Spaces of Weighted Measures for Conservation Laws with Singular Shock Solutions*

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April 16, 1993

Abstract

We study a model system of two strictly hyperbolic conservation laws which is genuinely nonlinear but for which the Riemann problem has no global solution. Singular solutions are defined by means of a generalized Rankine-Hugoniot relation and an overcompressive condition on the discontinuity. We show that approximate solutions which can be constructed by several standard methods converge in a weighted measure space and that the error in the approximation converges to zero. Viscous approximations satisfy approximate entropy inequalities which imply the overcompressive condition.

Contents

1 Introduction **2**
1.1 Formal Solution to the Riemann Problem Using Singular Shocks 4

*Journal of Differential Equations, to appear
†Research supported by NSF/AFOSR under grant number 89-03768; DOE under grant DE-FG05-91ER25102, NSF grant DMS-91-03560 and TARP grant 003652077, and by the Ontario Ministry of Education and Training and the Natural Sciences and Engineering Research Council of Canada while visiting the Fields Institute.
‡Partial support from the Research Office, Adelphi University.

2 Approximate Solutions **6**
2.1 Dafermos-DiPerna Viscosity Solutions 7
2.2 Functions in $\mathcal{G}_s$ 8
2.3 Box Approximations 15

3 Convergence of Approximate Solutions **19**
3.1 The spaces $\mathcal{F}_0$, $\mathcal{F}_0^*$, $\mathcal{F}_\alpha$, $\mathcal{F}_\alpha^*$ 20
3.2 Spaces for u^ϵ and v^ϵ 25
3.3 Weak and Strong Limits of the Error 28

4 Entropy and Admissibility Conditions **30**
4.1 A Convex Approximate Entropy Function 31

5 Conclusions **34**

6 Acknowledgements **35**

1 Introduction

In this paper we define a new kind of wave that can appear in hyperbolic conservation laws, a *singular shock*. In earlier papers, as summarized in Section 2.1, we found approximations to these waves; here we establish that singular shocks are well-defined objects in a space of weighted measures (Section 3), and that they appear consistently as limits of several different approximations. In Section 4, we discuss an admissibility criterion for singular shocks.

We have established the approximation, convergence and admissibility properties we claim here for only a single model equation – equation (1) below – but they most likely hold for all equations of a certain class. Membership in the class is determined by global structure (the property of *same variation*), and the class includes some nonstrictly hyperbolic equations, for example those analysed in [15]. This is discussed in Section 5. Although the error of the approximations converges to zero, determining the sense in which singular shock waves may be considered solutions of the equation remains an open question.

The system we consider is

$$\begin{array}{l} u_t + (\ u^2\ -\ v)_x \\ v_t + (\frac{1}{3}u^3 - u)_x \end{array} \equiv U_t + F(U)_x = 0. \tag{1}$$

The system is genuinely nonlinear; its Jacobian matrix

$$A = dF = \frac{\partial(f,g)}{\partial(u,v)} = \begin{pmatrix} 2u & -1 \\ u^2-1 & 0 \end{pmatrix}$$

has eigenvalues

$$\lambda_1(u) = u - 1, \qquad \lambda_2(u) = u + 1;$$

the right eigenvectors (with the usual normalization $\mathbf{r}_i \cdot \nabla\lambda_i > 0$) are $\mathbf{r}_1 = (1, u+1)$ and $\mathbf{r}_2 = (1, u-1)$. The eigenvalues depend on the first component, u, alone — in fact, the system is linear in the second variable, v. Both normalized eigenvectors point in the positive x direction: thus, if two states U_L and U_R can be joined by a rarefaction wave, with U_L the state on the left, then $u_R > u_L$ for waves of both families. The reverse is true for shocks.

System (1) has the property that the shock speeds are not separated for waves of amplitude greater than two units. This property is related to that of a type of nonstrictly hyperbolic system, which we first mentioned in [8], (see Example 3, there), which displays "same" rather than "opposite" variation of the wave speeds. Such systems lack classical solutions, even for waves of very small amplitude, when the states are close to the line where strict hyperbolicity fails. An example has been analysed by Schaeffer, Schecter and Shearer [15], using asymptotic techniques similar to those outlined in Section 2.1 below, and approximate solutions with the same behavior appear there.

Some physical models also lead to this equation. A nonconservative form of equation (1) appears in [2] as a model for a nonlinear elastic system. More recently, Rosenau, [14], has advanced the equation

$$V_{tt} + (V^2)_{tx} = [(1 - V^2)V_x]_x + \beta V_{xxtt}$$

as a model for the evolution of ion-acoustic waves. When written as a system with $I_t = (V - \frac{1}{3}V^3)_x$, this is equation (1), with the addition of a third order term, βV_{xtx}. Since this model contains a *dispersive*, rather than a dissipative term, its solutions need not converge to those of (1) as $\beta \to 0$. Rosenau derives a second equation:

$$V_{tt} + (V^2)_{tx} = [(1 - V^2)V_x]_x + \beta V_{xxtt} + \nu V_{xtt}$$

where ν represents ion viscosity, also considered to be small. The quantity β is the normalized Debye length, and the formal limit $\beta = 0$ corresponds to a collisionless plasma. Whether our solutions are useful for this problem (where it is recognized by Rosenau that large amplitude waves have singular, in fact, explosive, behavior) will be the subject of further study.

In the remainder of this section, we write down the formal singular shock solutions to (1), which allow us to give a unique solution to the Riemann problem for (1).

1.1 Formal Solution to the Riemann Problem Using Singular Shocks

The integral (*rarefaction*) curves of (1) are

$$\begin{aligned} R_1 &= \{U \mid v = \tfrac{1}{2}u^2 + u + c_1\} \\ R_2 &= \{U \mid v = \tfrac{1}{2}u^2 - u + c_2\}. \end{aligned}$$

The *Hugoniot locus*:

$$H(U_0) = \{U \mid \exists s \text{ such that } s(U - U_0) = F(U) - F(U_0)\},$$

is compact; it is given by

$$v - v_0 = (u - u_0)\left(\frac{u + u_0}{2} \mp \sqrt{1 - \frac{(u - u_0)^2}{12}}\right)$$

for $|u - u_0|^2 \leq 12$. The corresponding shock speeds are

$$s = u_0 + \frac{u - u_0}{2} \pm \sqrt{1 - \frac{(u - u_0)^2}{12}}.$$

(The lower signs refer to the slower, or 1-wave family.) For a given U_L, the *Riemann problem* with data

$$U(x, 0) = \begin{cases} U_L = U_0, & x < 0 \\ U_R = U, & x \geq 0 \end{cases}$$

has a classical solution consisting of a 1-wave followed by a 2-wave if and only if U lies in a curvilinear quadrant $Q(U_0)$ whose boundary is the union of three curves, $J(U_0)$, $J_1(U_0)$ and $J_2(U_0)$, given by the equations (see Figure 1)

$$\begin{aligned}
J(U_0) &= \{U \mid v = \tfrac{1}{2}u^2 + u + \tfrac{9}{2} + v_0 - \tfrac{1}{2}u_0^2 - u_0; u \geq u_0 - 3\} \\
J_1(U_0) &= \{U \mid U \in H(U_0); u \leq u_0 - 3\} \\
J_2(U_0) &= \{U \mid v = \tfrac{1}{2}u^2 - u - \tfrac{9}{2} + v_0 - \tfrac{1}{2}u_0^2 + u_0; u \geq u_0 - 3\}.
\end{aligned}$$

The exterior of Q is divided into three open regions by the curves

$$\begin{aligned}
D(U_0) &= \{U \mid v = v_0 + u^2 + (1 - u_0)u - u_0; u \leq u_0 - 3\} \\
E(U_0) &= \{U \mid v = v_0 + (u - u_0)(u_0 - 1); u \leq u_0 - 3\},
\end{aligned}$$

illustrated in Figure 1. As described in [9], we can construct, formally, a solution to the Riemann problem, which contains nonclassical elements in these three regions: there are singular shocks, called S_δ shocks, joining U_0 to any point U in the open set Q_7 bounded by D, J_1, and E. Singular shocks to points on E, followed by 2-rarefaction waves, provide a solution in the region Q_5 below E, while a 1-rarefaction to a state U_1 followed by a singular shock to $D(U_1)$ completes the solution in Q_6.

The solution so described is unique, if we impose the following requirements on singular shocks. The singular shocks satisfy a *generalized Rankine-Hugoniot condition*: if a pair of states $U_L = U_0$ on the left and $U_R = U$ on the right are joined by a singular shock, then the first equation in (1) is satisfied in the weak sense, while the second is not. This defines a shock speed, s:

$$s = \frac{[u^2] - [v]}{[u]} = u + u_0 - \frac{v - v_0}{u - u_0}, \tag{2}$$

and the *Rankine-Hugoniot deficit*

$$a^2 = s[v] - \tfrac{1}{3}[u^3] + [u] = [v]\left(\frac{[u^2] - [v]}{[u]}\right) - \tfrac{1}{3}[u^3] + [u], \tag{3}$$

which is positive in Q_7.

In the interior of Q_7 the condition

$$\lambda_2(u_0) > \lambda_1(u_0) > s > \lambda_2(u) > \lambda_1(u)$$

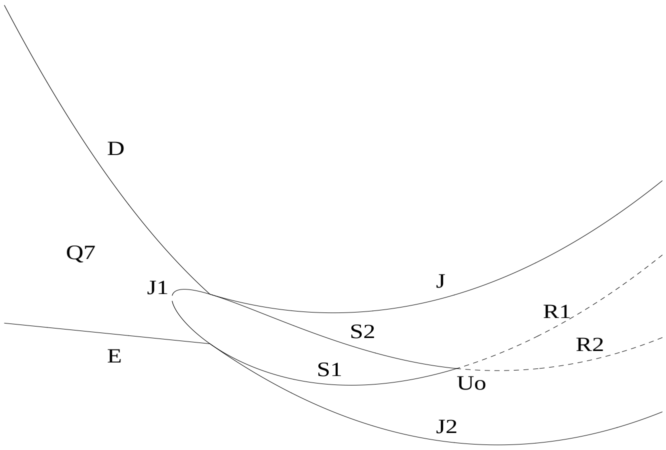

Figure 1: The Riemann Problem in Phase Space

holds (this means the wave is overcompressive), and the boundaries D and E are defined by $s = \lambda_1(u_0)$ and $s = \lambda_2(u)$ respectively. The conditions

$$\lambda_2(u_0) > \lambda_1(u_0) \geq s \geq \lambda_2(u) > \lambda_1(u), \tag{4}$$

which hold in the closure of Q_7 and nowhere else in the plane, constitute *geometric admissibility conditions* for singular shocks. Imposing these conditions is justified in Section 4.

