Free Boundary Problems for Nonlinear Wave Equations: Interacting Shocks

Barbara Lee Keyfitz[a]

Department of Mathematics

University of Houston

Houston, Texas 77204-3008

joint work with Sunčica Čanić and Eun Heui Kim

computations by Alexander Kurganov

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Similarity Analysis of Two-Dimensional Systems

$$U_t + F(U)_x + G(U)_y = 0, \quad U \in \mathbb{R}^n$$

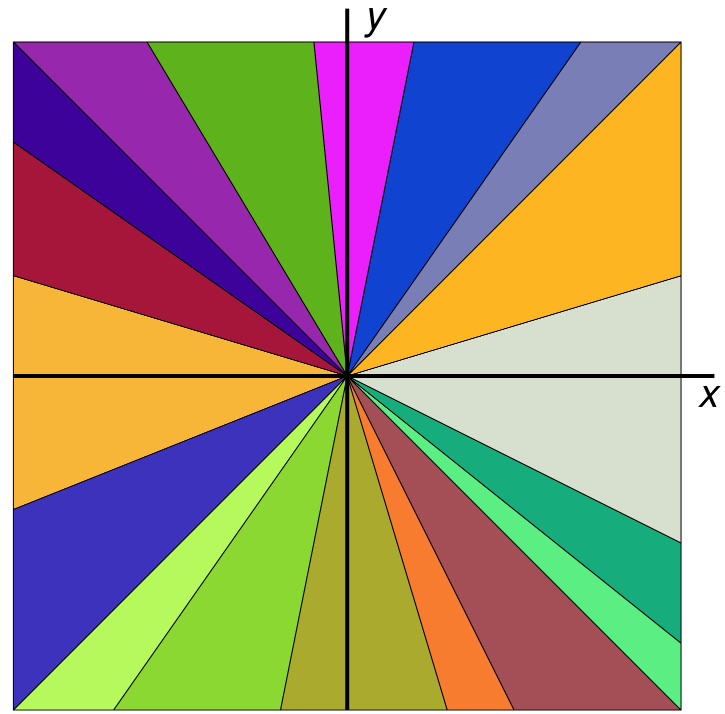
Data: $U(x, y, 0) = f\left(\frac{x}{y}\right)$

Similarity Variables:

$$\xi = \frac{x}{t}, \eta = \frac{y}{t} \qquad U = U(\xi, \eta)$$

Reduced System in Two Variables

$$\partial_\xi (F - \xi U) + \partial_\eta (G - \eta U)$$
$$\equiv \widetilde{F}_\xi + \widetilde{G}_\eta = -2U$$



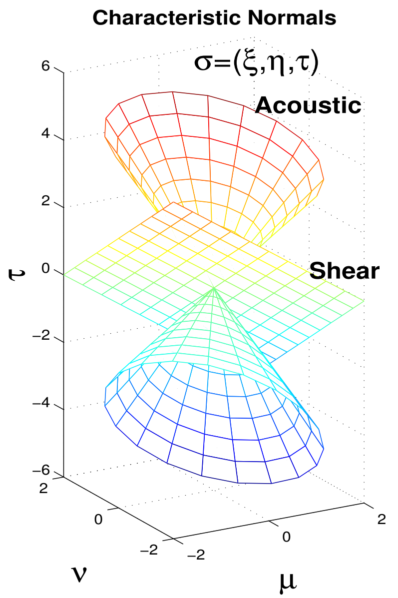
Sectorially Constant Data

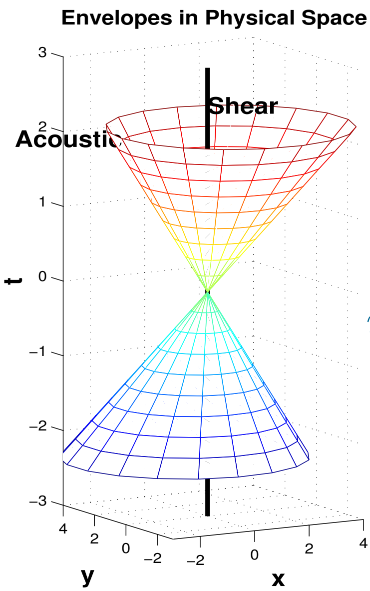
Method: resolve 1-D far-field discontinuities; solve as IV/BVP in 2-D

Type Changes: hyperbolic in far field; ‘subsonic’ region near origin

Acoustic-type Structure: Degenerate/Non-degenerate Characteristics

$$U_t + AU_x + BU_y = 0; \quad \det|I\tau + A\lambda + B\mu| = \left(\prod_{i=1}^{n-2} \ell_i \cdot \sigma\right)\sigma^T Q_N \sigma$$





$$\sigma = (\tau, \lambda, \mu)$$

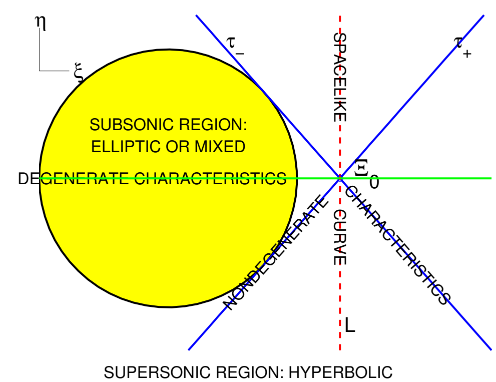
$$((A - \xi I)\partial_\xi + (B - \eta I)\partial_\eta)U = 0$$

$\Xi = (\xi, \eta)$ dual vector $\vec{\alpha} = (\alpha, \beta)$

$$\prod_{i=1}^{n-2} \ell_i \cdot (-\vec{\alpha} \cdot \Xi, \alpha, \beta)\, \underbrace{q(\sigma(\vec{\alpha}, \Xi), U)}_{\tilde{q}(\vec{\alpha}, \Xi, U)}$$

CHANGE OF TYPE THEOREM *Reduced equation hyperbolic* iff $x = (1, \xi, \eta)$ *outside acoustic wave cone* $\mathcal{C}_W = \{x^T Q_N^{-1} x = 0\}$.

