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Quasi-One-Dimensional Riemann Problems and Their Role in Self-Similar Two-Dimensional Problems

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Abstract. We study two-dimensional Riemann problems with piecewise constant data. We identify a class of two-dimensional systems, including many standard equations of compressible flow, which are simplified by a transformation to similarity variables. For equations in this class, a two-dimensional Riemann problem with sectorially constant data becomes a boundary-value problem in the finite plane. For data leading to shock interactions, this problem separates into two parts: a quasi-one-dimensional problem in supersonic regions, and an equation of mixed type in subsonic regions.

We prove a theorem on local existence of solutions of quasi-one-dimensional Riemann problems. For 2×2 systems, we generalize a theorem of Courant and Friedrichs, that any hyperbolic state adjacent to a constant state must be a simple wave.

In the subsonic regions, where the governing equation is of mixed hyperbolic-elliptic type, we show that the elliptic part is degenerate at the boundary, with a nonlinear variant of a degeneracy first described by Keldysh.

Key words. Quasi-one-dimensional Riemann problems – two-dimensional conservation laws – self-similar reduction – equations of mixed hyperbolic-elliptic type

1. Introduction; an Example

This paper outlines a framework for solving two-dimensional Riemann problems. For a reasonably wide class of two-dimensional conservation laws,

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solution of Riemann problems divides naturally into two parts, *supersonic* and *subsonic*. The supersonic theory can be completed by generalizations of one-dimensional results; the most straightforward are given in this paper.

There are several reasons to study two-dimensional Riemann problems. The first is that one-dimensional Riemann problems form the building blocks for the one-dimensional existence theory established by Glimm in 1965 [7]. A second stems from an observation of Rauch [13] that multidimensional systems are unlikely to satisfy the total variation estimates which are the backbone of Glimm's existence proof in one space dimension. Riemann problems may indicate what singularities are to be expected in multidimensional problems. In fact, the linear two-dimensional wave equation, $u_{tt} = c^2 \Delta u$, has two properties which characterize the systems we study in this paper:

1. The self-similar equation changes type, and
2. The self-similar solution has a singularity on the locus of change of type.

These facts are well-known. In the similarity variables $\xi = x/t$, $\eta = y/t$, the reduced equation,

$$(\xi^2 - c^2)u_{\xi\xi} + 2\xi\eta u_{\xi\eta} + (\eta^2 - c^2)u_{\eta\eta} + 2(\xi u_\xi + \eta u_\eta) = 0\,,$$

is hyperbolic or elliptic according as $\xi^2 + \eta^2$ is greater than or less than c^2. Furthermore [16, page 62], the fundamental solution of the wave equation, in similarity variables, has a square root singularity on the surface of the wave cone $\xi^2 + \eta^2 = c^2$.

Applications also motivate the study of self-similar problems. Shock reflection by a wedge results in a flow unchanged under a dilatation of the spatial coordinates $(x, y, t) \mapsto (kx, ky, kt)$. The flow field can be modeled by the isentropic or polytropic compressible Euler equations. Simplified models such as the unsteady transonic small disturbance (UTSD) equation are also used. These models have several properties in common.

In this paper, we show that change of type characterizes a class of systems when the similarity transformation $(x, y, t) \mapsto (x/t, y/t)$ is used to reduce the dimension. The reduced system resembles the steady transonic flow equation, and, in fact, change of type in self-similar systems is more like steady than like unsteady change of type, as contrasted, for example, in [10].

We show that our analysis of Riemann problems for the UTSD equation [4] is valid for a larger class of equations, including the nonlinear wave equation, the isentropic and polytropic Euler equations, and the transonic full potential equation. The class to which our analysis applies was identified in [3].

In the next section, we describe this class, giving its characteristic structure and its behavior under self-similar reduction, which always shows change of type.

In Section 3, we establish the nonlinear behavior of systems in this class, in particular the existence of a shock polar, Theorem 3.1; and for 2×2

systems we generalize a theorem of Courant and Friedrichs on simple waves, Theorem 3.2.

In Section 4, we give the proof of Theorem 3.1 and prove local existence of solutions to quasi-one-dimensional Riemann problems, Theorem 4.1.

In Section 5, we show that for the class of conservation laws we study, the equation in the subsonic region is of mixed hyperbolic-elliptic type (if there are more than two equations), and that the elliptic part is always degenerate at the boundary with a degeneracy which is a nonlinear version of a degeneracy first noted by Keldysh [9].

2. Self-Similar Reduction of Two-Dimensional Hyperbolic Systems

In this section, we define a class of systems whose behavior under self-similar reduction we can characterize, and we identify some well-posed Riemann problems.

2.1. A Class of Two-Dimensional Hyperbolic Systems

Systems of conservation laws in three independent variables (two space variables and time) are of the form

$$\partial_t H(U) + \partial_x F(U) + \partial_y G(U) = 0 \tag{2.1}$$

with $U = (u_1, u_2, \dots, u_n)$. The quasilinear form of (2.1) is

$$P(\partial, U)U \equiv (J\partial_t + A\partial_x + B\partial_y)U = 0 \tag{2.2}$$

where $J(U) = dH(U)$, $A(U) = dF(U)$ and $B(U) = dG(U)$ are $n \times n$ matrices depending on U.

The linearized symbol of P is

$$p(\sigma, U) \equiv \det |P(\sigma, U)|, \tag{2.3}$$

where $\sigma = (\tau, \mu, \nu)$ is a covector which lies in a space dual to $(t, x, y) = \boldsymbol{x}$.

We assume throughout that the operator P is hyperbolic; that is, there are timelike co-directions $\boldsymbol{N}$ such that $p(\boldsymbol{N}, U) > 0$; the equation $p(\lambda_i(\sigma)\boldsymbol{N} + \sigma, U) = 0$ has n real solutions $\lambda_i(\sigma)$ for all σ in $\mathbb{R}^3 \setminus \mathrm{span}\{\boldsymbol{N}\}$; and the symbol matrix $P(\sigma, U)$ has a complete set of eigenvectors. The solutions of $p(\sigma, U) = 0$ form the *normal cone.*

When the system is hyperbolic in time, then $(\tau, 0, 0)$ is timelike; this is not the case for the UTSD equation.

The basic assumption of this paper was introduced in [3].

Assumption 1: We can write p as a product of $n-2$ linear factors and a single quadratic factor:

$$p(\sigma, U) = \left(\prod_{i=1}^{n-2} \ell_i \cdot \sigma\right) q(\sigma, U). \tag{2.4}$$

The equations of inviscid, compressible gas dynamics, as well as the shallow water equations, linear and nonlinear wave equations, unsteady full-potential flow and the UTSD equation, all have this feature.

As shown in [3], q can be written

$$q(\sigma, U) = \sigma^T Q_N(U)\sigma$$

where Q_N is a 3×3 symmetric matrix with eigenvalues $\lambda_1 > 0 > \lambda_2, \lambda_3$.

We define the *nondegenerate normal cone*

$$\mathcal{C}_N(U) = \{\sigma | \sigma^T Q_N(U)\sigma = 0\},$$

a convex, quadratic cone through the origin in $\mathbb{R}^3$. The remaining conormals lie in $n-2$ planes (which may coincide) with normals ℓ_i. The *nondegenerate wave cone*,

$$\mathcal{C}_W(U) = \{\boldsymbol{x} | \boldsymbol{x}^T Q_W(U)\boldsymbol{x} = 0\},$$

with $Q_W = Q_N^{-1}$, is dual to $\mathcal{C}_N$: if $\boldsymbol{x} \in \mathcal{C}_W$, then $\exists \sigma \in \mathcal{C}_N$ such that $\sigma \cdot \boldsymbol{x} = 0$. (For nomenclature see Courant and Hilbert [6], or Rauch, [14].)

The axis of $\mathcal{C}_N$, $N_0 = (\tau_0, \mu_0, \nu_0)$, is the eigenvector of Q_N corresponding to λ_1. If we choose $\tau_0 > 0$, N_0 identifies one component of the complement of $\mathcal{C}_N$ in $\mathbb{R}^3$, the interior of the cone of positive timelike directions, $\mathcal{C}_N^+$.

The same direction, N_0, is the axis of $\mathcal{C}_W$, and identifies the interior of the forward wave cone.

2.2. Self-Similar Reduction and Change of Type

In the similarity variables $\Xi = (\xi, \eta) = (x/t, y/t)$, equation (2.2) reduces to

$$\begin{aligned}\widetilde{P}(\Xi, \partial, U)U &\equiv \big((A - \xi J)\partial_\xi + (B - \eta J)\partial_\eta\big)U \\ &\equiv \big(\widetilde{A}(\Xi, U)\partial_\xi + \widetilde{B}(\Xi, U)\partial_\eta\big)U = 0. \end{aligned} \quad (2.5)$$

This equation has a conservation form:

$$\begin{aligned}&\partial_\xi \widetilde{F}(\Xi, U) + \partial_\eta \widetilde{G}(\Xi, U) \\ &\qquad \equiv \partial_\xi\big(F(U) - \xi H(U)\big) + \partial_\eta\big(G(U) - \eta H(U)\big) = -2H(U). \end{aligned} \quad (2.6)$$

Let $\boldsymbol{\alpha}$ denote (α, β). The linearized symbol of $\widetilde{P}$ is

$$\tilde{p}(\Xi, \boldsymbol{\alpha}, U) = \det| - (\xi\alpha + \eta\beta)J + A\alpha + B\beta| = p(\sigma(\boldsymbol{\alpha}, \Xi), U),$$

where $\sigma(\Xi, \boldsymbol{\alpha}) = (-\Xi \cdot \boldsymbol{\alpha}, \alpha, \beta)$. Thus, $\tilde{p}$ factors as p did:

$$\tilde{p}(\Xi, \boldsymbol{\alpha}, U) = \left(\prod_{i=1}^{n-2} \ell_i \cdot (-\Xi \cdot \boldsymbol{\alpha}, \alpha, \beta)\right) q(\sigma(\Xi, \boldsymbol{\alpha}), U). \quad (2.7)$$

The linear factors, of the form $a_i\alpha + b_i\beta$, yield real roots of $\tilde{p}$. The reduced operator $\widetilde{P}$ is hyperbolic when $\tilde{q}(\Xi, \boldsymbol{\alpha}, U) \equiv q(\sigma(\Xi, \boldsymbol{\alpha}), U) = 0$ has real roots $\boldsymbol{\alpha}$. The following theorem, proved in [3], holds under Assumption 1 on the structure of p.