2 Approximate Solutions

In [9], [10], and [11] we found an asymptotic version of a regularized solution to (1) by considering the "Dafermos-DiPerna viscosity approximation":

$$\begin{aligned} u_t + (\,u^2\, - \,v)_x &= \epsilon t u_{xx} \\ v_t + (\tfrac{1}{3}u^3 - u)_x &= \epsilon t v_{xx}. \end{aligned} \tag{5}$$

We summarize those results in Section 2.1. We also describe two other procedures which yield approximate solutions; these do not correspond to a specific form of viscosity, but they give explicit, self-similar functions which behave like the asymptotic functions. We present the two new approximations in Section 2.2 and Section 2.3. The first is motivated by the theory of generalized distributions of Colombeau [3]; the second is a simple step-function approximation.

2.1 Dafermos-DiPerna Viscosity Solutions

We summarize the construction in [9]. Suppose that (5) has self-similar solutions $U(\xi)$, where $\xi = x/t$, which approach left and right states, U_L and U_R, as $\xi \to -\infty$ and $\xi \to \infty$. When $U_R \in Q_7(U_L)$ then the solution admits an asymptotic development with leading term

$$\tilde{U}(\xi) = \left(\frac{1}{\epsilon}\tilde{u}\left(\frac{\xi - s}{\epsilon^2}\right), \quad \frac{1}{\epsilon^2}\tilde{v}\left(\frac{\xi - s}{\epsilon^2}\right) \right).$$

Here $\tilde{u}$ and $\tilde{v}$ are solutions of a singular ordinary differential equation in $\eta \equiv (\xi - s)/\epsilon^2$:

$$\begin{aligned} x' &= x^2 - y \\ y' &= \tfrac{1}{3}x^3 \end{aligned} \qquad (6)$$

which correspond to homoclinic orbits in the upper half plane. The next order term in the expansion can be written like an ordinary shock profile,

$$U = \bar{U}\left(\frac{\xi - s}{\epsilon}\right)$$

where $\bar{U}$ is bounded and satisfies the equation for viscous profiles (with $\tau = (\xi - s)/\epsilon$):

$$\frac{d\bar{U}}{d\tau} - F(\bar{U}) + s\bar{U} = C_{\mp} \qquad (7)$$

with a different constant of integration on each side of the singularity. Finally, $C_+ - C_- = C = (0, a^2)$, where a^2 is defined in (3), and identifies the particular trajectory of the singular solution $\tilde{U}$. There is thus a two-parameter family of solutions (with parameters s and a^2) which gives an asymptotic solution for each $U_R \in Q_7(U_L)$. The solutions of (2) and (3) give a coordinate system in Q_7.

We did not give a proof of existence of self-similar solutions of (5) and the techniques used in [4] and [5] do not suffice to do so. In [10] we give some partial results, and in [11] some numerical evidence. However, the results of Section 2.2 show that we can choose $\bar{U}$, which is not uniquely specified by the conditions listed so far, so that the asymptotic functions define an approximate solution. The following result is proved in Section 2.2. For each $U_R \in Q_7(U_L)$ and each $\epsilon > 0$ define the *asymptotic solution* $U_A(\xi) \equiv \bar{U} + \tilde{U}$. Then there is a choice of $\bar{U}$ such that the following holds.

Proposition 1 *The asymptotic solution U_A is an approximate solution to (1) in the sense that*

$$\sup_{\xi \in \mathbf{R}} |\partial_t U_A(\xi) + \partial_x F(U_A(\xi))| \leq K\epsilon$$

where K depends only on $|\tilde{u}|$, $|\tilde{v}|$, $|\bar{U}|$ and their first and second derivatives with respect to their arguments.

The proof of Proposition 1, along with the remaining details of the construction of U_A, will be given at the end of Section 2.2.

2.2 Functions in $\mathcal{G}_s$

Colombeau's theory of generalized distributions has been used to discuss weak solutions of conservation laws (see [1], [2], and the review article [3]). One can find objects, using the calculus of generalized distributions, which satisfy equation (1) in the sense of association. Motivated by this, we define an approximation sequence in equation (8) below.

Proposition 2, which is the main result of the section, shows that the functions defined in (8) are approximate solutions, if h, ρ, and p are chosen correctly. We work in the ordinary calculus of $\mathcal{C}^\infty$ functions.

As suggested by the asymptotics in Section 2.1, we seek superpositions of bounded and unbounded functions. For convenience, scale the approximate solutions, assumed to be self-similar in the variable $\xi = x/t$, to depend in a simple manner on another parameter, ϵ.

For a given $U_L = U_0$, and $U_R = U_1 \in Q_7(U_0)$, write

$$\begin{aligned} u^\epsilon(\xi) &= u_0 + (u_1 - u_0)\, h\left(\frac{\xi - s}{\epsilon^p}\right) + \frac{a}{\sqrt{\epsilon}}\, \rho\left(\frac{\xi - s}{\epsilon}\right) \\ v^\epsilon(\xi) &= v_0 + (v_1 - v_0)\, h\left(\frac{\xi - s}{\epsilon^p}\right) + \frac{a^2}{\epsilon}\, \rho^2\left(\frac{\xi - s}{\epsilon}\right) \end{aligned} . \qquad (8)$$

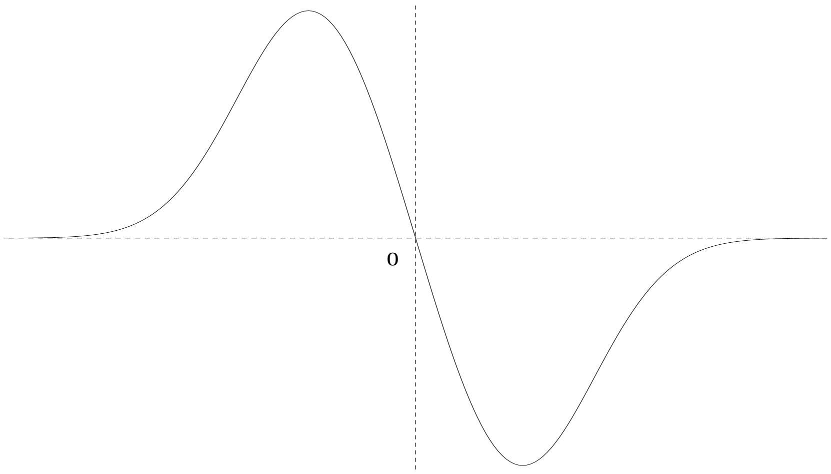

Figure 2: The Function ρ

Normalize $\rho \in \mathcal{C}_0^\infty([-1,1])$ by

$$\int_{-1}^{1} \rho^2 = 1, \qquad \int_{-1}^{1} \rho^3 = 0; \tag{9}$$

later, for admissibility, we shall also need

$$x\rho(x) \leq 0. \tag{10}$$

Figure 2 illustrates a typical function ρ.

The function h is a $\mathcal{C}^\infty$ monotonic approximation to the Heaviside function; we can suppose h' to have its support contained in $[-1,1]$. The positive number p is an index which determines whether the Heaviside is wide ($p < 1$) or narrow ($p \geq 1$) with respect to the singular part of the solution. Functions of both types provide approximations to equation (1). We are using ϵ where we used ϵ^2 in the previous section; as a consequence, the error is now $\mathcal{O}(\sqrt{\epsilon})$ instead of $\mathcal{O}(\epsilon)$. We change to this convention.

We abbreviate

$$\begin{aligned} u^\epsilon(\xi) &= h_1^\epsilon(\xi) + a\rho^\epsilon(\xi) \\ v^\epsilon(\xi) &= h_2^\epsilon(\xi) + a^2\delta^\epsilon(\xi), \end{aligned}$$

where $\delta^\epsilon \equiv (\rho^\epsilon)^2$ is an approximate Dirac δ-function supported at $\xi = s$.

Now we determine how the functions defined by (8) can satisfy equation (1) approximately. Since the solutions are functions of $\xi = x/t$ only, we introduce the notation

$$D = \frac{d}{d\xi}.$$

It is immediate that

$$U_t + F_x = \frac{1}{t}(-\xi DU + DF(U)) = \frac{1}{t}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \tag{11}$$

where

$$R_1 = -\xi Du^\epsilon + D((u^\epsilon)^2 - v^\epsilon) \tag{12}$$

$$R_2 = -\xi Dv^\epsilon + D(\tfrac{1}{3}(u^\epsilon)^3 - u^\epsilon). \tag{13}$$

Since it is standard for R_1 and R_2 to have bounds in a negative norm, we calculate

$$< R_i, \psi > \equiv \int_{-\infty}^{\infty} R_i(\xi)\psi(\xi)\, d\xi, \tag{14}$$

where ψ is a $\mathcal{C}_0^\infty$ test function. Weak solutions of (1) are, of course, defined with respect to test functions $\psi(x,t)$ with compact support in both variables. Except near $t = 0$, the two definitions are equivalent, and we shall ignore the factor $1/t$ in what follows (that is, we assume t is bounded away from zero – not an essential difficulty for Riemann problems). Our main result is

