SELF-SIMILAR PROBLEMS IN MULTIDIMENSIONAL CONSERVATION LAWS

SUNČICA ČANIĆ
Department of Mathematics, University of Houston, Houston, Texas 77204-3008, USA
E-mail: canic@math.uh.edu

BARBARA LEE KEYFITZ
Department of Mathematics, University of Houston, Houston, Texas 77204-3008, USA
E-mail: blk@math.uh.edu

EUN HEUI KIM
Department of Mathematics, California State University, Long Beach, CA 90840-1001, USA
E-mail: ehkim4@csulb.edu

We report on an approach to analysing hyperbolic conservation laws in several space variables by examining two-dimensional Riemann problems. Use of self-similar coordinates reduces the problem to a system of conservation laws in two variables; however, the system now changes type, and a complete analysis requires solving unusual boundary-value problems for degenerate elliptic and degenerate hyperbolic equations, as well as free-boundary problems for such equations. Recent work has resolved some of these difficulties. The talk illustrates this by solving some problems related to weak shock reflection in prototype equations.

1 Multidimensional Conservation Laws

Modeling by conservation principles is fundamental to fluid mechanics, and the importance of multidimensional systems is widely acknowledged. However, there are no general existence theorems for weak solutions of systems of conservation laws in more than one space dimension, as the tools which form the basis of a theory for hyperbolic conservation laws in a single space dimension do not extend to higher dimensions. To be specific, the principal method of analysis is through solution of the Riemann problem; this constitutes a nonlinear version of the method of characteristics. The role of characteristics in propagation of solutions of hyperbolic equations is complicated in several space dimensions, even for linear and semilinear problems, and a nonlinear formulation has not yet been found.

Recently, we have started to analyse two-dimensional Riemann problems. One goal of the research is to learn what sorts of singularities appear generically — that is, what are the two-dimensional analogues of shock discontinuities. Related to this, we hope to establish *a priori* bounds on weak solutions. In addition, a number of self-similar problems are of interest in themselves. For example, the so-called "von Neumann paradox" in weak shock reflection focuses on the failure of shock polar analysis to explain the nature of shock reflection when the waves are weak enough that the nonlinear acoustic waves dominate the linear entropy and vorticity waves. This problem can be studied in prototype equations which are simpler than the full equations of gas dynamics. We have examined the unsteady transonic small disturbance (UTSD) equation and the nonlinear wave system (NLWS).

2 Self-Similar Reduction

A working definition of a Riemann problem (not the only definition possible), is one for which the data depend only on x/y and hence self-similar solutions in x/t and y/t are expected. A system of conservation laws in two space dimensions and time,

$$U_t + F(U)_x + G(U)_y \equiv U_t + A(U)U_x + B(U)U_y = 0,$$

where $U(x, y, t) \in \mathbb{R}^n$ and F and G are smooth maps on $\mathbb{R}^n$, becomes a system in two variables $\xi = x/t$, $\eta = y/t$, which can also be written in conservation form:

$$\widetilde{F}_\xi + \widetilde{G}_\eta \equiv (F - \xi U)_\xi + (G - \eta U)_\eta = -nU.$$

A typical system of conservation laws, for example the equations of isentropic or polytropic compressible gas dynamics, is hyperbolic in space and time, with a pair of nonlinear acoustic wave speeds and a number of linear, degenerate characteristics corresponding to entropy or vorticity waves. The reduced system is hyperbolic only far from the origin and changes type at the sonic line, corresponding to the acoustic wave cone[1]; there is a bounded set $\{(\xi, \eta) \in \Omega\}$ in which the system is elliptic (if $n = 2$) or of mixed type (if $n > 2$). The reduced system is often called 'quasi-steady', and there is a close analogy with the equations of steady transonic flow, which are also much used in applications but for which there is not a complete theory. In the prototype systems we have studied, the UTSD equation and NLWS, the elliptic part can be written as a second-order equation which appears to be tractable. The Euler system is more complicated.

In any case, Ω is not known *a priori*, but depends on the solution U; typically the boundary of Ω is at most Lipschitz. In the hyperbolic region, solutions of the reduced system may be relatively simple. For example, for Riemann data which is piecewise constant in sectors, the far-field solution can be found by the elementary construction of solving one-dimensional Riemann problems. Interactions in the hyperbolic region of these one-dimensional waves can be analysed for small data (as a consequence of one-dimensional theory), and in some case have simple self-similar solutions by elementary constructions[1].

At least two types of behavior at $\partial\Omega$ have been identified. If U is continuous at $\partial\Omega$ then the elliptic equation is degenerate at $\partial\Omega$. This is the case even for linear equations such as the two-dimensional wave equation, whose fundamental solution has a square-root singularity at the wave cone. When U is also constant at $\partial\Omega$, the nonlinear equation possesses a nonlinear version of the same anisotropic degeneracy, which is of a type first analysed in work of Keldysh[2]; it is different from the Tricomi singularity, which appears when the steady transonic potential equation is written in hodograph variables. This nonlinear equation had not been previously studied. Čanić and Keyfitz[3,4], and Čanić and Kim[5] found solutions in weighted Sobolev spaces and in Hölder spaces (see also related work of Zheng[6]), and found that nonlinear Keldysh equations, as distinct from linear equations, may in addition have solutions which are continuously differentiable up to the degenerate boundary. Both singular and regular behavior occur, often in the same problem, on different parts of the boundary[7].

