# Why are Multidimensional Conservation Laws So Difficult?

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Research supported by the Department of Energy,

National Science Foundation,

and NSERC of Canada.

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$$u_{tt} - c^2 \Delta u = 0, \quad u_{tt} - \nabla \cdot (c^2 \nabla u) = 0$$

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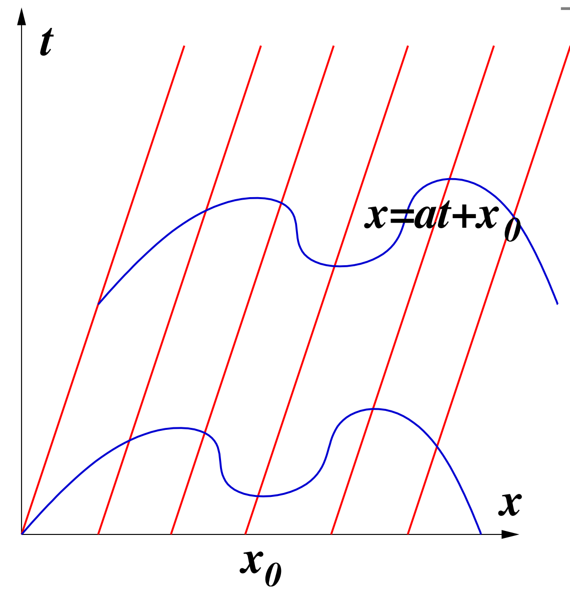
Nonlinear if $c = c(u)$ for example.

$$U_t + \sum \partial_{x_i} F_i(U) = 0; \quad U = (u_1, \ldots, u_n) \in \mathbf{R}^n, \quad F_i \in \mathbf{R}^n$$

# Hyperbolic vs Elliptic: Prototype

Hyperbolic $u_t + au_x = 0$, $u(x,0) = u_0(x)$
Solution $u = u_0(x - at)$
Features:

- IVP well-posed
- characteristics
- finite propagation speed
- no smoothness



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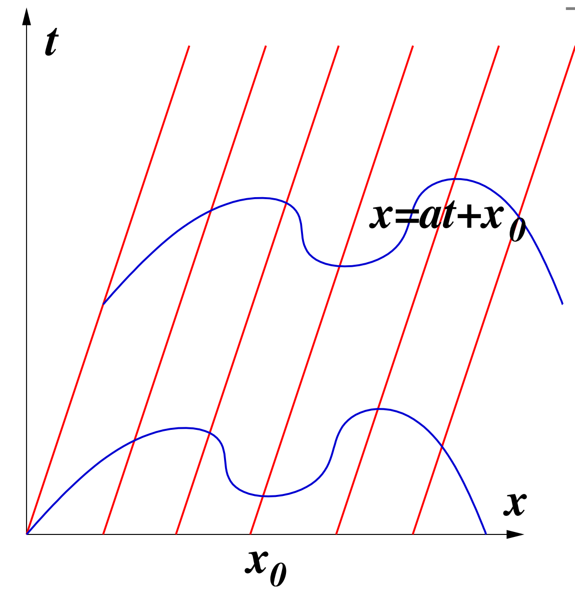
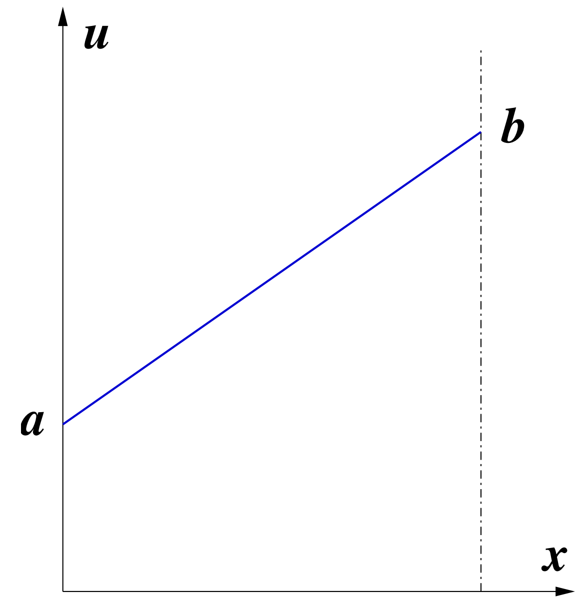



Elliptic $u_{xx} = 0$, $u(0) = a$, $u(1) = b$
Solution $u(x) = a + (b - a)x$
Features:

- BVP well-posed
- maximum principles
- apriori bounds on derivatives
- no notion of propagation



# Notation

Operator $P(D)$: $P(D)u = \sum c_\alpha D^\alpha u$

Multi-index: $x \in \mathbf{R}^n$, $\quad \alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$, $\alpha_i$ integers

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \quad \text{order of multi-index}$$

Deriv vector $D = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots \frac{\partial}{\partial x_n} \right) \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n}}$

Principal part of $k$-th order operator

$$P_k(D) = \sum_{|\alpha|=k} c_\alpha D^\alpha u$$

Example: $c^2 u_{xx} - u_{tt} + mu = 0$ (Klein-Gordon equation)