Proposition 2 *Let u^ϵ and v^ϵ be defined by equation (8), where s and a^2 are defined by (2) and (3) respectively, and ρ satisfies (9); if $p < 1$ then assume also*

$$h(0) = \frac{s - u_0}{u_1 - u_0},$$

while if $p = 1$ let ρ and h satisfy

$$\int \rho^2 h = \frac{s - u_0}{u_1 - u_0}.$$

If $p > 1$, then let ρ satisfy the additional condition

$$\int_0^1 \rho^2 = \frac{s - u_0}{u_1 - u_0}.$$

Define $q = min(|1-p|, 1/2)$ *if* $p \neq 1$ *and* $q = 1/2$ *if* $p = 1$. *Then the error terms,* R_i*, are bounded by*

$$| < R_1, \psi > | \leq k\, \epsilon^p (\|\psi\| + \|\psi'\|) + c\,\sqrt{\epsilon}(\|\psi\| + \|\psi'\|),$$

and

$$| < R_2, \psi > | \leq c\epsilon^q (\|\psi\| + \|\psi'\| + \|\psi''\|),$$

for any $\mathcal{C}_0^\infty$ *test function* ψ*. (Here* $\|\psi\| = \sup_\xi |\psi(\xi)|$ *;* k *and* c *depend on* U_0*,* U_1 *and* h*, and* c *also depends on* ρ*.)*

Proof: We integrate by parts in (14) using (12) and (13); the first term of R_1 becomes

$$< -\xi D u^\epsilon, \psi >= \int_{-\infty}^{\infty} u^\epsilon D(\xi\psi(\xi))\, d\xi.$$

The second term in $< R_1, \psi >$, after an integration by parts, is

$$\begin{aligned} - < (u^\epsilon)^2 - v^\epsilon, D\psi > &= \int (v^\epsilon - (u^\epsilon)^2)\psi'\, d\xi \\ &= \int [(h_2^\epsilon + a^2\delta^\epsilon - (h_1^\epsilon)^2 - 2ah_1^\epsilon\rho^\epsilon - a^2(\rho^\epsilon)^2)]\psi' \\ &= \int (h_2^\epsilon - (h_1^\epsilon)^2 - 2ah_1^\epsilon\rho^\epsilon)\psi' \end{aligned}$$

where we identify $(\rho^\epsilon)^2$ as an approximate Dirac mass, δ^ϵ. The most singular parts of $(u^\epsilon)^2$ and v^ϵ cancel identically here and do not contribute to the error. Thus,

$$\begin{aligned} < R_1, \psi > &= \int \left[h_1^\epsilon(\xi\psi)' + (h_2^\epsilon - (h_1^\epsilon)^2)\psi'\right] d\xi \\ &+ \; a\int [\rho^\epsilon(\xi\psi)' - 2h_1^\epsilon\rho^\epsilon\psi']\, d\xi. \end{aligned} \tag{15}$$

The first integral is what one obtains on approximating an ordinary jump discontinuity (a shock or other solution to the Rankine-Hugoniot relation) by a smooth function, with the difference that we have scaled h by an unspecified power p of ϵ. This term is bounded by ϵ^p, as long as s satisfies (2), and the second integral is also small (of order $\sqrt{\epsilon}$), because ρ^ϵ is small in the L_1 norm. Thus the presence of a singular contribution to U does not affect the first equation. This is not surprising, since singular shocks satisfy a Rankine-Hugoniot relation in the first component.

Specifically, the first integral in (15) is bounded by

$$k\epsilon^p(\sup_\xi |\psi(\xi)| + \sup_\xi |\psi'(\xi)|), \tag{16}$$

and the second integral by a bound of the same form with ϵ^p replaced by $\sqrt{\epsilon}$. Since $p = 1$ is the standard viscous profile smoothing for shocks, the singular approximation converges more slowly than an approximation to a standard shock, where the second integral in (15) is absent.

We now estimate R_2. After integration by parts,

$$< R_2, \psi >= \int_{-\infty}^{\infty} \left[v^\epsilon(\xi\psi)' - (\tfrac{1}{3}(u^\epsilon)^3 - u^\epsilon)\psi' \right] \, d\xi,$$

and we obtain

$$\begin{aligned} < R_2, \psi > \; &= \; \int \left[h_2^\epsilon(\xi\psi)' + (h_1^\epsilon - \tfrac{1}{3}(h_1^\epsilon)^3)\psi' + \; a^2\delta^\epsilon((\xi\psi)' - h_1^\epsilon\psi') \right] \, d\xi \\ &+ \; a \int \rho^\epsilon(1 - (h_1^\epsilon)^2)\psi' \, d\xi - \tfrac{1}{3}a^3 \int (\rho^\epsilon)^3\psi' \, d\xi \end{aligned} \tag{17}$$

The middle integral can be treated like the second integral in (15). The other two are more interesting. The last integral appears to be very singular: on substituting $\xi = s + \epsilon\tau$ it becomes

$$\frac{1}{\sqrt{\epsilon}} \int_{-1}^{1} \rho^3(\tau)\psi'(s + \epsilon\tau) \, d\tau.$$

However, if we expand ψ' in a Taylor series,

$$\psi'(s + \epsilon\tau) - \psi'(s) = \int_0^{\epsilon\tau} \psi''(s + \theta) \, d\theta = \epsilon\tau\psi''(s + \bar{\theta})$$

using the mean value theorem, then the second property of (9) implies that this entire term is bounded by $c\sqrt{\epsilon}\sup|\psi''(\xi)|$, where c is determined by ρ. This is the place where second derivatives of ψ appear.

We now turn to the first integral in (17). (We have already made the substitution $(\rho^\epsilon)^2 = \delta^\epsilon$.) Note that it contains two types of expressions. Those containing only h_1^ϵ, h_2^ϵ and $(h_1^\epsilon)^3$ can be estimated exactly like the first integral in (15): there will be an $\mathcal{O}(1)$ term

$$\psi(s) \left\{ \tfrac{1}{3}[u^3] - [u] - s[v] \right\} \tag{18}$$

and a remainder which can be estimated like (16).

The other part of the first integral in (17) can be rewritten as an integral in τ (where $\xi = s + \epsilon\tau$):

$$a^2 \int_{-1}^{1} \rho^2(\tau)(\xi\psi' + \psi - h_1(\epsilon^{1-p}\tau)\psi')\, d\tau,$$

and, ignoring the parts which are $\mathcal{O}(\epsilon)$ when ξ is expanded, and using the first relation in (9), we obtain

$$a^2\psi(s) + a^2 \left(s - \int_{-1}^{1} h_1(\epsilon^{1-p}\tau)\rho^2(\tau)\, d\tau\right) \psi'(s).$$

The first term exactly cancels (18) because of (3). The second must go to zero with ϵ. Note that

$$h_1 = u_0 + (u_1 - u_0)h(\epsilon^{1-p}\tau)$$

so we need

$$s = u_0 + (u_1 - u_0) \int_{-1}^{1} h(\epsilon^{1-p}\tau)\rho^2(\tau)\, d\tau \tag{19}$$

and there are three cases, depending on the size of p. If we define

$$H \equiv \int_{-1}^{1} h(\epsilon^{1-p}\tau)\rho^2(\tau)\, d\tau$$

and note that H is always between 0 and 1, then we find that in principle h and ρ can be chosen to accomplish (19), at least to highest order, because of the following Lemma, whose proof is omitted.

Lemma 1 *For any U_0 and $U_1 \in Q_7(U_0)$, and s defined by (2),*

$$0 < \frac{s - u_0}{u_1 - u_0} < 1.$$

When $p = 1$, we have immediately

$$H = \int_{-1}^{1} h(\tau)\rho^2(\tau)\, d\tau$$

and, by hypothesis, (19) is exactly satisfied in this case.

If $p < 1$, then expanding $h(\epsilon^{1-p}\tau)$ about 0 gives $H = h(0)$ and an error term; the error is bounded by ϵ^{1-p} times a factor involving ψ' and $h'(0)$.

If $p > 1$, then expand h around $|\tau| = \infty$, and

$$H = \int_0^1 \rho^2(\tau)\, d\tau,$$

with an error of order ϵ^{p-1}. In all cases, the form of the error term is given by an expression like (16). Thus we obtain the bounds stated in Proposition 2. ∎

Remark The functions defined by (8) have been shown to be approximations, when ρ and h satisfy the given restrictions and s and a^2 satisfy (2) and (3). The sign of a is not specified by any of our hypotheses. (Alternatively, ρ is shown, by (9), to behave roughly like an odd function, but replacing ρ by $-\rho$ would not affect the estimates in Proposition 2.) Only when ρ satisfies equation (10) and a is the positive root of a^2 do the $\mathcal{G}_s$ approximations resemble qualitatively the viscous approximations defined in Section 2.1. In other words, it cannot be seen from Proposition 2 whether these approximations satisfy (roughly) a forward or a backward heat equation, and, because our approximation by self-similar functions has removed the dynamics, both kinds of approximation will converge, as we shall prove in Section 3. It is the asymptotic behavior described in Section 2.1 which motivates equation (10) and the choice $a > 0$ in (8). As we discuss in Section 4, other criteria also point to this choice.

Proof of Proposition 1. Proposition 1 would follow from Proposition 2 if U_A were of the form (u^ϵ, v^ϵ) of this section. The difficulty that U_A is not compactly supported we ignore, since, as shown in [9], $\tilde{U}$ decays like ξ^{-2} or faster at infinity, while $\bar{U}$ approaches its asymptotic values exponentially fast. Now, since

$$\begin{aligned} u_A &= \frac{1}{\epsilon}\tilde{u}\left(\frac{\xi - s}{\epsilon^2}\right) + \bar{u}\left(\frac{\xi - s}{\epsilon}\right) \\ v_A &= \frac{1}{\epsilon^2}\tilde{v}\left(\frac{\xi - s}{\epsilon^2}\right) + \bar{v}\left(\frac{\xi - s}{\epsilon}\right) \end{aligned} \tag{20}$$

we see that this looks like (8) only if $(\tilde{u}^2 - \tilde{v})/\epsilon^2$ is bounded uniformly in ϵ.

