Partial Differential Equations

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Chapter 1

Introduction

Two aspects of partial differential equations form the thread of this book:

1. obtaining global from local information by solving the equation
2. relating the algebraic structure of a partial differential operator to analytic properties of its solutions.

In this introduction, we will try to explain what is meant by these statements. The second part of the chapter gives a brief review of material from ordinary differential equations and advanced calculus which will be used later in the book.

A partial differential equation (PDE) is an equation involving a function of several variables, $u(x, y)$, for example, and its partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}, \ldots.$$

Some examples of PDE which govern interesting phenomena in mathematics, physics and engineering are

Potential Equation $\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Wave Equation $\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$

Heat Equation $\frac{\partial u}{\partial t} = k \Delta u$

Burgers Equation $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}$

Korteveg-deVries Equation $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \mu \frac{\partial^3 u}{\partial x^3}$

Cauchy-Riemann Equations $u_x = v_y$, $v_x = -u_y$.

Like an ordinary differential equation, a partial differential equation expresses a relation between the values of a function at a point and its derivatives at the same point. This is what we mean by 'local' information. In the case of a PDE, the derivatives express different mathematical and physical properties of the function: if t represents time, then a time derivative is a rate of change, so if u in addition represents position in space then $\partial u/\partial t$ represents velocity, $\partial^2 u/\partial t^2$ is acceleration, and so on. If x represents a spatial variable, then $\partial u/\partial x$ represents a spatial gradient: it is a slope if u represents position in space, a thermal gradient if u represents temperature, and so on. If u is spatial position, then $\partial^2 u/\partial x^2$, the infinitesimal change in slope, is proportional to the curvature of u in the direction of x, while if u represents a state like temperature or density, then the interpretation of its second derivatives is less intuitive, though we shall see that second derivatives arise naturally in modeling physical problems. In any case, we can always graph u versus x, whatever u stands for, and the second derivative is (related to) the geometric curvature of the graph. The point is that differential equations (ordinary or partial) arise naturally as statements about processes that are governed by local influences. Newton's law of cooling, for example, predicts that the instantaneous rate of change of temperature at a point in space is governed by the temperature gradients in its immediate neighborhood, and is insensitive to what is occurring at distant points. Many rational deductions about the behavior of the physical world are based on balancing forces in a small neighborhood of a point, and lead to differential equations.

However, it is evident that a statement about what influences temperature at a point is less interesting (to someone trying to predict the weather, say) than a determination of what the temperature actually is at that point and, even more important, what it will be tomorrow. As we all know, future behavior of temperature depends on much more than just one point: even if we take the heat equation to be a complete statement about all the dynamics of temperature change, then we can be certain that temperature in the future will depend both on temperature now (initial conditions) and on the distribution of sources and sinks of heat (for example, controls imposed on the boundary of the region being considered – a silver spoon dipped into a pot of boiling water, say). The fact that the local PDE, given by physical or mathematical principles, and information about heat exchange at the boundary of the body, enables one to predict, fairly accurately, the future temperature distribution in a solid body (the spoon), based only on the temperature at a particular initial time, is what makes PDE a powerful tool for solving problems in science and engineering. The process of combining the local PDE with initial and boundary conditions to arrive at a function defined in the entire region of space and time where the local law holds is called, reasonably enough, 'solving the equation'.

The success of this approach to studying the physical world has made PDE a mainstay of applied mathematics and science for over a hundred years, so much so that in many fields people regard modeling phenomena by PDE as the natural way to do it, even in the case of phenomena like traffic flow or quantum mechanics that may not seem, at first inspection, to have the requisite features, such as a continuum in space and time or a local governing relation.