$k = 2$, variables $(x, t)$; $c_{20} = c^2$, $c_{02} = -1$, $c_{00} = m$

$$P_2 u = c^2 u_{xx} - u_{tt}$$

# Algebraic Structure & Classification

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First-order systems $\sum A_j \partial_{x_j} U + BU = 0$, $U \in \mathbf{R}^n$
Elliptic or hyperbolic: structure of

$$P(\xi) = \text{det} \sum \xi_j A_j$$

# The Paradox of Multidimensional CL

Systems of Conservation Laws

$$u_t + f(u)_x + g(u)_y = 0,$$

eg, compressible gas dynamics

$$u = (\rho, m, n, \ldots), \quad m = u\rho, \quad n = v\rho$$

Important in applications, simulations

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- smooth data lead to discontinuous solutions (need to study weak solutions)
- discontinuities in quasilinear equations propagate on shocks, not on characteristics
- Characteristics in multiD are complicated (WF sets)

# Weak Solutions: Linear Equations

Divergence form equations (conservation laws):

$$\nabla \cdot F(U) = 0 \Rightarrow \iint F \cdot \nabla \theta \, dx = 0, \quad \forall \theta$$

$U \in$ Sobolev space (or in $\mathcal{D}'$)
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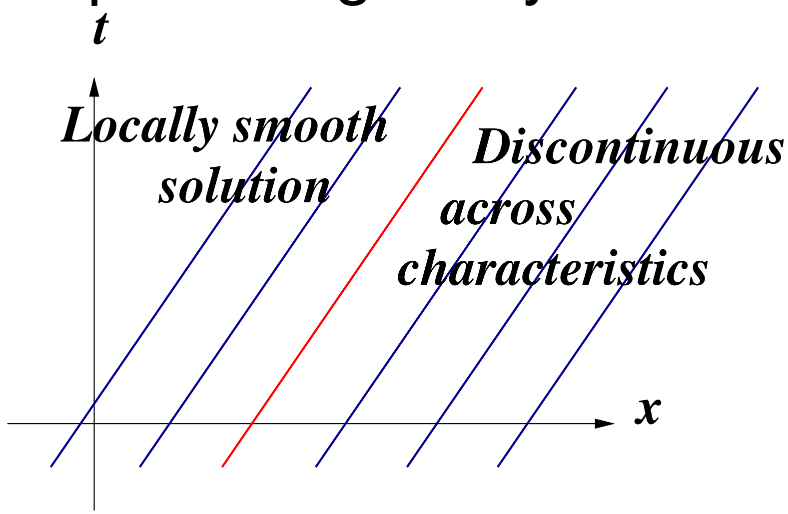

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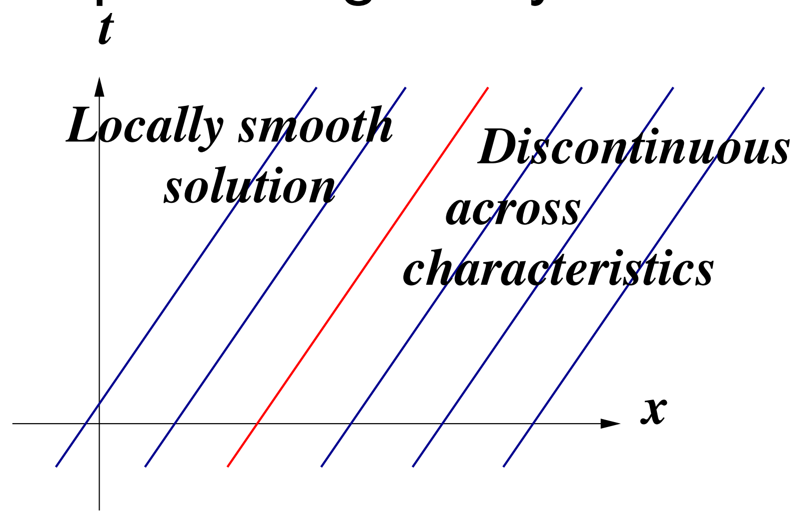


Higher dimensions: loss of regularity (focussing) when waves interact; multidim wave propagation subtle
Hyperbolic vs Elliptic: different properties of wk solns

# Quasilinear Equations/Systems

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Elliptic equations: Theory based on linear equations

Hyperbolic equations: New phenomena appear

Linear $u_t + au_x = 0, \quad \tau + a\xi = 0$, char $\lambda = -\tau/\xi = a$

Quasilinear $u_t + uu_x = 0, \quad \tau + u\xi = 0$, char $\lambda = -\tau/\xi = u$

$$u_t + (u^2/2)_x = 0 \quad \text{(Burgers equation)}$$

Discontinuities become shocks & RH replaces char. eqn

$$\sigma[u] = \left[\frac{u^2}{2}\right] \quad \text{or} \quad \sigma = \frac{u_L + u_R}{2}$$

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Incompatible difficulties:
loss of regularity in multidim (linear) wave propagation
nonlinear discontinuities do not propagate along char'cs

# Riemann Problems: Self-Similar Solution

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Our program: formulate and solve in 2D