Theorem 2.1. *For fixed $U \in \mathbb{R}^n$, $\widetilde{P}(\Xi, \partial, U)$ is hyperbolic if and only if $(1, \xi, \eta)$ is outside the nondegenerate wave cone $\mathcal{C}_W$ of P.*

Thus, for each U, $\widetilde{P}$ is hyperbolic outside the conic section

$$\mathcal{B}_U \equiv \mathcal{C}_W \cap \{t = 1\} = \{\Xi | \boldsymbol{x}^T Q_W(U) \boldsymbol{x} = 0 \text{ with } \boldsymbol{x} = (1, \xi, \eta)\},$$

which we call the *sonic line*. In the interior of $\mathcal{B}_U$, the reduced equation is of mixed type (if $n > 2$) or elliptic ($n = 2$).

2.3. The Sonic Line and the Sonic Locus

If the forward support cone of the operator P in space-time lies in the positive-time half-space or in its closure (the latter is the case for the UTSD or transonic full-potential equation), then the sonic line $\mathcal{B}_U$ is a circle, ellipse or parabola, and not a hyperbola. We assume this is the case.

Assumption 2: The sonic line $\mathcal{B}_U$ is connected.

The sonic line satisfies a simple equation. Define the nondegenerate normal cone of the reduced problem,

$$\mathcal{C}_R(\Xi, U) \equiv \{\boldsymbol{\alpha} | \sigma(\Xi, \boldsymbol{\alpha})^T Q_N(U) \sigma(\Xi, \boldsymbol{\alpha}) = 0\}, \tag{2.8}$$

in $\mathbb{R}^2$ and write

$$\sigma(\Xi, \boldsymbol{\alpha}) = \begin{pmatrix} -\alpha\xi - \beta\eta \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\xi & -\eta \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \equiv N\boldsymbol{\alpha}.$$

Then we have

$$\sigma^T Q_N(U) \sigma = \boldsymbol{\alpha}^T Q_R(\Xi, U) \boldsymbol{\alpha} \equiv \boldsymbol{\alpha}^T N^T Q_N(U) N \boldsymbol{\alpha}. \tag{2.9}$$

Proposition 2.1. *The sonic line is characterized by*

$$\mathcal{B}_U = \{\Xi | \det Q_R(\Xi, U) = 0\},$$

and the quadratic equation $\boldsymbol{\alpha}^T Q_R \boldsymbol{\alpha} = 0$ has real roots if and only if $\det Q_R \le 0$.

Proof. The result follows from Theorem 2.1 and some elementary matrix algebra. □

We call (Ξ, U) a *hyperbolic pair* if $\widetilde{P}(\Xi, \partial, U)$ is hyperbolic. Change of type occurs in a neighborhood of the set

$$\mathcal{B} \equiv \{(\Xi, U) | \det Q_R(\Xi, U) = 0\} \subset \mathbb{R}^2 \times \mathbb{R}^n.$$

The sonic line $\mathcal{B}_U$ is the projection of $\mathcal{B}$ into self-similar physical space.

Definition 2.1. *The projection of $\mathcal{B}$ into state space $\mathbb{R}^n$ is the* sonic locus*,*

$$\mathcal{B}_\Xi = \{U | (\Xi, U) \in \mathcal{B}\}\,.$$

The sonic locus is determined by the dependence of the characteristics on U, which we now examine.

The characteristics of $\widetilde{P}$ are of two types:

(i) *degenerate* characteristics arising from the linear factors of $\tilde{p}$, and
(ii) *nondegenerate* characteristics with normals in $\mathcal{C}_R$ (equation (2.8)).

Characteristics of type (i) descend from the characteristics ℓ_i of P which propagate signals in a different manner from the nondegenerate cones which give rise to the characteristics of type (ii). We make an assumption motivated by the equations of isentropic and polytropic gas dynamics, where the type (i) characteristics, representing shear and entropy waves, are also linearly degenerate.

> **Assumption 3:** Characteristics arising from the linear factors of $\tilde{p}$ are all linearly degenerate, and their speeds (or 'angles') are intermediate; the type (ii) characteristics are the extreme characteristics.

Under this assumption, the degenerate wave directions ℓ_i of the operator P lie inside the nondegenerate wave cone $\mathcal{C}_W$, which is then the boundary of the support cone of the operator P. In most applications which do not satisfy Assumptions 1 and 3, additional nondegenerate families are present, as in MHD, crystal optics, and elasticity; see [6].

The following proposition describes the characteristics of the reduced operator $\widetilde{P}$, linearized about a constant state, U.

Proposition 2.2. *If (Ξ_0, U) is a hyperbolic pair, $\widetilde{P}$ has two nondegenerate characteristics, which consist of straight lines tangent to $\mathcal{B}_U$. The degenerate characteristics are straight lines which, under Assumption 3, lie inside the cone containing $\mathcal{B}_U$ defined by the nondegenerate characteristics.*

Proof. Since the tangent line from Ξ_0 to $\mathcal{B}_U$ is the projection into similarity space of a characteristic plane of P, it is characteristic for $\widetilde{P}$. Since $\mathcal{B}_U$ is a conic section, there are two tangents, with slopes $\boldsymbol{\alpha}_\pm$, for each Ξ_0 outside $\mathcal{B}$.

Along any degenerate characteristic $\phi(\Xi) = 0$ with $\nabla\phi = \boldsymbol{\alpha}$,

$$\frac{d\xi}{d\eta} = -\frac{\beta}{\alpha} = \frac{b - a\xi}{c - a\eta}\,,$$

(where $\ell_i = (a, b, c)$ and $\ell_i \cdot \sigma(\Xi, \boldsymbol{\alpha}) = 0$) and the solution of this equation is a straight line through $(b/a, c/a)$ and Ξ_0. Assumption 3 implies that $(b/a, c/a)$ is inside $\mathcal{B}_U$, so the characteristic lies inside the cone, as claimed. □

Genuine nonlinearity of P and $\widetilde{P}$ are needed to define Rankine-Hugoniot relations, shocks and shock admissibility. It is convenient to use the following fact, which is a consequence of Assumption 2.

Proposition 2.3. *In a neighborhood of each* $(\mu, \nu) \in \mathbb{R}^2 \backslash \{0\}$*, with at most one exceptional direction, the equation* $q(\sigma, U) = 0$ *has a well-defined solution* $\tau(\mu, \nu)$.

The definition of genuine nonlinearity for P is as follows.

Definition 2.2. *A characteristic speed* $\tau(\mu, \nu, U)$ *of* $P(\partial, U)U = 0$ *is* genuinely nonlinear *at* U *if*

$$R_\sigma \cdot \nabla_U \tau \neq 0,$$

where $P(\sigma, U)R_\sigma = 0$.

Genuine nonlinearity of hyperbolic systems in two independent variables means that the characteristic speed shows nonzero change in the direction of the corresponding characteristic vector. For a quasi-steady system, characteristic speed is replaced by characteristic angle, measured by $-\beta/\alpha$, say. Since P and $\widetilde{P}$ have the same eigenvectors at corresponding eigenvalues, we have

Definition 2.3. *A characteristic angle* $\boldsymbol{\alpha}(\Xi, U)$ *of* $\widetilde{P}$ *is* genuinely nonlinear *if*

$$R_\sigma \cdot \nabla_U \left(\frac{\beta}{\alpha} \right) \neq 0\,,$$

where $\sigma = \sigma(\Xi, \boldsymbol{\alpha}) = (-\Xi \cdot \boldsymbol{\alpha}, \alpha, \beta)$.

The two concepts are the same.

Proposition 2.4. *Suppose the reduced equation is hyperbolic at* (Ξ, U)*, and* $\boldsymbol{\alpha}(\Xi, U)$ *is a characteristic normal there. Then, if* $\tau(\boldsymbol{\alpha}, U)$ *is genuinely nonlinear at* U*, the reduced operator* $\widetilde{P}$ *is genuinely nonlinear at* (Ξ, U).