By contrast to the situation in elementary ordinary differential equations, it turns out that there are relatively few types of PDE for which solutions can be written down explicitly in terms of the functions (polynomials, trigonometric, or transcendental) one studies in calculus. This, rather than leading to a defeatist attitude, has stimulated mathematical advances in two directions. First, there is an extensive study of qualitative properties of solutions. If one cannot write down a solution explicitly, it becomes all the more important to know whether a solution exists, and if it does, if it is unique. Other properties, such as the location of its maxima and minima, possible discontinuities, and its dependence on the initial and boundary data, become very interesting (by contrast, if the solution can be written down explicitly, then all this behavior can be verified using calculus), and form a large part of the study of PDE. A second field of study is the numerical simulation of solutions of PDE. While this book is not primarily about how to obtain numerical approximations to PDE, we shall develop a few straightforward methods, in part to help visualize the solutions we describe and in part as an application of the theory we will develop in this book. It turns out that a few simple principles, which derive from the theory, are very helpful in developing suitable approximations. As an additional advantage, theory also establishes principles which are useful in forming new models for physical processes.

An important feature in the study of PDE, which will be emphasized here, is the interesting relation between what one might call the algebraic structure of a partial differential operator and the analytic features of solutions of the equation, as referred to above (location of extrema and discontinuities, for example). A function of several variables has so many different derivatives that even finding a notation to write them all down requires some thought. We shall explore some convenient notations later. Then, partial derivatives can be combined into an equation in many different ways, as the examples on page 1 show. When faced with a collection of things that look superficially different, a mathematician's first reaction is to classify them into sets: objects within one set resemble each other closely, while those in different sets differ in some essential way. An example from ordinary differential equations will illustrate the point.

EXAMPLE 1 A second-order, linear, constant-coefficient homogeneous ODE can be written

$$ay'' + 2by' + cy = 0;$$

here $y = y(x)$, say, is the dependent variable, the prime $'$ denotes differentiation with respect to x, and a, b, and c are constants. (The factor of 2 in front of b is a convenience.) We may assume $a \neq 0$, or else the equation is not second-order. (In ODE it is not usually difficult to write down the general case of a type of equation, particularly for linear equations.) Now, in solving this equation, one learns to write down the auxiliary equation

$$a\lambda^2 + 2b\lambda + c = 0.$$

This is an algebraic (even polynomial) equation, and it provides an example of what will prove to be very useful in PDE: associating an algebraic object with a partial differential operator. In fact, this can be done formally by using a symbol like 'D' to denote differentiation; then the equation is written

$$aD^2y + 2bDy + cy \equiv (aD^2 + 2bD + c)y = 0,$$

and formation of the auxiliary equation is effected by substituting an algebraic symbol, λ, for the symbol for differentiation, D. The expression $aD^2 + 2bD + c$ is an example of an operator: it is a set of instructions for forming another function, $ay'' + 2by' + cy$, from the function y (that is why the last term in the operator is 'c', which is the instruction 'multiply by c' and produces the term 'cy' in the function). Operator notation is suggestive: it reminds us that there is a close connection between the equation and the algebraic object which is the symbol of the operator. In this example, we find the solution by solving the auxiliary quadratic equation. For the purposes of illustrating the connection between algebraic properties and properties of solution, the precise solution of the equation is less important than the classification into types of behavior: when $b^2 - ac < 0$ the quadratic equation has complex conjugate roots and the solutions of the differential equation are oscillatory; when a, b and c all have the same sign, all solutions of the equation decay to zero as $x \to \infty$, and so on. Thus solutions of a second-order, linear, constant-coefficient, homogeneous, ODE can be classified according to the relative sizes of the coefficients.

PROBLEM 1 Verify the statements in this example by finding the general solution of the equation.