From (6),

$$\frac{1}{\epsilon^2}(\tilde{u}^2 - \tilde{v}) = \frac{1}{\epsilon^2}\tilde{u}' = \frac{d}{d\xi}\tilde{u}\left(\frac{\xi - s}{\epsilon^2}\right),$$

which is bounded in the desired sense (multiply by a test function and integrate with respect to ξ). The compatibility condition of Proposition 2 is imposed on the bounded part of U_A, since the scalings of (20) correspond to $p = 1/2$, and is thus a condition on $\bar{U}(0)$. Now, note that (7) does not specify $\bar{U}$ uniquely, as no conditions have been imposed on $\bar{U}$ at $\tau = 0$. Since the conditions (4) imply that U_0 is an unstable and U_1 a stable node, it is clear that we may choose almost any initial and final conditions for $\bar{U}(0)$ to obtain a continuous solution of (7). In particular, we may choose

$$\bar{u}(0) = \frac{s - u_0}{u_1 - u_0}.$$

This still leaves one degree of freedom, $\bar{v}(0)$, to connect U_0 and U_1. ∎

2.3 Box Approximations

In this section we describe a third way to define approximate solutions to (1). This time, the candidates are piecewise constant ("box") functions, and satisfy the equation only in a weak, or distributional, sense. But unlike their limits, whose projections onto the class of distributions do not satisfy the equation in any sense, the box approximations are classical distributions, and we may calculate with them using the standard operational calculus. The idea behind the construction is very simple: as we have seen in the last two sections, the approximate solutions are a superposition of a bounded part (a type of Heaviside function), and an unbounded part, which is concentrated near the shock, and can be defined by scaling a single function. We saw in Section 2.2 that the relative widths of the singular function and the approximate Heaviside function did not matter. Box functions are a limiting case in which h is an exact Heaviside function, and ρ is not required to be smooth.

As in the last two sections, write

$$\begin{aligned} u^\epsilon(\xi) &= u_0 + (u_1 - u_0)\,h(\xi - s) + a\,u_b(\xi - s) \\ v^\epsilon(\xi) &= v_0 + (v_1 - v_0)\,h(\xi - s) + a^2\,u_b^2(\xi - s) \end{aligned} \tag{21}$$

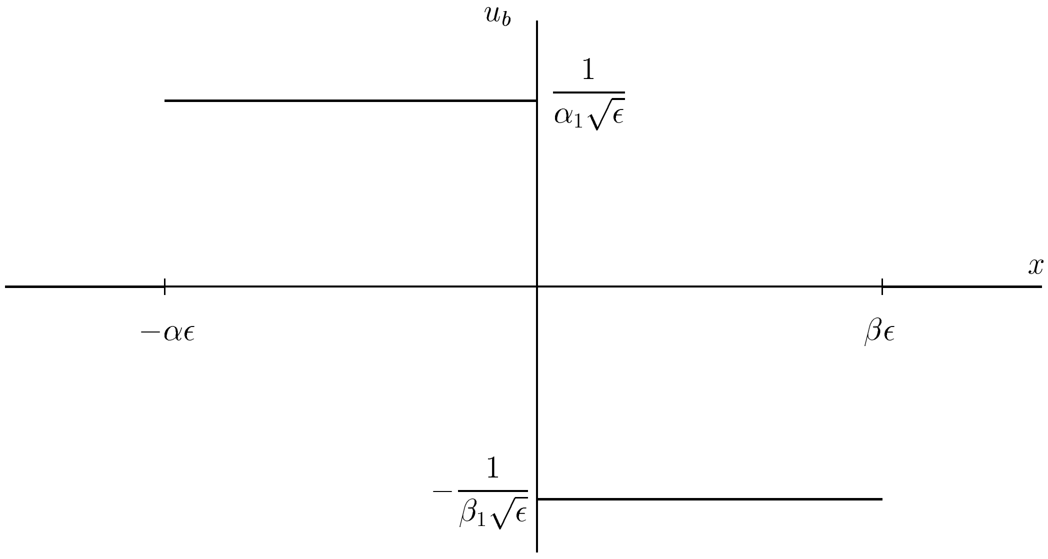

Figure 3: The Box Function u_b

where now

$$h(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

and

$$u_b = u_b^\epsilon(x) = \begin{cases} 0, & x \leq -\alpha\epsilon \\ 1/(\alpha_1\sqrt{\epsilon}), & -\alpha\epsilon < x \leq 0 \\ -1/(\beta_1\sqrt{\epsilon}), & 0 < x \leq \beta\epsilon \\ 0, & \beta\epsilon < x. \end{cases}$$

We sketch u_b in Figure 3. As usual, $U_1 \in Q_7(U_0)$, s and a satisfy (2) and (3), $a > 0$, and the constants which appear in the definition of u_b are determined by

$$\beta_1 = \frac{s - u_0}{u_1 - u_0}, \quad \alpha_1 + \beta_1 = 1, \quad \alpha = \alpha_1^3, \quad \beta = \beta_1^3. \tag{22}$$

By Lemma 1, $0 < \beta_1 < 1$; the same bounds hold for α_1, α and β.

Substituting (21) in (1), the error is again defined by (11), (12) and (13).

Write $u^\epsilon = h_1 + au_b$, with $h_1 = u_0 + [u]h(\xi - s)$; note that $h_1^2 = u_0^2 + [u^2]h$.

Similarly, $v^\epsilon = h_2 + a^2 u_b^2$. Powers of u_b and products of u_b with h_1 are equally easy to compute; for example

$$h_1 u_b = \begin{cases} 0, & x \le -\alpha\epsilon \\ u_0/(\alpha_1\sqrt{\epsilon}), & -\alpha\epsilon < x \le 0 \\ -u_1/(\beta_1\sqrt{\epsilon}), & 0 < x \le \beta\epsilon \\ 0, & \beta\epsilon < x. \end{cases}$$

The distributional calculus gives

$$Dh_1 = [u]\delta_s$$

where δ_s is the Dirac δ-function supported at s. Similarly,

$$Du_b = \frac{1}{\alpha_1\sqrt{\epsilon}}\delta_{s-\alpha\epsilon} - \left(\frac{1}{\beta_1} + \frac{1}{\alpha_1}\right)\frac{1}{\sqrt{\epsilon}}\delta_s + \frac{1}{\beta_1\sqrt{\epsilon}}\delta_{s+\beta\epsilon}.$$

We define some functionals which will be useful in the calculations below:

Definition 1 *Let M_{rs} be the distribution*

$$< M_{rs}, \psi >= \frac{1}{r-s}\int_s^r \psi'(\xi)\, d\xi;$$

let $M_- = M_{s-\alpha\epsilon,s}$ and $M_+ = M_{s+\beta\epsilon,s}$.

Then we have

Lemma 2 *For any r and s,*

$$\delta_r - \delta_s = (r-s)M_{rs}.$$

Corollary 1 *For any*

$$w = \begin{cases} 0, & x \le -\alpha\epsilon \\ A, & -\alpha\epsilon < x \le 0 \\ B, & 0 < x \le \beta\epsilon \\ 0, & \beta\epsilon < x \end{cases},$$

we have

$$Dw = A\alpha\epsilon M_- + B\beta\epsilon M_+.$$

The identity

$$\xi \delta_s = s\delta_s$$

is also valid in the sense of distributions. Then the computation of R_1 produces (after using (2) and (22))

$$\begin{aligned} R_1 &= \sqrt{\epsilon}\Big\{(2u_0 a - s)\alpha_1^2 M_- + (2u_1 a - s)\beta_1^2 M_+ \\ &\quad - \alpha_1^2 \delta_{s-\alpha\epsilon} - \beta_1^2 \delta_{s+\beta\epsilon}\Big\}, \end{aligned} \tag{23}$$

and R_1 is of order $\sqrt{\epsilon}$.

The computation of R_2 is similar to the $\mathcal{G}_s$ calculation: the most singular term vanishes only if

$$\int u_b^3 = 0,$$

which is equivalent to

$$\frac{\alpha}{\alpha_1^3} = \frac{\beta}{\beta_1^3}.$$

There is no loss of generality in letting the ratio be unity, since this just amounts to scaling ϵ. The identity in (3) is also necessary to cancel terms of order one, and another compatibility condition forces

$$\alpha_1(s - u_0) + \beta_1(s - u_1) = 0$$

which is the first condition in (22). (This replaces the conditions on h and ρ which were needed in Proposition 2.) Using these identities where necessary in the calculation, we obtain

$$\begin{aligned} R_2 &= \{-s[v] + \tfrac{1}{3}[u^3] - [u]\}\delta_s + \tfrac{1}{3}\frac{a^3}{\sqrt{\epsilon}}\{M_- - M_+\} \\ &\quad + a^2\{\alpha_1(u_0 - s)M_- + \beta_1(u_1 - s)M_+ + \alpha_1\delta_{s-\alpha\epsilon} + \beta_1\delta_{s+\beta\epsilon}\} \\ &\quad + a\sqrt{\epsilon}\{\alpha_1^2(u_0^2 - 1)M_- - \beta_1^2(u_1^2 - 1)M_+\}. \end{aligned}$$

We note that

$$\begin{aligned} < M_- - M_+, \psi > &= \frac{1}{\alpha\epsilon}\int_{s-\alpha\epsilon}^{s} (\psi'(\xi) - \psi'(s))\, d\xi \\ &\quad -\frac{1}{\beta\epsilon}\int_{s}^{s+\beta\epsilon} (\psi'(\xi) - \psi'(s))\, d\xi \\ &\equiv < M_1 - M_2, \psi > , \end{aligned}$$

where M_1 and M_2 are defined as the two integral distributions. Now,

$$< M_1, \psi >= \frac{1}{\alpha\epsilon} \int_{s-\alpha\epsilon}^{s} (s-\tau)\psi''(\tau)\, d\tau, \tag{24}$$

and a similar expression holds for M_2. These terms are thus of order ϵ, and so the a^3 term in R_2 is of order $\sqrt{\epsilon}$.

The coefficient of the a^2 term in R_2 is

$$\alpha_1(u_0 - s)M_1 + \beta_1(u_1 - s)M_2$$

and is $\mathcal{O}(\epsilon)$.