Proof. Since τ and $\boldsymbol{\alpha}$ are related by

$$\tau(\alpha, \beta, U) + \alpha\xi + \beta\eta = 0 \tag{2.10}$$

we can compute $\nabla_U(\beta/\alpha)$ implicitly; for convenience suppose $\alpha = 1$. Then

$$\nabla_U \beta = -\frac{1}{\tau_\nu + \eta} \nabla_U \tau$$

if the first factor on the right is not zero. Now, if $\Xi \in \mathcal{B}_U$, then $(1, \beta)$ is a double root of $\tilde{p}$ and hence of (2.10), and so differentiation with respect to β yields $\tau_\nu + \eta = 0$. Thus, on the sonic line, $\nabla_U \beta$ is undefined. But for $\Xi \notin \mathcal{B}_U$, $\tau_\nu + \eta \neq 0$, since β is not a double root. $\square$

The shape of the sonic line $\mathcal{B}_U$ reflects the shape of the wave cone of the linearized operator P. The shape of $\mathcal{B}_\Xi$, the sonic locus in state space, reflects the nonlinear dependence of the equation on the state U. In many examples, the sonic locus has simple global geometry. For example, it is a straight line, $\{U|u = \xi_0 + \eta_0^2/4\}$, for the UTSD. For the compressible Euler gas dynamics equations, the self-similar wave cone is

$$(u-\xi_0)^2 + (v-\eta_0)^2 = c^2$$

(with $c^2 = c^2(\rho)$ or $\gamma p/\rho$), and the sonic locus is the projection of this surface into state space.

If $U \in \mathcal{B}_\Xi$, then $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\Xi, U)$ is a double eigenvalue of $\widetilde{P}$. We have

Proposition 2.5. *At values (Ξ, U) in $\mathcal{B}$, the symbol matrix $\widetilde{P}$ has an eigenvector deficiency.*

Proof. The reduced equation $(\widetilde{A}\alpha + \widetilde{B}\beta)R = 0$ corresponds to the equation

$$(\tau J + \alpha A + \beta B)R = 0 \tag{2.11}$$

in the original system with $\tau = -\alpha\xi - \beta\eta$. Equation (2.11) does not have τ as a double root; indeed, if the pair (α, β) gives rise to two roots of the quadratic equation $\sigma^T Q_N(U)\sigma = 0$, then one is $-\alpha\xi - \beta\eta$, and the other corresponds to a different point, Ξ_1 say. Otherwise said, since the double eigenvalue is an artifact of the self-similar reduction, it does not yield a higher-dimensional eigenspace. □

Denote right and left eigenvectors of $\widetilde{P}$ on $\mathcal{B}$ by R_* and L_*.

We require R_* to be transverse to $\mathcal{B}_\Xi$. For steady systems of two equations, this was a consequence of genuine nonlinearity [10]. For operators of order $n > 2$, we impose an additional assumption.

> **Assumption 4:** At any point U_* on $\mathcal{B}_\Xi$, the right eigenvector R_* is transverse to $\mathcal{B}_\Xi$.

If a normal $\boldsymbol{\nu}$ is chosen pointing into the nonhyperbolic region, for example $\boldsymbol{\nu} = \nabla_U \det Q_R(\Xi, U)$, then we may normalize R_* with $\boldsymbol{\nu} \cdot R_* < 0$, so R_* points into the hyperbolic region.

We will use this information to find the shock polar in Section 3.1 (Theorem 3.1).

2.4. Well-Posed Riemann Problems

Several types of initial data lead to two-dimensional self-similar problems. The following have all been suggested by various authors.

1. *Data Constant in Sectors*:

$$U(0,x,y) = U_i, \text{ for } a_i \le \theta < a_{i+1}, i = 1, \dots, k\,,$$

where $(x, y) = (r\cos\theta, r\sin\theta)$.

Two special cases have received particular attention.

2. *Data Constant in Quadrants*:

$$U(0,x,y) = U_i, \text{ for } (x,y) \in Q_i, i = 1, \dots, 4.$$

The isentropic and polytropic compressible Euler equations with data of this form were examined by Zhang and Zheng, [18]; see also Schultz-Rinne, Collins and Glaz, [15].

3. *Oblique Shock Interaction Data*:

$$U(0,x,y) = U_i, \text{ for } a_i \le \theta < a_{i+1}, i = 1, k,$$

with U_i, U_{i+1} and a_{i+1} related by $U_{i+1} \in \mathcal{H}(U_i, a_{i+1})$, the Hugoniot locus with shock angle a_{i+1}. This is the case considered in [4]. The Hugoniot locus is given by Definition 3.1.

In principle, the only restriction on Riemann data is that U_0 be constant on rays.

4. *General Cauchy Data*:

$$U(0,x,y) = U_0(\theta)\,,$$

where $(x,y) = (\cos\theta, \sin\theta)$.

If the Riemann problem for (2.1) is well-posed, then the solution is self-similar: $U(t,x,y) = U(\xi,\eta)$. Solving the initial-value problem is then equivalent to solving the reduced equation with boundary data

$$U(\xi,\eta) \to U_0(\theta) \quad \text{as} \quad (\xi,\eta) = (|\Xi|\cos\theta, |\Xi|\sin\theta) \to \infty \tag{2.12}$$

corresponding to the given initial condition. Since $|\Xi| \to \infty$ as $t \to 0$ and $|\Xi| \to 0$ as $t \to \infty$, the forward timelike direction for the reduced problem is inward toward the origin. For the first three types of Riemann problem, we expect that (2.12) can be replaced by a finite curve.

Proposition 2.6. *Suppose that the Riemann problem for the operator P has the properties that*

(i) the forward wave cone $\mathcal{C}_W(U)$ and the degenerate directions ℓ_i intersect the plane $\{t = 1\}$ in a compact set K for all $|U| \le M$,
(ii) the solution to the Riemann problem with data $U_0(\theta)$ is bounded by M.

Then, for piecewise constant (sectorial) Riemann data U_0, the solution to the reduced problem is one-dimensional sufficiently far from the origin, and for the reduced problem, data equivalent to (2.12) *may be given on a circle $\xi^2 + \eta^2 = C$ at a finite distance from the origin.*

Proof. Outside K, the reduced equation is hyperbolic, and a curve is spacelike if, from every point on it, all the characteristics drawn in the direction of K lie on one side of it. We have proved that the nondegenerate characteristics are tangent to $\mathcal{C}_W$, and the degenerate characteristics, by hypothesis (i), point into K. Since K is bounded, any sufficiently large circle, $\xi^2 + \eta^2 = C$, will be spacelike. Hypothesis (ii) implies that one-dimensional Riemann problems for the original system are solvable. Thus, at each discontinuity on $\xi^2 + \eta^2 = C$, the one-dimensional Riemann problem can be solved locally with bounded wave speeds. Since the distance between successive discontinuities around the circle grows without bound as $C \to \infty$, there is a finite C which is large enough that waves arising at different discontinuities do not interact. $\square$

Since hypothesis (ii) cannot be verified a priori on the basis of our present knowledge of two-dimensional conservation laws, we can use this proposition only to justify the procedure a posteriori.

It does not appear that the fourth type of Riemann problem can be treated by the reduction of Proposition 2.6, since one-dimensional shock waves will form from smooth data arbitrarily far from the origin.

If we suppose that there exists a bounded solution to the Riemann problem, construction of the solution for sectorially constant Riemann data falls into three parts:

1. Along the finite curve $\xi^2 + \eta^2 = C$, there is a one-dimensional Riemann problem at each jump, whose solution propagates parallel to the sector boundary;
2. Hyperbolic problems for the reduced operator are created when one-dimensional waves from different sector boundaries interact;
3. Elliptic or mixed type boundary and free-boundary problems arise in a neighborhood of the origin.

We shall assume that solutions to the first type of problem can be constructed, and turn our attention in the remainder of this paper to the second.

We will show how to solve those problems which have a *quasi-one-dimensional* character: these are Riemann problems for the reduced system localized at the point of interaction of different one-dimensional shocks or linear waves.

3. Quasi-One-Dimensional Waves

In this section, we define the basic building blocks for solving $\widetilde{P}U = 0$ in the hyperbolic region.

3.1. Shocks and the Hugoniot Locus

A self-similar shock configuration consists of constant states U_0 and U separated by a shock line $\xi = \kappa\eta + \omega$ satisfying (2.6) in the weak sense.

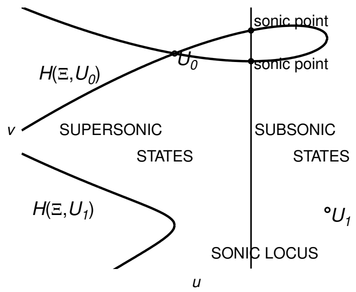

Fig. 3.1. Shock Polars for the UTSD Equation

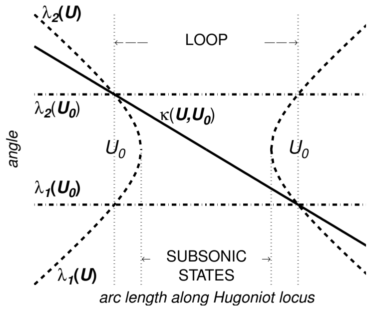

Fig. 3.2. Shock and Characteristic Angles on the Hugoniot Locus

Fixing a state U_0, consider shocks through a given point $\Xi_0 = (\xi_0, \eta_0)$; κ measures the shock angle ($\kappa = \cot\theta$ where θ is the angle to the horizontal) and $\omega = \xi_0 - \kappa\eta_0$. The Rankine-Hugoniot relation,

$$\big(F(U) - F(U_0)\big) - \kappa\big(G(U) - G(U_0)\big) - (\xi_0 - \kappa\eta_0)\big(H(U) - H(U_0)\big) = 0\,, \quad (3.1)$$

is parameterized by κ.

