Hold that Light! Modeling of Traffic Flow by Differential Equations

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Abstract. We show how the theory of hyperbolic conservation laws is used to develop and solve a simple continuum model for traffic flow. The solution is used to explain some common traffic phenomena.

1 Introduction: a Continuum Model for Traffic Flow

I begin this paper with a disclaimer: I am not a traffic engineer, nor an expert in the use of the model I am about to present to you. However, the basic model here, developed by Lighthill and Whitham [22], and by Richards [27], is well-regarded, at least by academic traffic engineering researchers. It has been presented in a number of elementary and advanced textbooks, for example [6, 9, 21, 29], as well as in introductions to mathematical modeling [5, 11]. The expository article by Gazis [8] briefly mentions this model, and the model is still used in traffic engineering research to treat certain situations such as bottlenecks, as in a technical paper by Newell [24].

My motivation in this paper is to show you something of the way applied mathematics works, by deriving a differential equation from physical principles and common sense, solving the equation, and then interpreting the answer as it refers to the phenomenon being modeled. In the example here, the equation can be solved by elementary methods: we do not get a closed-form solution, but we get enough information to visualize the solution.

There is a second reason for introducing the traffic flow model. It illustrates a widely-used approach to dynamic problems, an approach which often leads

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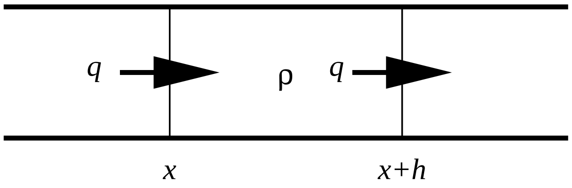

Figure 1.1: The Control Section for One-Way Traffic

to systems of partial differential equation in the form of conservation laws. The approach is to act as though a quantity which one wants to monitor is continuously distributed in space, rather than being a discrete variable. In the case of my example here, the quantity is cars on a highway — traffic — which is quite obviously discrete. However, for the sake of argument, suppose that we are not interested in the motion of individual cars, but only in some averaged quantities — for example, the carrying capacity of the road, which is the maximum number of cars per hour which the road can accommodate, or the number of times a car keeping up with the traffic will be stopped by a red light on a given stretch of road. In cases like this, one gets a good approximation, which appears to agree with the data, by considering, instead of individual cars, a continuously distributed quantity, ρ, the *linear density* of traffic on the road, measured in cars per mile. We suppose there is only one road, that it is straight and uniform, and that traffic moves along it in one direction (left to right): a straight one-way highway with no intersections. Other embellishments can be added.

Now, the equation describing the dynamics of traffic on this road is very simple to derive: it is the *law of conservation of cars*, which is a statement that the way the amount of traffic on a section of the road changes in time is simply by a net flux, that is the difference between the rate at which traffic enters the section of road and the rate at which traffic leaves. We may take the road to be the x-axis, and examine a control section between x and $x+h$, as in Figure 1.1. Assume the density is a function of x and of time, denoted by t. Since the amount of traffic in the control section is the integral of the density, the fundamental equation is

$$\frac{d}{dt}\int_x^{x+h} \rho(y,t)\,dx = q(x,t) - q(x+h,t)\,. \tag{1.1}$$

The quantity q is the *linear flux* of traffic. Flux may not be a familiar a concept as density. In this example, it is natural to think of it as the number of cars passing the point x in a unit of time. Of course, we are not looking at individual cars, so one should properly think of the units of q as the amount of traffic per hour, measured in cars per hour.

We have a simple way of relating the flux to more familiar quantities in traffic: if the speed of the traffic is v, in units of miles per hour, then the amount of traffic passing a given point x is just ρv cars per hour. Of course, if you think of actual traffic, composed of a lot of vehicles of different sizes, all traveling at different rates of speed, then v is a composite quantity which might be difficult to calculate. But in this simple model, just as there is a single quantity which represents density, so there is a single velocity, $v(x,t)$, at each point x and time t.

Applying the mean value theorem for integrals to the left side of (1.1), we obtain

$$\frac{\partial}{\partial t}\big(h\rho(x^*,t)\big) = q(x+h,t) - q(x,t)\,,$$

where the change in notation from an ordinary to a partial derivative with respect to t follows convention. The value x^* is a point in the interval $(x, x+h)$. Divide by h to get

$$\frac{\partial}{\partial t}\big(\rho(x^*,t)\big) = \frac{q(x+h,t) - q(x,t)}{h}\,,$$

and finally take the limit $h \to 0$ to obtain the fundamental conservation principle in the form of a partial differential equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0\,, \tag{1.2}$$

where we have substituted $q = \rho v$ to get the second form.

Now, the reasoning here was straightforward, and, as long as you accept that ρ represents an averaged quantity, and v the velocity that corresponds to it, we have not made any approximations: the total amount of traffic is exactly conserved, and that is all equation (1.2) says.

On the other hand, we have obtained only an incomplete model, because (1.2) is a single equation in two unknowns, ρ and v. For partial differential equations, as for any other kind of equation, you can not find a satisfactory solution unless the number of equations is the same as the number of unknowns. It is an interesting fact that modeling with conservation laws almost always brings one to this same place: it is straightforward in many physical situations to reason that certain quantities are conserved — mass, momentum, energy, and so on — but it always turns out that the number of unknowns is greater than the number of equations. In this example, the only physical quantity which is conserved is mass, which is why we obtain a single equation. A comparison with fluid dynamics modeling is given in Section 4.

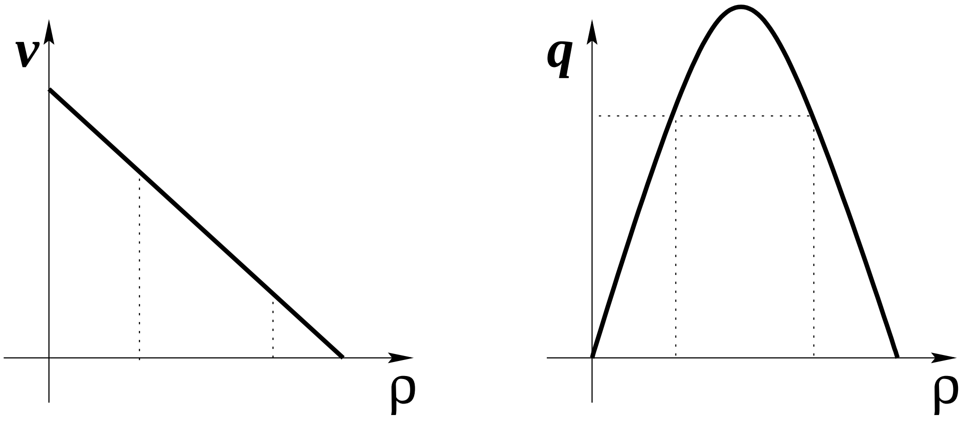

Figure 1.2: Idealized Velocity and Flux Curves

To obtain a solvable model from (1.2), one invokes a *closure* assumption. To be specific, in this case we assume that the velocity is a function of the density, that is, $v = v(\rho)$. This is a statement that, on this particular road, the speed at which traffic moves is completely determined by its density. Such a model is an oversimplification of actual traffic, as it does not take into account the variations of individual drivers or different kinds of vehicles. However, when we examine what the model predicts, we find it quite realistic in some respects.