RP in $2+1$ dim $\Rightarrow$ CP w. data at ∞



Far Field Solution and Wave Interactions

- 'Quasi-One-Dimensional' Riemann Problems in Hyperbolic Region
  - Shock-shock

    Rarefaction-rarefaction

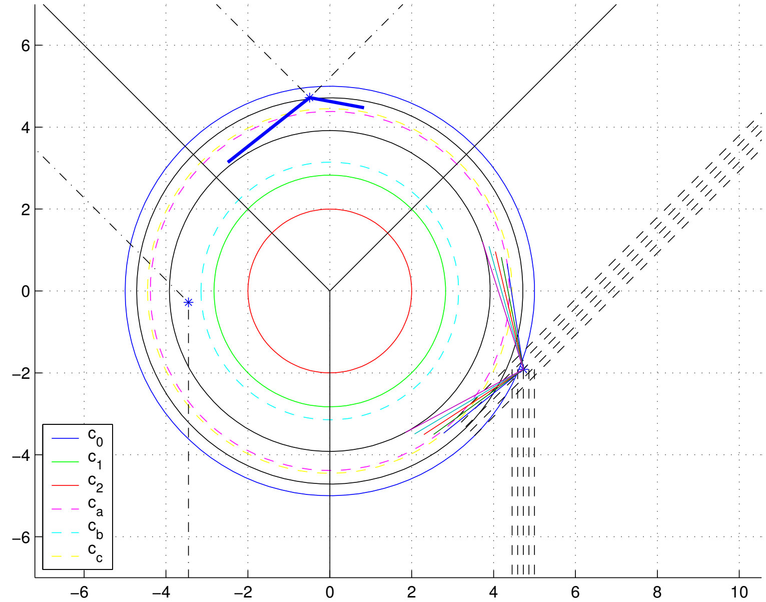
    Rarefaction-shock

  - No Q-1-D solns for some probs

    Mach stems

  - Sonic lines

    Sonic bdry not det. a priori



  - Current effort: examine simplified 'Nonlinear Wave System' — Nonlinear and Linear parts decouple & Linear waves stationary

Comparison: Isentropic Gas Dynamics & NLWS

Isentropic Gas Dynamics: $p = \rho^\gamma/\gamma$

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0$$
$$(\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0$$

Nonlinear Wave System:

$$\rho_t + m_x + n_y = 0$$
$$m_t + p_x = 0$$
$$n_t + p_y = 0$$

Second-order equation for nonlinear char. variable (ρ):

$$\big((c^2(\rho) - U^2)\rho_\xi - UV\rho_\eta\big)_\xi + \big((c^2(\rho) - V^2)\rho_\eta - UV\rho_\xi\big)_\eta + (Vu_\eta - Uv_\eta)\rho_\xi + (Uv_\xi - Vu_\xi)\rho_\eta + 2(v_\xi u_\eta - u_\xi v_\eta)\rho = 0$$

$U = u - \xi, \quad V = v - \eta$ ('pseudo-vel.')

$$\big((c^2(\rho) - \xi^2)\rho_\xi - \xi\eta\rho_\eta\big)_\xi + \big((c^2(\rho) - \eta^2)\rho_\eta - \xi\eta\rho_\xi\big)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0$$

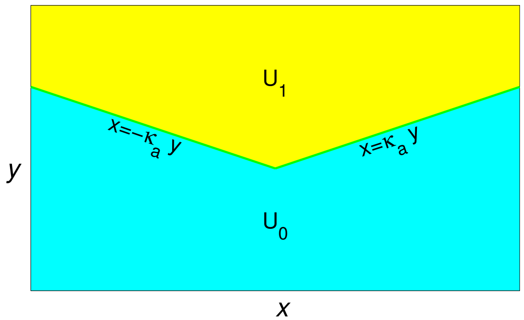
Transport equation for linear char. variable:

$$W = V_\xi - U_\eta = v_\xi - u_\eta$$
$$UW_\xi + VW_\eta + (U_\xi + V_\eta + 1)W = 0$$

$$w = n_\xi - m_\eta$$
$$(\xi, \eta) \cdot \nabla w + w = 0$$

or $\quad rm_r = p_\xi \qquad rn_r = p_\eta$

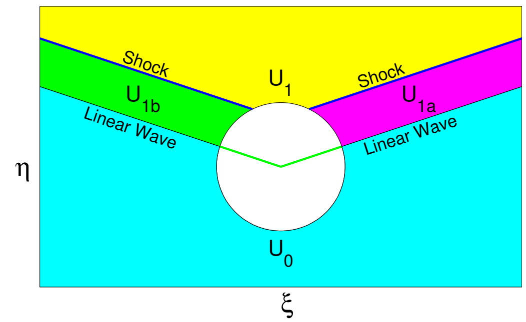
Converging Shocks: A Bifurcation Problem for NLWS



