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Prototypes for Nonstrict Hyperbolicity in Conservation Laws

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Abstract. Many conservation law systems that arise in continuum mechanics fail to be strictly hyperbolic; this situation may even arise generically in some dimensions. By examining simple systems in a single space variable, where the mathematical theory of weak solutions has been developed, one may try to gain insight into how more realistic systems will behave. However, even in simple systems, there is no standard model for the breakdown of strict hyperbolicity. This paper surveys a number of nonstrictly hyperbolic models. Some may be good prototypes for multidimensional problems in continuum mechanics; some appear frequently in models for multiphase flow; some are associated with change of type (loss of hyperbolicity). All have complicated Riemann solutions; for some, existence of solutions to the Cauchy problem has been proved, and interesting qualitative behavior has been observed.

1. Introduction

This article deals with failure of strict hyperbolicity in conservation laws. What happens when equations fail to be strictly hyperbolic? How prevalent is it? What causes this failure; can it be predicted or prevented? How is it related to change of type, or to failure of genuine nonlinearlity? What mathematical results exist?

1.1. Conservation Laws. We live in a multi-dimensional world; the theory of hyperbolic conservation laws tends to look at one-dimensional equations. To what extent are these good models?

1.2. Some Examples. As an example, consider the equations of a compressible ideal gas in two-dimensions:

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (\rho E + p) u \end{pmatrix} + \partial_y \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (\rho E + p) v \end{pmatrix} = 0\,,$$

where the variables are ρ, u, v and p and E is defined as $e + (u^2 + v^2)/2$ with $(\gamma - 1)\rho e = p$. The characteristic structure of this system is obtained by linearizing

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around a constant state $U = (\rho, u, v, p)$; letting $\sigma = (\tau, \mu, \nu)$ be the dual variable and $\lambda = \tau + u\mu + v\nu$, then

$$\lambda^2\big(\lambda^2 - c^2(\mu^2 + \nu^2)\big) = 0,$$

and the roots consist of a nondegenerate pair, $\lambda = \pm c\sqrt{\mu^2 + \nu^2}$ and a degenerate direction $\lambda = 0$ of constant multiplicity two.

The one-dimensional version of this consists of a system of three equations, whose characteristics are $u \pm c$ and u. The phenomenon of multiplicity has disappeared. It is a fact about the one-dimensional system that the "acoustic wave speeds", $u \pm c$, are genuinely nonlinear, while the third characteristic, u, is linearly degenerate. In more than a single space dimension, characteristic structure is more complicated, and the degeneracy in the shear and entropy waves is of two kinds: they are linearly degenerate, and they also give rise to wave propagation (in space-time) along one-dimensional lines, rather than, as in the case of the acoustic waves, $n-1$ dimensional cones. Freistuhler [**Fre91a**] showed that (in one dimension) characteristics of constant multiplicity greater than one are always linearly degenerate.

Multi-dimensional problems also present examples where the characteristics have variable multiplicity. Examples include an anisotropic version of crystal optics [**CH62**, page 602] and the equations of magnetohydrodynamics [**CH62**, page 612]. Other examples in elasticity and continuum mechanics have been presented in this workshop. In fact, a theorem of Lax [**Lax82**] states that in $\mathbf{R}^{3+1}$ a first-order system of n equations cannot be strictly hyperbolic if $n \equiv 2 \mod 4$. This was shown by Friedland, Robbin and Sylvester in [**FRS84**] to be true for other values of n as well. In the three-dimensional world in which we would like to pose most of these problems, either constant multiplicity greater than one or 'crossing' of characteristics will occur in all first-order systems of 2, 3, 4, 5, 6, or 10 equations (and the pattern repeats mod 8). It would be an interesting exercise to apply the topological methods of Lax and of Friedland et al. to find the analogous result in two dimensions.