There is no theoretical basis to determine the shape of the curve $v(\rho)$. The following postulate is intuitively reasonable: v is a monotone decreasing function of ρ, with a maximum at $\rho = 0$. At some value ρ_0, v becomes zero: ρ_0 is a critical density above which traffic cannot move. For single lane traffic, this has been measured at about 225 cars per mile [29, page 68]. The flux, $q(\rho) = \rho v(\rho)$ is thus zero at $\rho = 0$ and $\rho = \rho_0$. Some experimental data are quoted by Haberman [11, page 286]. A number of functional relationships have been proposed to approximate this empirical relationship; see [9, page 57]. We shall assume, because it makes the problem much simpler and because it is what the data show, that q is concave down: $q'' = 2v' + \rho v'' < 0$. We get a qualitative picture (this model is due to Greenshields [9]) by letting v decrease linearly with increasing ρ; we can choose units of measurement so that $v = 1 - \rho$ and $q = \rho - \rho^2$, as in Figure 1.2. The illustrations and examples will use this function. The theory, however, is exactly the same for any $v(\rho)$ which leads to a concave flux q.

Notice a feature of any concave flux function: to each value of q below the maximum, there correspond two values of density, one lower and one higher, and two values of velocity, one higher and one lower. That is, a road can process a given volume of traffic in one of two modes. At the lower density and greater vehicle speed, an individual driver will complete the trip more rapidly, and this is considered preferable by most of us. Section 3 gives an example of how a traffic configuration can slip from the more desirable mode into the other.

Now, with $q(\rho)$ a known, differentiable function of ρ, equation (1.2) becomes an example of a *first-order quasilinear partial differential equation in conservation form*, otherwise known as a *scalar conservation law*. Equations of this form were examined in the 1940's and 1950's for several reasons. First, Jan Burgers [4] had proposed a related equation as a model for the formation of coherent structures in turbulent fluids. Burgers' equation has the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = \nu \frac{\partial^2 u}{\partial x^2}\,, \tag{1.3}$$

which differs from (1.2) in that the flux function is convex rather than concave — this is an inessential difference, as one can replace u by $-u$ — and, more substantially, in having a second order term on the right hand side. The parameter ν is (in Burgers' model) proportional to the viscosity of the fluid. Burgers was interested in features of turbulence which appeared at very small viscosities, and so considered the case $\nu = 0$, which is just our traffic flow equation after a simple change of variables. A mathematical theory for scalar conservation laws was developed by Hopf [12], Lax [19, 20], Oleĭnik [25] and others.

In the next section, we will use this example to develop properties of solutions of first-order partial differential equations and of conservation laws. The mathematical properties have appealing interpretations as features of traffic flow.

2 Some Conservation Law Theory

As a first step, we rewrite (1.2), carrying out the differentiation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = \frac{\partial \rho}{\partial t} + a(\rho)\frac{\partial \rho}{\partial x} = 0\,. \tag{2.1}$$

Now $a(\rho) = dq/d\rho$ is another known function of the density, called the *wave speed*. Let us see why.

Consider first an unrealistic special case: suppose our traffic has the property that v is independent of ρ: that is, $v = v_0$ for all ρ and $q = \rho v_0$. Then $a = v_0$ and the equation becomes

$$\frac{\partial \rho}{\partial t} + v_0\frac{\partial \rho}{\partial x} = 0\,. \tag{2.2}$$

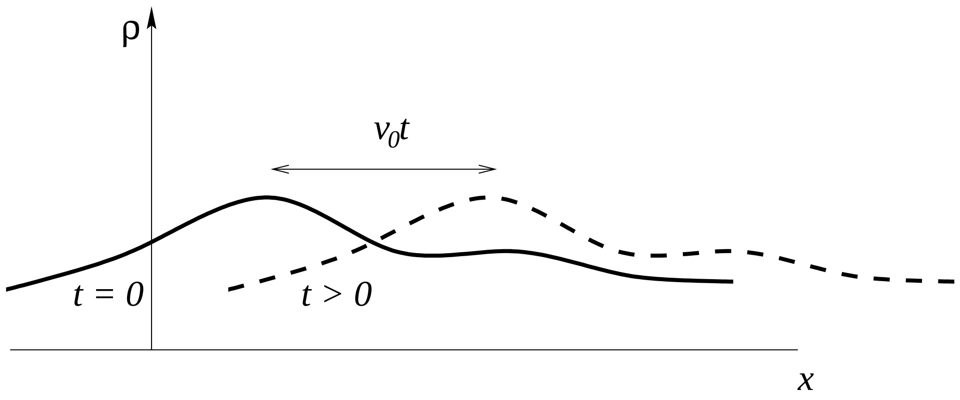

Figure 2.1: Density Profiles for the Linear Equation

This is a *linear* equation (which means simply that the principle of superposition holds and the solutions form a vector space). The following proposition is easily demonstrated.

Proposition 2.1 *Every differentiable solution of* (2.2) *is of the form*

$$\rho(x,t) = \rho_0(x - v_0 t)$$

for a differentiable function ρ_0 of one variable. Furthermore, every choice of ρ_0 yields a solution of the equation.