Definition 3.1. *For a given base point Ξ_0 in physical space, and a fixed state U_0, we say U is in the* Hugoniot locus of U_0 based at Ξ_0, *$U \in \mathcal{H}(\Xi_0, U_0)$, if* (3.1) *is satisfied for some shock angle κ.*

For a hyperbolic pair (Ξ_0, U_0), and with Ξ_0 fixed, equation (3.1) is a standard Rankine-Hugoniot relation: in a neighborhood of U_0 there are two branches of solutions, corresponding to the two nondegenerate characteristic angles $\boldsymbol{\alpha}_\pm$; κ is close to a characteristic angle on each branch, is monotonic along the branch, and parameterizes the curve locally. There are no solutions of this type near U_0 if (Ξ_0, U_0) is not hyperbolic. Solutions corresponding to the linearly degenerate branches form contact discontinuities (slip lines) which are discussed in Section 4.2.

Change of type in the operator $\widetilde{P}$ changes the problem. In many examples, $\mathcal{H}$ resembles the shock polar of steady transonic flow. That is, for any hyperbolic pair (Ξ_0, U_0), the locus forms a single curve, globally parameterized by κ, with a single self-intersection at U_0. A segment of the loop consists of nonhyperbolic points. If (Ξ_0, U_0) is not hyperbolic, the locus forms a single smooth curve, with no self-intersections, consisting entirely of hyperbolic points. The two cases are illustrated in Figure 3.1.

By assuming that $\|U - U_0\|$ is small, we obtain a general result for operators P satisfying Assumptions 1, 2, 3, and 4. The results are local in a second sense: the solutions may be valid only in a part of the Ξ plane. This is a consequence of the geometry: all time-like directions in the Ξ plane run into a neighborhood of the origin, so the domain of determinacy of data given on a line is smaller than a half-space.

The organizing feature in the local construction of shock polars is the sonic locus of Definition 2.1, where the equation changes type. Our main result is that the shock polar found in examples occurs in all problems near the locus of change of type.

Theorem 3.1. *Consider an operator P satisfying Assumptions 1, 2, and 3. Let the sonic locus $\mathcal{B}_\Xi$ satisfy Assumption 4. Then there is an open neighborhood $\mathcal{N}$ of $\mathcal{B}_\Xi$ in $\mathbb{R}^n$ such that the following are true. If $U_0 \in \mathcal{B}_\Xi$, then the shock polar $\mathcal{H}(\Xi, U_0)$ is a curve consisting of hyperbolic points, with a cusp at U_0. If U_0 is near the locus and (Ξ, U_0) is hyperbolic, then $\mathcal{H}(\Xi, U_0)$ forms a loop and contains subsonic points; if (Ξ, U_0) is subsonic, the shock polar is detached from U_0 and consists entirely of supersonic points.*

Corollary 3.1. *In the same neighborhood $\mathcal{N}$, the shock angle κ is monotone along the shock polar and the relative sizes of the shock and characteristic angles along the shock polar are as given in Figure 3.2.*

The proof is given in Section 4.1.

3.2. Admissibility of Quasi-Steady Shocks

From the local description of the shock angle and its relation to the characteristics, one can determine which shocks are admissible.

The Lax geometric entropy condition for admissibility of shocks is a one-dimensional criterion. However, the shocks in our problem are extreme shocks, by Assumption 3, and so the condition agrees with the multidimensional stability criterion of Majda [12].

From Figure 3.2 it is easy to verify the geometric criterion. On one of the semi-infinite segments of the Hugoniot locus,

$$\lambda_2(U) > \kappa(U, U_0) > \lambda_2(U_0)\,, \quad \kappa(U, U_0) > \lambda_1(U_0) > \lambda_1(U),$$

and on the other

$$\lambda_1(U_0) > \kappa(U, U_0) > \lambda_1(U)\,, \quad \lambda_2(U) > \lambda_2(U_0) > \kappa(U, U_0)\,.$$

From the geometry of the characteristics and the sonic line, it is clear that the shock in $\xi\eta$-space must cross one of $\mathcal{B}_U$ and $\mathcal{B}_{U_0}$ and not the other, as in Figure 3.3. At any point Ξ_0 on the shock, the shock itself is a spacelike curve with respect to one state and not the other. The Lax admissibility criterion, applied to hyperbolic states, implies that the shock is spacelike with respect to the state which is on its upstream side. The upstream condition is the standard admissibility criterion for steady transonic shocks [10], and extends the Lax admissibility criterion to the middle segment of the loop, where U is subsonic: in this segment, U_0 is upstream and U downstream from the shock, as is the case on the adjacent hyperbolic segments in the closed part of the loop. On the semi-infinite segments of the locus, U is the upstream state.

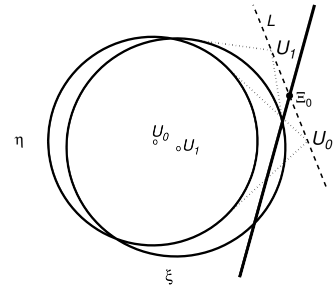

Fig. 3.3. The Shock and Sonic Lines: Gamma-Law Gas

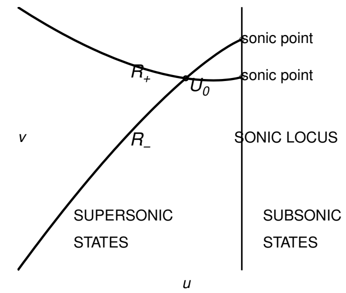

Fig. 3.4. Rarefaction Locus in State Space for the UTSD Equation

3.3. Simple Waves and Centered Rarefactions

Simple waves also occur in reduced systems.

Definition 3.2. *A solution of $\widetilde{P}(\Xi, \partial, U)U = 0$ is a* simple wave *if there exists a scalar-valued function h such that*

$$U(\xi, \eta) = \overline{U}(h(\xi, \eta)). \tag{3.2}$$

The basic properties of simple waves are given by the following proposition.

Proposition 3.1. *Suppose $\overline{U}$ is a simple wave. Then*

1. *(Ξ, U) is hyperbolic and $(h_\xi, h_\eta) = (\alpha, \beta)$ is a characteristic normal,*
2. *U_ξ and U_η are parallel and are eigenvectors of $\widetilde{P}$, and*
3. *the corresponding characteristic family forms straight lines.*

Proof. We have, substituting in (2.5),

$$\widetilde{P}\overline{U} = \widetilde{A}\overline{U}' h_\xi + \widetilde{B}\overline{U}' h_\eta = (\widetilde{A} h_\xi + \widetilde{B} h_\eta)\overline{U}' = 0.$$

Thus, $\overline{U}_\xi$ and $\overline{U}_\eta$ are both parallel to $\overline{U}'$, which must be an eigenvector, R, of $\widetilde{P}(\Xi, (h_\xi, h_\eta), U)$, with (h_ξ, h_η) the corresponding characteristic normal. This implies that the characteristics are real, and hence (Ξ, U) is hyperbolic.

To see that the characteristics of the corresponding family are straight lines, write $h \equiv \xi - \phi(\eta)$. Thus, $(h_\xi, h_\eta) = (\alpha, \beta)$ with $\alpha = 1$, and, along a curve $\overline{U} = const$, we compute

$$\frac{d\xi}{d\eta} = \phi'(\eta) = -\beta(\Xi, \overline{U}(h)) \, .$$

That is, along this curve, ϕ solves the ordinary differential equation

$$\phi' = -\beta\big((\phi(\eta) + h, \eta), \overline{U}(h)\big).$$

On the other hand, there is a straight line characteristic with the direction $(1, \beta)$ at Ξ, along which U has the value $\overline{U}(h)$. By the uniqueness theorem for ordinary differential equations, the characteristic through Ξ must be this straight line. □

Corollary 3.2. *If U_ξ and U_η are linearly dependent in an open set, then they are also eigenvectors of $\widetilde{P}$, and U forms a simple wave.*

To solve Riemann problems centered at a point Ξ_0 in a quasi-one-dimensional system, we define a *centered simple wave (centered rarefaction)* to be a simple wave of the form

$$U = U\left(\frac{\xi - \xi_0}{\eta - \eta_0}\right). \tag{3.3}$$

For a fixed state U_0, the locus in state space of points that can be joined to U_0 by a rarefaction wave, centered at Ξ_0, of the form (3.3) is a curve parameterized by $\kappa = (\xi - \xi_0)/(\eta - \eta_0)$. If we define R_i to be a unit eigenvector corresponding to a genuinely nonlinear root $\boldsymbol{\alpha}_i$ of $(\alpha\widetilde{A} + \beta\widetilde{B})R = 0$, and solve $U' = R_i(U)$, we obtain the *rarefaction locus* of states which can be joined to U_0 by a centered rarefaction wave, parameterized by arclength, μ. Thus $U(\mu)$ satisfies

$$\big(\widetilde{A}(\Xi, U(\mu)) - \kappa(\mu)\widetilde{B}(\Xi, U(\mu))\big)R(U(\mu)) = 0\,.$$

Genuine nonlinearity implies $\kappa' \neq 0$ and we can solve for $\mu(\kappa)$. Since, by Assumption 4, the directions R_i are transversal to $\mathcal{B}_\Xi$, so near $\mathcal{B}_\Xi$ the rarefaction loci run into $\mathcal{B}_\Xi$ and terminate there; see Figure 3.4.

Definition 3.3. *The collection of states U which can be joined to a given state U_0 by an admissible shock or rarefaction wave, with U_0 the upstream state, forms the* downstream locus, *$\mathcal{D}(\Xi_0, U_0)$.*

The following is a corollary of Theorem 3.1.