1.3. Hyperbolic Systems. Hyperbolicity is a desirable property of time-dependent partial differential equations. For example, the Cauchy problem for a constant-coefficient linear partial differential operator is well-posed in reasonable spaces if and only if the operator is hyperbolic [**Hor83**]. One should note that the theory of first-order systems is a little different from that developed for a single higher-order equation, though the concept of strict hyperbolicity is the same for both, [**RR93**, page 43]. Systems that are hyperbolic but not strictly hyperbolic arise in at least two natural ways in physical models:

- as *symmetric* or *symmetrizable* systems $PU = \sum A_i \partial_i U$, where the A_i are symmetric and A_0, corresponding to the time derivative, is also positive definite
- as *kinematic* systems $PU = \partial_t U + \sum \partial_i\big(\phi_i(U)\,U\big)$; systems of this form, as we shall see, have real eigenvalues and a complete set of eigenvectors.

Symmetry is often a consequence of conservation principles in physics, which accounts for the prevalence of the first type of system. The second type of model describes a system with a number of components, all of which are convected with velocity $\phi(U)$, where $\phi = (\phi_1, ..., \phi_n)$. This would be the case in the gas dynamics equations, for example, if one omitted the pressure terms — that is, if one ignored the dynamics for some reason. Such a crude approximation might not be easily justified for ideal gas flow; however, there are a number of situations in which the flow of a mixture of components is described in these terms. The miscible displacement problem in enhanced oil recovery has this form, [**Isa, JW88**]. The special case in one spatial variable in which ϕ is rotationally symmetric, $\phi = \phi(|U|)$ may be thought of as a one-way model for waves in magnetohydrodynamics or continuum mechanics [**Fre91b**]. This was also recognized in [**KK80**]. Related systems appear in models for chromatography, [**RAA86**].

For a kinematic system of n equations in m space variables, the symbol matrix is a rank one perturbation of a scalar matrix:

$$P(\tau, \xi) = (\tau + \xi \cdot \phi)I + Uq^T \quad \text{where} \quad q_i = \xi \cdot \partial_i \phi$$

and provided that the trace of Uq^T is nonzero, then $-\xi \cdot \phi$ is an eigenvalue of multiplicity $n-1$ and $-(\xi \cdot \phi + \operatorname{tr}(Uq^T))$ an eigenvalue of multiplicity one. This system has multiple characteristics if $n \geq 3$; there is a complete set of eigenvectors, provided $\operatorname{tr}(Uq^T) \neq 0$.

Both symmetric and kinematic systems are hyperbolic but not necessarily strictly hyperbolic. Each type of system provides a framework in which hyperbolicity is structurally stable; perturbation may change the multiplicity (in symmetric systems) but eigenvalues remain real, and the set of eigenvectors complete. (For kinematic systems, this holds only in the open set where $\operatorname{tr}(Uq^T) \neq 0$. More on this later.) However, for either type of system, perturbations outside the class lead to a system that changes type.

2. Types of Nonstrict Hyperbolicity

In the study of nonstrictly hyperbolic equations, the basic dichotomy is constant versus variable multiplicity. As our first example showed, one motivation for the study of eigenvalues of constant multiplicity is that they appear in common higher-dimensional problems, such as gas dynamics. The theory of linear hyperbolic problems with multiple characteristics, see for example Mizohata's monograph, [**Miz85**], focuses on well-posedness in Gevrey classes.

In nonlinear conservation law theory, the first nonstrictly hyperbolic examples which were studied came from elasticity and exhibited variable multiplicity.

2.1. Eigenvalues of Constant Multiplicity.

The most straightforward examples which show constant multiplicity in a single space dimension are those which come from kinematic models. It is not clear, of course, if experience in studying such models will be useful in multidimensional problems, where the characteristic multiplicity arises for a different reason. However, kinematic systems also arise independently. In a series of papers, see [**Fre94**] and the citations there, Freistühler and co-workers have performed extensive studies of kinematic systems with rotational symmetry, of the form

$$U_t + \big(\phi(|U|)U\big)_x = 0\,. \tag{2.1}$$

(We remark that systems of this form are both kinematic, from their structure, and symmetric hyperbolic, because of the rotational symmetry.) Such systems can be separated by a change of variables into a scalar nonlinear equation and an essentially linear system, and then solved by a variant of the Glimm scheme. However, their behavior is more complicated than this reduction would suggest. First, the system possesses anomalous solutions which are not Riemann solutions of the reduced system; their importance and physical significance are unclear. Second, viscous approximations are not uniformly stable. Numerical studies by Freistühler and Pitman, [**FP92, FP94**], illustrate both of these points. Freistühler also extended some of his results to systems of the form

$$U_t - V_x = 0 \tag{2.2}$$

$$V_t - \big(\phi(|U|)U\big)_x = 0\,. \tag{2.3}$$

of which he found examples in isentropic magnetohydrodynamics and elasticity. Systems of this form still have multiple characteristics. Of course, they describe more physically complex and realistic situations than the simple kinematic equations mentioned above. We comment that (2.2), (2.3) is not symmetric hyperbolic, but if the eigenvalues of $d(\phi U)$ are positive it is symmetrizable.

