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BOUNDS FOR VISCOSITY PROFILES FOR 2 $\times$ 2 SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. Given a system of two conservation laws which is admissible and satisfies the half-plane condition introduced by Keyfitz and Kranzer, the existence of a unique travelling-wave solution of the associated parabolic system, $U_t + F(U)_x = \varepsilon U_{xx}$, which approximates a given shock, is proved. The shock profile trajectory is a convex curve in phase space, bounded by forward and backward shock curves.

1. Introduction. In [7], an existence theorem for solutions to the Riemann problem was proved for a class of genuinely nonlinear 2 $\times$ 2 conservation laws that is somewhat larger than those previously considered (see [9] and the references in [7]). In this article, we show that viscosity shock profiles in the form of travelling wave solutions to the associated parabolic system

$$U_t + F(U)_x = \varepsilon U_{xx} \tag{1}$$

can be constructed for this same class of equations. This enlarges the class considered by Conley and Smoller in [2] and does away with the need for any additional assumptions such as appear in their paper. We also obtain more satisfactory bounds on the trajectories of the travelling wave solutions, and show that the trajectories are convex curves.

One consequence of this result is that the shock wave solutions of

$$U_t + F(U)_x = 0 \tag{2}$$

for the class of equations considered in [7] are the limits of solutions of (1) as ε tends to zero. This verifies the admissibility condition of Gel'fand [5] for this larger class and without additional assumptions. There are also implications for the construction of solutions to the Cauchy problem for (1) by a vanishing viscosity method, although formidable difficulties remain in carrying out this procedure. The bounds obtained here may be useful.

We begin with some background. In (2), $U = (u_1, u_2)$ is a function of x

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and t, and $F = (f_1, f_2)$ a function of U. System (2) is called *an admissible system of hyperbolic conservation laws*, or, briefly, *admissible*, if

(i) the matrix $A = \partial F/\partial U$ has real distinct eigenvalues $\lambda_1 < \lambda_2$ (strict hyperbolicity);

(ii) $\ell_i d^2F(r_i, r_i) \equiv r_i \nabla \lambda_i > 0$, $i = 1$ and 2, (genuine nonlinearity), where ℓ_i and r_i are left and right eigenvectors of A chosen so that $\ell_i r_i > 0$ and $r_i \nabla \lambda_i > 0$ (thus the basic assumption is that $r_i \nabla \lambda_i \neq 0$); and

(iii) $\ell_i d^2F(r_j, r_j) > 0$, $i \neq j$ (the Johnson-Smoller interaction condition **[9]**), under the same normalization as in (ii). Here $d^2F(r, r)$ is the second Fréchet derivative of F in the direction r.

The *Riemann problem* for conservation laws is (2) with the initial data

$$U(x, 0) = \begin{cases} U_\ell & x \leqq 0 \\ U_r & x > 0 \end{cases} \tag{3}$$

where U_ℓ and U_r are constants. If solutions exist, they will also be homogeneous (functions of x/t alone) and consist of shocks and centered rarefaction waves.

While admissibility is sufficient to give solutions to the Riemann problem for U_ℓ and U_r very close together **[8]**, and even for the Cauchy problem with small initial oscillation **[6]**, some additional condition is necessary in the large **[1]**. In [7] we proposed the *half-plane condition*:

(iv) there is a fixed vector w such that $r_1 \cdot w < 0$ and $r_2 \cdot w > 0$ for all U.

The stronger condition of *opposite variation* ($r_i \nabla \lambda_j < 0$ if $i \neq j$) implies the half-plane condition [7].

In [7], we showed that the Riemann Problem was well-posed for systems satisfying (i)–(iv). In the notation of [7], the *Rarefaction Curve* $R_i(U_0)$ is the integral curve of the vector-field $r_i(U)$ through U_0. For an admissible system R_1 is convex toward $-r_2$ and R_2 is convex toward r_1. The *Shock Curves* $S_i(U_0)$ and $S_i^*(U_0)$ are curves of solutions, U, to the Rankine-Hugoniot relation

$$s(U - U_0) = F(U) - F(U_0) \tag{4}$$

for some s. By definition $S_i(U_0)$ is the branch of solutions emanating from U_0 tangent to $r_i(U_0)$ along which s decreases from $\lambda_i(U_0)$ at U_0. The term *Hugoniot locus* of U_0, refers to all the solutions to (4) for fixed U_0. The half-plane condition was used in [7] to prove the following three theorems, which are the only uses we will make of this condition.