Thus R_2 takes the form

$$\begin{aligned} R_2 &= \tfrac{1}{3}\frac{a^3}{\sqrt{\epsilon}}\{M_1 - M_2\} + a^2\{\alpha_1(u_0 - s)M_1 + \beta_1(u_1 - s)M_2\} \\ &\quad + a\sqrt{\epsilon}\{\alpha_1^2(u_0^2 - 1)M_- - \beta_1^2(u_1^2 - 1)M_+\}. \end{aligned} \tag{25}$$

We summarize this result in

Proposition 3 *Let $U_1 \in Q_7(U_0)$ be given, and suppose s and a^2 are defined by (2) and (3). Let the box approximations u^ϵ and v^ϵ be given by (21), where the parameters in u_b are defined by (22). Then u^ϵ and v^ϵ are approximate solutions to (1) with error $(R_1/t, R_2/t)$, where R_1 and R_2 are given by (23) and (25).*

3 Convergence of Approximate Solutions

We shall write any of the approximations of Section 2 as U^ϵ, and define the error

$$E^\epsilon \equiv U_t^\epsilon + F(U^\epsilon)_x \,. \tag{26}$$

In this section we consider the convergence of U^ϵ and of E^ϵ as $\epsilon \to 0$. Although the functions U^ϵ depend on the approximation used, their limit, which lies in a space of measures, is well-defined.

For simplicity, we consider only the case that the approximate solution is a "pure" singular shock joining states U_0 and U_1. We use the previously defined quantities

$$s = \frac{[u^2] - [v]}{[u]}, \quad a^2 = s[v] - \tfrac{1}{3}[u^3] + [u] > 0,$$

where

$$U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}.$$

and $[U] = U_1 - U_0$. The solution is a self-similar singular shock with its singularity along the line $x = st$; hence the similarity variable is $x/t - s$. In this section we redefine

$$\xi = \frac{x}{t} - s.$$

In this section, we establish appropriate function spaces for the approximate solutions. These spaces are dual to spaces of test functions. The approximate solutions converge to singular objects – functionals – which are not equivalent to pointwise-defined functions. The function spaces are most easily described when U^ϵ is a $\mathcal{C}^\infty$ function as in Section 2.2, and we give the proofs in this case only. Analogous results for the asymptotic functions of Section 2.1 or box approximations of Section 2.3 can be established.

Section 3.1 will discuss ρ^ϵ, the most singular part of u^ϵ.

3.1 The spaces $\mathcal{F}_0$, $\mathcal{F}_0^*$, $\mathcal{F}_\alpha$, $\mathcal{F}_\alpha^*$

We consider the functions

$$\rho^\epsilon(\xi) = \frac{1}{\sqrt{\epsilon}} \rho \left(\frac{\xi}{\epsilon} \right).$$

In the limit, the functions are concentrated at $\xi = 0$; it is necessary to consider one-sided limits, $\xi \downarrow 0$ and $\xi \uparrow 0$. Thus, we construct spaces dual to test functions on $[0, 1]$ and $[-1, 0]$ respectively, and locate ρ^ϵ in a space which is a direct sum of these spaces. For the moment, consider the restriction of ρ^ϵ to $[0, 1] \equiv I$.

We use the standard definitions of $\mathcal{C}(I)$ with norm $\|f\| = \sup_{x \in I} |f(x)|$, and dual space $\mathcal{M}(I) = \mathcal{C}^*(I)$, the space of signed Radon measures, μ. For now, we omit the notation I for the interval of definition. If $f \in \mathcal{C}$ and $\mu \in \mathcal{M}$ then

$$< \mu, f >= \mu(f) \equiv \int_I f(x) \, d\mu$$

and $\|\mu\| = \sup_{f : \|f\| \le 1} |\mu(f)|$.

Now define

$$\mathcal{F}_0 = \{ f \mid f \in \mathcal{C}((0, 1]) \text{ and } \sqrt{x} f(x) \text{ extends to be continuous at } 0 \}$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{F}_0 \arrow[r, shift left, "\Phi"] \arrow[d, leftrightarrow, "{<,>_{\mathcal{F}_0}}"'] & \mathcal{C} \arrow[l, shift left, "\Phi^{-1}"] \arrow[d, leftrightarrow, "{<,>_{\mathcal{C}}}"] \\
\mathcal{F}_0^* \arrow[r, shift right, "(\Phi^*)^{-1}"'] & \mathcal{C}^* = \mathcal{M} \arrow[l, shift right, "\Phi^*"']
\end{tikzcd}
$$

Figure 4: The Commutative Diagram for the Function Spaces

and

$$\|f\|_{\mathcal{F}_0} = \sup_{0<x\leq 1} |\sqrt{x} f(x)|.$$

Define a map Φ by

$$\Phi(f)(x) = \sqrt{x} f(x).$$

Proposition 4 *$\mathcal{F}_0$ is a complete normed space and the map $\Phi : \mathcal{F}_0 \to \mathcal{C}$ is an isometry. Its adjoint, Φ^*, defines an isometric isomorphism from $\mathcal{C}^* = \mathcal{M}$ onto $\mathcal{F}_0^*$, the dual space of $\mathcal{F}_0$.*

Proof: It is simple to check that Cauchy sequences are convergent in $\mathcal{F}_0$; also

$$\|f\|_{\mathcal{F}_0} = \sup |\sqrt{x} f(x)| = \|\Phi(f)\|_{\mathcal{C}};$$

Φ gives a linear isometry, and $\Phi^{-1} : \mathcal{C} \to \mathcal{F}_0$ is defined by

$$\Phi^{-1}(h)(x) = h(x)/\sqrt{x} \in \mathcal{F}_0.$$

Now, Φ^* is defined by

$$< \Phi^* \mu, f >_{\mathcal{F}_0} = < \mu, \Phi f >_{\mathcal{C}}$$

(see Figure 4), and is an isometry. In particular, we have a useful representation for $\mathcal{F}_0^*$, since $\psi \in \mathcal{F}_0^*$ implies the existence of a $\mu = (\Phi^*)^{-1}\psi \in \mathcal{M}$ such that

$$< \psi, f >_{\mathcal{F}_0} = < \Phi^* \mu, f >_{\mathcal{F}_0} = < \mu, \Phi f >_{\mathcal{C}} = \int_I \sqrt{x} f(x)\, d\mu$$

for all $f \in \mathcal{F}_0$. ∎

Remark Integrable functions are naturally embedded in $\mathcal{F}_0^*$ as follows: if $\mu \in \mathcal{M}$ is a continuous measure, then $d\mu = g(x)\,dx$, where g is an integrable function; hence, if $\psi = \Phi^*\mu$, then

$$< \psi, f >_{\mathcal{F}_0} = \int \sqrt{x}\, f(x) g(x)\, dx = \int f(x)[\sqrt{x} g(x)]\, dx = \int f\, d\psi$$

so that ψ has a representation also as a *weighted measure*: $d\psi = g(x)\sqrt{x}\, dx$, with the same integrable function. If we use the Stieltjes representation $d\mu = d\alpha(x)$ where α is a function of bounded variation [13, page 110] for μ, then $d\mu = \sqrt{x} d\alpha(x)$. This motivates the terminology *space of weighted measures* for $\mathcal{F}_0^*$.

We now set the restriction of ρ^ϵ to the interval $[0, 1]$ in this space. We have

Proposition 5 *For every $\epsilon > 0$, ρ^ϵ defines an element of $\mathcal{F}_0^*$, and, as $\epsilon \to 0$, $\rho^\epsilon \rightharpoonup \rho^0$ in $\mathcal{F}_0^*$ weak-$*$, where*

$$< \rho^0, f >= \rho_+ \lim_{x \to 0} \sqrt{x} f(x)$$

and

$$\rho_+ = \int_0^1 \frac{\rho(x)}{\sqrt{x}}\, dx.$$

We also have the identification $(\Phi^*)^{-1}\rho^0 = \rho_+ \delta^0$, where δ^0 is the (one-sided) Dirac mass supported at $x = 0$:

$$< \delta^0, f >= f(0)$$

for all $f \in \mathcal{C}$.

Proof: We identify ρ^ϵ, as in the Remark, with an element of $\mathcal{F}_0^*$, which we also call ρ^ϵ, by the identity

$$< \rho^\epsilon, f >= \int_0^1 \rho^\epsilon(x) f(x)\, dx = \int_0^1 \left(\frac{\rho^\epsilon(x)}{\sqrt{x}} \right) (\sqrt{x} f(x))\, dx$$

for all $f \in \mathcal{F}_0$; by Hölder's inequality,

$$| < \rho^\epsilon, f > | \leq \sup_x |\sqrt{x} f(x)| \int_0^1 \frac{|\rho^\epsilon(x)|}{\sqrt{x}}\, dx.$$

Since the support of ρ is contained in $[0,1]$, let $y = x/\epsilon$ and then

$$| < \rho^\epsilon, f > | \le \|f\|_{\mathcal{F}_0} \int_0^1 \frac{|\rho(y)|}{\sqrt{y}}\, dy \le C \|f\|_{\mathcal{F}_0},$$

uniformly in ϵ. Clearly, $\rho^0 \in \mathcal{F}_0^*$. Now,

$$\begin{aligned} < \rho^\epsilon - \rho^0, f > &= \int_0^1 \frac{1}{\sqrt{\epsilon}} \rho\left(\frac{x}{\epsilon}\right) f(x)\, dx - \rho_+ \lim_{x\to 0} \sqrt{x} f(x) \\ &= \int_0^1 \frac{\rho(y)}{\sqrt{y}} \sqrt{\epsilon y} f(\epsilon y)\, dy - \rho_+ \lim_{x\to 0} \sqrt{x} f(x). \end{aligned} \tag{27}$$

As $\epsilon \to 0$, $\sqrt{\epsilon y} f(\epsilon y)$ converges pointwise to $\lim_{x\to 0} \sqrt{x} f(x)$. Hence the integrand is majorized by $G(y) = \|f\|_{\mathcal{F}_0} |\rho(y)|/\sqrt{y}$, so the Lebesgue dominated convergence theorem [13, page 37] shows $\lim_{\epsilon\to 0} < \rho^\epsilon - \rho^0, f > = 0$.