Andrea Reiff studied both types of systems in her thesis. She showed that Glimm's scheme can be easily modified to solve the Cauchy problem for a kinematic system without rotational symmetry in the original variables [**Rei97, Rei99a**]. Reiff also solved the Cauchy problem for the system, analogous to (2.2), (2.3),

$$U_{tt} - \big(\phi(|U_x|)U_x\big)_x = 0$$

which models an elastic string [**Rei97, Rei99b**]. When U is the position of a point on the string in three-dimensional space, then this system has two sets of characteristics with constant multiplicity two.

This is just a beginning for research on conservation law systems with constant multiplicity. Some of Reiff's work was motivated by numerical results of Smadar Karni [**Kar94**]; Karni observed that characteristic methods such as Roe solvers displayed a loss of accuracy in treating the contact discontinuities associated with double characteristics in multidimensional flows. Karni was able to demonstrate the difficulties by means of some one-dimensional examples, which contained kinematic type subsystems. These examples are not covered by the kinematic or elastic string systems, and it remains an open question to investigate whether their behavior is qualitatively the same, or different. Eventually, of course, one would want to look directly at two-dimensional problems. At the moment, there are many unsolved problem associated with multidimensional flow; multiple characteristics may not be the most significant.

Yuxi Zheng, [**Zhe98**], has investigated models which display eigenvalue coincidence and eigenvector deficiency everywhere. Among kinematic equations, these are the systems that also have trace zero (See the second to last paragraph of Section 1.). Zheng's algorithm finds flux functions, both kinematic and general, with this property. Although these equations look somewhat artificial, some examples have been suggested as models for density fluctuations in gases under extremely low pressure, for example on astronomical scales, [**ERS96, SZ89**]. Typically, the characteristics are linearly degenerate, and singularities appear in the solutions which resemble Dirac δ-functions supported on curves. An example is analysed by Bouchut in [**Bou94**]; see also the discussion and references in [**Zhe98**].

2.2. The Crossing of Eigenvalues. Twenty years ago, H. C. Kranzer and B. L. Keyfitz solved two Riemann problems which were almost the first examples of nonstrictly hyperbolic systems. They were both systems of two equations. It is convenient to use the notation $U = (u_1, u_2) = (r\cos\theta, r\sin\theta)$. The first system

Keyfitz and Kranzer considered was

$$U_t + \big(\phi U\big)_x = 0$$

with $\phi = \phi(|U|)$ a convex function of its argument, r, and $\phi'(r_0) = 0$ at a point $r_0 \neq 0$. This corresponds to the case $n = 2$ in (2.1), and to violating the trace condition on a sphere in state space. Keyfitz and Kranzer also considered kinematic systems, as in Section 1, without rotational symmetry. In this case, eigenvalue coincidence again occurs when $\operatorname{tr}(Uq^T) = u_1\partial_1\phi + u_2\partial_2\phi = 0$. This condition is equivalent to $\partial_r\phi = 0$, and eigenvalue coincidence occurs on a codimension one submanifold of state space. The paper, [**KK80**], referred to both systems as "non-strictly hyperbolic", although only the first is actually hyperbolic at the coincidence locus. For the non-symmetric system, $\phi = \phi(U)$, we showed that the Jacobian matrix typically is nilpotent and the system has an eigenvector deficiency. Some authors, such as Liu and Osher, [**LO98**], now use the term "weakly hyperbolic" for systems with real eigenvalues and an eigenvector deficiency. The paper, [**KK83**], used the term "parabolic degeneracy", which seemed to make an appropriate analogy with a linear system

$$\begin{aligned} u_t + (au+v)_x &= 0 \\ v_t + av_x &= 0\,, \end{aligned}$$

which is of parabolic type in the standard classification. Solutions of the Riemann problem for the two systems are quite similar despite this difference. However, some of the similarity disappears when one looks at the Cauchy problem, as was pointed out by Temple, [**Tem82**]: in the parabolically degenerate case, initial data lying on the coincidence locus do not lead to a solution with bounded total variation. Temple was able to solve the Cauchy problem for this model problem by weighting the total variation with a weight that became zero on the coincidence locus.