Proof: After a change of independent variables

$$y = x - v_0 t, \qquad s = t,$$

the equation becomes

$$\frac{\partial \rho}{\partial s} = 0,$$

so the solution in the new coordinates must be a function of y alone. Direct differentiation shows that every function of y satisfies the equation. ■

What information about traffic is conveyed by this solution? The message is that any pattern of densities simply moves to the right with speed v_0, as in Figure 2.1. The function ρ_0 corresponds to a distribution of densities along the road at a particular time; for example, if $\rho_0(x)$ is the density at position x at time $t = 0$, then $\rho(x,t) = \rho_0(x - v_0 t)$ is the solution of the *initial value problem* consisting of equation (2.2) and the initial condition

$$\rho(x,0) = \rho_0(x)\,, \qquad -\infty < x < \infty\,. \tag{2.3}$$

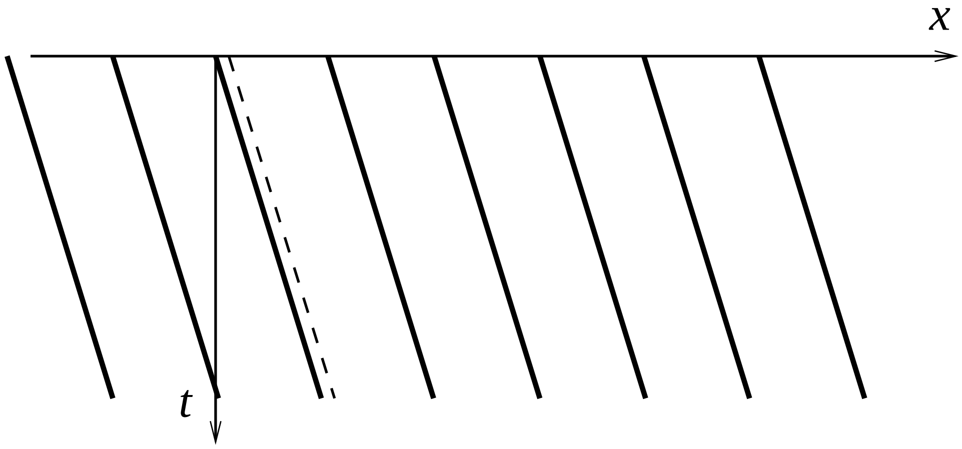

Figure 2.2: Characteristics for the Linear Equation

Another way of picturing the way traffic patterns change is by means of a space-time diagram, as in Figure 2.2. The lines $x - v_0 t = \text{const.}$ are contours of constant density. What Proposition 2.1 shows is that the lines $x - v_0 t = \text{const.}$ play a special role in solving the equation. These distinguished curves are called *characteristics* for the equation.

We return now to equation (2.1). This equation is not linear; rather, it is *quasilinear*, a term which refers to the fact that it is linear in the derivative terms, $\partial\rho/\partial t$ and $\partial\rho/\partial x$, but the coefficients may depend on ρ. The *method of characteristics* can be used to solve this equation, by seeking curves $x(t)$ with special properties. In fact, if we differentiate ρ with respect to t, assuming x to be a function of t, we obtain

$$\frac{d}{dt}\rho\bigl(x(t),t\bigr) = \rho_x \frac{dx}{dt} + \rho_t$$

(where subscripts denote partial derivatives) and now, comparing with (2.1), we see that if we choose $x(t)$ to satisfy $dx/dt = a(\rho)$ then this becomes

$$\frac{d}{dt}\rho\bigl(x(t),t\bigr) = \rho_t + a(\rho)\rho_x = 0\,,$$

so that ρ is constant along the curve $x(t)$ which then must be a straight line (since its reciprocal slope is $a(\rho)$ which is also constant). In fact, if (x_0, t_0) is any point on the line, and the value of the density is ρ_0 there, then the equation of the line is $x(t) = x_0 + a(\rho_0)(t - t_0)$, and $\rho = \rho_0$ at every point on the line. If we suppose a function $\rho_0(x)$ given by means of an initial condition

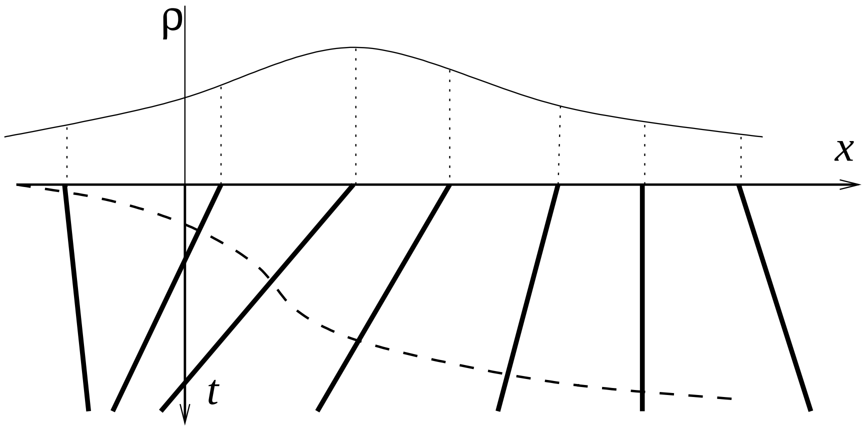

Figure 2.3: Characteristics for the Quasilinear Equation

(2.3), then $t_0 = 0$ and we can write the solution by means of an equation

$$\rho\Big(x_0 + a\big(\rho_0(x_0)\big)t, t\Big) = \rho_0(x_0) \tag{2.4}$$

which gives $\rho(x,t)$ implicitly as long as we can solve the equation

$$x = x_0 + a\big(\rho_0(x_0)\big)t \tag{2.5}$$

to find x_0 as a function of x (for fixed t). This we can do if x is a monotonic increasing function of x_0 (again for a fixed value of t). Now,

$$\frac{dx}{dx_0} = 1 + a'\rho_0' t\,,$$

by the chain rule, where $a' = a'\big(\rho_0(x_0)\big)$ and $\rho_0' = \rho_0'(x_0)$. At $t = 0$ the right side of the equation equals unity, and so the expression remains positive, at least for small t.

Although $\rho(x,t)$ cannot be found explicitly as in the linear case, it is possible to visualize how the densities evolve with time by means of a diagram analogous to Figure 2.2. Given an initial density distribution, $\rho_0(x_0)$, from each point x_0 on the x-axis draw the straight line (2.5), as in Figure 2.3. Notice that $a(\rho_0)$ is the slope of the line drawn in a conventional t,x-plane (that is, with the t-axis horizontal). The convention in partial differential equations is to make the x-axis the horizontal axis; a further convention in traffic engineering is to draw the t-axis downward (so you can see a as the slope if you rotate the picture). For reference, the initial density distribution that was used