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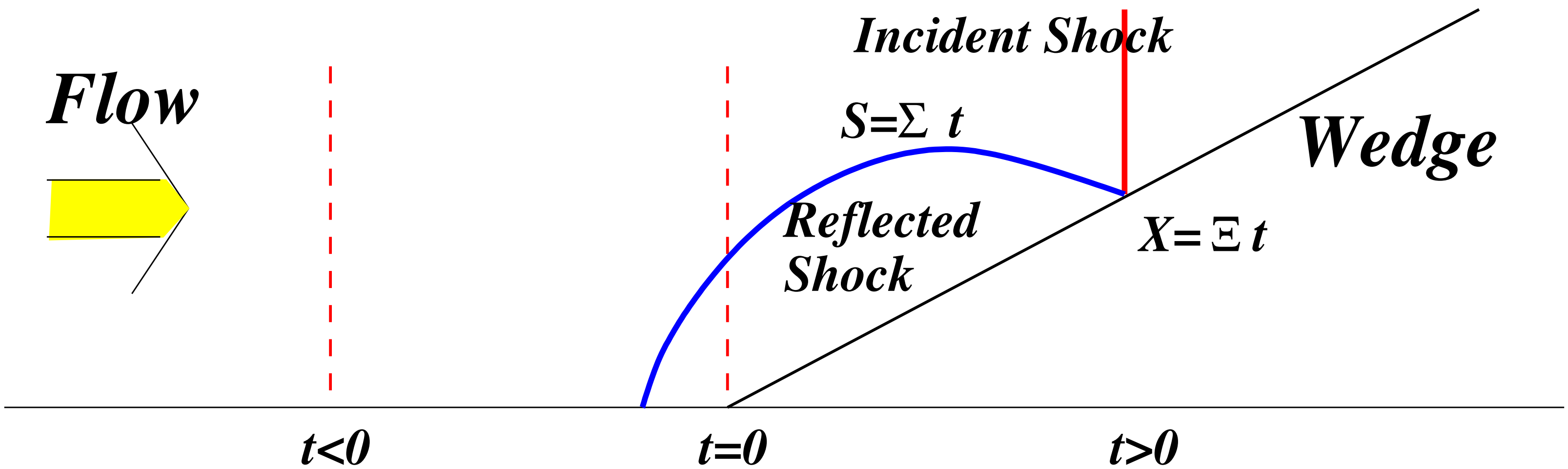
  Shock reflection by a wedge



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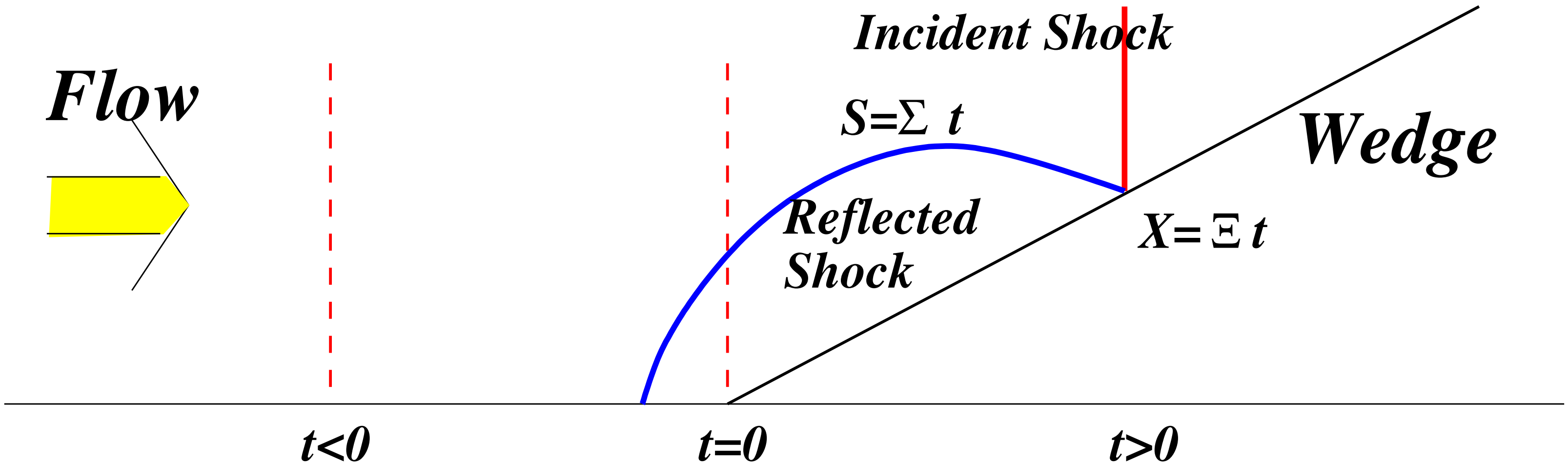


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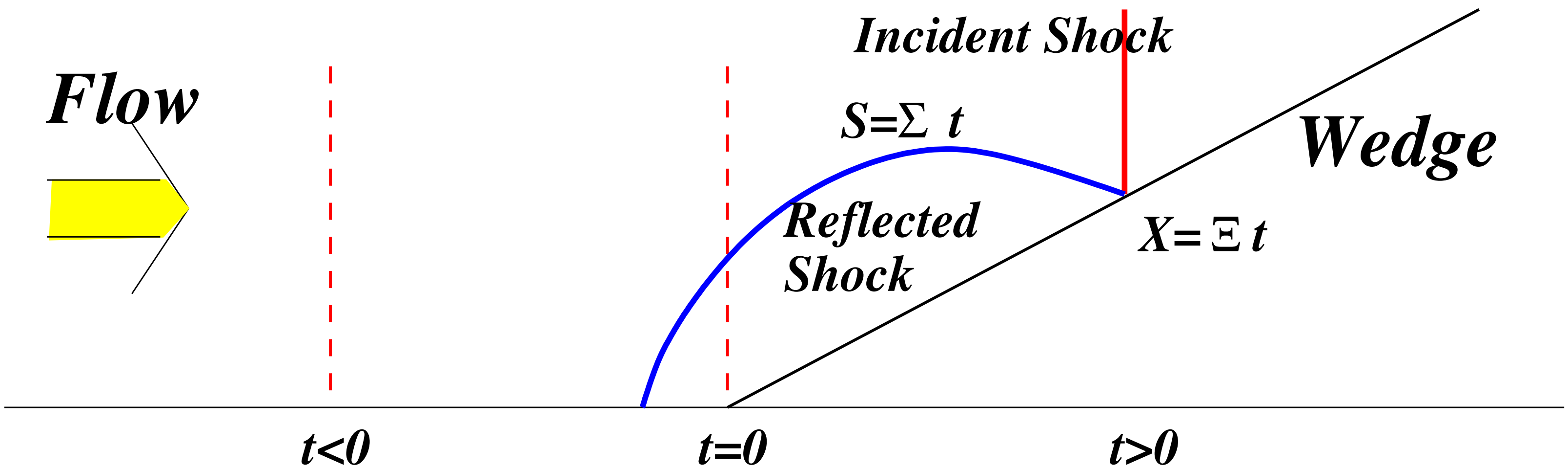