The equations in [**KK80**] were motivated by the model of an elastic string in the plane. In this model, the characteristic speeds correspond to speeds of longitudinal and transverse waves, and one can conceive of a material in which these speeds, both amplitude-dependent, become coincident and reverse order for some states. The kinematic model was intended to be a guide to solving the actual string problem. At the time, there was no interpretation for the second model, the one which had an eigenvector deficiency. However, soon afterwards Eli Isaacson discovered that the equations for the so-called miscible displacement problem in enhanced oil recovery could be cast in this form, [**Isa**]. In fact, these equations are precisely a kinematic system, and have this form for the principal reason kinematic systems are studied: a solute is transported passively by a solvent, but its concentration affects the velocity of the mixture. The simplified limit in which concentration is taken to be the only variable affecting the velocity leads to the kinematic system.

Isaacson and Temple investigated these equations in some depth. In [**IT86**] and [**IT90**] they found, along with other evidence of ill-posedness, that the time-asymptotic states may not coincide with the Riemann solutions. Isaacson and Temple also pointed out a very interesting direction in which their results could be extended. They introduced the term "resonance" to describe a system of conservation laws

$$U_t + \big(F(a(x), U)\big)_x = 0\,,$$

in which eigenvalues of $d_U F$ could assume the value zero. The relation with non-strict hyperbolicity is seen by writing the same equation as a system of $n+1$ equations:

$$U_t + \big(F(a,u)\big)_x = 0 \tag{2.4}$$

$$a_t = 0\,, \tag{2.5}$$

which exhibits eigenvalue coincidence if $d_U F$ has a zero eigenvalue. In [**IT92**] they solved the Riemann problem for such a system.

2.3. Crossing Avoidance and Genericity. Now, Peter Lax has pointed out the phenomenon of "crossing avoidance"; this is apparently well-known to physicists, and the term refers to the fact that the set of degenerate $n \times n$ matrices (those with a double eigenvalue) is of codimension two in the space of real symmetric $n \times n$ matrices. This fact is the 'cover story' in Lax's recent linear algebra text, [**Lax97**]. For example, if $n = 2$, then a symmetric matrix $A(U)$ will generally have coincident eigenvalues at isolated points, called *umbilics*. This leads to what we might call the "symmetric prototype" of quadratic systems,

$$U_t + Q(U)_x = 0,$$

where Q is a homogeneous quadratic flux vector. Originally classified and studied by Schaeffer, Shearer, Marchesin and Paes-Leme, [**SS87a, SSMP87**], quadratic systems have spawned a huge literature and are a topic in themselves. Some motivation for this study came from applications: kinematic models for the immiscible displacement problem in three-phase reservoir flow turned up systems of conservation laws that were different from any that had been studied, and this led to an interest in the mathematical behavior of quadratic systems. In this article, we will discuss the three-phase flow model briefly in Section 4. The arresting fact about quadratic systems is that they can be extremely complicated: if one restricts attention to quadratic fluxes with real characteristic speeds, then they form a two-parameter family with four regions of distinct qualitative behavior. Some papers which handle solution of the Riemann problem for some of the many cases that occur include [**IMPT88, IT88a, IT88b, She89, SS87b, SS91**]. Furthermore, a number of interesting features: nonuniqueness of Riemann solutions, and solutions involving nonclassical (over- and under-compressive) shocks, were first studied in the context of quadratic systems. The references above, while far from complete, will give some indication of the phenomenology of the subject. In connection with the application to multiphase flow, as we shall mention further below, one also needs to consider perturbations of quadratic systems, including, typically, perturbed systems which exhibit change of type.