The final statement follows from the isometry of Proposition 4. ∎

The proof of Proposition 5 shows that the convergence is definitely weak-$*$ and not strong; from equation (27),

$$\|\rho^\epsilon - \rho^0\|_{\mathcal{F}_0^*} = \sup_{\|f\|_{\mathcal{F}_0} \le 1} \left| \int_0^1 \frac{\rho(y)}{\sqrt{y}} (\sqrt{\epsilon y} f(\epsilon y) - \lim_{x \downarrow 0} \sqrt{x} f(x))\, dy \right|$$

and for a given $\epsilon > 0$ one can find $f \in \mathcal{F}_0$ which makes the right-hand side arbitrarily large. The lack of strong convergence may not be important here, since even if the convergence were strong the limit functional, ρ^0, is not equivalent to a function and so expressions such as u^2 and u^3 cannot be evaluated. However, it is useful to identify the failure of compactness as caused by singularities in u rather than by oscillations.

In addition, $\mathcal{F}_0^*$ can be compactly embedded in another space as follows. Let $\mathcal{C}_\alpha(I)$ be the space of Hölder continuous functions with exponent α, where $0 < \alpha < 1$, with norm

$$\|f\|_{\mathcal{C}_\alpha} = \sup_x |f(x)| + \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

The following is easy to check.

Proposition 6 *For any $\alpha > 0$, the identity map embeds $\mathcal{C}_\alpha$ compactly in $\mathcal{C}$.*

$$
\begin{tikzcd}
\mathcal{F}_\alpha \arrow[r, "\Phi"] \arrow[d, leftrightarrow] & \mathcal{C}_\alpha \arrow[d, leftrightarrow] \\
\mathcal{F}_\alpha^* & \mathcal{C}_\alpha^* \arrow[l, "\Phi^*"']
\end{tikzcd}
$$

Figure 5: The Hölder and Weighted Hölder Spaces

We quote another result (see [16, p 100]):

Proposition 7 *If K is a compact linear operator from X to Y, then $K^* : Y^* \to X^*$ is also compact.*

Write $\mathcal{C}_\alpha^*$ as the dual space to $\mathcal{C}_\alpha$; then this proposition implies $\mathcal{M} \subset\subset \mathcal{C}_\alpha^*$.

We also define the weighted Hölder spaces, $\mathcal{F}_\alpha$, and their duals, $\mathcal{F}_\alpha^*$, as the (isometric) preimages and images of $\mathcal{C}_\alpha$ and $\mathcal{C}_\alpha^*$ under Φ and Φ^* respectively. See Figure 5.

Definition 2 *For $0 < \alpha < 1$, the space $\mathcal{F}_\alpha$ is defined as*

$$\mathcal{F}_\alpha = \{f \mid \Phi f \in \mathcal{C}_\alpha\},$$

and

$$\|f\|_{\mathcal{F}_\alpha} = \|\Phi f\|_{\mathcal{C}_\alpha} = \sup_x |\sqrt{x} f(x)| + \sup_{x \neq y} \frac{|\sqrt{x} f(x) - \sqrt{y} f(y)|}{|x-y|^\alpha}.$$

The following three propositions have straightforward proofs, which we omit.

Proposition 8 *$\mathcal{F}_\alpha$ is a Banach space and Φ an isometry from $\mathcal{F}_\alpha$ onto $\mathcal{C}_\alpha$; Φ^* is an isometric isomorphism of $\mathcal{F}_\alpha^*$ onto $\mathcal{C}_\alpha^*$.*

Proposition 9 *The identity map embeds $\mathcal{F}_\alpha$ compactly into $\mathcal{F}_0$ and $\mathcal{F}_0^*$ compactly into $\mathcal{F}_\alpha^*$.*

Proposition 10 *The sequence ρ^ϵ converges to ρ^0 in the norm of $\mathcal{F}_\alpha^*$.*

To consider convergence of derivatives of u^ϵ, we study $d\rho^\epsilon/d\xi$. Define

$$\mathcal{F}_1 \equiv \{f \mid f \in \mathcal{F}_0 \text{ and } \frac{df}{d\xi} \in \mathcal{F}_0\}$$

with $\|f\|_{\mathcal{F}_1} = \|f\|_{\mathcal{F}_0} + \|df/d\xi\|_{\mathcal{F}_0}$. For any ϵ, including zero, we have

$$\left\langle \frac{d}{d\xi}\rho^\epsilon, f \right\rangle = -\left\langle \rho^\epsilon, \frac{df}{d\xi} \right\rangle \tag{28}$$

and hence $d\rho^\epsilon/d\xi \in \mathcal{F}_1^*$, where $\mathcal{F}_1^*$ is the dual space of $\mathcal{F}_1$. We also define $\mathcal{F}_{1+\alpha}^*$ as the space dual to $\mathcal{F}_{1+\alpha}$, where

$$\begin{aligned} \mathcal{F}_{1+\alpha} &= \{f \mid f \in \mathcal{F}_\alpha \text{ and } \tfrac{df}{d\xi} \in \mathcal{F}_\alpha\}, \\ \|f\|_{1+\alpha} &= \|f\|_{\mathcal{F}_\alpha} + \|\tfrac{df}{d\xi}\|_{\mathcal{F}_\alpha}. \end{aligned}$$

Proposition 11 *The sequence $d\rho^\epsilon/d\xi$ converges weak-$*$ in $\mathcal{F}_1^*$ and strongly in $\mathcal{F}_{1+\alpha}^*$ for any $\alpha > 0$.*

Proof: This follows from the definition (28) of the weak derivative and from compactness and isometry. ∎

Remark The Hölder spaces $\mathcal{C}_{1+\alpha}$ and $\mathcal{C}_{1+\alpha}^*$ are standard; however they are not the isomorphic images and preimages of $\mathcal{F}_{1+\alpha}$ and $\mathcal{F}_{1+\alpha}^*$, even for $\alpha = 0$, since differentiation does not commute with Φ.

3.2 Spaces for u^ϵ and v^ϵ

To consider limits of functions ρ^ϵ defined on $[-1, 1]$, write $\rho^\epsilon(\xi) = \rho_+^\epsilon(\xi) + \rho_-^\epsilon(\xi)$, where

$$\rho_+^\epsilon(\xi) = \begin{cases} \rho^\epsilon(\xi) & \xi \geq 0 \\ 0 & \xi < 0 \end{cases}$$

and $\rho_-^\epsilon = \rho^\epsilon - \rho_+^\epsilon$. Now ρ_-^ϵ is defined on $\mathcal{F}_\alpha^*([-1, 0])$, the dual space to $\mathcal{F}_\alpha([-1, 0])$; for any α, $0 \leq \alpha < 1$, $\mathcal{F}_\alpha([-1, 0])$ is isometrically isomorphic to $\mathcal{C}_\alpha([-1, 0])$ under the isometry $\tilde{\Phi}(f)(x) = \sqrt{-x}f(x)$. For the $\mathcal{G}_s$ approximations of Section 2,

$$\rho^\epsilon \in \mathcal{F}_\alpha^*([-1, 0]) \oplus \mathcal{F}_\alpha^*([0, 1]) \equiv \mathcal{N}_\alpha.$$

The space $\mathcal{N}_\alpha$ is dual to $\mathcal{F}_\alpha([-1, 0]) \oplus \mathcal{F}_\alpha([0, 1])$.

Proposition 12 *For any of the approximations of Section 2, $u^\epsilon \in \mathcal{N}_\alpha$ for any $0 \le \alpha < 1$.*

Proof: We give the proof for the $\mathcal{G}_s$ approximations,

$$u^\epsilon = [u]h^\epsilon + u_0 + a\rho^\epsilon,$$

where $[u] = u_1 - u_0$, $h^\epsilon = h(\xi/\epsilon^p)$ and $h = h_+ + h_-$ is an approximation to the Heaviside function. Write $u^\epsilon = u_+^\epsilon + u_-^\epsilon$. We compute $\|u_+^\epsilon\|_{\mathcal{F}_0^*}$ from

$$< u_+^\epsilon, f >= \int_0^1 ([u]h^\epsilon + u_0) f(x)\, dx + a \int_0^1 \rho^\epsilon f\, dx.$$

The first term is bounded by

$$\left| \int_0^1 \frac{[u]h^\epsilon + u_0}{\sqrt{x}} \sqrt{x} f(x)\, dx \right| \le \|f\|_{\mathcal{F}_0} \int_0^1 \frac{|u_1|h^\epsilon + |u_0|(1-h^\epsilon)}{\sqrt{x}}\, dx.$$

For all ϵ, $0 \le h^\epsilon \le 1$, so this expression is bounded by

$$\|f\|_{\mathcal{F}_0^*} \left[(|u_1| + |u_0|) \int_0^1 \frac{1}{\sqrt{x}}\, dx \right] = 2(|u_1| + |u_0|)\|f\|_{\mathcal{F}_0^*};$$

hence

$$\|u_+^\epsilon\|_{\mathcal{F}_0^*} \le 2(|u_1| + |u_0|) + |a| \int_0^1 \frac{|\rho(y)|}{\sqrt{y}}\, dy,$$

using Proposition 5. Similarly, (with $\mathcal{F}_0^* = \mathcal{F}_0^*([-1, 0])$ now),

$$\|u_-^\epsilon\|_{\mathcal{F}_0^*} \le 2(|u_1| + |u_0|) + |a| \int_{-1}^0 \frac{|\rho(y)|}{\sqrt{-y}}\, dy,$$

and so $u^\epsilon \in \mathcal{N}_0$ with a norm which is uniformly bounded, independent of ϵ. Now, $\mathcal{N}_0 \subset\subset \mathcal{N}_\alpha$, $0 < \alpha < 1$, as a consequence of Proposition 9, so $u^\epsilon \in \mathcal{N}_\alpha$, $0 < \alpha < 1$. ∎