THEOREM 1. (Theorem 4.5 of [7]). *Each curve $S_i(U_0)$ consists of a simple arc extending from U_0 to infinity. It is star-shaped with respect to U_0, and lies entirely inside $R_1(U_0)$ and outside $R_2(U_0)$. At U_0, S_i is tangent to $R_i(U_0)$. As $S_i(U_0)$ is traversed in the direction away from U_0, it crosses all R_i curves from outside to inside and all R_2 curves from inside to outside. The associated*

$s(U)$ from (4) *is monotonic as U traverses the curve. At each point $U \neq U_0$ on $S_i(U_0)$,*

$$\lambda_i(U_0) > s(U) > \lambda_i(U) \tag{5}$$

and the shock speed on $S_1(U_0)$ also satisfies

$$s(U) < \lambda_2(U). \tag{6}$$

THEOREM 2. (Theorem 4.6 of [7]). *Each curve $S_i^*(U_0)$ consists of a simple arc extending from U_0 to infinity. It has the same properties as $S_i(U_0)$ in Theorem* 1, *with the subscripts* 1 *and* 2 *permuted and the inequalities on the shock speeds reversed.*

THEOREM 3. (Theorem 5.1 of [7]). *The Hugoniot locus is precisely the union of the four shock loci $S_i(U_0)$, $S_i^*(U_0)$, $i = 1, 2$.*

In what follows we may consider either the class of admissible systems satisfying the half-plane condition, or the (presumably larger) class of admissible systems for which Theorems 1–3 hold. These three theorems imply the next two, which are needed for this paper.

THEOREM 4. (Theorem 5.4 of [7]). *A point U is in $S_i(U_0)$ if and only if $U_0 \in S_i^*(U)$.*

THEOREM 5. (Lemma 6.2 of [7]). *The speed s_2 of the* 2*-shock joining $\bar{U}$ to an arbitrary point $U \in S_2(\bar{U})$ is always greater than the speed s_1 of the* 1*-shock joining $\bar{U}$ to U_0.*

Finally we note that for a given U_0 and $U_1 \in S_i(U_0)$, it is shown in [7] that U_1 can be joined to U_0, with U_0 on the left (in the x-t plane), by a shock satisfying the Lax Entropy Condition [8]; if $U_1 \in S_1^*(U_0)$, then the same is true with U_1 on the left.

2. Viscosity shock profiles. *A shock profile* is a travelling wave solution to (1), the parabolic system obtained from (2) by the addition of a particular kind of dissipation term. Other types of viscosity term were considered as well in [2], [4]. In [2] it was shown, for example, that if the right hand side in (1) is of the form εBU_{xx} where B is a 2×2 constant matrix, then similar shock profiles exist for B close to the identity, but the situation is qualitatively different for a class of positive definite B's which are not close to I. Analogous results can be recovered in the present problem, but we shall not consider it further.

Of interest here are travelling waves that converge to shocks as $\varepsilon \to 0$. (By *shock* we shall always mean a discontinuity that satisfies the Lax Entropy Condition.) That is, for any U_0 and $U_1 \in S_i(U_0)$ or $S_i^*(U_0)$, we have s defined by (4) and the associated travelling wave will be a function

$$W(\xi) = W\left(\frac{x - st}{\varepsilon}\right) \tag{7}$$

with

$$\lim_{\xi \to -\infty} W(\xi) = U_0, \lim_{\xi \to \infty} W(\xi) = U_1 \tag{8a}$$

if $U_1 \in S_i(U_0)$, and

$$\lim_{\xi \to -\infty} W(\xi) = U_1, \lim_{\xi \to \infty} W(\xi) = U_0 \tag{8b}$$

if $U_1 \in S_i^*(U_0)$.

By Theorem 4, the cases $U_1 \in S_i(U_0)$ and $U_1 \in S_i^*(U_0)$ are really the same, and we will assume from now on that $U_1 \in S_i(U_0)$, for $i = 1$ or 2.

Substituting W into (1) and integrating once results in the system

$$\frac{dW}{d\xi} = V(W) \equiv F(W) - sW + C \tag{9}$$

where

$$C = sU_1 - F(U_1) = sU_0 - F(U_0) \tag{10}$$

by (4).