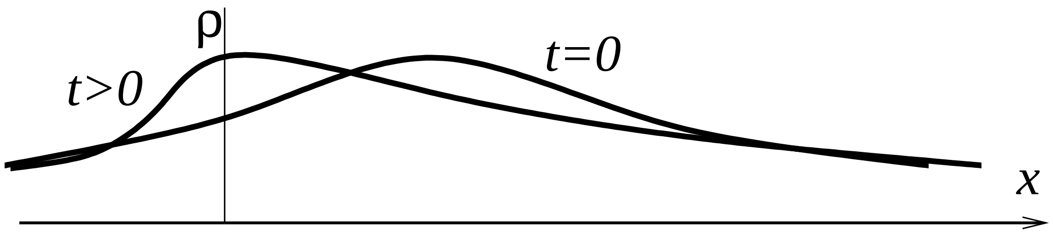

FIGURE 2.4: Density Profiles for the Quasilinear Equation

to draw the picture is sketched above the x-axis: it contains a density wave, with a peak value near the maximum value $\rho = 1$, in the center. The concentration is initially symmetric front to back, decreasing to a value $\rho = 1/3$ at the edges of the picture. If we assume $q(\rho) = \rho - \rho^2$, so $a(\rho) = 1 - 2\rho$, then the slopes dx/dt vary between 1/3 and -1. The evolution of the density wave with time is shown in Figure 2.4. One sees that the wave remains a wave, but it no longer preserves its shape. The peak moves more slowly than the wave as a whole; in fact, the peak may even move backwards. After some time, a wave that was originally symmetric, front to back, has become skewed. It is steeper at the back (engineers call this a compression wave) and slopes more gently at the front (this is called a rarefaction). Remembering the assumption that denser traffic moves more slowly, we see that the picture is a simple consequence of the modeling assumption. A patch of traffic in which congestion is greater at the rear will gradually stretch out, as the cars at the front race away, while a situation in which low-density traffic is spread out behind a slow-moving traffic jam will tend to become more concentrated as the cars in the back catch up with the crowd.

It is time to say a word about what is moving in the wave. In our first, linear model, the line $x = x_0 + v_0 t$ in space-time represented both the position at any time of the car which had been at x_0 at the initial time, and the position of the point on the wave (peak, front, back) as it evolves in time. Such is no longer the case in the quasilinear model. In fact, every point in the wave with density $\rho > 1/2$ is actually moving backwards in time, but every car is moving forward or at worst standing still! Recall that at a given density ρ, the velocity of a car is $v = q/\rho$, in terms of the flux function q, while the velocity of that part of the wave (that is, the part with value ρ) is $a = q' = v + \rho v'$, which is less than v if q is not constant. See Figure 2.5, in which both speeds are depicted as slopes. This brings up an interesting point, which is central to the study of wave motion: what travels when a wave moves? Think, for example, of the waves that fans make in a sports arena: the wave travels around the arena, but

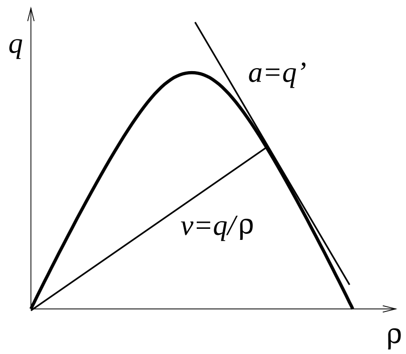

Figure 2.5: Vehicle Velocity and Wave Velocity

the arms that generate it move up and down. It might be reasonable to say that what is traveling is information. Indeed, traffic modeling that takes into account individual drivers and their reaction times reaches the same conclusion about wave motion by postulating that drivers react to a patch of dense traffic just ahead by slowing down (thus contributing to increasing density of traffic and passing the signal on to drivers further in the rear). It is not difficult to trace the route of an individual driver through the density wave shown in Figures 2.2 and 2.3. A sample space-time path is shown by a dotted line. Wave motion in which points in different parts (or phases) or the wave have different velocities is said to have a *dispersive* character (Whitham [29]).

However, examination of the situation in Figure 2.3 for times beyond the end of the drawing shows that the theory so far — in particular the implicitly given solution (2.4) — will fail eventually. For the straight lines in Figure 2.3 are not parallel, and so they must intersect. (The sole exception is the case of data where all the lines are spreading apart.) Fortunately, this does not mean a collision, since the lines are not the trajectories of individual cars, but it does mean that the mathematical function obtained by the method of characteristics no longer solves the problem. The resolution of this difficulty is interesting, and is at the heart of the subject of conservation laws. To understand it, one must go back to the beginning of the derivation of the model, to the calculus assumption under which the differential equation (1.2) was derived from the integral equation (1.1). In writing the differential equation we assumed, without stating this explicitly, that ρ and q were differentiable functions of x and t. But there are *discontinuous* functions for which (1.1) holds. For example the piecewise constant function

$$\rho(x,t) = \begin{cases} \rho_0, & x < st \\ \rho_1, & x > st \end{cases}, \quad q(x,t) = \begin{cases} q_0, & x < st \\ q_1, & x > st \end{cases}, \quad s = \frac{q_1 - q_0}{\rho_1 - \rho_0} \tag{2.6}$$

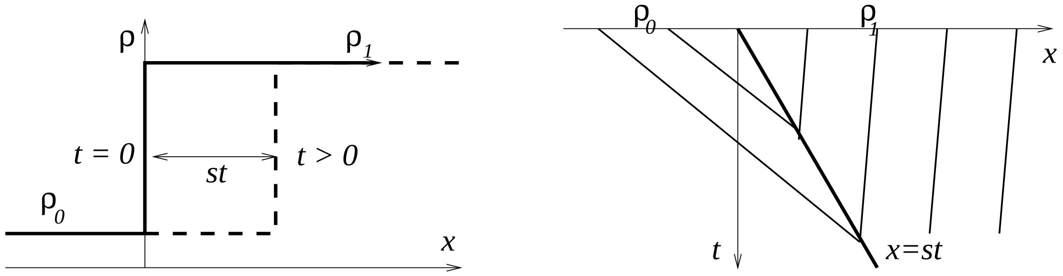

FIGURE 2.6: Piecewise Constant Solution; Profile and Characteristics

solves equation (1.1) exactly. This function also represents a traffic configuration; its profile is shown in Figure 2.6, along with the space time diagram of the characteristics in the case $\rho_0 < \rho_1$ (we shall see shortly why this is the only interesting case).

What is the interpretation of a discontinuous solution like this? The discontinuity, called a shock, is a demarcation line between regions of heavy and of light traffic. For example, if cars are backed up behind a red light, then $\rho_0 = 1$ and $\rho_1 = 0$; the corresponding fluxes in our model are $q_0 = q_1 = 0$, so $s = 0$ and traffic sitting at a light is a solution to the equation. It is conventional to refer to discontinuous solutions as *weak* solutions, because they do not satisfy the differential equation (1.2) in the classical sense, but instead the equation (1.1). (An even more general notion of a weak solution is conventionally used, but will not be needed in this paper.) The situation $0 < \rho_0 < \rho_1 \leq 1$ also admits an all-too-familiar interpretation: cars free-wheeling on the expressway suddenly come upon a patch of dense slow (or stopped) traffic, slam on their brakes and join the queue. Depending on the relative densities of cars before and behind the shock, the discontinuity may move forward or backward, or stand still.

