BOUND FOR VISCOSITY PROFILES FOR 2 × 2 SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. Given a system of two conservation laws which is admissible and satisfies the half-plane condition introduced by Keyfitz and Kranzer, the existence of a unique travelling-wave solution of the associated parabolic system, \( U_t + F(U)_x = \varepsilon U_{xx} \), which approximates a given shock, is proved. The shock profile trajectory is a convex curve in phase space, bounded by forward and backward shock curves.

1. Introduction. In [7], an existence theorem for solutions to the Riemann problem was proved for a class of genuinely nonlinear 2 × 2 conservation laws that is somewhat larger than those previously considered (see [9] and the references in [7]). In this article, we show that viscosity shock profiles in the form of travelling wave solutions to the associated parabolic system

\[
U_t + F(U)_x = \varepsilon U_{xx}
\]

can be constructed for this same class of equations. This enlarges the class considered by Conley and Smoller in [2] and does away with the need for any additional assumptions such as appear in their paper. We also obtain more satisfactory bounds on the trajectories of the travelling wave solutions, and show that the trajectories are convex curves.

One consequence of this result is that the shock wave solutions of

\[
U_t + F(U)_x = 0
\]

for the class of equations considered in [7] are the limits of solutions of (1) as \( \varepsilon \) tends to zero. This verifies the admissibility condition of Gel'fand [5] for this larger class and without additional assumptions. There are also implications for the construction of solutions to the Cauchy problem for (1) by a vanishing viscosity method, although formidable difficulties remain in carrying out this procedure. The bounds obtained here may be useful.

We begin with some background. In (2), \( U = (u_1, u_2) \) is a function of \( x \)
and \( t \), and \( F = (f_1, f_2) \) a function of \( U \). System (2) is called an **admissible system of hyperbolic conservation laws**, or, briefly, **admissible**, if

(i) the matrix \( A = \frac{\partial F}{\partial U} \) has real distinct eigenvalues \( \lambda_1 < \lambda_2 \) (strict hyperbolicity);

(ii) \( \zeta_i \frac{\partial^2 F}{\partial r_i \partial r_i} \equiv r_i \nabla \lambda_i > 0 \), \( i = 1 \) and \( 2 \), (genuine nonlinearity), where \( \zeta_i \) and \( r_i \) are left and right eigenvectors of \( A \) chosen so that \( \zeta_i r_i > 0 \) and \( r_i \nabla \lambda_i > 0 \) (thus the basic assumption is that \( r_i \nabla \lambda_i \neq 0 \)); and

(iii) \( \zeta_i \frac{\partial^2 F}{\partial r_i \partial r_j} > 0 \), \( i \neq j \) (the Johnson-Smoller interaction condition [9]), under the same normalization as in (ii). Here \( \frac{\partial^2 F}{\partial r \partial r} \) is the second Fréchet derivative of \( F \) in the direction \( r \).

The **Riemann problem** for conservation laws is (2) with the initial data

\[
U(x, 0) = \begin{cases} 
U_r & x \leq 0 \\
U_l & x > 0 
\end{cases}
\]

where \( U_r \) and \( U_l \) are constants. If solutions exist, they will also be homogeneous (functions of \( x/t \) alone) and consist of shocks and centered rarefaction waves.

While admissibility is sufficient to give solutions to the Riemann problem for \( U_r \) and \( U_l \) very close together [8], and even for the Cauchy problem with small initial oscillation [6], some additional condition is necessary in the large [1]. In [7] we proposed the **half-plane condition**:

(iv) there is a fixed vector \( w \) such that \( r_1 \cdot w < 0 \) and \( r_2 \cdot w > 0 \) for all \( U \).

The stronger condition of **opposite variation** (\( r_i \nabla \lambda_j < 0 \) if \( i \neq j \)) implies the half-plane condition [7].