- Shock interactions
- Numerical simulations

# Similarity Reduction in Two-D Systems

$$U_t + F(U)_x + G(U)_y = 0, \quad U \in \mathbf{R}^n, \quad \text{hyperbolic}$$

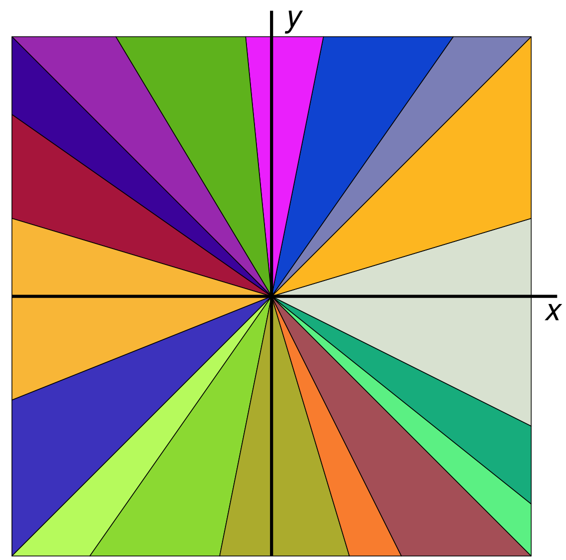
Riemann Data: $U(x, y, 0) = f\left(\frac{x}{y}\right)$

Similarity Variables:

$$\xi = \frac{x}{t}, \quad \eta = \frac{y}{t} \qquad U = U(\xi, \eta)$$

Reduced System in Two Variables

$$\partial_\xi(F - \xi U) + \partial_\eta(G - \eta U) = -2U$$




Sectorially Const Data

Method: resolve 1-D far-field discont; IV/BVP in 2-D

RP in $2+1$ dim $\Rightarrow$ CP in 2 ind. vbles. w. data at $\infty$

Reduced to a previously solved problem

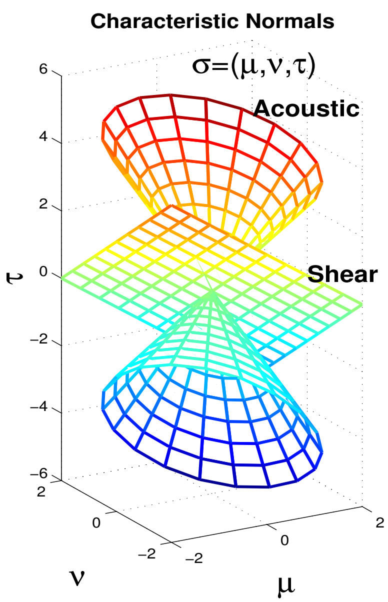
BUT

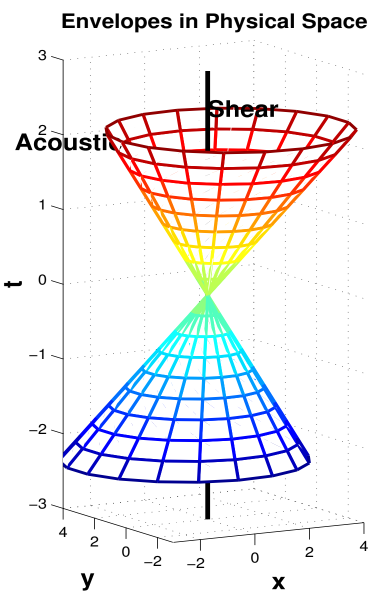
Type Changes: hyperb in far field; 'subsonic' region near 0

# Acoustic-type Structure

$$U_t + AU_x + BU_y = 0; \quad \det|I\tau + A\lambda + B\mu| = \left(\prod_{i=1}^{n-2} \ell_i \cdot \sigma\right)\sigma^T Q_N \sigma$$

$$\sigma = (\tau, \lambda, \mu)$$



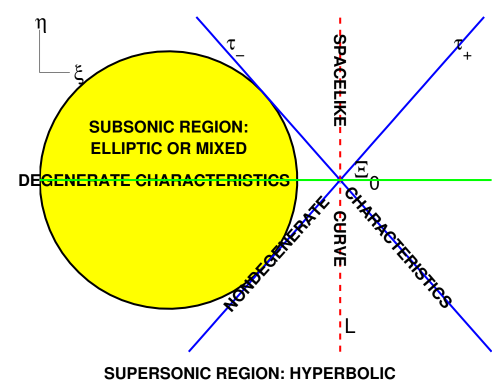

$$((A - \xi I)\partial_\xi + (B - \eta I)\partial_\eta)U = 0$$

$$\Xi = (\xi, \eta)$$

dual vector $\vec{\alpha} = (\alpha, \beta)$

$$\prod_{i=1}^{n-2} \ell_i \cdot (-\vec{\alpha}\cdot\Xi, \alpha, \beta) \underbrace{q(\sigma(\vec{\alpha}, \Xi), U)}_{\tilde{q}(\vec{\alpha},\Xi,U)}$$

CHANGE OF TYPE THEOREM *Reduced equation hyperbolic iff* $x = (1, \xi, \eta)$ *outside acoustic wave cone* $\mathcal{C}_W = \{x^T Q_N^{-1} x = 0\}$.