The segments of $\partial\Omega$ at which U is continuous and constant correspond to space-like surfaces; that is, the problem of posing Dirichlet data on $\partial\Omega$ is well-posed. However, there are configurations in which locally well-posed solutions outside of Ω are not constant on $\partial\Omega$ and do not extend to a solution in all of $\mathbb{R}^2$. Thus, even when a solution which is continuous across the sonic line is expected (from the absence of compression waves in the data, for example), it is not always possible to predict the location of Ω based on the supersonic solution alone.

A second type of behavior occurs when transonic shocks appear in the solution. In this case, the solution is discontinuous across $\partial\Omega$. The equation may be strictly hyperbolic on one side and the elliptic part of the operator strictly elliptic on the other; however, the boundary itself is now unknown *a priori*. This leads, then, to a free boundary problem in which the position of the shock and the subsonic flow are coupled by means of the Rankine-Hugoniot equations, a system of nonlinear equations relating the shock slope, the (known) state outside the shock and the unknown state inside Ω. In simple cases, the equation governing the subsonic flow is strictly elliptic, the shock may change continuously from supersonic to transonic, crossing a degenerate part of $\partial\Omega$ as it does so. Even without this additional complication, the free boundary problem is not of a standard type, as the underlying elliptic equation is quasilinear and the coupling between the shock slope and the states is highly nonlinear. This has turned out to be the principal challenge of the project up to this point.

3 Oblique Derivative Free Boundary Problems

In work with Lieberman[8] which proves a stability result for steady transonic flow, and which we have extended to establish weak[9] and strong[10] regular reflection patterns in the UTSD equation, at least in a neighborhood of the interaction point, we have found a method to prove existence of the free boundary and the corresponding subsonic solution. The method is classical, but seems well-adapted to quasilinear equations and nonlinear boundary conditions. It is based on formulating the elliptic equation as a second-order equation $Q(u) = 0$, whose coefficients do not involve the derivatives of u (here u is one state variable); and on casting the Rankine-Hugoniot as an evolution equation for the shock position and an oblique derivative boundary condition, $\beta \cdot \nabla u = 0$, on the free portion of the boundary. Taking an approximate position for the shock in an appropriate Hölder space $\mathcal{K}$ of curves, a mapping on $\mathcal{K}$ is defined by solving the quasilinear fixed boundary problem for u and then solving the evolution equation to define a new curve. The key is is a gain of regularity in this mapping, due principally to estimates one can obtain in the oblique derivative problem; we can show that the mapping is compact and has a fixed point. Kim has shown that in some cases the solution is unique[11]. The lack of regularity of $\partial\Omega$ requires the use of weighted Hölder norms. The lack of uniform ellipticity in the case of a shock adjacent to a continuous sonic boundary is handled by elliptic regularization.

We have solved two prototype problems for the UTSD equation[9,10], but we expect the method to work quite generally. Up to this point we have assumed that the oblique derivative boundary condition is uniformly oblique. This is true in cases where the shock itself is oblique and never normal. However, in many interesting problems, such as the formation of a Mach stem, the shock is normal at one point (the foot or symmetry point), and such appears to be, in fact, the generic situation for transonic shocks. For example, a uniform planar shock spanning a subsonic region has this property at its mid-point. Our current work focuses on adapting the compactness estimates to include this degeneracy.

Acknowledgments

Research of the first author (SC) supported by the National Science Foundation (NSF), grant DMS-9970310 and by the Texas Advanced Research Program (TARP) grant 003652-0112-2001. Research of the second author (BLK) supported by the Department of Energy, grant DE-FG-03-94-ER25222 and TARP grant 003652-0076-2001. Research of the third author (EHK) supported by NSF grant DMS-0103823.

References

1. S. Čanić and B. L. Keyfitz. Quasi-one-dimensional Riemann problems and their role in self-similar two-dimensional problems. *Archive for Rational Mechanics and Analysis*, 144:233–258, 1998.
2. M. V. Keldysh. On some cases of degenerate elliptic equations on the boundary of a domain. *Doklady Acad. Nauk USSR*, 77:181–183, 1951.
3. S. Čanić and B. L. Keyfitz. An elliptic problem arising from the unsteady transonic small disturbance equation. *Journal of Differential Equations*, 125:548–574, 1996.
4. S. Čanić and B. L. Keyfitz. A smooth solution for a Keldysh type equation. *Communications in Partial Differential Equations*, 21:319–340, 1996.
5. S. Čanić and Eun Heui Kim. A class of quasilinear degenerate elliptic equations. *Journal of Differential Equations*, to appear.
6. Yuxi Zheng. Existence of solutions to the transonic pressure-gradient equations of the compressible Euler equations in elliptic regions. *Communications in Partial Differential Equations*, 22:1849–1868, 1997.
7. S. Čanić, B. L. Keyfitz, and E. H. Kim. Mixed hyperbolic-elliptic systems in self-similar flows. *Boletim da Sociedade Brasileira de Matemática*, 32:1–23, 2002.
8. S. Čanić, B. L. Keyfitz, and G. M. Lieberman. A proof of existence of perturbed steady transonic shocks via a free boundary problem. *Communications on Pure and Applied Mathematics*, LIII:1–28, 2000.
9. S. Čanić, B. L. Keyfitz, and E. H. Kim. A free boundary problem for a quasilinear degenerate elliptic equation: Regular reflection of weak shocks. *Communications on Pure and Applied Mathematics*, LV:71–92, 2002.
10. S. Čanić, B. L. Keyfitz, and E. H. Kim. Free boundary problems for the unsteady transonic small disturbance equation: Transonic regular reflection. *Methods and Applications of Analysis*, 7:313–336, 2000.
11. Eun Heui Kim. Boundary behaviors and uniqueness of solutions for a class of quasilinear degenerate elliptic equations. *Submitted*, 2002.