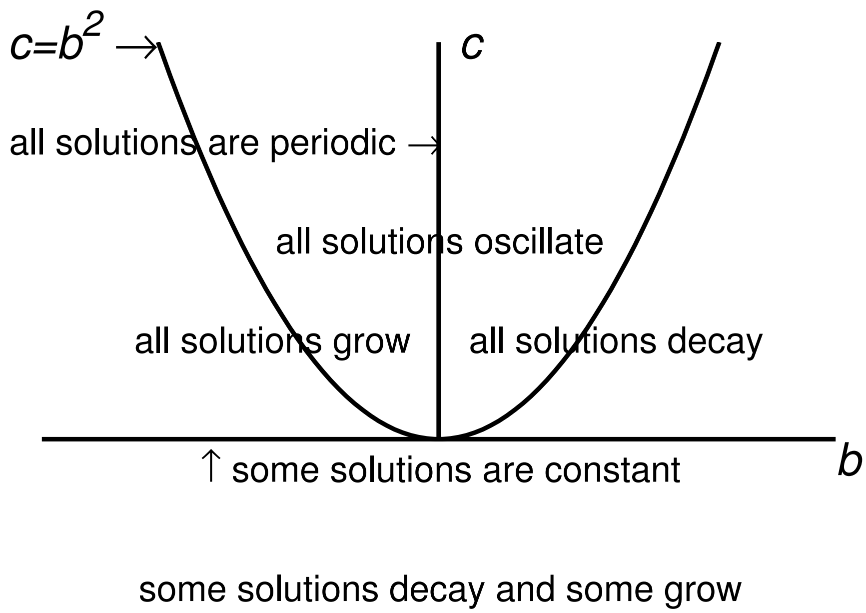

Figure 1.1: Regions of Different Qualitative Behavior of $y'' + 2by' + cy = 0$

PROBLEM 2 Since we have assumed $a \neq 0$, there is no harm in dividing the equation (and hence the operator) by a, or, what is equivalent, assuming $a = 1$. Then verify, using the solution of the previous problem, that the b, c-plane can be divided into five open regions of different qualitative behavior, as shown in Figure 1.1.

A second theme of the subject of PDE, and of this book, is that operators can be classified into certain broad types ('hyperbolic', 'parabolic', 'elliptic') on the basis of algebraic criteria, and membership in one of these classes determines many qualitative properties of the solutions: not only location of extrema and existence of discontinuities in the solutions, but even such fundamental questions as what sorts of initial and boundary value problems have solutions at all. The situation is much more complicated than it is for ODE. We will begin to explore this in Chapter 2.

1. Definitions and Notation

We are studying functions of several independent variables; when we want to emphasize generality we will consider the variable to be $\mathbf{x} = (x_1, x_2, \ldots, x_n)$; the collection of n-tuples is denoted $\mathbb{R}^n$, real n-dimensional Euclidean space. When we are dealing with specific examples, or to simplify or to be concrete, we will write the independent variable as $\mathbf{x} = (x, y)$ or as $\mathbf{x} = (x, y, z)$.

The standard measure of distance between two points $\mathbf{x}$ and $\mathbf{y}$ is Euclidean distance,

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}.$$

The open ball with center $\mathbf{x}^0$ (it is convenient, when we are using subscripts to denote coordinate functions, to use superscripts to denote specific points) and radius r is the subset of $\mathbb{R}^n$:

$$B(\mathbf{x}^0, r) = \{\mathbf{x} \mid d(\mathbf{x}, \mathbf{x}^0) < r\}.$$

The closed ball is the closure of the open ball:

$$\overline{B}(\mathbf{x}^0, r) = \{\mathbf{x} \mid d(\mathbf{x}, \mathbf{x}^0) \leq r\},$$

while the sphere of radius r is the surface of the ball:

$$S(\mathbf{x}^0, r) = \{\mathbf{x} \mid d(\mathbf{x}, \mathbf{x}^0) = r\}.$$

We recall the definitions of open and closed sets in $\mathbb{R}^n$, and the boundary of a set. A set A in $\mathbb{R}^n$ is open if around every point $\mathbf{x}$ in A we can find some open ball $B(\mathbf{x}, r)$ which is also in A; if A is open its complement, $\mathbb{R}^n \backslash A$ is closed. For any set A, the interior of A, $\mathring{A}$, is the largest open set contained in A, the closure of A, $\overline{A}$, is the smallest closed set containing A and the boundary of A, ∂A, is the closure of A minus the interior of A.

A neighborhood of a point $\mathbf{x}^0$ in $\mathbb{R}^n$ is an open set containing $\mathbf{x}^0$.

We recall the definition of a limit: if f is a function of n variables, then

$$\lim_{\mathbf{x} \to \mathbf{x}^0} f(\mathbf{x}) = L$$

means that for any $\epsilon > 0$ there exists a $\delta > 0$, depending on ϵ, such that

$$|f(\mathbf{x}) - L| < \epsilon \text{ whenever } d(\mathbf{x}, \mathbf{x}^0) < \delta.$$

A function f is continuous at $\mathbf{x}^0$ when $\lim_{\mathbf{x} \to \mathbf{x}^0} f(\mathbf{x}) = f(\mathbf{x}^0)$.