Proposition 13 *As $\epsilon \to 0$, $u^\epsilon \rightharpoonup u$ weak-$*$ in $\mathcal{N}_0$, where*

$$u = [u]h^0 + u^0 + a\,\rho^0 \tag{29}$$

and h^0, u^0 *and* $\rho^0 \in \mathcal{N}_0$ *are defined by*

$$\begin{aligned} < h^0, f > &= \int_0^1 f(x)\,dx \\ < u^0, f > &= u_0\{\int_{-1}^0 f(x)\,dx + \int_0^1 f(x)\,dx\} \\ < \rho^0, f > &= \rho_+ \lim_{x\downarrow 0} \sqrt{x} f(x) + \rho_- \lim_{x\uparrow 0} \sqrt{-x} f(x) \end{aligned}$$

and

$$\rho_- = \int_{-1}^0 \frac{\rho(y)}{\sqrt{-y}}\,dy.$$

Proof: From the direct sum decomposition of $\mathcal{N}_0$, we can work separately on the intervals $[-1,0]$ and $[0,1]$, with $f(x) = f_+(x) + f_-(x)$; each $f_i \in \mathcal{F}_0$ and so limits of $\sqrt{x} f_+(x)$ and $\sqrt{-x} f_-(x)$ exist. The computation on $[0,1]$ has been done for ρ_+^ϵ in Proposition 5, while the weak-$*$ limits of $h_\pm^\epsilon$, for any scaling factor p, and of $u_\pm^0$ are clear. Finally, we have

$$\begin{aligned} < \rho_-^\epsilon, f_- > &= \int_{-1}^0 \frac{1}{\sqrt{\epsilon}} \rho(\frac{x}{\epsilon}) f(x)\,dx = \sqrt{\epsilon} \int_{-1}^0 \rho(y) f(\epsilon y)\,dy \\ &= \int_{-1}^0 \frac{\rho(y)}{\sqrt{-y}} \sqrt{-\epsilon y} f(\epsilon y)\,dy \\ &\to \rho_- \lim_{x\uparrow 0} \sqrt{-x} f(x) \end{aligned}$$

as in the proof of Proposition 5.

The integrals in Proposition 13 are bounded since $f(x)$ is absolutely integrable. ∎

We now consider v^ϵ, given by

$$v^\epsilon = [v] h^\epsilon + v_0 + a^2 (\rho^\epsilon)^2.$$

Since $(\rho^\epsilon(x))^2 = \rho^2(x/\epsilon)/\epsilon$, and $\int_{-1}^1 \rho^2 = 1$, v^ϵ is an element of $\mathcal{M} = \mathcal{M}([-1,1]) = \mathcal{C}^*([-1,1])$. Alternatively, we may consider a direct sum of spaces $\mathcal{M}([-1,0]) \oplus \mathcal{M}([0,1])$. In the first case, $(\rho^\epsilon)^2 \rightharpoonup \delta^0$, the Dirac mass at 0; in the second,

$$(\rho^\epsilon)^2 \rightharpoonup (\int_{-1}^0 \rho^2)\delta_-^0 + (\int_0^1 \rho^2)\delta_+^0,$$

a weighted sum of one-sided Diracs. The first, which we will use for the remainder of this paper, is simpler. However, the second is a better analogue to the space for u^ϵ, and is required for the definition of nonuniform singular shocks. This will be discussed in a future paper, [7].

Proposition 14 *As $\epsilon \to 0$,*

$$v^\epsilon \rightharpoonup [v]h^0 + v^0 + a^2\,\delta^0 \tag{30}$$

weak-$*$ on $\mathcal{M}$, where h^0 and δ^0 are the Heaviside and Dirac distributions, respectively and v^0 the constant function. Furthermore, v^ϵ converges strongly in $\mathcal{C}^*_\alpha([-1,1])$ for any α, $0 < \alpha < 1$, to the limit in (30). The derivatives $dv^\epsilon/d\xi$ converge weak-$*$ and strongly in $\mathcal{C}^*_1([-1,1])$ and in $\mathcal{C}^*_{1+\alpha}([-1,1])$ respectively.

The proof is a routine computation.

Thus, we have shown that the approximations U^ϵ and their derivatives converge in spaces of measures. We summarize Proposition 13 and Proposition 14 in the following convergence theorem, which is the main result of this paper.

Theorem 1 *For any of the approximate solutions of Section 2, $u^\epsilon \rightharpoonup u$ weak-$*$ in $\mathcal{N}_0$ and strongly in $\mathcal{N}_\alpha$, for any $\alpha > 0$; and $v^\epsilon \rightharpoonup v$ weak-$*$ in $\mathcal{M}$ and strongly in $\mathcal{C}^*_\alpha$.*

3.3 Weak and Strong Limits of the Error

From (26), we define the two components of E^ϵ,

$$\begin{aligned} E^\epsilon_u &= \partial_t u^\epsilon + \partial_x((u^\epsilon)^2 - v^\epsilon) \\ E^\epsilon_v &= \partial_t v^\epsilon + \partial_x(\tfrac{1}{3}(u^\epsilon)^3 - u^\epsilon). \end{aligned}$$

We begin with the observation:

Proposition 15 *If $t \geq t_0 > 0$, then E^ϵ_u is uniformly bounded in $\mathcal{C}^*_1([-1,1])$, and E^ϵ_v is uniformly bounded in $\mathcal{C}^*_2([-1,1])$.*

Proof: We expect E^ϵ_u to be bounded in $\mathcal{M}_1$, and E^ϵ_v in a larger dual space, because of the presence of the terms $(u^\epsilon)^2$ and v^ϵ in the first expression and

$(u^\epsilon)^3$ in the second. However, there is cancellation between $(u^\epsilon)^2$ and v^ϵ in E_u^ϵ and self-cancellation of $(u^\epsilon)^3$ in E_v^ϵ — specifically between the contributions of u_+^ϵ and u_-^ϵ in the latter case. These cancellations were established in Section 2, and this Proposition simply restates the results of Propositions 1, 2, and 3 in the function space language of Section 3.

Finally, we have

Theorem 2 *The error terms in any of the approximations of Section 2 converge as follows:*

$$E_u^\epsilon \rightharpoonup 0 \textit{ weak-}* \textit{ in } \mathcal{C}_1^*([-1,1])$$

$$E_v^\epsilon \rightharpoonup 0 \textit{ weak-}* \textit{ in } \mathcal{C}_2^*([-1,1]).$$

The error converges strongly to zero in the norms of $\mathcal{C}_{1+\alpha}^*$ and $\mathcal{C}_{2+\alpha}^*$ respectively.

Proof: We give the calculation for the box functions (using R_1, from (23), and R_2, from (25)), which is especially simple. From Definition 1, we have (with $\|\cdot\|_\infty$ the norm on $\mathcal{C}$):

$$|<M_-,\psi>| \leq \|\psi'\|_\infty$$

and similarly for M_+, so

$$|<R_1,\psi>| \leq \sqrt{\epsilon}\left\{c_1\|\psi\|_\infty + c_2\|\psi'\|_\infty\right\},$$

where c_1 and c_2 are constants determined by the Riemann data U_0 and U_1. From (24) we get the bound

$$|<M_1,\psi>| \leq \alpha\epsilon\|\psi''\|_\infty$$

and hence (c_3 and c_4 also depend only on U_0 and U_1)

$$|<R_2,\psi>| \leq \tfrac{1}{3}a^3\sqrt{\epsilon}\|\psi''\|_\infty + a^2\epsilon c_3\|\psi''\|_\infty + a\sqrt{\epsilon}c_4\|\psi'\|_\infty.$$

Thus we obtain weak convergence in the indicated spaces, and, from Propositions 6 and 7, strong convergence in the dual Hölder spaces. ∎

4 Entropy and Admissibility Conditions

This section discusses admissibility of singular shock solutions. As we showed in [9] and [10], the Riemann problem for (1) has a unique solution in the class of shocks, singular shocks and rarefaction waves for all initial data pairs U_0, U_1 if and only if singular shocks joining a point U to points in the set $Q_7(U)$ and its boundary, and no other singular shocks, are allowed. Thus a demonstration that these shocks are distinguished from other objects (u, v) of the form (29), (30), with $u \in \mathcal{N}_0$ and $v \in \mathcal{M}$, satisfying (2) and (3), but with $U_1 \notin Q_7(U_0)$, would complete the discussion of the Riemann problem begun in [9].

One distinguishing feature of the singular shocks with $U_1 \in Q_7(U_0)$ is that they are overcompressive. (The states on the boundary components D and E are only weakly so, since one characteristic speed is equal to the shock speed. The result we present below will be given for states U_1 in the interior of $Q_7(U_0)$, but can be modified to include the case that $U_1 \in D \cup E$.)

A second feature of the singular shocks is that the singular measure carries a particular sign, so that replacing, say, ρ^ϵ by $-\rho^\epsilon$ produces a different shock, which we would like to term inadmissible.

Since admissibility criteria based on parabolic regularizations of the underlying equations (Dafermos-DiPerna type viscosity) motivated these restrictions, we consider alternative criteria in this section.

There are several established techniques for justifying admissibility conditions. Examining linearized stability of the uniform shock to perturbations of Cauchy data gives the immediate result that only if the shock is overcompressive does one obtain a well-posed boundary-value problem without a Rankine-Hugoniot condition. This observation does not depend on the nature of the singular shock. It is possible to show that one actually gets a smooth evolution of the singular shock by giving a careful definition of nonuniform singular shocks. This is done in a separate paper, [7].

Within the framework of inviscid perturbation, one can approximate shocks by smooth profiles, and then admissible shocks are those which are limits of compressive waves; the others dissipate into rarefactions under time evolution. In [6], it is shown that if the ϵ-approximate solutions are taken as initial data for the hyperbolic problem, then the candidates for admissible waves do steepen, while other choices (for example with a taken to be negative) do not.

In this section, we use a third method of studying admissibility in the inviscid equations: the construction of entropy functions for the original system, (1). We show that approximate solutions with $U_1 \in Q_7(U_0)$ satisfy an entropy inequality.

4.1 A Convex Approximate Entropy Function

It can be shown, using the geometric optics construction in [12], that convex entropy functions for system (1) exist when U is in any bounded subset of $\mathbf{R}^2$, or, in fact, in any set in which v is bounded above. A convex entropy for our class of approximate solutions, in which $\sup v \to +\infty$ as $\epsilon \to 0$, does not exist.