# Prototype Systems: UTSD & NLWS

## Comparison of Isentropic Gas Dynamics & NLWS

Isentropic Gas Dyn: $p = \rho^\gamma/\gamma$

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0$$
$$(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0$$

Nonlinear Wave System:

$$\rho_t + m_x + n_y = 0$$
$$m_t + p_x = 0 \qquad m = \rho u$$
$$n_t + p_y = 0 \qquad n = \rho v$$

Self-sim 2nd-order eqn for nonlinear charac vble ($\rho$):

$$\big((c^2(\rho) - U^2)\rho_\xi - UV\rho_\eta\big)_\xi + \big((c^2(\rho) - V^2)\rho_\eta - UV\rho_\xi\big)_\eta + \ldots = 0$$

$U = u - \xi, \quad V = v - \eta$ (‘$\psi$-vel.’)

$$\big((c^2(\rho) - \xi^2)\rho_\xi - \xi\eta\rho_\eta\big)_\xi + \big((c^2(\rho) - \eta^2)\rho_\eta - \xi\eta\rho_\xi\big)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0$$

Transport equation for linear characteristic variable:

$W = V_\xi - U_\eta = v_\xi - u_\eta =$ vorticity

$$UW_\xi + VW_\eta + (U_\xi + V_\eta + 1)W = 0$$

Nonlinear evolution equation

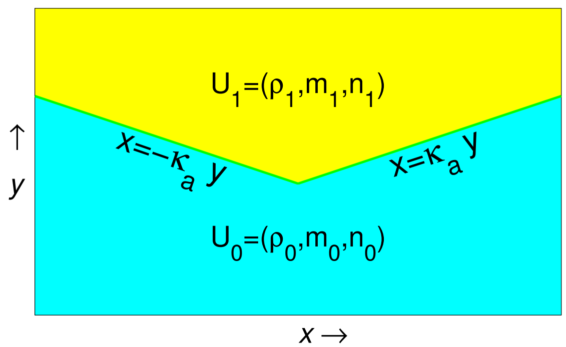
$w = n_\xi - m_\eta \qquad w_t = 0$

$(\xi, \eta) \cdot \nabla w + w = 0$ Linear

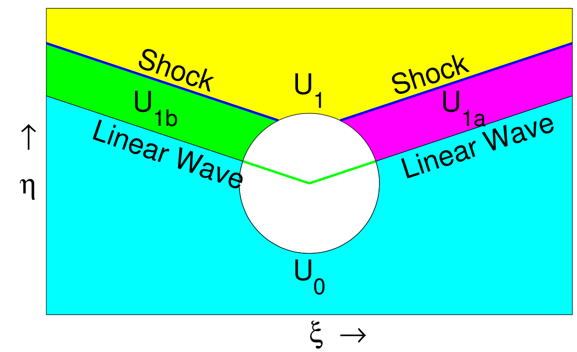
or: $rm_r = p_\xi \qquad rn_r = p_\eta$

# Prototype Data

## Interacting Shocks: A Bifurcation Problem for NLWS



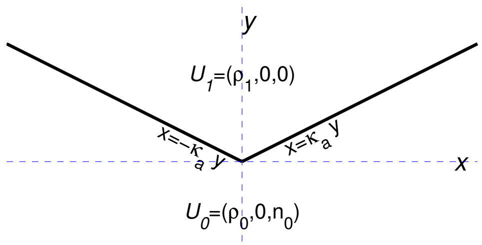

2-state data: $U_0$, $U_1$
Data give 2 shocks
Far field soln: 4 waves



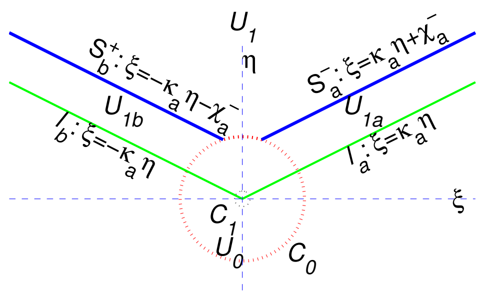

- Symmetric prototype for converging sector boundaries
- ‘Weak shock reflection’, von Neumann paradox
- Features

1. 2 parameters: $\rho_0/\rho_1 > 1$ and $\kappa_a$ (Mach # & wedge angle)
2. Incident shocks: $\xi = \kappa_a \eta - \chi$, $\xi = -\kappa_a \eta + \chi$
3. Small $\kappa$: two local solns –‘weak’ & ‘strong’ reg refl
4. Large $\kappa$: ‘Mach reflection’
5. Intermed $\kappa$: no sol’n from shock polars (Q1D RP)