Proposition 3.2. *If $U_0 \in \mathcal{N}$, then $\mathcal{D}(U_0, \Xi_0)$ consists of the loop part of the Hugoniot locus $\mathcal{H}(\Xi_0, U_0)$ and the portion located in $\mathcal{N}$ of the semi-infinite segments of the rarefaction locus for U_0 centered at Ξ_0.*

Simple waves arise, typically, next to regions of constant states. In the next subsection, we generalize a result of Courant and Friedrichs [5, page 60 ff] to reduced systems of two equations.

3.4. The Theorem of Courant and Friedrichs

To give a precise statement, we formalize the notion of adjacency.

Definition 3.4. *A state U is* adjacent *to a state U_0 if there are two open sets Ω and Ω_0 in physical space whose boundaries have a common curve D, such that U and U_0 are classical C^1 solutions of a conservation law system in Ω and Ω_0 respectively, and are continuous across D.*

When one state, U_0, say, is constant, Courant and Friedrichs, [5], prove a 'Fundamental Lemma' and 'Fundamental Theorem' for systems in two variables of the form $A(U)U_\xi + B(U)U_\eta = 0$. Although our systems do not have this form, we can still prove the result for systems of two equations.

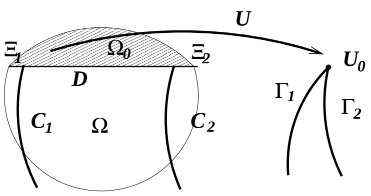

Fig. 3.5. Physical Space and State Space

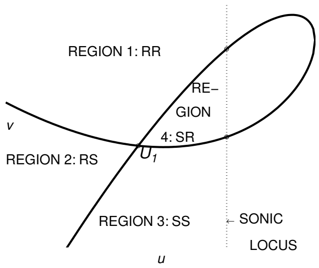

Fig. 3.6. The Downstream Locus and Complementary Regions

Theorem 3.2. *Let*

$$\widetilde{P}U = \big(A(U) - \xi J(U)\big)U_\xi + \big(B(U) - \eta J(U)\big)U_\eta = 0$$

be a quasi-one-dimensional system of two equations. If U is a hyperbolic state adjacent to a hyperbolic, constant state U_0, then U is a simple wave.

Proof. Consider the hodograph mapping of an open subset of $\mathbb{R}^2$ to $\mathbb{R}^2$, given by a solution $U(\Xi)$. If U is a simple wave, then the image of this mapping is one-dimensional, as each straight-line characteristic is mapped to a point $U(\kappa)$.

If U is *not* a simple wave, then U_ξ and U_η are not linearly dependent in Ω, by Corollary 3.2. Suppose this is the case for a supersonic solution U. Now (Ξ, U_0) is also supersonic on D; since the system is strictly hyperbolic, any discontinuity in a derivative must lie on a characteristic, and so D is characteristic. Hence, since U_0 is constant, D is a straight line.

We can rule out the possibility that U_ξ or U_η becomes infinite as $U \to U_0$. For, if $\widetilde{P}$ remains strictly hyperbolic up to D, then blow-up of derivatives occurs only as a consequence of shock formation, when two characteristic curves of the same family approach each other as $\Xi \to D$. Shock formation is inconsistent with the assumption that U is continuous. Hence if U_ξ and U_η blow up, then strict hyperbolicity must fail, and this contradicts the assumption that U is hyperbolic.

Now, taking limits in Ω, we see that D is also characteristic for U. There are two characteristic families; call them 1 and 2, and suppose for definiteness that D is a 1-characteristic. Let Ξ_1 and Ξ_2 be distinct points on D and let C_1 and C_2 be the 2-characteristics in Ω through them, as in Figure 3.5. Since $[U_\xi, U_\eta]$ has rank two in Ω, the image of each $C_i \cap \Omega$ under the map U is a curve $\Gamma_i = U(C_i)$ in $\mathbb{R}^2$. Along C_i the equation

$$\frac{d\xi}{d\eta} = \lambda_2(\Xi, U) \tag{3.4}$$

defines a curve $\xi(\eta)$. (Here λ is the characteristic angle $-\beta/\alpha$.) We seek an equation for Γ_i. Unless both $\det(\widetilde{A})$ and $\det(\widetilde{B})$ are identically zero in

Ω, we may assume, say, $\det(\widetilde{B}) \neq 0$ in Ω, and then write the system as $U_\eta + MU_\xi = 0$, where $M = \widetilde{B}^{-1}\widetilde{A}$. Then, along Γ_i, with $\dot{\xi}$ given by (3.4) and with η parameterizing Γ_i,

$$\dot{U} = U_\xi \dot{\xi} + U_\eta = (\lambda_2 - M)U_\xi \tag{3.5}$$

in the interior of Ω. Equation (3.5) implies $L_2^T \dot{U} = 0$, where L_2 is a unit left 2-eigenvector of M. This in turn implies that $\dot{U}$ has the direction R_1, a unit right 1-eigenvector of M. We reparametrize (3.5) so that η is replaced by arc length, s, along the curve Γ_i. Then, we rewrite (3.5) as

$$\frac{dU}{ds} = R_1(U, \Xi(U)) \equiv \mathcal{F}(U)\,; \tag{3.6}$$

where R_1 is given as a function of U and Ξ by

$$(\widetilde{A} - \lambda_1 \widetilde{B})R_1 = 0, \quad \|R_1\| = 1 \tag{3.7}$$

and Ξ is further given as a function of U by inverting the relation $U = U(\Xi)$, valid in Ω. Equation (3.6) is the representation we sought for Γ_i. We claim that $\mathcal{F}$ is a Lipschitz vector field on the closure of $U(\Omega)$; for this it is sufficient to show that $d\mathcal{F}$ is bounded. Denote the components of U by (u, v). Since $\partial R_1/\partial U$ is clearly bounded, the only thing to verify is that the composite derivative,

$$\begin{pmatrix} \partial_\xi R_1 & \partial_\eta R_1 \end{pmatrix} \begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix} = \frac{1}{\mathcal{J}} \begin{pmatrix} \partial_\xi R_1 & \partial_\eta R_1 \end{pmatrix} \begin{pmatrix} v_\eta & -u_\eta \\ -v_\xi & u_\xi \end{pmatrix}$$

remains bounded as $\Xi \to D$ and $U \to U_0$. (Here, $\mathcal{J} = u_\xi v_\eta - u_\eta v_\xi$ is the Jacobian of the mapping U.)

Now, $\partial_\xi R_1$ and $\partial_\eta R_1$ are calculated from (3.7) by implicit differentiation with the result

$$\partial_\xi R_1 = \left\{ \frac{1}{\lambda_2 - \lambda_1} L_2^T \widetilde{B}^{-1} J R_1 \right\} L_2\,, \qquad \partial_\eta R_1 = -\lambda_1 \partial_\xi R_1\,.$$

Ignoring some bounded factors, the result of the calculation of the composite derivative is the matrix

$$\frac{1}{\mathcal{J}} L_2 \big(v_\eta + \lambda_1 v_\xi \quad -(u_\eta + \lambda_1 u_\xi) \big).$$

But, finally, it can be verified that $\mathcal{J}$ factors, up to a bounded scale factor, into a product $(u_\eta + \lambda_1 u_\xi)(u_\eta + \lambda_2 u_\xi)$ or $(v_\eta + \lambda_1 v_\xi)(v_\eta + \lambda_2 v_\xi)$. The first factor of each product, but not the second, tends to zero as $\Xi \to D$. Since these factors cancel exactly with terms in the numerator, the entire expression is bounded in the closure of $U(\Omega)$.

Hence the vector field $\mathcal{F}(U)$ in (3.6) satisfies a Lipschitz condition.

With this we can now prove that the image of the mapping U is, in fact, one-dimensional, by showing that two solutions U_1 and U_2 of (3.6) must be identical. Let us denote by A the Lipschitz constant, so

$$|\mathcal{F}(U_1) - \mathcal{F}(U_2)| \leq A|U_1 - U_2|. \tag{3.8}$$

Let U_1 and U_2 be two solutions such that $U_1(0) = U_2(0) = U_0$. Each solution satisfies

$$U_i(s) = U_0 + \int_0^s R_1(U_i, \Xi(U_i))ds. \tag{3.9}$$

Subtracting and using (3.8) we get

$$|U_1 - U_2| \leq A \int_0^s |U_1 - U_2||ds| = A \int_0^s |U_1 - U_2|ds. \tag{3.10}$$

Gronwall's inequality immediately yields $U_1 = U_2$. Therefore, U_ξ and U_η are linearly dependent, and U must be a simple wave. □

4. The Shock Polar; Quasi-One-Dimensional Riemann Problems

In this section, we prove Theorem 3.1, and then show how the shock polar can be used to construct solutions to quasi-one-dimensional Riemann problems with data in a neighborhood of $\mathcal{B}$.

4.1. Proof of Existence of the Shock Polar

In Theorem 3.1, Ξ is fixed, so the situation is similar to that described in an earlier paper [10] for the case $n = 2$.