However, we justify the geometric admissibility condition (4) for singular shocks by introducing an approximate entropy function and deriving an entropy-like inequality which is satisfied by viscous approximations to singular shocks. We prove that this inequality, when applied to a similarity solution of (5) implies our geometric admissibility condition.

For each positive number k, define the *approximate entropy function*

$$\eta(u, v) = \exp[k(u^2/2 \pm u - v)]$$

and the corresponding entropy-flux function

$$q(u, v) = (u \pm 1)\eta.$$

Observe that $\eta(U)$ is convex for all $U \in \mathbf{R}^2$, and that if U is a differentiable solution of (1), then

$$\partial_t \eta(u, v) + \partial_x q(u, v) = \eta u_x.$$

For large k the terms on the left-hand side of this equation are $\mathcal{O}(k)$ while the right-hand side is $\mathcal{O}(1)$; this justifies calling η an approximate entropy function.

Suppose now that $U^\epsilon = (u^\epsilon, v^\epsilon)$ is a solution of the Dafermos-DiPerna viscosity approximation (5). Let $\hat{\eta} = \eta(U^\epsilon)$ and $\hat{q} = q(U^\epsilon)$. The usual entropy inequality is replaced by

$$\hat{\eta}_t + \hat{q}_x \leq \epsilon t \hat{\eta}_{xx} + \hat{\eta} u_x.$$

We shall use the integrated form:

$$\int_{x_1}^{x_2} (\hat{\eta}_t + \hat{q}_x) dx \leq \epsilon t \int_{x_1}^{x_2} \hat{\eta}_{xx} dx + \int_{x_1}^{x_2} \hat{\eta} u_x dx. \tag{31}$$

Now choose $k = 1/\epsilon$. Since each integrand in (31) contains an exponential factor of the form $\exp[-\phi(x)/\epsilon]$, where $\phi(x) = \phi_\pm(x) = v - u^2/2 \mp u$, and the coefficients of these factors are all bounded by powers of $1/\epsilon$, the behavior of the integrals when ϵ is small can be determined by the method of steepest descent. The dominant terms in any interval (x_1, x_2) come from the point(s) x at which $\phi(x)$ is smallest, and all remaining terms will be less by a factor of $\mathcal{O}(\epsilon)$.

How does ϕ behave on $\mathbf{R}$? If U^ϵ is a self-similar solution of (5), a direct calculation shows that $\phi_{xx} < 0$ whenever $\phi_x = 0$. Therefore $\phi(x)$ cannot have a local minimum at any finite value of x; either it is strictly monotone for all x, or else there is a separating point x_0 such that $\phi(x)$ is strictly increasing for $-\infty < x < x_0$ and strictly decreasing for $x_0 < x < +\infty$. Monotone behavior may (and usually does) occur for regular shocks, but for singular shocks there is a separating point.

To see this, suppose that U^ϵ is an approximation to a singular shock profile, which means that $U^\epsilon(-\infty) = U_0$, $U^\epsilon(+\infty) = U_1$, and

$$\lim_{\epsilon \to 0} = \begin{cases} U_0 & x < st \\ U_1, & x > st \end{cases} \tag{32}$$

but $U^\epsilon(x,t)$ does not remain uniformly bounded as $\epsilon \to 0$. Since the values of ϕ at $x = \pm\infty$ are independent of ϵ, a function $\phi(x)$ which was strictly monotone would have to be uniformly bounded. Neither of the $\phi_\pm(x)$ can be uniformly bounded, however, because for small ϵ the asymptotic form U_A of the approximate solution U^ϵ requires that v^ϵ be of order ϵ^{-2} and u^ϵ of order ϵ^{-1} in such a manner that the difference $v^\epsilon - (u^\epsilon)^2$ is only $\mathcal{O}(\epsilon^{-1})$. Therefore $v - u^2/2$ must be of order ϵ^{-2}, and must dominate $\phi_\pm$.

Thus in a singular shock the separating point x_0, where ϕ has its maximum, must exist for each sufficiently small positive ϵ; since $U^\epsilon(x_0)$ becomes unbounded as $\epsilon \to 0$, we see that x_0 approaches st, because of (32). So if we apply inequality (31) with one endpoint of integration at $x = st$ and the other at $x = (s \pm \delta)t$ for a fixed $\delta > 0$, the dominant terms come from the endpoint $x = (s \pm \delta)t$. We can then derive four inequalities, corresponding to the two choices of interval and the two choices of sign in $\phi_\pm(x)$.

We proceed as follows. On the left-hand side of (31), the self-similarity of U^ϵ implies that $\hat{\eta}_t = -(x/t)\hat{\eta}_x = [-s + \mathcal{O}(\delta)]\hat{\eta}_x$. Therefore

$$\int_{x_1}^{x_2} (-s\hat{\eta}_x + \hat{q}_x)dx \le \epsilon t \int_{x_1}^{x_2} \hat{\eta}_{xx} dx + \int_{x_1}^{x_2} \hat{\eta} u_x dx + \mathcal{O}(\delta) \int_{x_1}^{x_2} |\hat{\eta}_x| dx.$$

Integrating, we get

$$(-s\hat{\eta} + \hat{q})|_{x_1}^{x_2} \le \epsilon t \hat{\eta}_x|_{x_1}^{x_2} + \int_{x_1}^{x_2} \hat{\eta} u_x dx + \mathcal{O}(\delta) \int_{x_1}^{x_2} |\hat{\eta}_x| dx,$$

which we evaluate by steepest descent. For the interval $[(s-\delta)t, st]$, the principal contribution comes from the left endpoint x_1 and yields the estimate

$$s - (u(x_1) \pm 1) \le t\phi_x(x_1) + \epsilon[u_x(x_1) + \mathcal{O}(\delta)] + \mathcal{O}(\epsilon)$$

after division by the common exponential factor $\exp[-\phi(x_1)/\epsilon]$. Let $\epsilon \to 0$ for a fixed δ. Since $x_1 < st$, the limiting profile (32) shows that the derivatives ϕ_x and u_x on the right-hand side approach zero, while $u(x_1)$ converges to u_0. Thus we obtain two inequalities:

$$s - (u_0 \pm 1) \le 0.$$

The other interval $[st, (s+\delta)t]$ leads in the same fashion to two more inequalities:

$$-s + (u_1 \pm 1) \le 0.$$

Observing that $u \pm 1$ are the characteristic speeds $\lambda(U)$ of (1), we combine these four inequalities:

$$\lambda_2(U_0) > \lambda_1(U_0) \ge s \ge \lambda_2(U_1) > \lambda_1(U_1).$$

In other words, singular shocks which are limits of Dafermos-DiPerna viscosity solutions must satisfy the condition of *overcompressiveness.* This condition is the same as the geometric admissibility condition (4) and is equivalent to $U_1 \in Q_7(U_0)$.

We have proved the following theorem.

Theorem 3 *Let U_0 and U_1 be the left and right states of a self-similar solution $U^\epsilon(x,t)$ of the Dafermos-DiPerna equation (5) which is not uniformly bounded and which approaches a constant as $\epsilon \to 0$ except on the line $x = st$. Then $U_1 \in Q_7(U_0)$ and U_0 and U_1 may be joined by an admissible singular shock.*

5 Conclusions

This paper completes a program, begun in [9], to study shocks in a hyperbolic model equation with no classical Riemann solution. We have shown that several approximation methods provide, consistently, approximate solutions to the system, and we have now established that these approximations converge, and have described their limits. Many questions remain open. Perhaps the most important is to give a sense in which the limit measures satisfy the equation. We note that the singular nature of these weak solutions is quite different from the singularities arising from, say, weak convergence of oscillatory approximations (as in dispersive approximations to hyperbolic problems).

Another interesting question is extension of the results here to some class of equations beyond the single model. One such class consists of non-strictly hyperbolic conservation laws with cubic flux functions and 'same variation' at the coincidence locus, studied in [15]. Because the nature of the singularity is identical in the two cases, the nonstrictly hyperbolic shocks are conjectured to have the same limiting behavior.

Beyond that, the class of hyperbolic problems with this behavior is likely to be quite limited. Since the approximate solutions have large amplitude, the nature of the approximate solutions and of the convergence is closely tied to the limits of the fluxes at infinity. Thus the only case to which this approach would generalize is one in which the flux functions scale appropriately for all wave-amplitudes larger than the moderate one at which the wave speeds begin to overlap. The key would seem to be producing a singular ordinary differential equation analogous to (6) at large amplitudes; that is, an equation with a nonhyperbolic rest point which admits a homoclinic orbit. It would be interesting to study the relation of such equations to the phenomenon of 'same variation', which is also a feature defined with reference to large amplitude waves.

Examining the solutions we have constructed leads to a few conclusions. First, the role of the Rankine-Hugoniot equations in describing weak solutions is clarified: this relation applies only to solutions which are piecewise smooth or at least have this property almost everywhere. For problems in a single space dimension, most physically interesting equations have solutions in this class. However, the nature of singularities in more than one dimension is still an open question, and the singular solutions presented here may be relevant.

We have not given a sense in which the limiting measure satisfies the equation. The theory of generalized distributions of Colombeau, [1], [3], which motivated the introduction of the functions in Section 2.2, does not answer this question. The difficulty is that, even for classical weak solutions, the equation is satisfied only in the sense of association; hence the assertion that functions in $\mathcal{G}_s$ satisfy the equation does not hold in the strongest sense, and taking a limit in ϵ is not justified within that theory. In fact, there are numerous examples of generalized distributions which satisfy equations formally but in a completely vacuous sense (see [3]). Theorem 2, which is independent of the $\mathcal{G}_s$ theory, shows that our functions are true approximations to the equation.

6 Acknowledgements

This research was supported by the NSF, AFOSR and DOE, and by grants from the Texas Advanced Research Program, the Energy Laboratory at the University of Houston and the President of Adelphi University. We would also like to thank Vern Paulsen who helped us to sort out the function spaces in Section 3, and Jean-François Colombeau, who clarified the role of generalized distributions in this problem. We extend special thanks to the Institute for Mathematics and its Applications at the University of Minnesota, where this paper was begun during a visit by both authors, and to the Fields Institute for Research in the Mathematical Sciences, which hosted the first author while the research was completed.

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