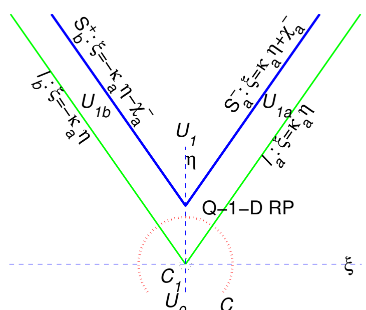
# Bifurcation of Interacting Shocks



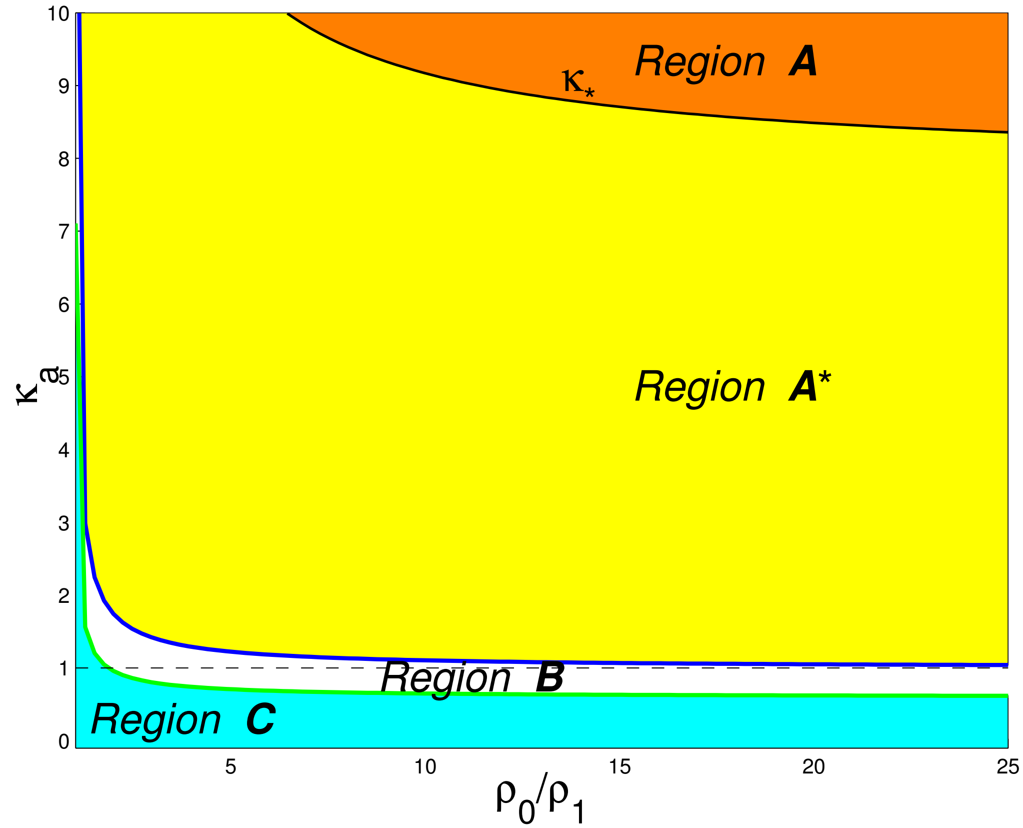

Data




**A+A***: Shock meets $C_0$




**C**: Q-1-D RP solvable

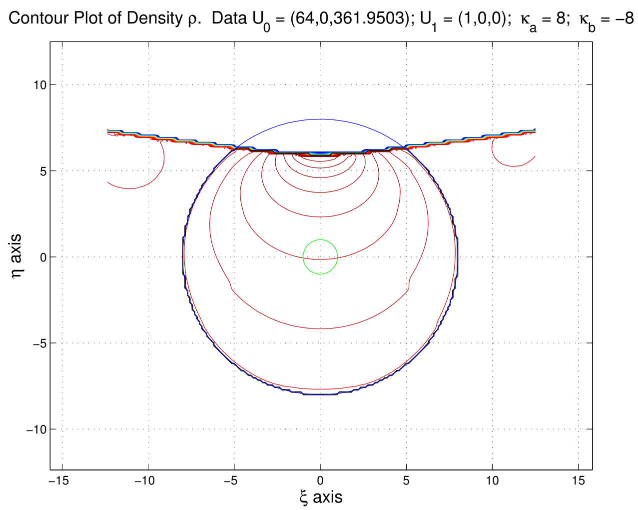


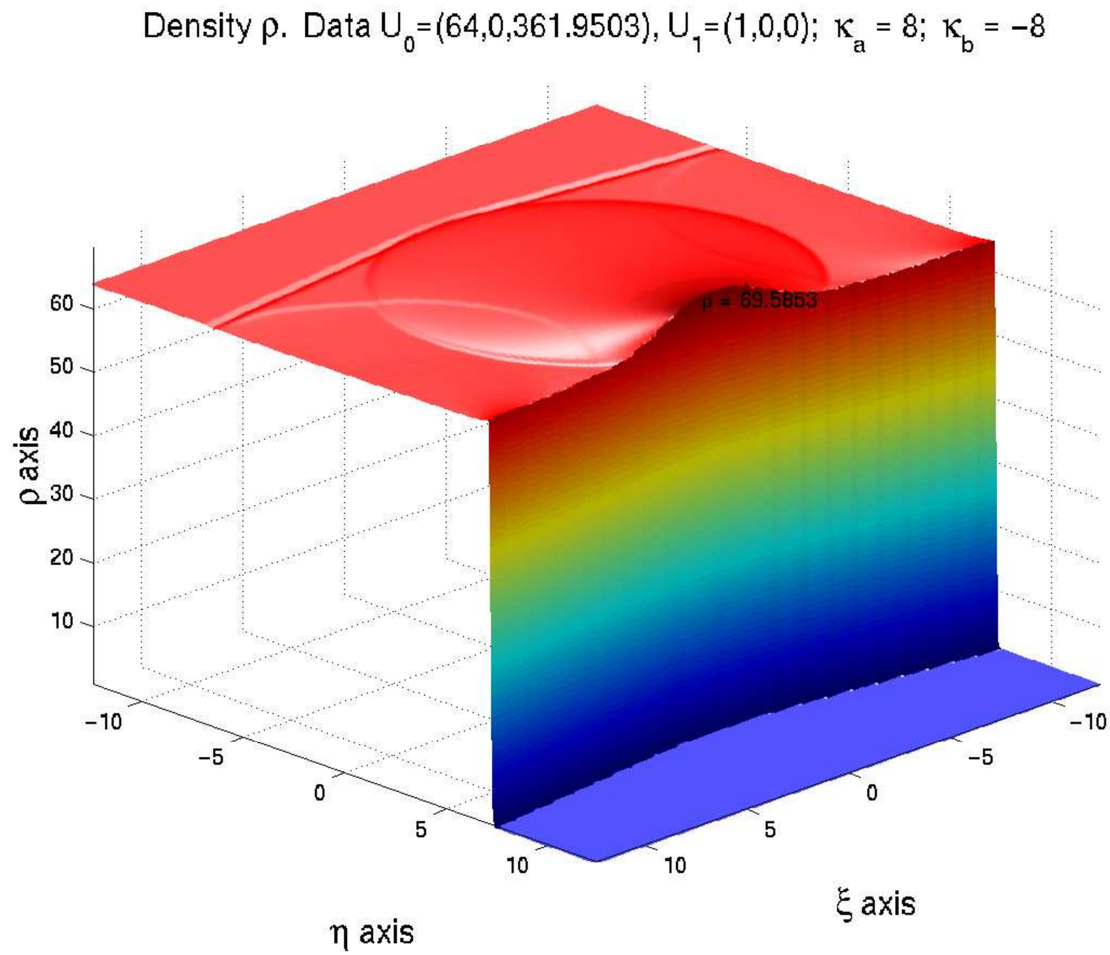

3 regions: **A+A*** MR possible

**C** RR possible

**B** neither possible

# Simulation of the Solution: Region A




Sonic circle

$$C_0 = \{\xi^2 + \eta^2 = c^2(\rho_0)\}$$

$$c^2(\rho) = \rho^{\gamma - 1}$$

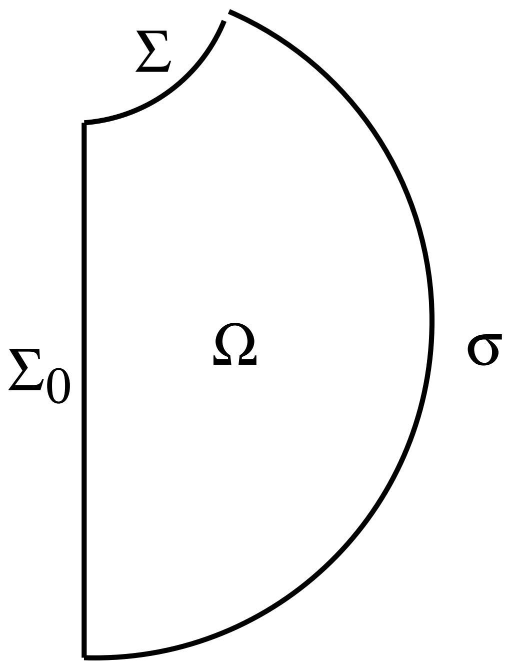
Supersonic soln known

Simulation indicates $U$ continuous at $C_0$, $\partial U/\partial r$ singular (not quite the case)

# Subsonic Flow with Mach Stem

Degenerate Elliptic Free Boundary Problem
Existence theorem for global problem for NLWS

$Q \equiv \big((c^2(\rho)-\xi^2)\rho_\xi - \xi\eta\rho_\eta\big)_\xi + \big((c^2(\rho)-\eta^2)\rho_\eta - \xi\eta\rho_\xi\big)_\eta + \xi\rho_\xi + \eta\rho_\eta$




$Q(\rho) = 0$ (degenerate elliptic) in $\Omega$

$\rho = \rho_0$ on $\sigma$

(degenerate boundary, continuous soln)

$\rho_\xi = 0$ (symmetry) on $\Sigma_0$

Free boundary from RH equations:

$N(\rho) \equiv \beta \cdot \nabla\rho = 0$ (oblique deriv) on $\Sigma$

$$\frac{d\eta}{d\xi} = \frac{\eta^2 - s^2}{\xi\eta + \sqrt{s^2(\xi^2+\eta^2-s^2)}} \qquad s^2 = \frac{[p]}{[\rho]}$$

$\rho = \rho_{\max}$ at $\Sigma \cap \Sigma_0$ (part of D. bdry)

Approach: Fixed Point Theorem (CK & Lieberman, CKK)

- Difficulties: $N$ not unif oblique; est. at degenerate corner

# Free Boundary as a Fixed Point

Formulate as 2nd order PDE for density, $\rho$ (not potential);
Rewrite RH conditions as (1) evolution eqn for shock and
(2) ODBC for $\rho$
Problem is quasilinear, degenerate elliptic PDE, mixed BC
Regularize PDE (parameter $\varepsilon$)