Quadratic fluxes arose from considering the singular case for symmetric systems. However, other systems, such as rank-one systems (kinematic equations), if they admit eigenvalue coincidence, do so along a codimension-one submanifold; that is, along a curve in the plane when $n = 2$. Furthermore, this behavior is generic: within the framework of kinematic models, eigenvalue coincidence is a codimension one phenomenon. One algorithm for constructing coincidence-line fluxes is given in [**Key93**].

If one examines eigenvalue coincidence in systems which are neither symmetric nor of rank one type, then it is often the case that the eigenvalue coincidence is accompanied by change of type. This is not a theorem; in fact, there are many examples to the contrary. Kranzer and Keyfitz [**KK83**] looked at a set of equations whose simplest example is

$$u_{tt} - (u_x)^3_x = 0$$

. This has an eigenvalue coincidence on the line $u_x = 0$, with a nilpotent matrix. Like the nonsymmetric kinematic system, it is not hyperbolic on the coincidence locus; this is another example of a parabolic degeneracy. Recently, Zhuangzhi Zhang [**Zha97**] studied another interesting set of nonstrictly hyperbolic equations in which both families are genuinely nonlinear, and there is a coincidence line, and the system is hyperbolic (not parabolically degenerate) on the line. However, we do not know of another class of models which leads always to real eigenvalues, whether nonstrictly hyperbolic or parabolically degenerate.

2.4. Prototype Coincidence-Line Systems. A convenient way of constructing systems with a prescribed characteristic structure is to look at systems of the form

$$u_t + \big(f(u) - v\big)_x = 0 \tag{2.6}$$

$$v_t + g(u)_x = 0\,. \tag{2.7}$$

If f begins with terms of quadratic or higher order at $u = 0$, and g begins with cubic terms, then $u = 0$ is a coincidence locus, and the system has a parabolic degeneracy there. In fact, one may take $f = \alpha u^2$ and $g = \gamma u^3/3$, with $\alpha^2 > \gamma$. The eigenvalues are $\lambda = \alpha u \pm |u|\sqrt{\alpha^2 - \gamma}$, and there are three qualitatively different cases.

1. $\gamma < 0$. Kranzer and Keyfitz called this case 'opposite variation'. If the eigenvalues are reordered to $\lambda = (\alpha \pm \sqrt{\alpha^2 - \gamma})u$, then both families are genuinely nonlinear, and at the coincidence locus the direction of variation of one eigenvalue along the common eigenvector is the opposite of that of the other. We solved the Riemann problem in [**KK83**].
2. $\gamma = 0$. One family is now linearly degenerate. The solutions are similar to those in the kinematic system of [**KK80**]. This is, of course, an example of resonance, first introduced by Isaacson and Temple in [**IT92**]; see (2.4), (2.5).
3. $\gamma > 0$. In this case, both characteristic families vary in the same direction at the coincidence locus. The Riemann problem has no classical solution, and, in a sense, no bounded solution at all for some states. Schaeffer, Schecter, and Shearer [**SSS93**] found approximate solutions by considering a viscous perturbation of this system. Much of their construction is similar to the theory in [**KK95**] on the construction of singular shocks; in fact, the model introduced in [**KK95**] can be thought of as a variation of this nonstrictly hyperbolic same-variation case.

3. Loss of Genuine Nonlinearity and Change of Type

The prototypes defined in the previous section are not 'generic'; that is to say, the systems may change their properties completely under small perturbations of the coefficients. For example, if we focus on the first case, of opposite variation, then we may take $\gamma = -1/3$ and $\alpha = 0$; a perturbation of this system is

$$u_t - v_x = 0 \tag{3.1}$$

$$v_t - \left(\frac{1}{3}u^3 + \epsilon u\right)_x = 0\,. \tag{3.2}$$

When $\epsilon > 0$, this system is strictly hyperbolic, with characteristic speeds $\lambda = \pm\sqrt{u^2+\epsilon}$. However, it fails to be genuinely nonlinear along the line $u = 0$. This behavior seems to have acquired the name "genuinely not genuinely nonlinear", or *g.n.g.n.*, at this conference. Tai-Ping Liu found admissibility criteria and proved that the Riemann problem for such systems could be solved in [**Liu74**]. In [**Key87**], the construction is carried out explicitly for this example, to show that the solution is qualitatively identical to the solution to the parabolically degenerate problem with $\epsilon = 0$, and becomes that solution in the limit.