In [7], we showed that the Riemann Problem was well-posed for systems satisfying (i)–(iv). In the notation of [7], the **Rarefaction Curve** \( R_1(U_0) \) is the integral curve of the vector-field \( r_i(U) \) through \( U_0 \). For an admissible system \( R_1 \) is convex toward \( -r_2 \) and \( R_2 \) is convex toward \( r_1 \). The **Shock Curves** \( S_i(U_0) \) and \( S_i^*(U_0) \) are curves of solutions, \( U \), to the Rankine-Hugoniot relation

\[
s(U - U_0) = F(U) - F(U_0)
\]

for some \( s \). By definition \( S_i(U_0) \) is the branch of solutions emanating from \( U_0 \) tangent to \( r_i(U_0) \) along which \( s \) decreases from \( \lambda_i(U_0) \) at \( U_0 \). The term **Hugoniot locus** of \( U_0 \), refers to all the solutions to (4) for fixed \( U_0 \). The half-plane condition was used in [7] to prove the following three theorems, which are the only uses we will make of this condition.

**Theorem 1.** (Theorem 4.5 of [7]). **Each curve** \( S_i(U_0) \) **consists of a simple arc extending from** \( U_0 \) **to infinity. It is star-shaped with respect to** \( U_0 \), **and lies entirely inside** \( R_1(U_0) \) **and outside** \( R_2(U_0) \). **At** \( U_0 \), **\( S_i \) is tangent to** \( R_1(U_0) \). **As** \( S_i(U_0) \) **is traversed in the direction away from** \( U_0 \), **it crosses all** \( R_1 \) **curves from outside to inside and all** \( R_2 \) **curves from inside to outside. The associated**
s(U) from (4) is monotonic as U traverses the curve. At each point U \neq U_0 on S_1(U_0),

\[ \lambda_2(U_0) > s(U) > \lambda_1(U) \]

and the shock speed on S_1(U_0) also satisfies

\[ s(U) < \lambda_2(U). \]

Theorem 2. (Theorem 4.6 of [7]). Each curve \( S_1^*(U_0) \) consists of a simple arc extending from \( U_0 \) to infinity. It has the same properties as \( S_1(U_0) \) in Theorem 1, with the subscripts 1 and 2 permuted and the inequalities on the shock speeds reversed.

Theorem 3. (Theorem 5.1 of [7]). The Hugoniot locus is precisely the union of the four shock loci \( S_i(U_0), S_i^*(U_0), i = 1, 2 \).

In what follows we may consider either the class of admissible systems satisfying the half-plane condition, or the (presumably larger) class of admissible systems for which Theorems 1–3 hold. These three theorems imply the next two, which are needed for this paper.

Theorem 4. (Theorem 5.4 of [7]). A point \( U \) is in \( S_1(U_0) \) if and only if \( U_0 \in S_2^*(U) \).

Theorem 5. (Lemma 6.2 of [7]). The speed \( s_2 \) of the 2-shock joining \( \tilde{U} \) to an arbitrary point \( U \in S_2(\tilde{U}) \) is always greater than the speed \( s_1 \) of the 1-shock joining \( \tilde{U} \) to \( U_0 \).

Finally we note that for a given \( U_0 \) and \( U_1 \in S_1(U_0) \), it is shown in [7] that \( U_1 \) can be joined to \( U_0 \), with \( U_0 \) on the left (in the \( x-t \) plane), by a shock satisfying the Lax Entropy Condition [8]; if \( U_1 \in S_1^*(U_0) \), then the same is true with \( U_1 \) on the left.

2. Viscosity shock profiles. A shock profile is a travelling wave solution to (1), the parabolic system obtained from (2) by the addition of a particular kind of dissipation term. Other types of viscosity term were considered as well in [2], [4]. In [2] it was shown, for example, that if the right hand side in (1) is of the form \( \varepsilon BU_{xx} \) where \( B \) is a \( 2 \times 2 \) constant matrix, then similar shock profiles exist for \( B \) close to the identity, but the situation is qualitatively different for a class of positive definite \( B \)'s which are not close to \( I \). Analogous results can be recovered in the present problem, but we shall not consider it further.

Of interest here are travelling waves that converge to shocks as \( \varepsilon \to 0 \). (By shock we shall always mean a discontinuity that satisfies the Lax Entropy Condition.) That is, for any \( U_0 \) and \( U_1 \in S_1(U_0) \) or \( S_1^*(U_0) \), we have \( s \) defined by (4) and the associated travelling wave will be a function
228  B.L. KEYFITZ

(7) \[ W(\xi) = W\left(\frac{x - st}{\varepsilon}\right) \]

with

(8a) \[ \lim_{\xi \to -\infty} W(\xi) = U_0, \quad \lim_{\xi \to \infty} W(\xi) = U_1 \]

if \( U_1 \in S_1(U_0) \), and

(8b) \[ \lim_{\xi \to -\infty} W(\xi) = U_1, \quad \lim_{\xi \to \infty} W(\xi) = U_0 \]

if \( U_1 \in S^*_1(U_0) \).