Proof. The proof uses the qualitative bifurcation theory of Golubitsky and Schaeffer [8]. We claim that the model is a one-state-variable bifurcation problem with normal form

$$h(x, \lambda) = x(x - \lambda^2) \tag{4.1}$$

when $U_0 \in \mathcal{B}_\Xi$, and is found using Liapunov-Schmidt reduction. The appropriate equivalence relation is t-equivalence, [8, page 129]. Equation (4.1) is a codimension one bifurcation problem, whose universal unfolding is

$$h(x, \lambda, a) = x(x - \lambda^2 + a)\,. \tag{4.2}$$

We find the organizing center by choosing a fixed $U_0 = U_* \in \mathcal{B}_\Xi$; then, with R_* the right eigenvector of P corresponding to the double root κ_*, taking $x = (U - U_*) \cdot R_*$ and $\lambda = \kappa - \kappa_*$. Finally, we obtain the unfolding by moving U_0 away from $\mathcal{B}_\Xi$; for example, by choosing $a = (U_0 - U_*) \cdot R_*$.

An explicit change of coordinates will prove that (4.1) and (4.2) give the normal form for the bifurcation problem when we make these choices. The assertions of Theorem 3.1 and Corollary 3.1 are consequences of the normal form.

There are several steps to the proof.

In the next subsection, we show that the defining conditions for the singularity are $h_x(0,0) = 0 = h_{x\lambda}(0,0)$ and the nondegeneracy conditions are $h_{xx}(0,0) \neq 0$ and $h_{x\lambda\lambda}(0,0) \neq 0$.

For the Liapunov-Schmidt reduction we follow the recipe in [8, page 26ff].

The parameterized bifurcation problem is, from (3.1),

$$V(U,\kappa;U_0) \equiv F(U) - \kappa G(U) - (\xi_0 - \kappa\eta_0)H(U) \\ - \left[F(U_0) - \kappa G(U_0) - (\xi_0 - \kappa\eta_0)H(U_0)\right].$$

Since $V(U_0,\kappa;U_0) \equiv 0$, a trivial solution exists for all κ, but there is no other symmetry; hence the appropriate equivalence relation is t-equivalence. Note also that

$$d_U V = A(U) - \kappa B(U) - (\xi_0 - \kappa\eta_0)J = \widetilde{A}(U) - \kappa\widetilde{B}(U)\,. \tag{4.3}$$

Now, we fix $U_0 = U_*$ in $\mathcal{B}_\Xi$, and let $\kappa_* = -\beta/\alpha$ be the double eigenvalue of $d_U V$ corresponding to the double characteristic $\boldsymbol{\alpha}$. The equation

$$(\widetilde{A}(U_*) - \kappa\widetilde{B}(U_*))R_* = 0$$

defines a unique right eigenvector R_*; similarly there is a unique left eigenvector L_*, as defined following Proposition 2.5. Now $\mathrm{Rank}(d_U V|_{(U_*,\kappa_*)}) = n-1$; $\ker(d_U V(0)) = \mathrm{span}\{R_*\}$. Also, L_* is orthogonal to the range of $d_U V(0)$. We have $\mathbb{R}^n = \mathrm{span}\{R_*\} \oplus M$, say, and also $\mathbb{R}^n = \mathrm{span}\{L_*\} \oplus \mathrm{Range}(d_U V(0))$. Let E be the projection of $\mathbb{R}^n$ onto $\mathrm{Range}(d_U V(0))$, so that $\ker(E) = L_*$. The projection $I - E$ has range L_* and its kernel is the range of $d_U V(0)$. We use the fact that $V(U,\kappa,U_*) = 0$ if and only if

$$EV(U,\kappa,U_*) = 0, \tag{4.4}$$

$$(I-E)V(U,\kappa,U_*) = 0. \tag{4.5}$$

Now, the idea is that (4.4) gives a map of rank $n-1$ which can be inverted, using the implicit function theorem, to solve for $n-1$ of the components of U, and then the bifurcation behavior is found from (4.5). To be specific, if we write $U - U_* = v + w$, where $v = xR_*$ and $w \in M$, then (4.4) becomes

$$EV(U_* + xR_* + w, \kappa; U_*) = 0\,,$$

where we now consider a map from $(x,w,\kappa) \mapsto \mathrm{Range}(d_U V(0))$, or $\mathrm{span}\{R_*\} \times M \times \mathbb{R} \to \mathrm{Range}(d_U V(0))$. If we define $f(x,w,\kappa) \equiv EV(U_* + xR_* + w, \kappa; U_*)$ then $d_w f(0) = E(d_U V(0)) = d_U V(0)$ since E is the identity on the range of $d_U V(0)$. That is, as a map from M to $\mathrm{Range}(d_U V(0))$, $d_U V(0)$ is invertible, and so (4.4) has a unique solution $w = W(x,\kappa)$ near 0, defined by

$$EV(U_* + xR_* + W(x,\kappa), \kappa; U_*) \equiv 0\,, \quad W \cdot R_* = 0 \tag{4.6}$$

in a neighborhood of (U_*, κ_*). We now define g from (4.5) by

$$g(x,\kappa) \equiv L_*^T V(U_* + xR_* + W(x,\kappa), \kappa; U_*)$$

and our task is complete if we can show that g satisfies the defining and nondegeneracy conditions for h given above. To compute derivatives of g, however, we need to know derivatives of W, which are given implicitly by (4.6). Differentiating with respect to x, we find

$$E d_U V(U_* + xR_* + W, \kappa; U_*)[R_* + W_x(x,\kappa)] = 0 \tag{4.7}$$

and setting $x = \kappa = 0$ we get $W_x(0,0) = 0$ since $d_U V(0)$ is invertible on $R_*^\perp$. Similarly, differentiating (4.6) with respect to κ gives

$$E[d_U V(U_* + xR_* + W, \kappa; U_*) W_\kappa(x,\kappa) - G(U) + \eta H(U)] = 0 \tag{4.8}$$

and at $(0,0)$ we have

$$d_U V(0) W_\kappa(0,0) + E\{-G(U_*) + \eta H(U_*)\} = 0\,.$$

To simplify the calculations, assume that $G(U_*) - \eta H(U_*) = 0$, and so we get $W_\kappa(0,0) = 0$.

Using (4.3), we derive

$$\begin{aligned}
g_x &= L_*^T(\widetilde{A} - \kappa\widetilde{B})(R_* + W_x),\\
g_{xx} &= L_*^T(\widetilde{A} - \kappa\widetilde{B})[W_{xx}] + L_*^T d_U^2 V[R_* + W_x, R_* + W_x], && (4.9)\\
g_{x\kappa} &= L_*^T(-\widetilde{B})(R_* + W_x) + L_*^T(\widetilde{A} - \kappa\widetilde{B})W_{x\kappa}\\
&\quad + L_*^T d_U^2 V[R_* + W_x, W_\kappa], && (4.10)\\
g_{x\kappa\kappa} &= 2L_*^T(-\widetilde{B})W_{x\kappa} + L_*^T(-d_U^2 G)[R_* + W_x, W_\kappa]\\
&\quad + L_*^T(\widetilde{A} - \kappa\widetilde{B})W_{x\kappa\kappa} + L_*^T d_U^3 V[R_* + W_x, W_\kappa, W_\kappa]\\
&\quad + L_*^T d_U^2 V[W_{x\kappa}, W_\kappa] + L_*^T d_U^2 V[R_* + W_x, W_{\kappa\kappa}]\,.
\end{aligned}$$

Now, $L_*^T(\widetilde{A}(0) - \kappa_*\widetilde{B}(0)) = 0$, so we immediately get $g_x(0,0) = 0$. Also, this relation and the fact that $W_x(0,0) = W_\kappa(0,0) = 0$ take care of most of the other terms; in (4.10) we are left with only $g_{x\kappa}(0,0) = -L_*^T\widetilde{B}R_*$; below, we demonstrate that this is zero. In (4.9), we have

$$g_{xx}(0,0) = L_*^T d_U^2 V[R_*, R_*] \neq 0$$

as a consequence of genuine nonlinearity. Finally,

$$g_{x\kappa\kappa} = -2L_*^T\widetilde{B}(U_*)W_{x\kappa} + L_*^T d_U^2 V[R_* + W_x, W_{\kappa\kappa}]$$

and here we claim that the second term is zero and the first is not. To see that $W_{\kappa\kappa}(0,0) = 0$, which makes the second term zero, we differentiate (4.8) with respect to κ, which gives

$$E d_U V[W_{\kappa\kappa}] + E d_U^2 V[W_\kappa, W_\kappa] + E\{-\widetilde{B}\}W_\kappa = 0$$

and since $Ed_U V$ is invertible on M and the other terms are zero when $x = \kappa = 0$, this gives $W_{\kappa\kappa}(0,0) = 0$.