Now, in his existence proof for the Cauchy problem for the miscible displacement problem, Temple noted that the total variation of approximate solutions may be unbounded, [**Tem82**]. Since the reason for this is simply the eigenvector deficiency at the coincidence locus, the parabolically degenerate system here has the same property. (We have not checked to see whether Temple's existence proof, with the modifications of the Glimm functional introduced there, would apply to this problem, but it seems likely.) Chern ([**Che**]) has probed existence of solutions of the Cauchy problem of *g.n.g.n.* systems; constructing Glimm approximations to (3.1), (3.2) with $\epsilon > 0$ indicates a growth in the total variation of order $1/\epsilon$, consistent with the approximation. One might conjecture that this example, and *g.n.g.n.* systems in general, will share the property of resonant systems, that the time-asymptotic states differ from the Riemann solutions.

On the other hand, when $\epsilon < 0$, system (3.1), (3.2) is a prototype for *change of type*: the characteristics are complex in a strip in phase space centered on $u = 0$. Models with this property first appeared in some applications more than fifteen years ago: Richard James suggested such a model in examining propagating phase boundaries in two-phase elasticity [**Jam80**] and Marshall Slemrod saw it as the inviscid limit in the dynamics of a van der Waals gas [**Sle83**]. Shearer [**She82, She83, She86a**] began a systematic study of the Riemann problem and admissibility conditions for this model. Research on this and related problems has also grown beyond the compass of this short survey.

It is worth noting that the solution to the Riemann problem for (3.1), (3.2), except in a strip of width $\sqrt{\epsilon}$, is very similar, whether ϵ is positive, negative, or zero, [**Key95**]. This suggests that conservation law systems with coincidence line fluxes might be good models for the qualitative features of both *g.n.g.n.* and change of type systems.

A number of researchers, beginning with Helge Holden and his co-workers, [**Hh87, HH89, HHR90**], have looked at perturbations of quadratic systems by linear terms. By contrast with the perturbation in (3.1), (3.2), these perturbed systems always exhibit change of type, this time near the origin. Typically, solving the Riemann problem for such a system is very complicated, since even the homogeneous quadratic case is complex, and in some cases still unsolved. Although the genesis of the two types of problem (perturbation of quadratic fluxes and perturbation of coincidence line fluxes) is different, a number of the mathematical issues that arise are the same. In both cases, one must deal with undercompressive shocks, because there are states for which there is no Riemann solution in the class of Lax shocks. Admissibility then becomes more than a formality. There is more than one type of Riemann solution, and the possibility occurs of bifurcation between them. One can look at these problems from the viewpoint of bifurcation theory and the associated dynamics, based on adding viscosity and seeking traveling waves. A picture is beginning to emerge: shocks of nonclassical type enjoy some sorts of stability properties, and may be important in understanding both the mathematical structure and the physical significance of models in which change of type occurs. For some idea of the variety of approaches, we refer the reader to some of the papers on this subject: [**AMPZ, CP95, KL94, SMP96, Vin92**].

4. An Example from Three Phase Porous Medium Flow

Study of nonstrictly hyperbolic and change of type systems was given great impetus in the mid 1980's when such systems showed up in models for multiphase and multicomponent porous medium flow. In this section, we examine a numerical study by Bell, Trangenstein and Shubin of a specific model. In [**BTS86**], the authors assume that the flow is one-dimensional and incompressible. They also ignore gravitational effects and assume a homogeneous medium.

Letting s_w, s_o, and s_g represent water, oil, and gas saturation, respectively, and assuming that the flow is saturated, so that $s_w + s_o + s_g = 1$, then standard assumptions (Darcy's Law, conservation of mass, no capillary forces) yield a system

$$s_t + F_x = 0 \tag{4.1}$$

which expresses conservation of mass for the saturations $s = (s_w, s_g)$; the fluxes $F = (F_w, F_g)$ are empirically determined functions of s. To be specific, the model postulates $F_i = k_i/k_t$, where k_i is the relative permeability of the phase for $i = o, w, g$, scaled by the phase viscosity, and $k_t = k_w + k_o + k_g$. Stone's model gives

$$k_w = k_w(s_w), \quad k_o = k_o(s_w, s_g), \quad k_g = k_g(s_g)\,. \tag{4.2}$$

The relative permeability functions have the form

$$\begin{aligned} k_i(s_i) &= \alpha s_i^a - \beta s_i^b \quad i = w, g, ow, og, \\ k_o(s_w, s_g) &= (1 - s_w - s_g)\, k_{ow}(1 - s_w)\, k_{og}(1 - s_g)\,, \end{aligned}$$

where different constants α, β, a, and b are chosen in k_g, k_w, k_{ow}, and k_{og}, to fit experimental data for two-phase flows.