As the graph of the characteristics in Figure 2.3 makes clear, one does not expect there to exist a differentiable solution of (2.1) for all $t > 0$ if the initial data look like a wave with a density maximum in the center. In fact, the implicitly defined solution given by equations (2.4) and (2.5) ceases to exist at the first time where $dx/dx_0 = 0$, which is the smallest positive value of $-1/a'\rho_0'$. Typically, the profile ρ develops a vertical tangent at this value of t, which grows into a shock for later t (the precise development of this singularity is interesting and was given by a number of people; for a recent description, see Nakana [23]). Far away from the shock, the method of characteristics solution is still valid, but after a while more and more characteristics run into the shock and terminate; the size of the jump in ρ also grows initially, but then stops

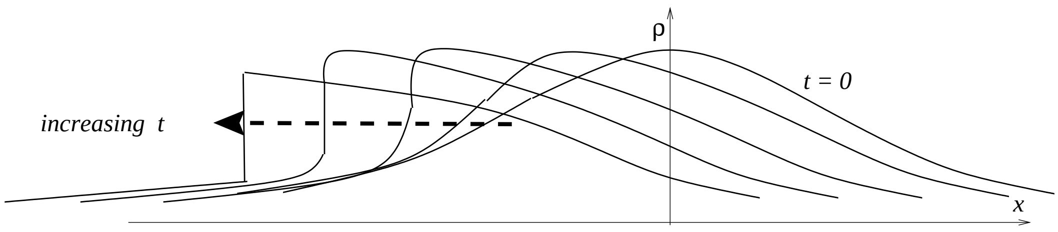

Figure 2.7: Development and Decay of a Shock

growing and begins to decrease, as sketched in Figure 2.7. (The asymptotic form of the wave for large t, often called an N-wave, is given by Lax, [20].)

We now consider a final feature of solutions of conservation law equations, which must be resolved by mathematical theory. Let us return to the initial-value problem with piecewise constant data

$$\rho(x,0) = \begin{cases} \rho_0\,, & x < 0 \\ \rho_1\,, & x > 0 \end{cases}$$

with $\rho_0 > \rho_1$. This type of data is called *Riemann data* in the field of conservation laws. (We saw that $\rho_0 = 1$ and $\rho_1 = 0$ might represent cars stopped at a red light. One could set $t = 0$ at the moment the light turns green. You can imagine an interpretation for other values of ρ_i: perhaps a flagman has been slowing the traffic and has just walked away.) We saw that the piecewise constant function ρ given in equation (2.6) is a solution. But when $\rho_0 > \rho_1$ there is a second solution, as you can check:

$$\rho(x,t) = \begin{cases} \rho_0\,, & x < (1-2\rho_0)t \\ \frac{1}{2}\left(1-\frac{x}{t}\right), & (1-2\rho_0)t < x < (1-2\rho_1)t \\ \rho_1\,, & x > (1-2\rho_1)t \end{cases}. \qquad (2.7)$$

The solution profile and characteristic lines are shown in Figure 2.8. This fanlike solution is called a *centered rarefaction wave*. It is an idealized portrait of an everyday phenomenon: when a line of congested traffic is faced with an emptier road ahead, the congestion breaks up as the first cars race off into the open space; the opportunity to accelerate gradually moves back through the line.

Mathematically, the existence of two solutions for the same initial condition is a sign of *ill-posedness*: the problem is badly or incompletely specified. In fact, much modeling, especially of complicated situations, results in ill-posed problems, and much effort in applied mathematics goes into finding satisfactory resolutions of the dilemmas which result. An *ad hoc* fix is available here:

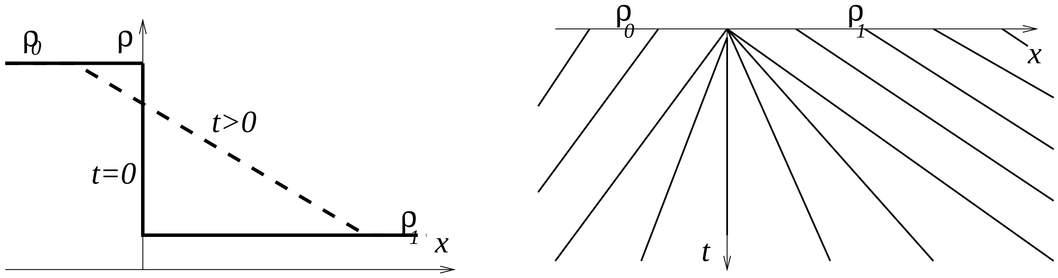

Figure 2.8: A Centered Rarefaction Wave

reasoning from the situation faced when a light turns green, one sees that the two solutions represent different cases: the traffic remains at the intersection (the shock wave), or begins to move (the rarefaction). The second is what actually happens (the 'physical' solution). A good deal of research has gone into finding what are called *admissibility conditions* for shocks — conditions which will, ideally, be strong enough to eliminate spurious solutions but not so restrictive as to eliminate all solutions.

Peter Lax has given a condition which tells which discontinuities are admissible for our model (the condition can be extended to systems of conservation laws also). Called the *geometric entropy condition*, Lax's condition states that shocks are admissible if the characteristics on either side run into the shock in the direction of forward time. Thus, the shock in Figure 2.6 is admissible, but a shock drawn in the situation of Figure 2.8 would not be, because the characteristics on either side flow away from each other. A heuristic argument for why this is a reasonable condition is that, under this condition, every point (x, t) in space-time, except points that are actually on shocks, can be connected to the initial data by a unique characteristic line, and hence the solution is uniquely given there. As long as most points are not on shocks, then this is enough to determine a weak solution, since the integral equation satisfied by a weak solution is not affected by the values of ρ at a discrete set of points.

To justify the model, one wants a theory that gives both existence and uniqueness. For the traffic-flow model, and in fact for any scalar conservation law, such a theory exists. It was first developed by Lax [19, 20] for a convex flux function and extended by Oleinik [25]. When q is neither convex nor concave, new phenomena appear; see also [13]. A complete theory for a single equation was given by Kružkov [17] in 1970. There is also extensive theory for systems; the first existence theorem was given by Glimm [10] in 1965. Recent work by Bressan [3] is finally tackling questions of uniqueness, but many problems remain open.