The remaining claims follow from the eigenstructure of $d_U V$ at U_*. Since, R_* is the only null vector of $\widetilde{A}_* - \kappa\widetilde{B}_*$, there is a generalized null vector, which satisfies

$$(\widetilde{A}_* - \kappa\widetilde{B}_*)R_1 = \gamma\widetilde{B}_* R_*$$

for some $\gamma \neq 0$. (To see that this is the correct condition, suppose $\widetilde{B}_*$ to be invertible.) Now we write

$$g_{x\kappa}(0,0) = -L_*^T\widetilde{B}_* R_* = -\frac{1}{\gamma}L_*^T(\widetilde{A}_* - \kappa\widetilde{B}_*)R_1 = 0 \tag{4.11}$$

which verifies the other defining condition. To show $g_{x\kappa\kappa} \neq 0$, we first find $W_{x\kappa}$, by differentiating (4.7) with respect to κ. This gives

$$Ed_U^2 V[R_* + W_x, W_\kappa] + E\{-\widetilde{B}\}[R_* + W_x] + Ed_U V[W_{x\kappa}] = 0$$

whence

$$Ed_U V(0)W_{x\kappa}(0,0) = E\widetilde{B}_* R_* \,.$$

Again, $Ed_U V(0)$ is invertible on M and E is the identity there, so we may write $W_{x\kappa}(0,0)$, abusing notation somewhat, as

$$W_{x\kappa}(0,0) = (\widetilde{A}_* - \kappa\widetilde{B}_*)^{-1}\widetilde{B}_* R_* \,.$$

So then

$$\begin{aligned} g_{x\kappa\kappa} &= -2L_*^T\widetilde{B}_*(\widetilde{A}_* - \kappa\widetilde{B}_*)^{-1}\widetilde{B}_* R_* \\ &= -2L_*^T\widetilde{B}_*(\widetilde{A}_* - \kappa\widetilde{B}_*)^{-1}\frac{1}{\gamma}(\widetilde{A}_* - \kappa\widetilde{B}_*)R_1 = -\frac{2}{\gamma}L_*^T\widetilde{B}_* R_1 \,, \end{aligned}$$

and we claim this is nonzero. For $L_*^T\widetilde{B}_* \neq 0$; on the other hand $L_*^T\widetilde{B}_*$ is orthogonal to R_*, as we just saw in (4.11); and it is also orthogonal to the $n-2$ eigenvectors R_j corresponding to the other roots κ_j and hence it cannot also be orthogonal to R_1.

The Liapunov-Schmidt construction gives the normal form for the bifurcation diagram in terms of projections on R_* and a complementary space. In the normal form, the factors of $h(x,\lambda)$ give the trivial solution $U \equiv U_0$ and the Hugoniot locus, represented by $x = \lambda^2$ in (4.1) or $x = \lambda^2 - a$ in (4.2). To complete the proof and obtain the qualitative description of the Hugoniot locus given in Figures 3.1 and 3.2, we use Assumption 4, which says that R_* is transverse to $\mathcal{B}_\Xi$. The shock polar for a point U_* on $\mathcal{B}_\Xi$ is obtained by projecting the curve $U = U_* + xR_* + W(x,\kappa)$ into $\mathbb{R}^n$; it is a curve lying in the hyperbolic region with a cusp point at U_*. The projection of the unfolding of this bifurcation diagram, represented by $x = \lambda^2 - a$, gives shock polars for points U_0 near $\mathcal{B}_\Xi$ as claimed.

The relative sizes of the characteristic angles are as in Figure 3.2 in a small neighborhood of U_*. For, if $U_0 = U_* \in \mathcal{B}$, then $\lambda_1(U_*) = \lambda_2(U_*)$, and for $U = U_* + xR_* + W$ with $x = \lambda^2 > 0$ we have $|\nabla\lambda_i \cdot R_i| \to \infty$ as $x \to 0$ (from the proof of Proposition 2.4). When this picture is unfolded, by taking $U_0 = U_* + aR_*$ with $a > 0$, then Figure 3.2 is obtained, on taking into account that, at $U = U_0$, we have $\kappa(U, U_0) = \lambda_2(U_0) = \lambda_2(U)$ on one branch of the curve and $\kappa(U, U_0) = \lambda_1(U_0) = \lambda_1(U)$ on the other. □

4.1.1. Normal Form of the Shock Polar Equation In the notation of [8], we solve the recognition problem for the singularity $h = x(x - \lambda^2)$, where λ is a distinguished parameter and the notion of equivalence is contact equivalence under transformations which preserve the trivial solution.

We let $h = xf$, so $f = x - \lambda^2$. The tangent space is

$$T_t(h) = \{x\}[< f, xf_x > + \mathcal{E}_\lambda\{f_\lambda\}]$$

and after a calculation we have

$$< f, xf_x > + \mathcal{E}_\lambda\{f_\lambda\} = < x + \lambda^2, x > + \mathcal{E}_\lambda\{\lambda\} = < x, \lambda > .$$

This is an intrinsic ideal of codimension one, as only the generator 1 is missing. The defining conditions for f are $f = 0$ and $f_\lambda = 0$; the nondegeneracy conditions are $f_x \neq 0$ and $f_{\lambda\lambda} \neq 0$. Since $h = xf$, we have $h_x = f + xf_x$ and $h_{x\lambda} = f_\lambda + xf_{x\lambda}$ and so the defining conditions for h are $h_x = 0 = h_{x\lambda}$. Since $h_{xx} = 2f_x + xf_{xx}$ and $h_{x\lambda\lambda} = f_{\lambda\lambda} + xf_{x\lambda\lambda}$, the nondegeneracy conditions for h are

$$h_{xx} \neq 0\,, \quad h_{x\lambda\lambda} \neq 0\,.$$

All other derivatives either are zero or do not affect the normal form.

4.2. Linear Waves

In constructing solutions of Riemann problems, we must include discontinuities which arise from the other $n - 2$ families of characteristics. By Assumption 3, the additional families are linearly degenerate and are intermediate in speed between the nonlinear characteristics. The loci of curves of linear waves (slip lines or contact discontinuities) are also found by integration of direction fields of eigenvectors. States U_0 and U_1 will be separated by a slip line with characteristic angle λ_i if U_1 lies on the integral curve of the direction field R_i corresponding to λ_i.

When $n \geq 4$, a linearly degenerate eigenvalue may have high multiplicity (as is the case for the Euler equations of a polytropic gas), but the corresponding eigenspace is complete, because of the hyperbolicity of the original problem. The locus of contact discontinuities will be $(n - 2)$-dimensional, and will include both hyperbolic and nonhyperbolic states.

4.3. Quasi-One-Dimensional Riemann Problems

One-dimensional Riemann problems at infinity or on $\xi^2 + \eta^2 = C$ give rise to quasi-one-dimensional Riemann problems if two shocks intersect at a point Ξ_0 where the equation is hyperbolic with respect to the two outside states U_1 and U_2. This occurs in some oblique shock interaction problems for the UTSD, as described in [4], and for oblique shock or shock-contact interactions in systems of $n > 2$ equations.

From Assumption 3 on the degenerate characteristics, we know that the *forward support cone* at any hyperbolic point $\Xi_0 \in \mathbb{R}^2$ is the sector of the plane bounded by the two characteristic segments tangent to $\mathcal{B}_{U(\Xi_0)}$. A spacelike line L through Ξ_0 does not intersect the interior of the support cone. The side of L containing the cone is the *downstream* side.

Definition 4.1. *A* quasi-one-dimensional Riemann problem *with data triple* (Ξ_0, U_1, U_2) *for the self-similar equation* $\widetilde{P}U = 0$ *comprises data given on a line* L *through* Ξ_0 *which is spacelike with respect to both states. We make the convention that* $U = U_1$ *below* Ξ_0 *and* $U = U_2$ *above, where 'below' and 'above' refer to smaller and larger angles subtended at* Ξ_0 *by a ray projecting into the future (towards the forward support cone).*

Locally, the problem is like a one-dimensional Riemann problem; one expects the solution to be a function of $t = (\xi - \xi_0)/(\eta - \eta_0)$, and we seek an admissible solution downstream from L. This problem was solved for the UTSD in [4].

We describe the solution by means of the downstream locus of a state near $\mathcal{B}_{\Xi_0}$, Definition 3.3.

Theorem 4.1. *Suppose we are given quasi-one-dimensional Riemann data* (Ξ_0, U_0, U_n) *for the system* $\widetilde{P}U = 0$ *with state variable* $U \in \mathbb{R}^n$. *Suppose Assumptions 1, 2, 3, and 4 hold and that the states* U_0 *and* U_n *are hyperbolic and in a neighborhood* $\mathcal{N}$ *of* $\mathcal{B}_{\Xi_0}$. *Then the Riemann problem has none, one or two admissible solutions. For* U_n *sufficiently close to* U_0 *there is always at least one solution, and in a curvilinear sector of* $\mathcal{N}$ *with vertex* U_0 *there is exactly one solution.*

Proof. We use the Lax construction [11]. Given Ξ_0 and U_0, we can construct a forward sequence of wave curves, parameterized by increasing wave angles $\kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_n$, connecting U_0 successively to intermediate states $U_1, \ldots, U_{n-1}$, and U_{n-1} to a final state V. The first and last wave curves are the downstream loci of U_0 and V respectively; the intermediate curves correspond to linear waves. Since the linear and nonlinear waves correspond to complementary eigenspaces, the range of the mapping $(\kappa_1, \kappa_2, \ldots, \kappa_n) \mapsto V$ is an open neighborhood of U_0. In fact, if U_n is in a sufficiently small neighborhood of U_0, bounded away from the sonic line, then the situation is exactly like that of a one-dimensional Riemann problem (with the variations that the intermediate families are linear, and their speeds are not necessarily distinct), and there is a unique solution contained in that neighborhood.