PROPOSITION 1. *The vector field $V(W)$ is zero at $W = U_0$ and U_1 and has no other singularities.*

PROOF. Clearly $V(U_0) = V(U_1) = 0$. If $V(W) = 0$, then $s(W - U_0) = F(W) - F(U_0)$ and so, if $W \neq U_0$, W is on the Hugoniot locus of U_0, with the same value of s as U_1. By Theorems 1 and 2, s is monotonic on $S_1(U_0) \cup S_i^*(U_0)$ and on $S_2(U_0) \cup S_2^*(U_0)$ so there is at most one point on each pair of curves with a given value of s. But now Theorem 5 implies that the maximum of s on $S_1 \cup S_1^*$ is less than the minimum of s on $S_2 \cup S_2^*$. Hence $W = U_1$.

The vector field V has nondegenerate singularities at U_0 and U_1.

PROPOSITION 2. *If $U_1 \in S_1(U_0)$, then U_0 is an unstable node and U_1 is a saddle; if $U_1 \in S_2(U_0)$, then U_0 is a saddle and U_1 a stable node.*

PROOF. At U_j, $j = 0$ or 1, $\partial V/\partial W = A(U_j) - sI$. The eigenvalues of this matrix are $\lambda_1 - s$, $\lambda_2 - s$. Suppose $U_1 \in S_1(U_0)$. Then from Theorem 1, $\lambda_2(U_0) - s > \lambda_1(U_0) - s > 0$, so U_0 is an unstable node and $\lambda_2(U_1) - s > 0 > \lambda_1(U_1) - s$, so U_1 is a saddle. If $U_1 \in S_2(U_0)$, then, using Theorem 5 for the second inequality, $\lambda_2(U_0) - s > 0 > \lambda_1(U_0) - s$, so U_0 is a saddle, and $0 > \lambda_2(U_1) - s > \lambda_1(U_1) - s$, so that U_1 is a stable node.

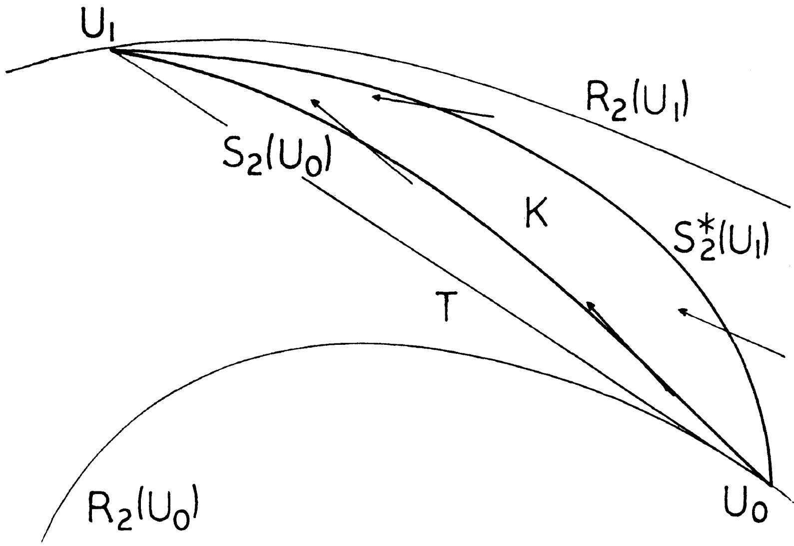
We can now state and prove the main result of the paper.

THEOREM 6. *Suppose that* (2) *satisfies conditions* (i)–(iii) *and either* (iv) *or* *the conclusions of Theorems* 1–3. *Then for any* U_0 *and* U_1, *with* U_1 *in the Hugoniot locus of* U_0, *there is a travelling wave solution* $W((x - st)/\varepsilon)$ *of* (1) *joining the states* U_0 *and* U_1, *and satisfying* (8a) *or* (8b). *Furthermore, the trajectory*

$$\Gamma = \{W(\xi) | -\infty < \xi < \infty\} \tag{11}$$

is a convex curve lying between the two curves $S_i(U_0)$ *and* $S_i^*(U_1)$ if $U_1 \in S_i(U_0)$, *or between* $S_i^*(U_0)$ *and* $S_i(U_1)$ *if* $U_1 \in S_i^*(U_0)$.