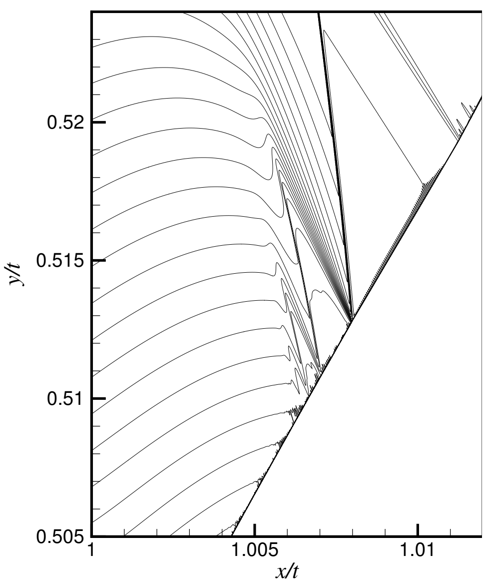
**Step 1** Fix approx $\eta = \eta(\xi)$, defines $\Sigma \in \mathcal{K}^{\varepsilon} \subset H_{1+\alpha_1}$ (Hölder)
**Step 2** Solve (fixed) mixed BVP for $\rho$
Lieberman's Mixed BVP theory + linearization
+ modifications for loss of obliqueness
**Step 3** Map $\eta \to \tilde{\eta} = J\rho$ by other RH cond (shock evolution)
Schauder F. P. Thm: Compactness $\Rightarrow$ fixed pt for $J$

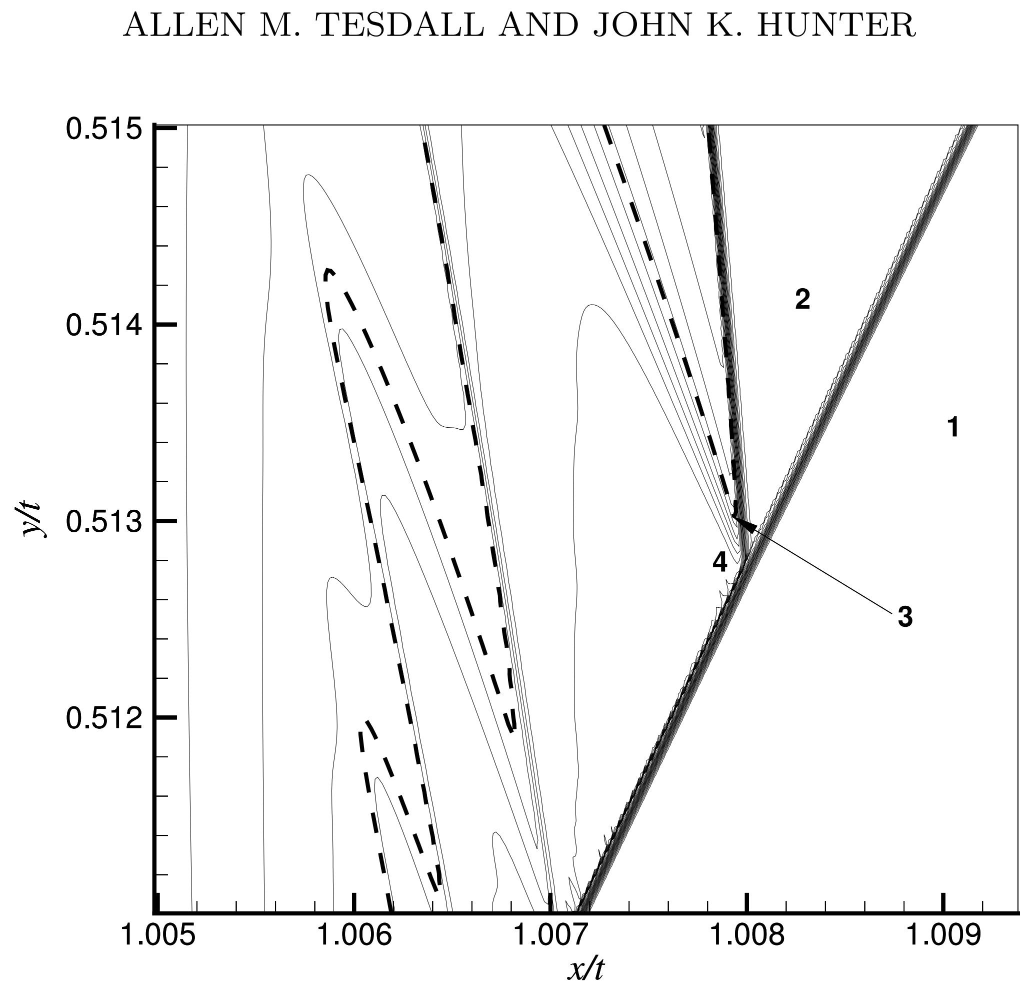
$$J : \mathcal{K} \subset H_{1+\alpha_1} \to \mathcal{K} \cap H_{1+\alpha}, \ \alpha > \alpha_1$$

**Step 4** Show $\eta$ and $\rho$ solve the problem.

# Supersonic Patch (Region B)

- Numerical results of Tesdall and Hunter on UTSD eqn
- SIAP, 2003
- Quasi-steady simulation
- Cascade of embedded supersonic regions



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