By Theorem 4, the cases \( U_1 \in S_1(U_0) \) and \( U_1 \in S_1^*(U_0) \) are really the same, and we will assume from now on that \( U_1 \in S_1(U_0) \), for \( i = 1 \) or 2.

Substituting \( W \) into (1) and integrating once results in the system

(9) \[ \frac{dW}{d\xi} = V(W) = F(W) - sW + C \]

where

(10) \[ C = sU_1 - F(U_1) = sU_0 - F(U_0) \]

by (4).

**PROPOSITION 1.** The vector field \( V(W) \) is zero at \( W = U_0 \) and \( U_1 \) and has no other singularities.

**PROOF.** Clearly \( V(U_0) = V(U_1) = 0 \). If \( V(W) = 0 \), then \( s(W - U_0) = F(W) - F(U_0) \) and so, if \( W \neq U_0 \), \( W \) is on the Hugoniot locus of \( U_0 \), with the same value of \( s \) as \( U_1 \). By Theorems 1 and 2, \( s \) is monotonic on \( S_1(U_0) \cup S_1^*(U_0) \) and on \( S_2(U_0) \cup S_2^*(U_0) \) so there is at most one point on each pair of curves with a given value of \( s \). But now Theorem 5 implies that the maximum of \( s \) on \( S_1 \cup S_1^* \) is less than the minimum of \( s \) on \( S_2 \cup S_2^* \). Hence \( W = U_1 \).

The vector field \( V \) has nondegenerate singularities at \( U_0 \) and \( U_1 \).

**PROPOSITION 2.** If \( U_1 \in S_1(U_0) \), then \( U_0 \) is an unstable node and \( U_1 \) is a saddle; if \( U_1 \in S_2(U_0) \), then \( U_0 \) is a saddle and \( U_1 \) a stable node.

**PROOF.** At \( U_j, j = 0 \) or 1, \( \partial V/\partial W = A(U_j) - sI \). The eigenvalues of this matrix are \( \lambda_1 - s, \lambda_2 - s \). Suppose \( U_1 \in S_1(U_0) \). Then from Theorem 1, \( \lambda_2(U_0) - s > \lambda_1(U_0) - s > 0 \), so \( U_0 \) is an unstable node and \( \lambda_2(U_1) - s > 0 > \lambda_1(U_1) - s \), so \( U_1 \) is a saddle. If \( U_1 \in S_2(U_0) \), then, using Theorem 5 for the second inequality, \( \lambda_2(U_0) - s > 0 > \lambda_1(U_0) - s \), so \( U_0 \) is a saddle, and \( 0 > \lambda_2(U_1) - s > \lambda_1(U_1) - s \), so that \( U_1 \) is a stable node.

We can now state and prove the main result of the paper.

**THEOREM 6.** Suppose that (2) satisfies conditions (i)-(iii) and either (iv) or
the conclusions of Theorems 1–3. Then for any $U_0$ and $U_1$, with $U_1$ in the Hugoniot locus of $U_0$, there is a travelling wave solution $W((x - st)/\varepsilon)$ of (1) joining the states $U_0$ and $U_1$, and satisfying (8a) or (8b). Furthermore, the trajectory

$$I' = \{W(\xi) | -\infty < \xi < \infty\}$$

is a convex curve lying between the two curves $S_2(U_0)$ and $S_2^*(U_1)$ if $U_1 \in S_2(U_0)$ or between $S_2^*(U_0)$ and $S_2(U_1)$ if $U_1 \in S_2^*(U_0)$.