We are ready to define partial derivatives. Let us consider a function $f(x, y)$ of two variables. Then

$$\left.\frac{\partial f}{\partial x}\right|_{(x^0, y^0)} = \lim_{h \to 0} \frac{f(x^0 + h, y^0) - f(x^0, y^0)}{h}$$

if the limit exists. Similarly,

$$\left.\frac{\partial f}{\partial y}\right|_{(x^0, y^0)} = \lim_{h \to 0} \frac{f(x^0, y^0 + h) - f(x^0, y^0)}{h}.$$

A partial derivative of a function is another function, and so we can define second and higher order derivatives.

Example 2 If $f(x,y) = 2x^2 + xy^3 - \frac{1}{x}$, then

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x + y^3 + \frac{1}{x^2} \\ \frac{\partial f}{\partial y} &= 3xy^2 \\ \frac{\partial^2 f}{\partial x^2} &= 4 - \frac{2}{x^3} \\ \frac{\partial^2 f}{\partial y^2} &= 6xy \\ \frac{\partial^2 f}{\partial x \partial y} &= 3y^2 = \frac{\partial^2 f}{\partial y \partial x}. \end{aligned}$$

The set of points where a function is defined is called its domain of definition, or its domain. The function in this example is defined for all points (x,y) in $\mathbb{R}^2$ with $x \neq 0$; that is, everywhere except on the y-axis. The same is true of its derivatives. (In this example, the y-derivatives of f are functions defined everywhere in $\mathbb{R}^2$; however, they are not the derivatives of f at points where f is undefined.) We say f is of class $\mathcal{C}^k$ on a set A, or $f \in \mathcal{C}^k(A)$, if f has k continuous derivatives at every point in A, and $f \in \mathcal{C}^\infty(A)$ if all partial derivatives of every order are continuous. A function can have partial derivatives without those derivatives being continuous.

Example 3 Here is a classic example in a single variable. Let f be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right), & x \neq \\ 0, & x = 0 \end{cases}.$$

(The derivative for $x \neq 0$ is found by applying the usual rules of calculus; to find the derivative at zero, one calculates the difference quotient and takes the limit.) It can be seen that f' is defined for all x, but is discontinuous at zero.

Example 4 By contrast to functions of a single variable, where one proves early in a calculus course that if a function is differentiable then it is continuous, a function of several variables may have partial derivatives but be discontinuous. The function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not continuous at zero, but

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

is defined, as is $\partial f/\partial y$.

2. Review of Background Material

2.1 Curves and Surfaces in Space

In this book we will be talking about functions of several variables. Although usually these functions do not represent geometric objects (they may be temperature or density, for example) and although the variables may not always be spatial variables (one variable may be time, or they may all be abstract placeholders), it helps in developing intuition about the differential equations they satisfy to think of them geometrically. We begin with a function of a single variable. We can view $y = f(x)$ as a function or a curve in the x, y-plane. Considered as a curve, the equation $y = f(x)$ is the same as $F(x, y) \equiv f(x) - y = 0$. However, an equation $F(x, y) = 0$ may not correspond to a function. For example, the equation $x^2 + y^2 - 4 = 0$ is the equation of a circle. Parts of the solution set of $F(x, y) = 0$ may be written as functions: $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$ in this example. See Figure 1.2. The Implicit Function Theorem (discussed below) states that if $\partial F/\partial y \neq 0$ at a point (x^0, y^0) in the solution set, then there is a neighborhood of (x^0, y^0) where the solution set is given by a function $y = f(x)$. In the case of the function $F(x, y) = x^2 + y^2 - 4$, the places where $F = 0$ and $\partial F/\partial y = 0$ are $(2, 0)$ and $(-2, 0)$, and those are exactly the places which do not have neighborhoods in which $F = 0$ can be written as a single-valued function of x. The roles of x and y are symmetric. Any point (x^0, y^0) at which $\partial F/\partial x \neq 0$ has a neighborhood in which we can solve for $x = f(y)$. In this example, the exceptions are the points $(0, 2)$ and $(0, -2)$. Notice that unless $\partial F/\partial y$ and $\partial F/\partial x$ are both zero at the same point, we can always solve for one of x or y. The points where $\nabla F \equiv (\partial F/\partial x, \partial F/\partial y) = 0$ are called singularities of the curve.