PROOF. For definiteness, assume $U_1 \in S_2(U_0)$, the other three cases are similar. The geometry is as follows (illustrated in Figure 1): $S_2(U_0)$ lies outside of $R_2(U_0)$; hence $R_2(U_1)$ lies outside of $R_2(U_0)$; $S_2^*(U_1)$ lies inside $R_2(U_1)$. Since $S_2(U_0)$ is tangent to$R_2(U_0)$ at U_0 and $S_2^*(U_1)$ cuts it transversally, $S_2(U_0)$ lies between $R_2(U_0)$ and $S_2^*(U_1)$ near U_0; similarly $S_2^*(U_1)$ lies between $R_2(U_1)$ and $S_2(U_0)$ near U_1. The secant $T = T(U_0, U_1)$ joining U_0 to U_1 does not cross either $S_2(U_0)$ or $S_2^*(U_1)$ between U_0 and U_1, since these curves are star-shaped with respect to U_0 and U_1 respectively. For the same reason, T lies entirely inside $R_2(U_1)$ and cuts $R_2(U_1)$ transversally at U_1. Hence the segment of $S_2(U_0)$ from U_0 to U_1 lies between T and $S_2^*(U_1)$. Note that $S_2(U_0)$ and $S_2^*(U_1)$ do not intersect except at U_0 and U_1, for if they did, the distinctness of shock speeds and the Rankine-Hugoniot relations would imply that the three intersection points were collinear. Let K be the interior of the region formed by $S_2(U_0)$ and $S_2^*(U_1)$ between U_0 and U_1.



Now we show that inside K there is a trajectory leaving U_0 along one of the unstable directions and approaching U_1. We note that the unstable manifold of V at U_0 has the direction r_2, and hence is tangent to $S_2(U_0)$ at U_0. Suppose $W \in S_2(U_0)$, strictly between U_0 and U_1. Then there is an $\bar{s} > s$ such that $\bar{s}(W - U_0) = F(W) - F(U_0)$. Hence

$$\begin{aligned} V(W) &= F(W) - sW + sU_0 - F(U_0) = (\bar{s} - s)(W - U_0) \\ &= (\bar{s} - s)T(W, U_0) \end{aligned}$$

where $T(W, U_0)$ is the secant joining W to U_0. By the star-shaped property of $S_2(U_0)$, with orientation given by $T = T(U_1, U_0)$, we see that $V(W)$ is directed strictly into K along $S_2(U_0)$.

Similarly, if $W \in S_2^*(U_1)$, then for $\bar{s} < s$, we have $\bar{s}(W - U_1) = F(W) - F(U_1)$. Hence

$$\begin{aligned} V(W) &= F(W) - sW + sU_1 - F(U_1) = (\bar{s} - s)(W - U_1) \\ &= (\bar{s} - s)T(W, U_1). \end{aligned}$$

Thus $V(W)$ has the opposite direction to the secant joining W to U_1; Since $S_2^*(U_1)$ is again star-shaped, with orientation given by T, we see that V is directed strictly into K along $S_2^*(U_1)$.

Hence the unstable trajectory leaving U_0 in the $-r_2$ direction cannot escape from K and must approach the node U_1. Thus there is a unique orbit Γ joining U_0 to U_1.

To show that Γ is convex we prove the sufficient condition that Γ is star-shaped with respect to both U_0 and U_1. To see this, observe that if the secant $T(W, U_0)$ joining W to U_0 is ever parallel to $V(W)$ in the interior of K, then $k(W - U_0) = F(W) - sW + sU_0 - F(U_0)$, or $F(W) - F(U_0) = (k - s)(W - U_0)$, so W is on the Hugoniot locus of U_0, which is impossible in $\mathring{K}$. Hence Γ is never tangent to the secant $T(W, U_0)$ at W, and so it is star-shaped with respect to U_0. Similarly, $T(W, U_1)$ is never parallel to $V(W)$ for $W \in \mathring{K}$.

This completes the proof. We conclude by observing that, for shocks that are not too large, the region K is quite narrow, and thus gives a good bound on the location of the shock profile. Also, for weak shocks, the shock profile trajectory coincides, to third order, with a rarefaction wave. This generalizes to shock profiles the observation made for shock curves in Courant-Friedrichs [**3**].

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