Let me state, without proof, Lax's theorem for the system examined in this paper.

Theorem 2.1 *The solution to*

$$\rho_t + (\rho - \rho^2)_x = 0\,, \qquad \rho(x,0) = \rho_0(x)\,,$$

for $t > 0$ is given by

$$\rho(x,t) = \frac{1}{2}\left(1 - \frac{x - y_0}{t}\right)$$

where y_0 is the value of y at which the function

$$G(x,y,t) = \int_{-\infty}^{y} \rho_0(s)\,ds + \frac{1}{2}\left(x - y - \frac{(x-y)^2}{2t}\right)$$

achieves its minimum. Provided that the integral of ρ_0 exists, this formula gives the solution at almost every point (x,t).

It is straightforward to verify that this formula gives the function (2.4) defined by the method of characteristics, for times before the first shock forms. One can also check that any shock solution given by this formula will satisfy Lax's condition. Lax's proof in [20] that the formula gives a solution for all time, and for all integrable data, uses a connection between the traffic-flow equation and Burgers' equation, (1.3).

At this point one can establish, via several steps, a connection with the calculus of variations. First note that if we define a new quantity u by $u_x = \rho$ or

$$u = \int_{-\infty}^{x} \rho(s,t)\,ds\,,$$

then, whenever ρ satisfies equation (2.1), u will satisfy

$$u_t + q(u_x) = 0\,. \tag{2.8}$$

An equation of this form is called a *Hamilton-Jacobi* equation. The calculus of variations provides an elegant method to solve (2.8), from which the formulas of Theorem 2.1 can be derived. An excellent exposition can be found in the text of Evans [7]. The last third of Evans's book also contains extensions of this example problem. The relation with Burgers' equation, (1.3), plays a role in the theory of viscosity solutions of Hamilton-Jacobi equations, as is also discussed in [7].

3 An Application of the Model: The Timing of Traffic Lights

One of things this model can do is predict, again under rather idealized conditions, what will happen at a red light, and how lights should be timed, depending on how heavy the traffic is. It also explains why, for an individual driver, there is no perfect staging of lights. The calculation was done by Richards [27], one of the originators of the model, and has been reproduced since, for example in Haberman [11] and Whitham [29].

Let us consider, first, a single traffic light which is red for a time interval T_R and then green for an interval of duration T_G. The density of traffic everywhere on the road can be calculated explicitly if we make a few simplifying assumptions: assume that the incoming density has the constant value ρ_0 and that the velocity-density relation is linear, $v = 1 - \rho$. Assuming that traffic is running free when the light changes, the red cycle is described by the first picture in Figure 3.1: cars back up behind the light, and a shock forms and moves up the street with speed

$$s = \frac{q_1 - q_0}{\rho_1 - \rho_0} = 1 - (\rho_0 + \rho_1) = -\rho_0 \,.$$

When the light turns green, traffic begins to move at the light, and a rarefaction forms; if we take this instant to be the origin of time, then the rarefaction is centered at the origin, and has head and tail speeds 1 and -1 respectively. The tail of the rarefaction runs into the shock at

$$t_0 = \frac{\rho_0}{1-\rho_0} T_R \,, \qquad x_0 = -\frac{\rho_0}{1-\rho_0} T_R \,,$$

and now the speed of the shock changes. If we let $\phi(t)$ denote the position of the shock, then the shock speed $s = d\phi/dt$ and ϕ satisfies the differential equation

$$\frac{d\phi}{dt} = 1 - \left(\rho_0 + \frac{1}{2}(1 - \frac{\phi}{t})\right)$$

since ρ_1, the density to the right of the shock, is given by equation (2.7). Rearranging terms gives a linear, first-order ordinary differential equation

$$\frac{d\phi}{dt} - \frac{\phi}{2t} = \frac{1}{2} - \rho_0 \,,$$

which we can solve. Applying the initial condition $\phi(t_0) = x_0 = -t_0$ we get

$$\phi(t) = (1 - 2\rho_0)t - 2\sqrt{\rho_0(1-\rho_0)T_R t}$$

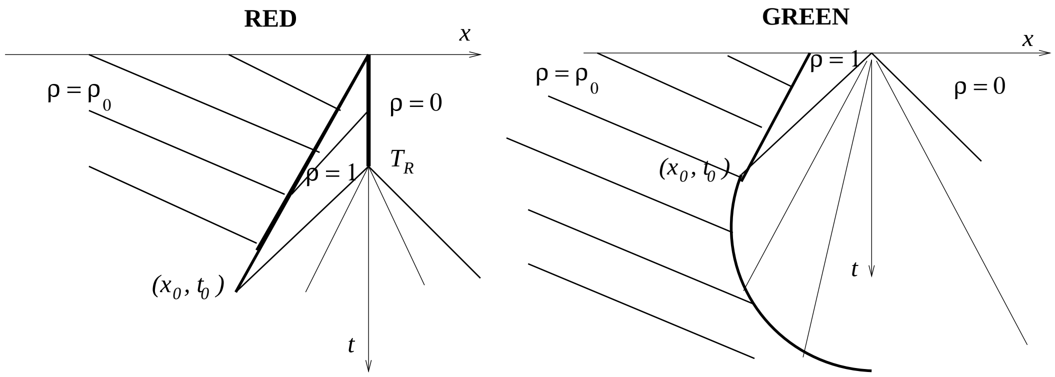

Figure 3.1: The Traffic Light Cycle

as the equation of the arc the shock describes in space-time. If $\rho_0 \geq \frac{1}{2}$, that is, if the road is at greater than half its carrying capacity, then the shock continues to move to the left for all time. Thus, even if the light never turns red again, the effect of the delay propagates back up the road forever. (The influence does, however, decay.)

On the other hand, if $\rho_0 < \frac{1}{2}$, the shock speed eventually becomes positive, and the shock crosses the point $x = 0$ at a time

$$t_r = \frac{\rho_0(1-\rho_0)T_R}{(\frac{1}{2}-\rho_0)^2}\,,$$

provided the light is still green.

It turns out to make a big difference whether the shock gets through the intersection or not before the light changes again. The discovery of this effect, and its quantification (which can be improved from the calculation here by taking a more realistic velocity-density equation) are a major contribution of the model. From a mathematical viewpoint, this example illustrates a difference between nonlinear and linear models. Linear problems "scale up": if the effect of doubling the density is to multiply the waiting time at a light by a factor K, then quadrupling the density multiplies it by $2K$, and proportionally smaller changes will have smaller effects. But in this model, the effect of increases in the density is small until a threshold is reached, after which the nature of the traffic flow changes dramatically.