Bell, Trangenstein and Shubin were the first to note that dF may have complex eigenvalues for some physically feasible states, for some reasonable choices of the constants. In [**BTS86**], they chose the constants in such a way as to maximize the size of the nonhyperbolic region.

Now, it is tempting to regard the system (4.1) as, qualitatively, a perturbation of a homogeneous quadratic flux. A topological argument for this was presented by Shearer and Trangenstein in [**ST89**]. One can motivate the argument as follows. The deviator of a matrix A is defined to be the trace-free matrix

$$\mathrm{dev} A = A - \frac{1}{2}\mathrm{tr}(A) I \tag{4.3}$$

which can be expressed as

$$\mathrm{dev} A = \begin{bmatrix} e & b \\ c & -e \end{bmatrix} = \begin{bmatrix} e & f+g \\ f-g & -e \end{bmatrix}.$$

The mapping $\operatorname{dev} dF$ from s into the space of trace-free matrices defines a surface in $\mathbb{R}^3$. The eigenvalues of dF are real and distinct if $\operatorname{dev} dF$ lies outside the cone $e^2 + f^2 = g^2$, complex if $\operatorname{dev} dF$ lies inside the cone, and coincident on the cone. In homogeneous quadratic systems, the deviator map is a plane which passes through the vertex of the cone, and a perturbation of such a quadratic system by linear terms will typically result in a nonhyperbolic region, as the shifted plane will cut the cone in an elliptical conic section. Furthermore, one can establish, topologically, whether this will be the case for an actual three-phase flow model, by considering the deviator map for the boundary of the physically feasible set of non-negative saturations. This was done by Shearer and Trangenstein [**ST89**], who established a set of conditions on the two-phase flows, which comprise the boundary states, that guarantee a deviator map which crosses the cone. The relative permeability functions in Stone's model give ambiguous results, because the single-phase data (corners of the feasible region) also have coincident eigenvalues.

One can give a symmetry argument for why a saturation model should satisfy the Shearer-Trangenstein conditions: under the (unrealistic) assumption that all the permeability functions in (4.2) are identical, then one gets an umbilic point, and, now regarding the actual model as a perturbation of this symmetry, one gets a nonhyperbolic region which is, topologically, an elliptical conic section. This was first noted by Marchesin, and was the genesis of the interest in quadratic flux functions, [**SSMP87**].

However, we graphed the cone and the $\operatorname{dev} dF$ surface for the parameter values used by Bell, Trangenstein and Shubin, and found that the large nonhyperbolic region is not of this type: the deviator surface intersects the cone surface in a curve which is homotopic to a point in the open nappe of the cone. In fact, by changing the parameters in the model, one can eliminate this region completely, supporting L. Holden's result in [**Hl90**] that Stone's model may have several removable elliptic regions. This does not contradict the results of Shearer and Trangenstein, because nonhyperbolic behavior can still occur near the boundary of the physically feasible region. Also, Holden [**Hl90**] showed that Stone's model almost always gives a nonremovable elliptic region.

What is the nature of (4.1) when the nonhyperbolic region is minimized? It appears that there is still a curve in phase space along which genuine nonlinearity fails. That is, one characteristic family is *g.n.g.n.* — genuinely not genuinely nonlinear. In fact, Bell, Trangenstein and Shubin already found this failure of genuine nonlinearity for their choice of parameters: it occurred in one family, along two curves which run into the nonhyperbolic region. In Section 3, we observed one example in which loss of genuine nonlinearity or loss of hyperbolicity occurred according as a parameter, ϵ was positive or negative. The three-phase flow example differs from the example of Section 3 in an interesting way: loss of genuine nonlinearity occurs in only one family. It would be interesting to fomulate a prototype model for this phenomenon.

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