2-state data: U_0, U_1

Data give 2 shocks

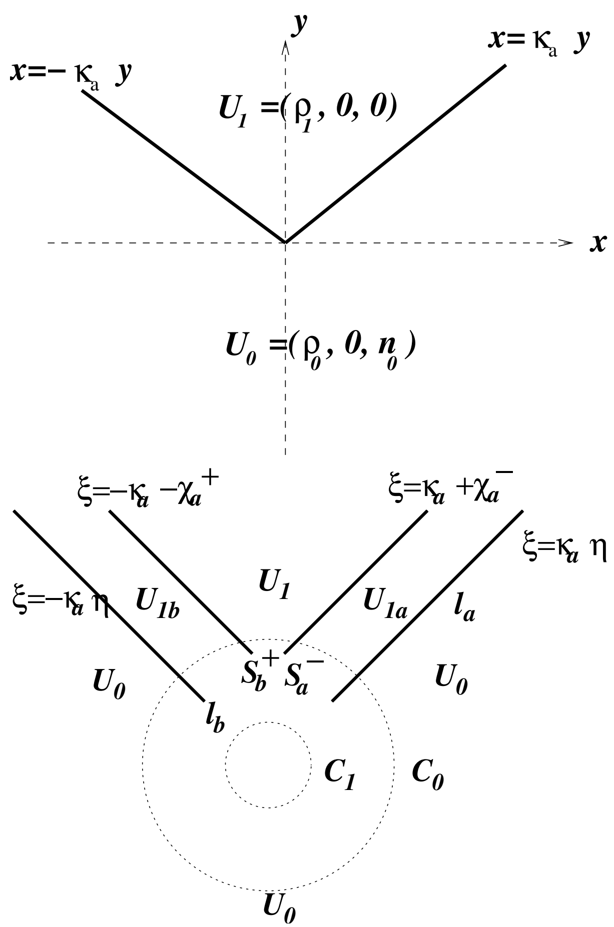
Far field soln: 4 waves

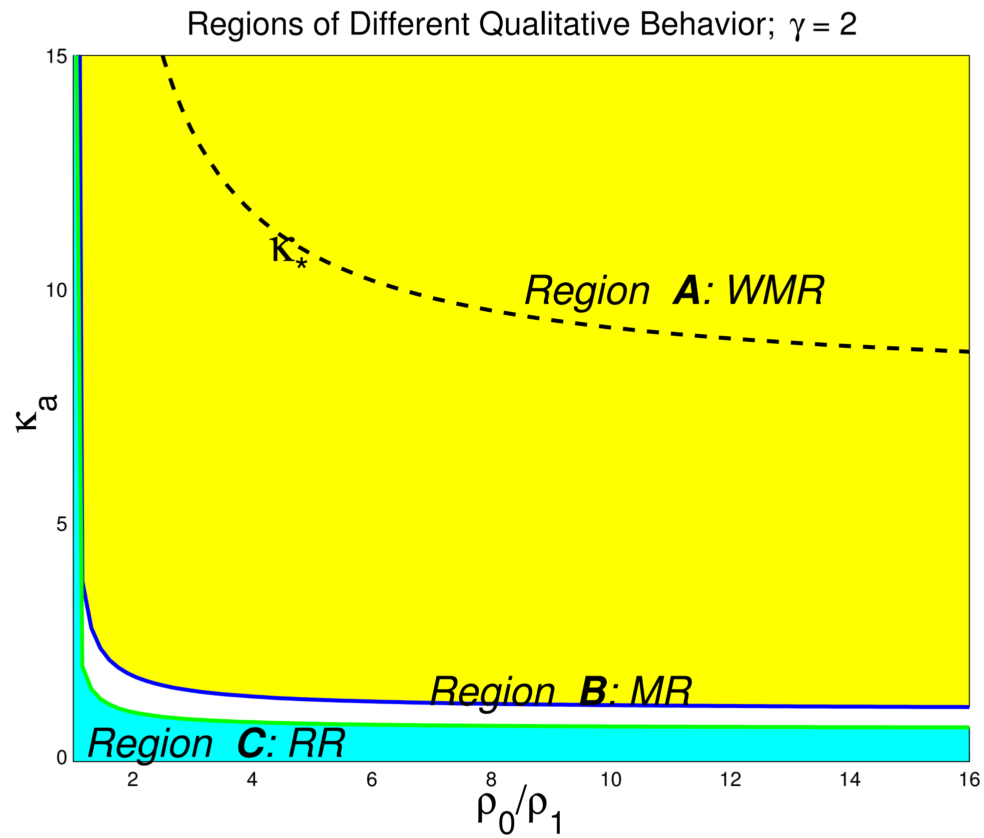


- Symmetric prototype for converging sector boundaries
- 'Weak shock reflection', von Neumann paradox

1. Parameterized by $\rho_0/\rho_1 > 1$ and by κ_a
2. Incident shocks: $\xi = \kappa\eta - \chi$, $\xi = -\kappa\eta + \chi$
3. Linear waves: angle of incidence $=$ angle of reflection
4. Small κ: two local solutions –'weak' and 'strong' regular reflection
5. Large κ: curved shock, weak reflected wave (cf. $\kappa = \infty$)
6. Intermediate values of κ: no solution from shock polar analysis