To see how, and why, look again at the density-flux diagram, Figure 1.2, and recall that a desired traffic flux which is less than the maximum value of q can be realized at two different densities. From the point of view of the road, the service factor is the same, but individual drivers will notice a great difference in the time it takes them to make the trip.

Let us first suppose that the incoming density, ρ_0, and the timing of red and green lights are such that the shock crosses the intersection before the light turns red again; that is, $T_G \geq t_r$. If we assume the cycle of lights is fixed, this means

$$\rho_0 \leq \frac{1}{2}\left(1 - \sqrt{\frac{T_R}{T_R + T_G}}\right) . \tag{3.1}$$

Since the density behind the shock is again ρ_0 when the light turns red again, then in this situation after the red-green cycle we have the same initial condition as before, and to the left of the light the flow is periodic in time. It is also periodic beyond the light, and the average density a short distance downstream must also be ρ_0: the effect of the light is to impose on the traffic a temporary slowdown, from which it recovers.

However, let us now suppose that the shock does not get through the intersection. We can make the following calculation of the flux at the light itself: during the green cycle, the density is precisely $\frac{1}{2}$ (since it is the center of the rarefaction), while during the red cycle it is zero, so the average flux over a period is

$$\bar{q} = \frac{1}{4}\frac{T_R}{T_R + T_G} .$$

Now, this must be the average flux anywhere along the road (since there are no sources or sinks) and so the average density is found by solving $\rho - \rho^2 = \bar{q}$. This equation has two roots, but the smaller is precluded by the fact that inequality (3.1) is violated. Hence we have

$$\bar{\rho} = \frac{1}{2}\left(1 + \sqrt{\frac{T_R}{T_R + T_G}}\right)$$

as the value of the average density. The effect of a small change in incoming density from a value satisfying (3.1) to one just violating it has been to move the average density from the lower to the upper half of the density-flux diagram, greatly increasing the travel time for drivers.

The actual numbers are dramatic: Richards [27] points out that if $T_R = T_G$ then the greatest incoming density that permits free flow at the light is $(1 - 1/\sqrt{2})/2 \approx .15$, while the average density when the shock does not get through the light is $(1 + 1/\sqrt{2})/2 \approx .85$.

A second feature of roads with traffic lights, also familiar to drivers, can be derived from this model. This has to do with the timing of successive lights along a road. If we look at the flow behind a well-timed light then it is periodic in time, with period $T_R + T_G$. However, beyond the light the mass that has

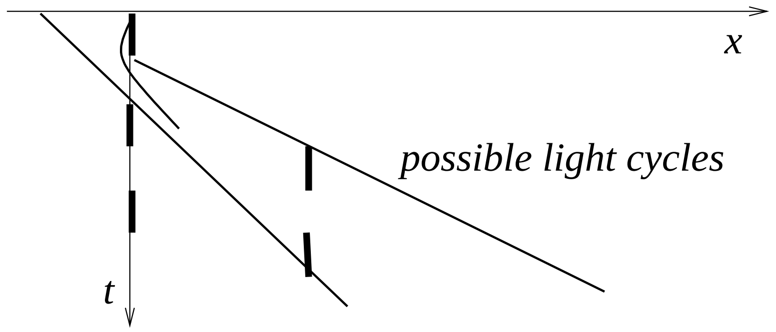

Figure 3.2: A Road With More Than One Light

crossed the light in a single green cycle (which corresponds to a set of drivers, as well) spreads out, since the velocities vary between 1 and $1-2\rho_0$. See Figure 3.2. If there is a second light on the road, and its cycle has the same period, then the only way to arrange its cycle so that the entire mass gets through the second light on a single cycle is to make the ratio T_G/T_R greater for the second light. Even then, this works only if the second light is sufficiently close to the first. Realistically, a series of lights will be timed so that only a part of this packet will get through later lights. Which part? Reasoning on the basis of the fluxes, it is optimal to plan so that the later part of the packet, where the density is greater, will cross the second light without waiting. Thus it is to be expected that the first drivers through a green light will get caught at the next light. A light that is timed for the benefit of the first phalanx of drivers will have the effect of cutting off the last drivers through the previous light, as in the title of this paper.

4 Extensions and Other Models

One source of interest in the continuum traffic flow equation is its resemblance to the *compressible Euler equations* used to study the dynamics of gases. A simple version of the equations, for one-dimensional flow, is

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + (\rho v^2 + p)_x &= 0\,. \end{aligned}$$

The first equation represents conservation of mass for a gas with density ρ and velocity v, and it is exactly the same as the traffic flow equation, except that we no longer assume v to be a function of ρ. The second equation is conservation of momentum, and is the expression, in a continuum, of Newton's law of dynamics, $F = ma$. (Models in which v is postulated to be a function of

ρ, instead of being determined from a second equation, are called *kinematic*, as contrasted with *dynamic*.)

One sees, in this system, that there is again a closure problem: the two equations contain a third variable, the pressure, p. Under some conditions (roughly that there is not much energy exhange in the flow), one can establish p as a function of ρ from thermodynamic principles. One then has a system which can be written in the form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \tag{4.1}$$

where $\mathbf{u}$ is the vector of states, $\mathbf{u} = (\rho, \rho v)$, and $\mathbf{f}$ the flux vector. This system can also be written in quasilinear form,

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = 0\,, \tag{4.2}$$

where A is the Jacobian matrix, $\partial\mathbf{f}/\partial\mathbf{u}$. Just as for the single equation, it is useful to look at the linear problem one obtains on replacing A by a constant or by a known function of x and t.

When do the methods established in the previous sections help in solving equation (4.1) or (4.2)? The system behaves like the scalar equation in important ways precisely when it is *hyperbolic*, that is, when the matrix A has real eigenvalues and is diagonalizable. In fact, it is a simple exercise in linear algebra to show that if A is constant and $P^{-1}AP = D$ is diagonal for a constant matrix P, then in a new coordinate system, $\mathbf{w} = P^{-1}\mathbf{u}$, equation (4.2) separates into a set of uncoupled equations, each of the form of the scalar linear equation, (2.2), with an eigenvalue of A playing the role of velocity. The eigenvalues are called *characteristics* for the system, and using this change of coordinates one can write down the general solution for this linear, constant-coefficient system. In the nonlinear case, the system does not decouple in any coordinate system, and there is no analogue to the implicit formula (2.4) for the solution to an initial-value problem. However, one can show that in many ways the solutions to a system behave like solutions to the equation we have studied: waves steepen, shocks form, and disturbances propagate through the flow, with velocities equal to one of the characteristic speeds.

