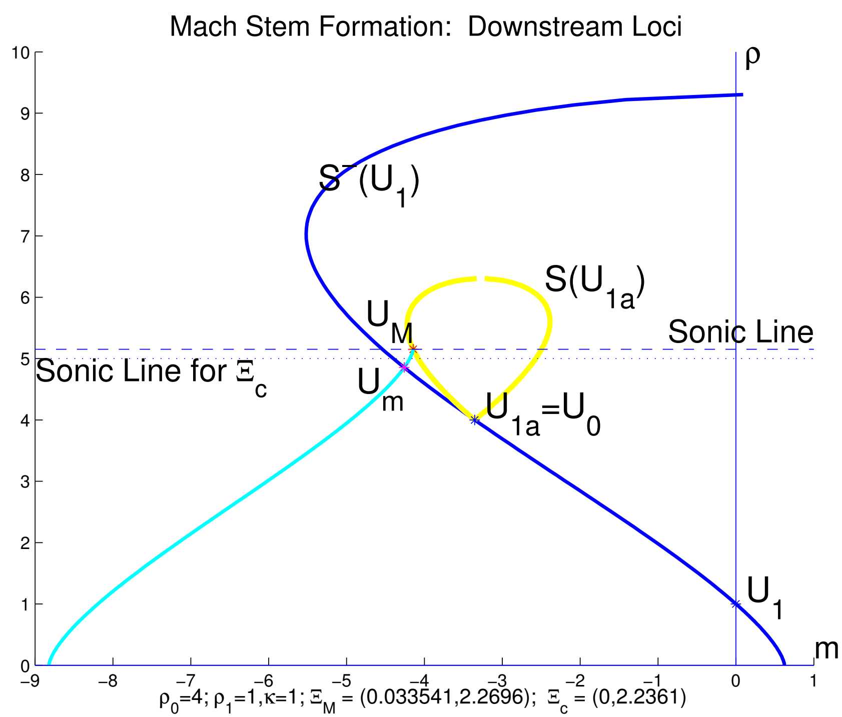
Embedded Supersonic Region in NLWS



- Construction of states U_M (sonic), U_m (supersonic)
- Supersonic patch not a domain of determinacy (no analysis done yet for this case)

Embedded Supersonic Region in NLWS

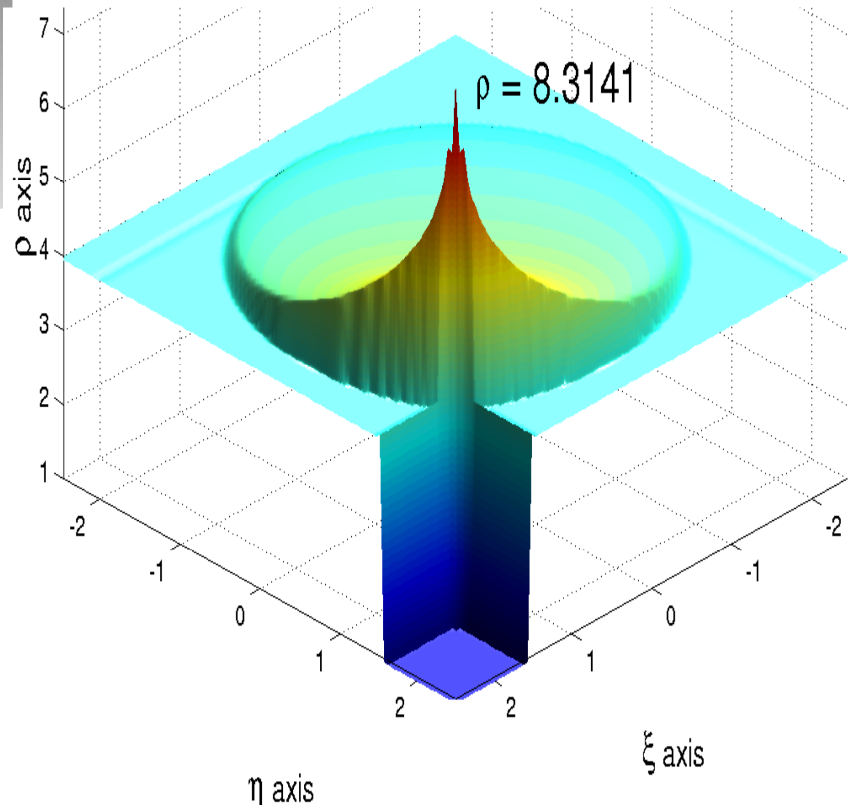


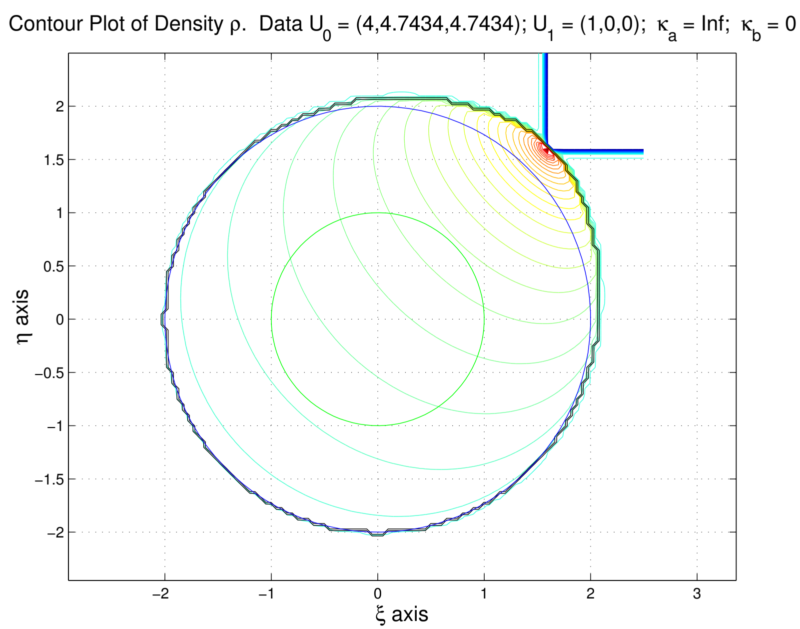


- Construction of states U_M (sonic), U_m (supersonic)
- Supersonic patch not a domain of determinacy (no analysis done yet for this case)
- Numerical evidence lacking in NLWS

Simulations of NLWS by Kurganov

Angle $\kappa_a = 1$, $\rho_0 = 4$: Region **B**: Apparent Triple Point





- Simulations show triple point (linear wave?)

Close-up of 'Triple Point'

Same Case, $\kappa_a = 1$, $\rho_0 = 4$