Converging Shock Data for the Nonlinear Wave System





Three regions: **A** Weak MR possible

C RR possible

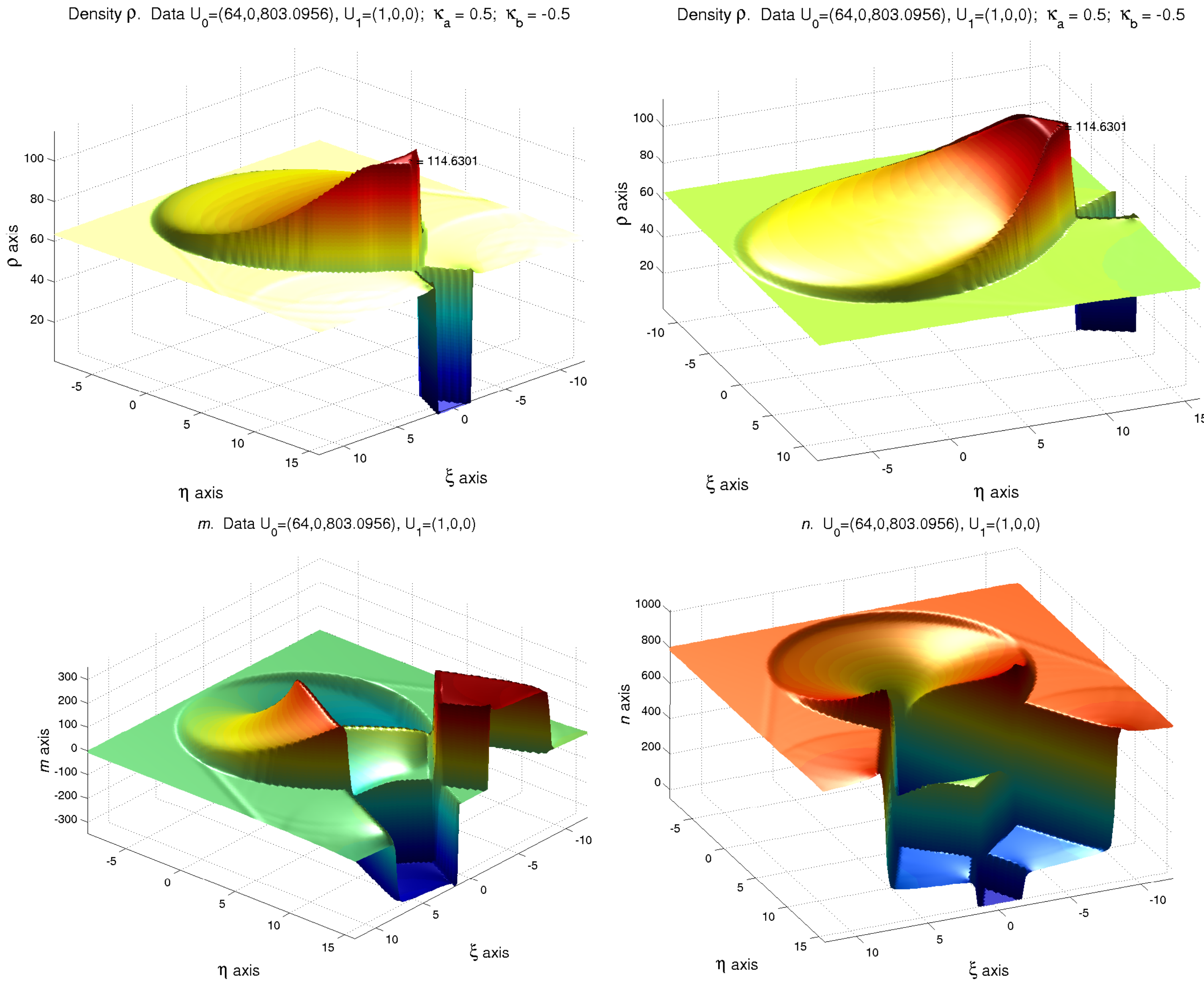
B neither possible

Some Numerical Simulations of the Problem

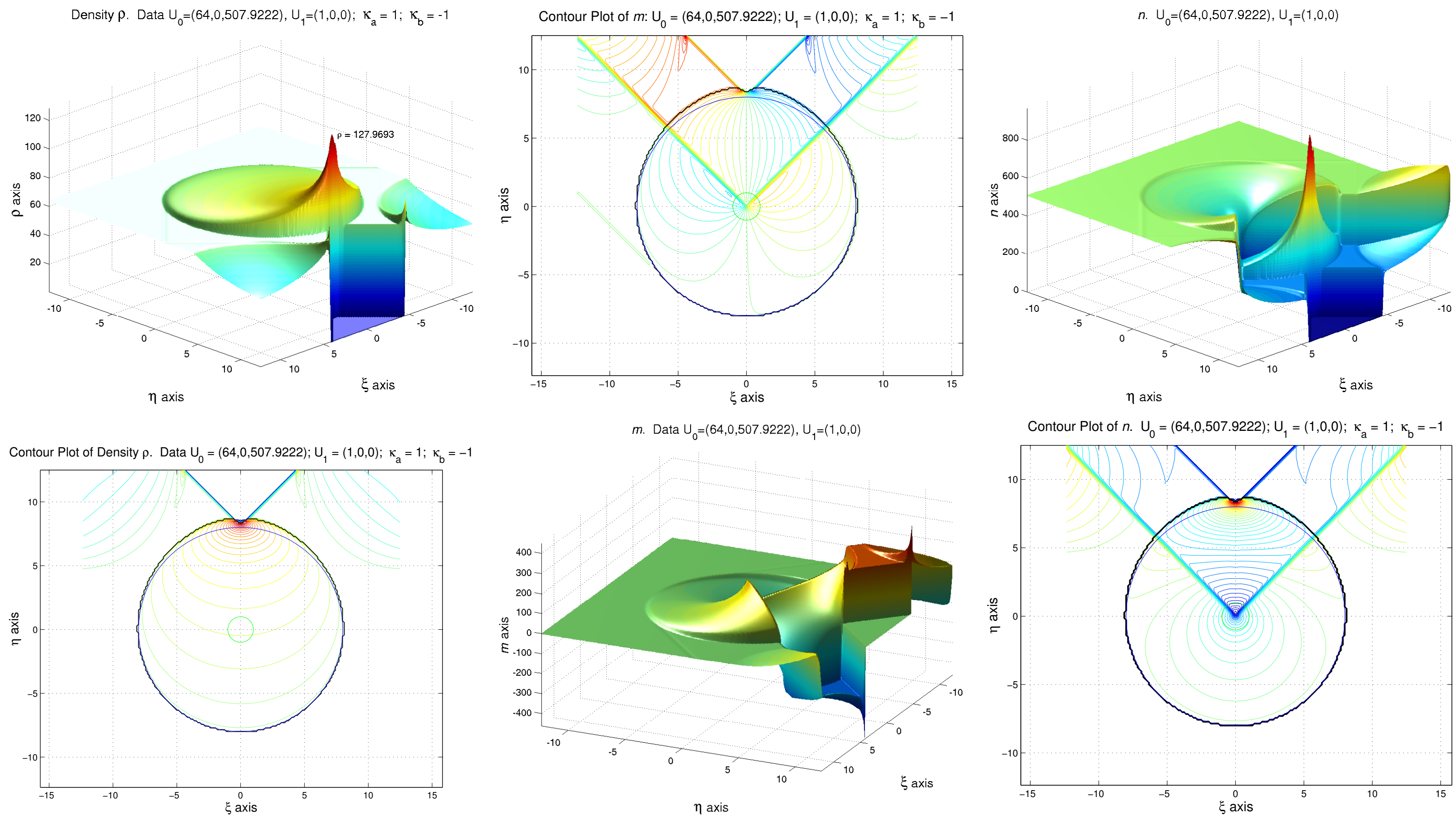
Alex Kurganov, Tulane University

- Godunov-type central scheme, 'central-upwind scheme', Alexander Kurganov, Sebastian Noelle and Guergana Petrova; modified version, Kurganov and Chi-Tien Lin. *SISC*, 23, 2001, pp 707-740. (`http://math.tulane.edu/~kurganov`).
- Central schemes (eg. Lax-Friedrichs scheme): avoid solving RP by integrating over local Riemann fans (at cell interface).
- Second-order staggered Nessyahu-Tadmor scheme (1990), later extended to higher orders and to multi-D.
- Kurganov & Tadmor, 1999: nonstaggered central schemes, with lower dissipation, simple semi-discrete form.
- New, central-upwind schemes: more accurate estimate of size of Riemann fans and more accurate projection step.