It may not be immediately apparent that a system with complex eigenvalues behaves differently. For example, Laplace's equation, $\Delta v \equiv v_{xx} + v_{yy} = 0$, when written as a first-order system for $\mathbf{u} = (v_x, v_y)$ is a coupled linear system with eigenvalues $\pm i$. Laplace's equation is a perfectly reasonable equation, and serves as a model for many physical processes. However, none of these involves time explicitly; a difficulty arises, not with the equation itself but with the problem of trying to solve it along with data given on one axis, say $y = 0$. This prototype example is known as *Hadamard's counterexample* and can be

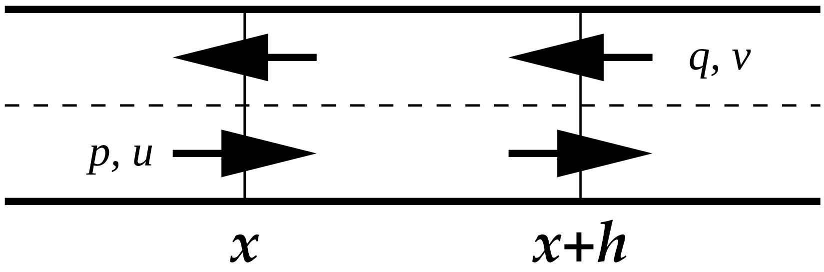

Figure 4.1: Variables in the Two-Way Traffic Model

found in most partial differential equations textbooks, for example Evans [7, page 233]. It is simple to demonstrate that data of small amplitude and high frequency (functions like $\sin nx/n$) lead to solutions of very large amplitude for non-zero values of y (functions like $\sin nx \sinh ny$). In the limit $n \to \infty$ the data tend to zero but the solution does not. Thus, the solution does not depend continuously on the data, which is another way an equation can be ill-posed.

Extension from a scalar equation to a system of conservation laws, so important for gas dynamics, also occurs in modeling traffic flow. First, there have been attempts to find two-equation models for the one-way traffic flow problem we have been describing. This is motivated partly by a desire to find solutions that agree better with actual traffic. In addition, the velocity-density relation which closed the single equation model does not have a theoretical justification, and people have sought a more scientific way of deriving it. A recent paper of Aw and Rascle [1] surveys the progress of this attempt and offers a new suggestion. A related use of two-equation models is to provide a reduction to the single-equation model by the use of what are called relaxation methods, as is done by Lattanzio and Marcati [18]. (A different attempt find $v(\rho)$ was made by Prigogine and Herman [26], using ideas from statistical mechanics and kinetic theory. Their book is an engaging introduction both to traffic theory and to statistical mechanics.)

However, I would like to close this paper by mentioning a very simple two-equation model which is closely related to the one-way model we have been discussing. Bick and Newell [2], almost forty years ago, proposed to extend the continuum model to two-way traffic on a road, by supposing that a density for traffic in each direction can be assigned, with a corresponding flux function, and that the two directions of traffic interfere with each other in some way.

Here is the two-way traffic model. (See Figure 4.1.) Changing notation somewhat, we let $p(x,t)$ and $q(x,t)$ be the densities of east-bound and west-bound traffic, respectively, and let u and v be the respective velocities. We assume that there are no U-turns, so conservation of mass holds for each direction of travel separately, and we obtain the system

$$\begin{aligned} p_t + (pu)_x &= 0 \\ q_t + (qv)_x &= 0\,. \end{aligned}$$

If we assume that each lane completely ignores the other and acts as a one-way road, then the closure assumptions from before are appropriate:

$$u = u(p) = 1 - p\,, \qquad v = v(q) = -(1-q)\,,$$

where v is negative since the west-bound traffic moves in the direction of decreasing x. This system is hyperbolic; in fact, the characteristic speeds are $1-2p$ and $2q-1$. The system is not *strictly hyperbolic*; that is to say, the speeds are not everywhere distinct. They coincide when $p+q=1$; if we think of (p,q) as a point in a *phase space*, then the characteristics coincide on a line in two-dimensional phase space. However, the system remains hyperbolic there, since there is no eigenvector deficiency.

Now, the idea explored by Bick and Newell was to suppose that the two lanes of traffic interfere with each other. No specific mechanism need be mentioned; perhaps passing is inhibited, perhaps a sense of crowding discourages higher speeds. To study the effect of this assumption, we can look at the simplest type of dependence:

$$u = u(p,q) = 1 - p - \beta q\,, \qquad v = v(p,q) = -(1 - q - \beta p)\,,$$

where β is a small coupling parameter. The equations are coupled if $\beta > 0$. In addition, the feasible region of phase space, which had been a square $\{0 \le p \le 1, 0 \le q \le 1\}$ now becomes a quadrilateral bounded by $p=0$, $q=0$, $p+\beta q=1$, and $q+\beta p=1$. But the most significant difference is that the coupled system is not hyperbolic for all states: computing the eigenvalues of the Jacobian matrix $A(p,q)$ shows that there is a region near the line $p+q=1$, elliptical in shape, with width of the order of β, in which the eigenvalues of A are complex conjugate. Bick and Newell performed some studies on this system; they identified some of the shocks and solved some Riemann problems. However, they could not explain away this surprising feature, nor solve the problem for all initial conditions. More recently, Vinod, while a graduate student at the University of Houston, explored this problem further, and showed that in a number of cases one could find solutions for $\beta > 0$ which approached the easily calculated solutions for the uncoupled case, $\beta = 0$, [28].

It is surprising that the two-way model, which seems like a simple and natural extension of the one-way model, turns out to have unexpected mathematical features. It is, at one level, completely ill-posed, in the sense that the corresponding linear problem may suffer from the Hadamard instability. Indeed, a complete solution to the mathematical problem is not known at this time. I came across the model because it is related to research I have been doing on other systems of conservation laws, used in physics and engineering, which are, similarly, derived from what appear to be reasonable principles and analogies to well-known and well-behaved models, and which are ill-posed in exactly the same way. This work is reviewed in [14, 15, 16].

A note of caution is in order here. Often one is presented with equations which are said to be mathematical models for something, and one is given the impression that there is not much depth to the modeling exercise — understand the physics well enough, and understand the mathematics well enough, and everything will be straightforward. In examples like the two-way traffic model, the interplay between "modeling" and "mathematics" is intricate and subtle. The process goes both ways: as more is understood about the mathematics, people return to the models to take another look. The one-way traffic model was not useful to engineers until the mathematical theory was developed in the 1950s. Perhaps future theory will make sense of two-way traffic models.

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