However, unlike the Lax theorem, we do not get uniqueness in a neighborhood large enough to contain the entire loop of the shock polar, and we can predict that solutions will fail to exist in a subset of $\mathcal{N}$. To see this, set $\kappa_2 = \ldots = \kappa_{n-1} = 0$, and examine the one-parameter families of solutions constructed using the downstream loci of U_0 (parameterized by κ_1) and and U_n (parameterized by κ_n). The downstream locus of U_0 divides $\mathcal{N}$ into four open sets (refer to Figure 3.6). To fix ideas, suppose κ increases along the downstream locus in the direction which goes counterclockwise around the loop, and suppose U_n lies in the region labeled '1' in Figure 3.6. Superimposing $\mathcal{D}(U_n)$ on the picture, one sees there is precisely one intersection at a point where $\kappa_1 < \kappa_n$, for $\mathcal{D}(U_n)$ is qualitatively similar to $\mathcal{D}(U_0)$. This intersection gives the intermediate state; in region '1', it corresponds to a rarefaction wave in each family, as indicated on the Figure. One finds a similar result in each of the other open regions, except for region '3', where there are two admissible solutions when U_n is close to U_0; in one of these the intermediate state is also close to U_0 while in the other it is a subsonic state, across $\mathcal{B}_\Xi$. However, if U_n is in region '3' but sufficiently far from U_0, the two intermediate states will coalesce and then for more distant states U_n there will be no admissible intersections of $\mathcal{D}(U_0)$ and $\mathcal{D}(U_n)$.

Thus, when $\kappa_2 = \ldots = \kappa_{n-1} = 0$ the situation is exactly that of the UTSD equation [4]. If we now perturb this solution by allowing linear waves, then for sufficiently small values of $\kappa_2, \ldots, \kappa_{n-1}$ the solutions persist, as does the nonexistence of solutions in region '3'.

Thus we obtain all the conclusions of the theorem. $\square$

There are hyperbolic wave interactions which do not give rise to quasi-one-dimensional Riemann problems, since if a rarefaction wave is present then its interaction with another wave will not be localized at a point Ξ_0. This situation arises in the two-dimensional Riemann problem for a scalar conservation law, as in the thesis of David Wagner [17]. It is possible that the techniques developed there could be applied to this problem.

5. The Subsonic Region

The self-similar equation is elliptic in the region inside $\mathcal{B}_U$ in the case $n = 2$, and is of mixed type, with $n - 2$ real characteristics there, if $n > 2$. The elliptic part of the equation degenerates on the boundary in a manner that is common to all the systems satisfying our assumptions.

5.1. The Linearized Equation

We show first that the type of degeneracy found for the UTSD equation ('Keldysh' type) is common to quasisteady reductions of hyperbolic equations under our structural Assumption 1.

Proposition 5.1. *If $n \geq 2$, the operator $\widetilde{P}$, linearized at a constant state, can be written as a second-order operator which changes type, coupled through lower-order terms to a first-order hyperbolic system of order $n-2$. The canonical form of the second-order equation at the change of type locus $x = 0$ is*

$$x\phi_{xx} + \phi_{yy} + g(x, y, \phi, \nabla\phi) = 0\,. \tag{5.1}$$

Proof. If the operator P is linearized about a constant state U_0, it becomes a constant-coefficient operator, which is equivalent to a single n-th order operator with the same characteristic form [6, page 14]. This equation can, in turn, be rewritten as a coupled system whose second-order part has the characteristic form $q(\sigma, U_0)$ of Assumption 1.

If we now apply the similarity reduction, the principal part of the second-order equation becomes

$$\mathcal{L}\phi \equiv (\partial_\xi \partial_\eta) Q_R(\Xi, U_0) \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix} \phi \tag{5.2}$$

where ϕ could be expressed in terms of U.

Now, $Q_R = N^T Q_N N$, where Q_N is the 3×3 symmetric matrix of the quadratic form q (see equation (2.9)). The sonic line $\mathcal{B}_U$ is given by $\{\Xi \mid \det Q_R = 0\}$. Assume $\mathcal{B}_U$ is an ellipse rather than a parabola. (The case of a parabola, exemplified by the UTSD equation, is discussed in our papers [1] and [2].) An affine mapping $(\xi, \eta) \mapsto (\tilde{\xi}, \tilde{\eta})$ will move $\mathcal{B}_U$ to the unit circle centered at the origin. Since this map preserves the form of P and its relation to $\widetilde{P}$, we may assume without loss of generality that $\mathcal{B}_U$ has this form in the first place. In that case $Q_N = \mathrm{diag}(1, -1, -1)$, and the operator $\mathcal{L}$ of equation (5.2) is

$$\mathcal{L}\phi = (\xi^2 - 1)\phi_{\xi\xi} + 2\xi\eta\phi_{\xi\eta} + (\eta^2 - 1)\phi_{\eta\eta} + \dots\,.$$

None of the nonlinear structure of the problem is preserved by these transformations, but we can establish the type of degeneracy of the linearized equation there by straightening the boundary. For this, introduce polar coordinates

$$\xi = r\cos\theta\,, \quad \eta = r\sin\theta \tag{5.3}$$

and then the further transformation $x = 1 - r$, $y = \theta$, so that the degenerate boundary becomes the y-axis. Then

$$\mathcal{L}\phi = (r^2 - 1)\phi_{rr} - \frac{1}{r^2}\phi_{\theta\theta} + \dots = x(2 - x)\phi_{xx} + \frac{1}{(1 - x)^2}\phi_{yy} + \dots\,.$$

Near $x = 0$ this has the form (5.1).

Linear equations of this form were first investigated by Keldysh [9], who proved that any solution which assumes the boundary condition $\phi(0, y) = 0$ is of the form $\sqrt{x}w(x, y)$, where $w(0, y) \neq 0$. We thus have the following Corollary.

Corollary 5.1. *Any nontrivial solution of* (5.1) *which is zero at $x = 0$ has a square-root singularity at the x-axis. Further, if* (5.2) *can be transformed to* (5.1) *by a transformation which is independent of ϕ, then any solution of* (5.2) *which is nonconstant in the interior of $\mathcal{B}_U$ has a square-root singularity on $\mathcal{B}_U$.*

5.2. Reduction of the Nonlinear Wave Equation

In [2] we found a solution to a nonlinear version of the Keldysh equation which was smooth at the degenerate boundary. This type of solution, which arises from the nonlinearity of the equation, may play a role in shock interaction problems. For the general class of equations discussed in this paper we do not have a description of the nonlinear behavior at the sonic boundary. However, for the nonlinear wave equation, $\rho_{tt} - \nabla(c^2(\rho)\nabla\rho) = 0$ with $c^2 = dp/d\rho$, the reduction can be performed explicitly. If this equation is written as a 3×3 system, the matrix $Q_N = \operatorname{diag}\,(1, -c^2, -c^2)$, and $\mathcal{B}_U$ is the circle $\xi^2 + \eta^2 = c^2(\rho)$. The transformation (5.3) to polar coordinates then yields a second-order equation for ρ:

$$\left(r^2 - c^2\right)\rho_{rr} - \frac{c^2}{r^2}\rho_{\theta\theta} + \left(2r - \frac{c^2}{r}\right)\rho_r - h(\rho)\left(\rho_r^2 + \frac{1}{r^2}\rho_\theta^2\right) = 0\,,$$

where $h = dc^2/d\rho$. We may seek a solution in a fixed subsonic region, supposing that $\rho = \rho_0$ at $r = c_0 = c(\rho_0)$. To a first approximation, there appear to be smooth solutions which depend on r. Rewrite the equation as

$$\frac{d}{dr}\left((c^2 - r^2)\rho_r\right) + \frac{c^2}{r}\rho_r = 0\,. \tag{5.4}$$

Let $x = c_0 - r$ and $u = \rho - \rho_0$ and retain linear powers of x and u, so

$$c - r = (\rho - \rho_0)c'(\rho_0) - (r - c_0) + \ldots = c'(\rho_0)u + x + \ldots$$

and

$$c^2 - r^2 = (c + r)(c - r) = 2c_0(c'(\rho_0)u + x) + \ldots$$

A solution of (5.4) of the form $u = \gamma x + \ldots$ for some nonzero γ exists only if $\gamma = -1/2c'(\rho_0)$. Thus, we can find a linear approximation precisely when $c'(\rho_0) \neq 0$; that is, when the equation is genuinely nonlinear. We have not proved that a smooth solution exists, but we conclude

Proposition 5.2. *The existence of a solution to the nonlinear wave equation which has a finite, nonzero slope at the degenerate boundary is consistent with the form of the equation there if and only if the equation is genuinely nonlinear at the degenerate boundary.*

6. Conclusions

We have presented a systematic approach to analyzing two-dimensional Riemann problems with sectorially constant data, for operators with the characteristic structure given by Assumption 1. This particular structure is undoubtedly useful in other applications; it was observed by Courant and Hilbert [6, page 596] that equations with reducible symbols occur frequently in mathematical physics.

Solving Riemann problems proceeds in two steps: first, wave interactions in the hyperbolic region of the reduced system; then solution in the subsonic region. Hyperbolic wave interactions lead to quasi-one-dimensional Riemann problems, whose distinguishing features are the non-autonomous nature of the equations and the existence of a hypersurface in state space across which change of type occurs.

Analysis of quasi-one-dimensional flows in the supersonic region leads to three main conclusions: first, that the geometry of the shock locus near the sonic boundary is that of a shock polar; second, solutions exist locally for quasi-one-dimensional Riemann problems. Finally, for systems of two equations, we generalize a result of Courant and Friedrichs. Other supersonic results, such as interactions involving rarefaction waves, are the subject of ongoing research.

Analysis of solutions in the subsonic region is still an open problem. Our results [1] and [2] on the subsonic region for the UTSD equation show that both singular and smooth solutions exist. A subsonic solution is also found by Yuxi Zheng for a particular form of the nonlinear wave equation, [19]. Shocks bounding the subsonic region are described by free boundary problems, which we are now studying for the UTSD equation.

What we have tried to show in this paper is that the UTSD equation has many features in common with a wider class of equations, and that analysis of two-dimensional Riemann problems for the former contains lessons which will apply to the latter.

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