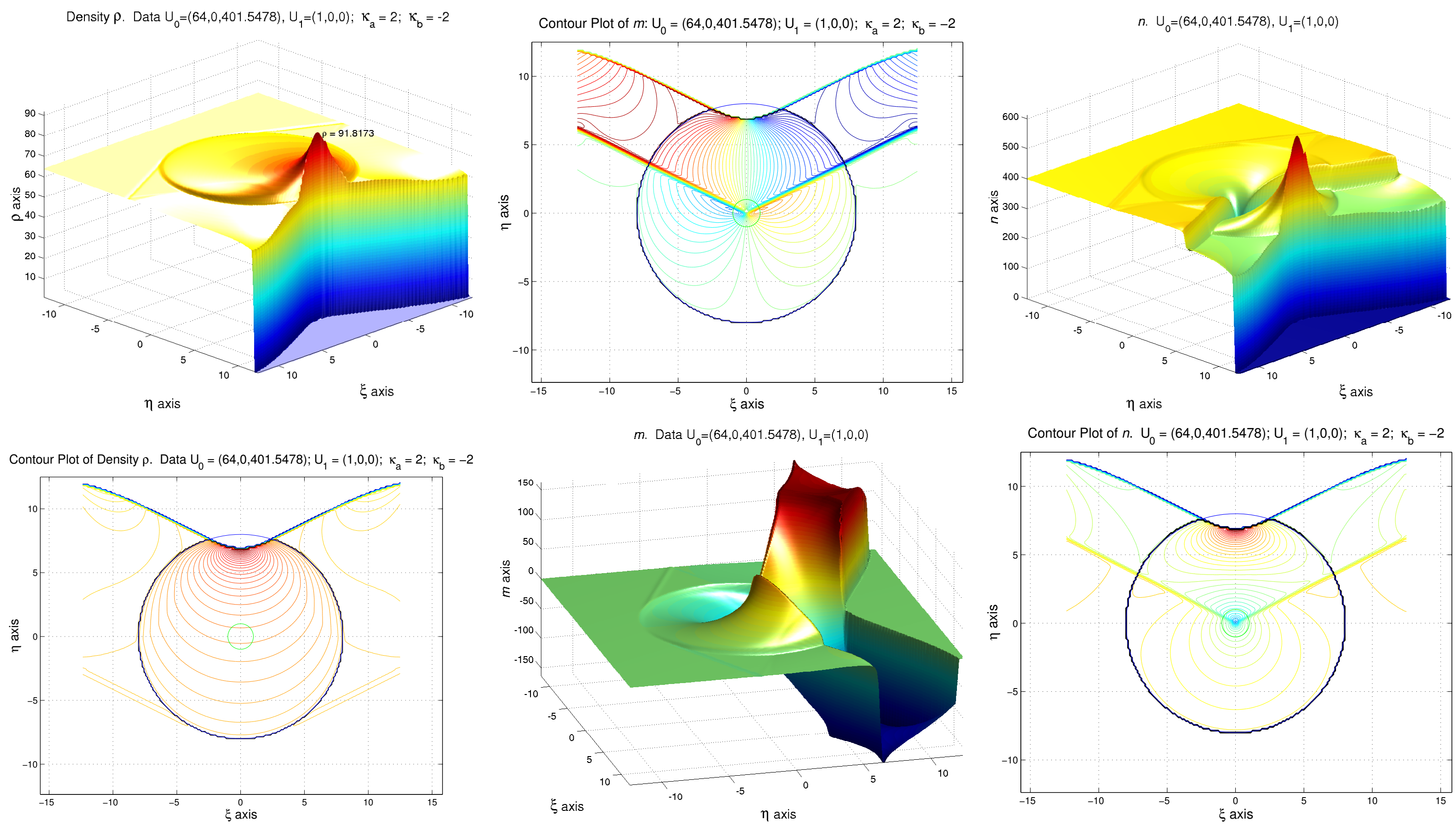
Regular Reflection: $\kappa_a = 0.5$



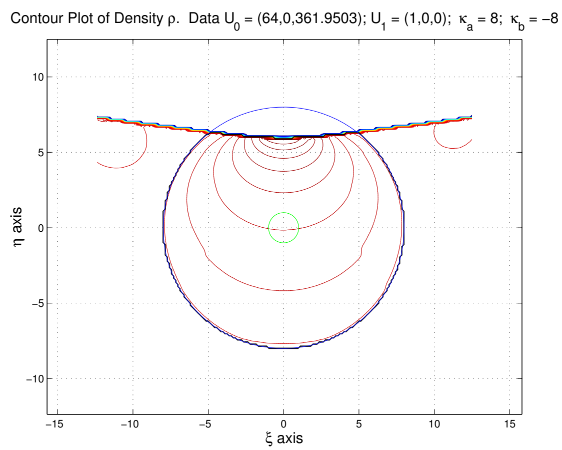
Intermediate Angle, $\kappa_a = 1$, Region **B**



An Intermediate Angle, $\kappa_a = 2$: Weak or Mach Reflection?