**Proof.** For definiteness, assume $U_1 \in S_2(U_0)$, the other three cases are similar. The geometry is as follows (illustrated in Figure 1): $S_2(U_0)$ lies outside of $R_2(U_0)$; hence $R_2(U_1)$ lies outside of $R_2(U_0)$; $S_2^*(U_1)$ lies inside $R_2(U_1)$. Since $S_2(U_0)$ is tangent to $R_2(U_0)$ at $U_0$ and $S_2^*(U_1)$ cuts it transversally, $S_2(U_0)$ lies between $R_2(U_0)$ and $S_2^*(U_1)$ near $U_0$; similarly $S_2^*(U_1)$ lies between $R_2(U_1)$ and $S_2(U_0)$ near $U_1$. The secant $T = T(U_0, U_1)$ joining $U_0$ to $U_1$ does not cross either $S_2(U_0)$ or $S_2^*(U_1)$ between $U_0$ and $U_1$, since these curves are star-shaped with respect to $U_0$ and $U_1$ respectively. For the same reason, $T$ lies entirely inside $R_2(U_1)$ and cuts $R_2(U_1)$ transversally at $U_1$. Hence the segment of $S_2(U_0)$ from $U_0$ to $U_1$ lies between $T$ and $S_2^*(U_1)$. Note that $S_2(U_0)$ and $S_2^*(U_1)$ do not intersect except at $U_0$ and $U_1$, for if they did, the distinctness of shock speeds and the Rankine-Hugoniot relations would imply that the three intersection points were collinear. Let $K$ be the interior of the region formed by $S_2(U_0)$ and $S_2^*(U_1)$ between $U_0$ and $U_1$. 
Now we show that inside $K$ there is a trajectory leaving $U_0$ along one of the unstable directions and approaching $U_1$. We note that the unstable manifold of $V$ at $U_0$ has the direction $r_2$, and hence is tangent to $S_2(U_0)$ at $U_0$. Suppose $W \in S_2(U_0)$, strictly between $U_0$ and $U_1$. Then there is an $\bar{s} > s$ such that $\bar{s}(W - U_0) = F(W) - F(U_0)$. Hence

$$V(W) = F(W) - sW + sU_0 - F(U_0) = (\bar{s} - s)(W - U_0) = (\bar{s} - s)T(W, U_0)$$

where $T(W, U_0)$ is the secant joining $W$ to $U_0$. By the star-shaped property of $S_2(U_0)$, with orientation given by $T = T(U_1, U_0)$, we see that $V(W)$ is directed strictly into $K$ along $S_2(U_0)$.

Similarly, if $W \in S_2^*(U_1)$, then for $\bar{s} < s$, we have $\bar{s}(W - U_1) = F(W) - F(U_1)$. Hence

$$V(W) = F(W) - sW + sU_1 - F(U_1) = (\bar{s} - s)(W - U_1) = (\bar{s} - s)T(W, U_1).$$

Thus $V(W)$ has the opposite direction to the secant joining $W$ to $U_1$; Since $S_2^*(U_1)$ is again star-shaped, with orientation given by $T$, we see that $V$ is directed strictly into $K$ along $S_2^*(U_1)$.

Hence the unstable trajectory leaving $U_0$ in the $-r_2$ direction cannot escape from $K$ and must approach the node $U_1$. Thus there is a unique orbit $\Gamma$ joining $U_0$ to $U_1$.

To show that $\Gamma$ is convex we prove the sufficient condition that $\Gamma$ is star-shaped with respect to both $U_0$ and $U_1$. To see this, observe that if the secant $T(W, U_0)$ joining $W$ to $U_0$ is ever parallel to $V(W)$ in the interior of $K$, then $k(W - U_0) = F(W) - sW + sU_0 - F(U_0)$, or $F(W) - F(U_0) = (k - s)(W - U_0)$, so $W$ is on the Hugoniot locus of $U_0$, which is impossible in $K$. Hence $\Gamma$ is never tangent to the secant $T(W, U_0)$ at $W$, and so it is star-shaped with respect to $U_0$. Similarly, $T(W, U_1)$ is never parallel to $V(W)$ for $W \in \hat{K}$.

This completes the proof. We conclude by observing that, for shocks that are not too large, the region $K$ is quite narrow, and thus gives a good bound on the location of the shock profile. Also, for weak shocks, the shock profile trajectory coincides, to third order, with a rarefaction wave. This generalizes to shock profiles the observation made for shock curves in Courant-Friedrichs [3].

REFERENCES


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