'Weak' Mach Reflection (Large κ_a): Region $\mathbf{C}$



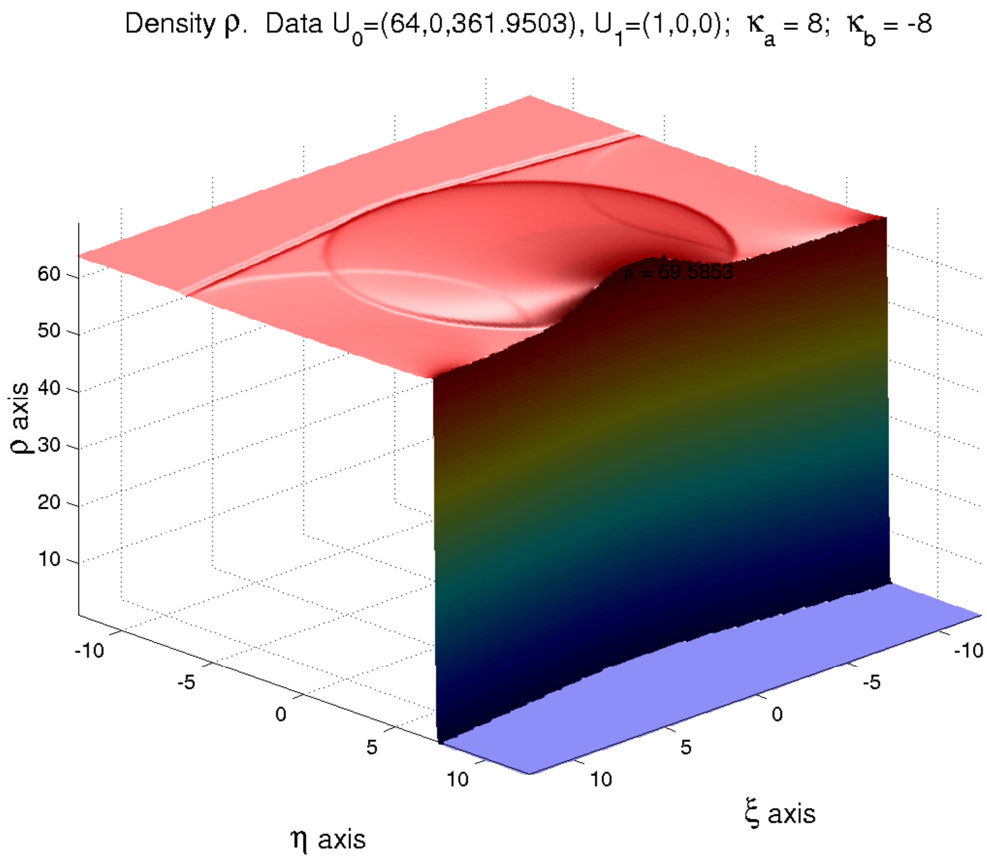
Sonic circle

$$C_0 = \{\xi^2 + \eta^2 = c^2(\rho_0)\}$$

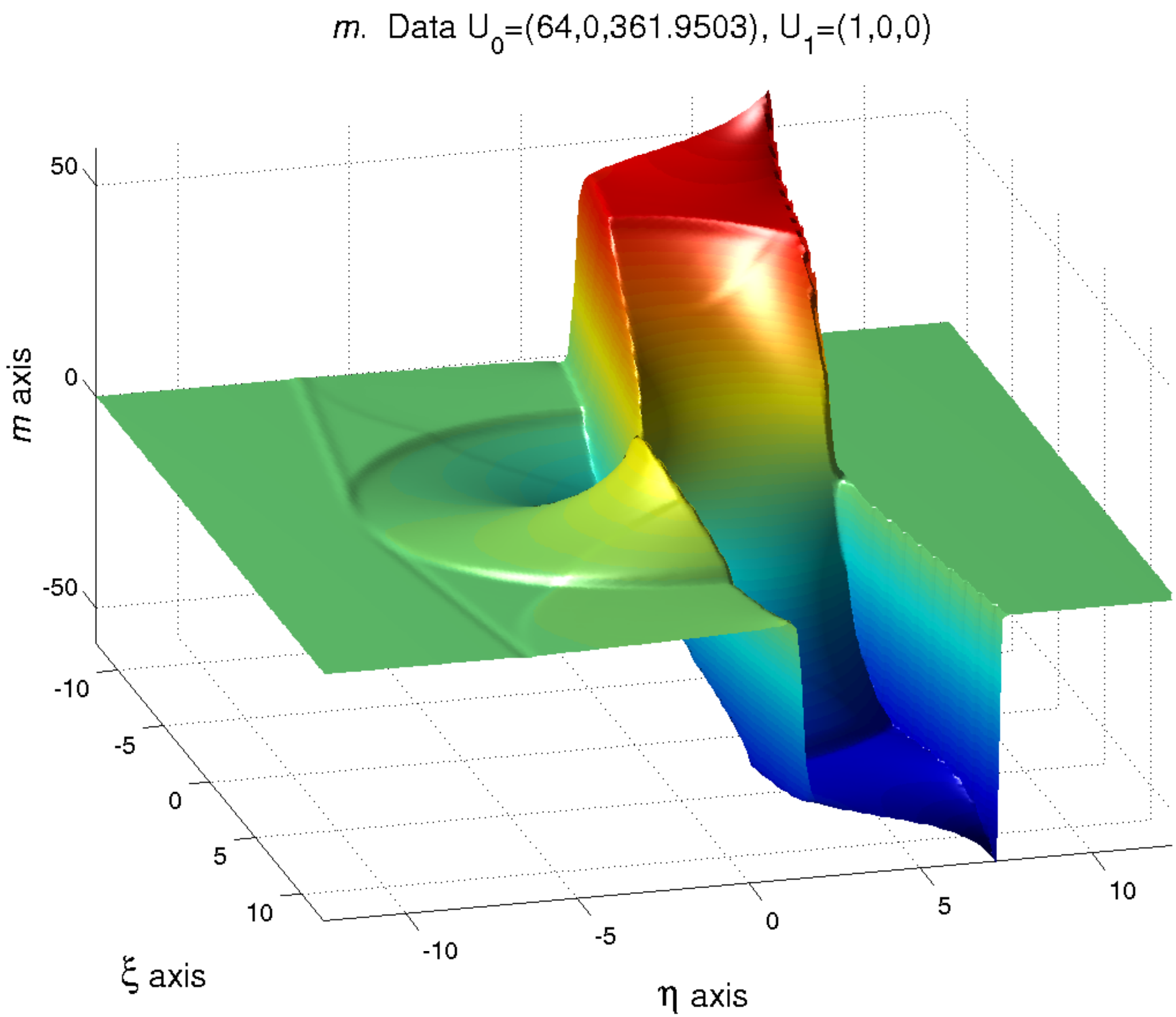
Supersonic soln known

U continuous at C_0

$\partial U / \partial r$ singular



'Weak' Mach Reflection (Large κ_a): Momentum Component m

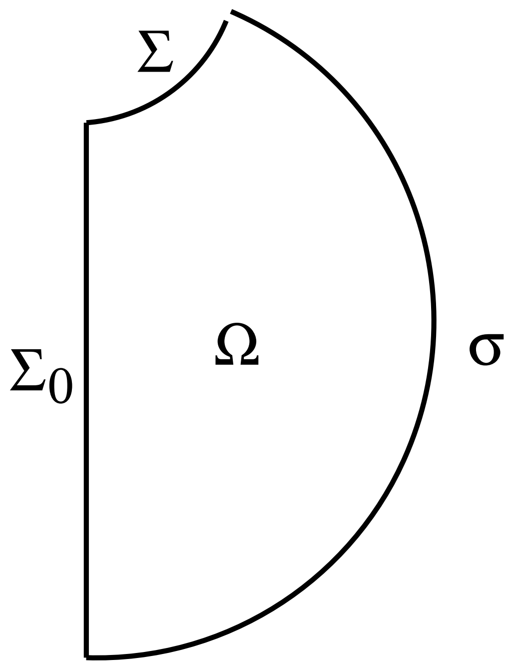


Analysis of Weak Mach Stem Problem (Large κ)

Degenerate Elliptic Free Boundary Problem

Existence theorem for global problem for NLWS

$$Q \equiv \left((c^2(\rho)-\xi^2)\rho_\xi - \xi\eta\rho_\eta\right)_\xi + \left((c^2(\rho)-\eta^2)\rho_\eta - \xi\eta\rho_\xi\right)_\eta + \xi\rho_\xi + \eta\rho_\eta$$



$Q(\rho) = 0$ (degenerate elliptic) in Ω

$c^2(\rho) = \xi^2 + \eta^2 = c^2(\rho_0)$ on σ

(degenerate boundary, continuous solution)

$\rho_\xi = 0$ (symmetry) on Σ_0

Free boundary from RH equations:

$N(\rho) \equiv \beta \cdot \nabla\rho = 0$ (oblique deriv) on Σ

$$\frac{d\eta}{d\xi} = \frac{\eta^2 - s^2}{\xi\eta + \sqrt{s^2(\xi^2+\eta^2-s^2)}} \qquad s^2 = \frac{[p]}{[\rho]}$$

$\rho = \rho_{\mathsf{max}}$ at $\Sigma \cap \Sigma_0$ (part of D. bdry)

Approach: Fixed Point Theorem (CK & Lieberman, CKK)

- New difficulties: N not uniformly oblique; est. at degenerate corner

Previous Results on Related Free Boundary Problems

Fixed point approach developed in joint work with Gary Lieberman
(F B problem in steady TSD equation: C, K & Lieberman)
Partial solution for regular reflection in UTSD model, $(a > \sqrt{2})$:

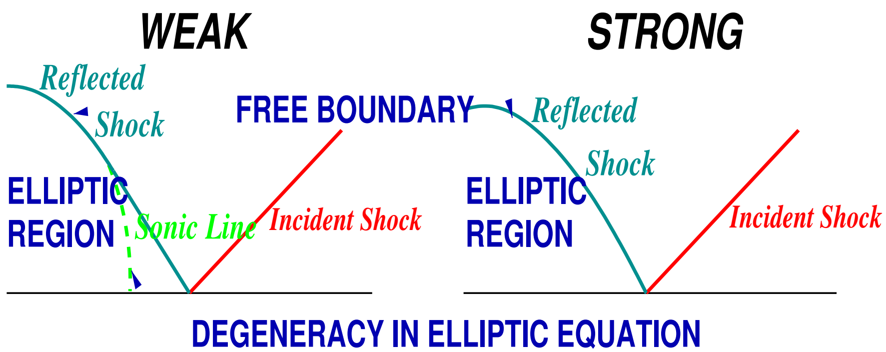
2 types of regular reflection

weak & *strong*

$a^* = (1 + \sqrt{5}/2)^{1/2}$

$(\sqrt{2}, a^*)$: both subs

$a > a^*$: 1 sub, 1 sup



Riemann data:

$$U_0 = (0,0), \quad x > a|y|,$$
$$U_1 = (1,-a), \; y > 0, x < ay$$
$$U_1^* = (1,a), \; y < 0, x < -ay$$

Complete description of flow needs:

- asymptotic behavior at $\xi = -\infty$
- solving degen elliptic prob
- solving F B prob for shock

Unbounded subsonic region: artifact of UTSD model

Procedure: Find the Free Boundary as a Fixed Point

Formulate as 2nd order PDE for density, ρ (not potential);

Rewrite RH conditions as (1) evolution eqn for shock and
(2) ODBC for ρ

Problem is quasilinear, degenerate elliptic PDE, mixed BC

Regularize PDE (parameter ε)

Step 1 Fix approx. $\eta = \eta(\xi)$, defines $\Sigma \in \mathcal{K}^\varepsilon \subset H_{1+\alpha_1}$ (Hölder)

Step 2 Solve (fixed) mixed BVP for ρ

Lieberman's Mixed BVP theory + linearization
+ modifications for loss of obliqueness

Step 3 Map $\eta \to \tilde{\eta} = J\rho$ by other RH condition (shock evolution)

Schauder F. P. Thm: Compactness $\Rightarrow$ fixed pt for J

$$J : \mathcal{K} \subset H_{1+\alpha_1} \to \mathcal{K} \cap H_{1+\alpha},\ \alpha > \alpha_1$$

Step 4 Show η and ρ solve the problem.

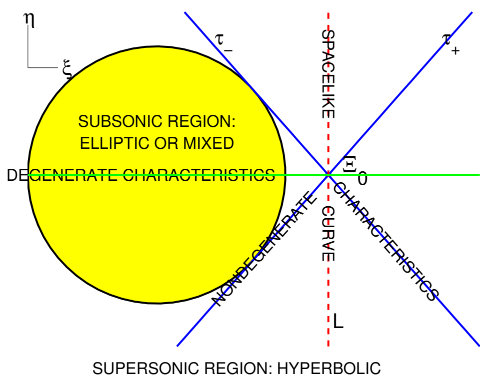
Degenerate Elliptic Equations

Principal part:

$$\left((c^2(\rho) - U^2)\rho_\xi - UV\rho_\eta\right)_\xi + \left((c^2(\rho) - V^2)\rho_\eta - UV\rho_\xi\right)_\eta$$

with $U = u - \xi$, $V = v - \eta$ or $U = -\xi$, $V = -\eta$

Linear prototype: Tricomi, $u_{xx} + xu_{yy}$, or Keldysh, $xu_{xx} + u_{yy}$



Differ on Hyperb. side ($x < 0$): dir. of char.

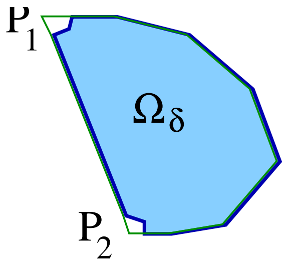
Differ on Ellliptic side ($x > 0$):

Fichera fn $b \equiv (b^k - a^{kj}_{x_j})n_k$ (NB: l. o. terms)

(Nonlinear definition ambiguous)

- Keldysh eqns permit Dirichlet BC (need in nonlin. eqn)
- Linear Keldysh solns have singularities at $x = 0$: $u = x^\gamma$
- Nonlinear Keldysh solns are regular OR singular

Mixed BC: Weighted Hölder Norms for Corner Singularities



$C^{k,\alpha} = H_{k+\alpha} : |u|_{k+\alpha} = \sum_{j<k} |D^j u|_0 + |D^k u|_\alpha$

Partially interior or weighted norms (corners)

$|u|^{(b)}_{a;\Omega\cup S^c} = \sup_\delta \delta^{a+b}|u|_{a;\Omega_\delta}\,, \quad H^{(b)}_{a;\Omega\cup S^c}$

Example: $S = V \equiv \{P_1, P_2\}$; $u = r^\gamma \in H^{(-\gamma)}_{1+\alpha}$

Lin Prob, $z \in H^{(-\gamma_1)}_{1+\epsilon}$: $Lu = D_i\big(a^{ij}(z)D_j u\big),\ Mu = \beta(z)\cdot\nabla u$

Estimates indep of z: $|u|^{(-\gamma)}_{1+\alpha} \le C_1\left(|f|_\gamma + |u|_0\right)$

Nonlinear problem: Harnack ineq $\Rightarrow$ est with α indep of α_1

Schauder Fixed Point Theorem: $\mathcal{B} = H_{1+\alpha_1}$; $\alpha_1 \equiv \min\left\{\frac{\alpha}{2}, \frac{\gamma}{2}\right\}$

$\mathcal{K}$ closed, bdd., cvx. $\mathcal{K} \subset \mathcal{B}$; $J : \mathcal{K} \to \mathcal{K}$, $J\mathcal{K} \subset\subset \mathcal{K} \Rightarrow J$ has fixed pt.

γ dep. on corners only; α indep. of α_1; $J\rho \in H_{1+\alpha}\,,\ \ \alpha > \alpha_1$

A Priori Estimates (following Lieberman, Zheng, Canic&Kim, CKL)

Regularize: $Q^\varepsilon \rho = Q\rho + \varepsilon\Delta\rho$.

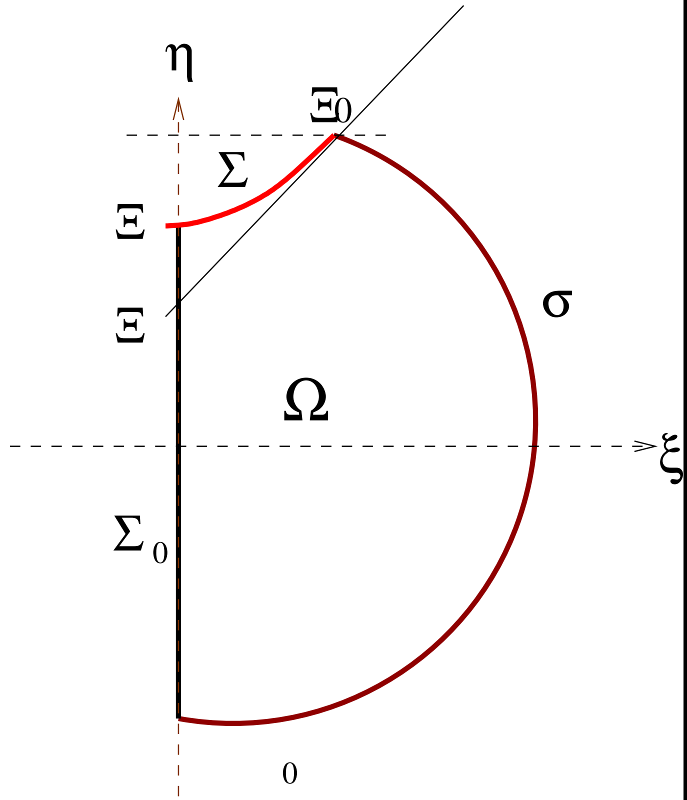
Obliqueness fails at Ξ_s.

STEP 1: $\exists$ for FBP for Q^ε; a priori bds on ρ^ε, $\rho^\epsilon(\eta)$ unif. in ϵ.

STEP 2: Local lower barrier for ρ^ε independent of ε, $\Rightarrow$ unif. local ellipticity $\Rightarrow$ local compactness.

STEP 3: Convergent subsequence; limit solves the problem in $\overline{\Omega} \setminus \{\Xi_0\}$: applying regularity, compactness and a diagonalization.

STEP 4: Convergence at Ξ_0 requires more.

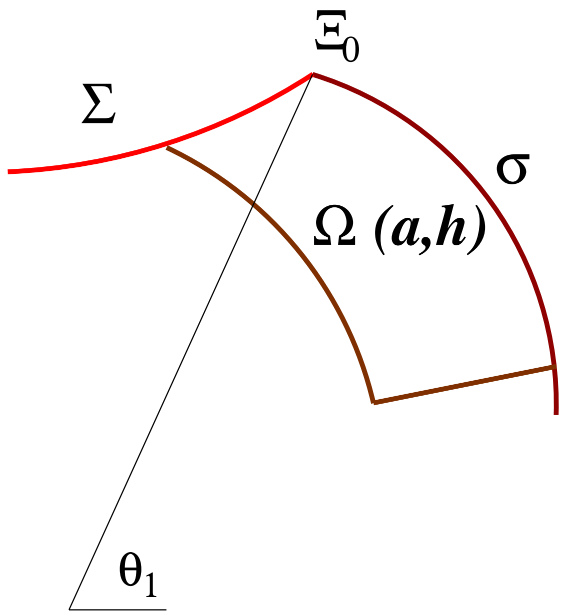


The Upper Barrier at Ξ_0

Barrier construction (supersolution):

$$Q\psi \le 0,\ N\psi \le 0,\ \psi \ge \rho \text{ on bdry of } \Omega(a,h)$$

$$Q^\varepsilon \rho = (c^2 - r^2 + \varepsilon)\rho_{rr} + \frac{c^2}{r^2}\rho_{\theta\theta} + p''(\rho_r^2 + \frac{1}{r^2}\rho_\theta^2) + (\frac{c^2}{r} - 2r)\rho_r$$



Upper barrier:

$$\psi(r,\theta) = \rho_0 + A(c_0 - r)^b + B(\theta_1 - \theta)^2$$

A, B, $b \in (0,1)$ to be determined from

$$Q^\varepsilon \psi = (c_0 - r)^{b-2} p'' B(\theta_1 - \theta)^2 b(b-1)A + \ldots \le 0$$

Singular barriers in degenerate eqns:

Choi-McKenna &Canic-Kim.

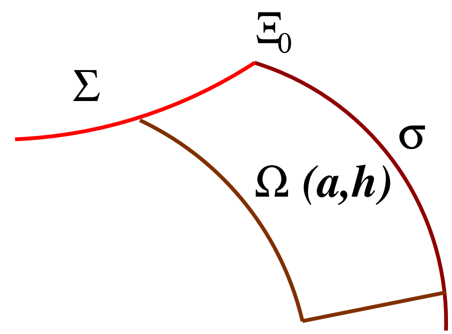
Conjecture: solution has $\sqrt{}$ sing. at C_0; b ($< 1/2$) optimal

$$(c_0 - r)^{2b-2} : A^2 b\left(p''(\overline{\rho})(b-1) + p''(\rho)b\right) \le A^2 k_0 < 0 \text{ if } b < \frac{\min p''}{2\max p''},$$

so $A >> 1$, $\Rightarrow Q^\varepsilon \psi < 0$.

Also need $\psi - \rho^\varepsilon \ge 0$ on $\partial\Omega(a,h)$.

Completing the Upper Barrier at Ξ_0



On σ: $\rho^\varepsilon = \rho_0 < \psi$.
On $\{\theta = \theta_1 - a\}$: Choose $Ba^2 > \rho_M$.
On $\{r = c_0 - h\}$: Choose A so $Ah^b > \rho_M$.

On Σ, ODBC: if $N_1(\rho^\varepsilon)\psi \equiv \beta(\rho^\varepsilon) \cdot \nabla\psi \le 0$, then $\psi - \rho^\varepsilon > 0$.
Now $\nabla\psi = (\psi_r, \psi_\theta) \to \infty$ as $r \to c_0$: $\psi_n \to \infty$ AND $\psi_t \to \infty$:

usual barrier construction fails.

But $N_1\psi = \beta^r\psi_r + \beta^\theta\psi_\theta, \qquad \beta^r = \frac{1}{r}(\beta_1\xi + \beta_2\eta),$

Now $\beta_1\xi + \beta_2\eta \equiv \underbrace{(\eta - \eta'\xi)(\xi + \eta'\eta)}_{>0 \text{ by geometry}} \underbrace{\left(r^2(c^2(\rho) + 3s^2(\rho, \rho_1)) - 4c^2s^2\right)}_{>0 \text{ at } r=c_0, \rho=\rho_0}$.

A priori bound: need $\beta_1\xi + \beta_2\eta > 0$ for all $\rho \in (\rho_0, \rho_M)$.
But $\rho_M = \rho_0 + \mathcal{O}(1/\kappa_a)$.
If $\rho_0 >> \rho_1$, then $\kappa_a \sim 8$ suffices.