Problem 3 Where are the singularities of the equation $F(x, y) = 0$ when $F = x^2 - y^2$? Describe the curve(s) that satisfy this equation; what happens to them at a singularity?

If $F(x, y) = 0$ is the graph of a function $y = y(x)$, then we can find derivatives of y (these are ordinary derivatives) by implicit differentiation,

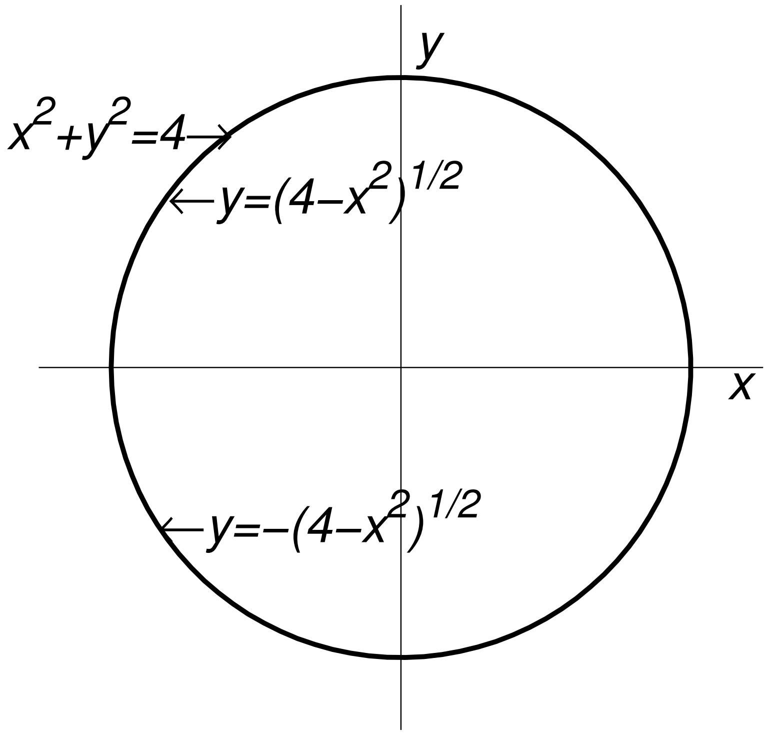

FIGURE 1.2: The Graph of a Circle

a technique learned in calculus: we differentiate the equation $F(x, y(x)) = 0$ with respect to x and obtain

$$\frac{d}{dx}F(x, y(x)) \equiv \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0. \tag{1.1}$$

We adopt the notation

$$F_x = \frac{\partial F}{\partial x}, \qquad F_y = \frac{\partial F}{\partial y},$$

and now see that (1.1) can be solved for dy/dx:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

For example, if $F = x^2+y^2-4$ then this yields $dy/dx = -x/y$. As a practical matter, knowing that $dy/dx = -x/y$ may not be very useful unless we also know the value of y at a given value of x, but the relation between dy/dx and the partial derivatives of F turns out to be very useful. We turn to this now.

In the x, y-plane, dy/dx is the slope of the tangent line to $y(x)$. A tangent vector to the curve $y(x)$ is given by $\mathbf{t} = (1, \frac{dy}{dx})$. (We say "a" tangent vector because there are many: any vector whose slope is $\frac{dy}{dx}$ — that is, any multiple of $\mathbf{t}$ — is a tangent vector.) Now the gradient of F is defined as $\nabla F = (F_x, F_y)$; this is also a vector, and the quantity $F_x + F_y \frac{dy}{dx}$ is the dot product of the two vectors $\mathbf{t}$ and ∇F. Recall that the dot product of two vectors is zero when the vectors are orthogonal. Hence the equation $F_x + F_y \frac{dy}{dx} = 0$ states that the gradient of a function F of two variables is orthogonal to the zero-set of F (which is the function $y(x)$). This is well known; in fact, more is true: the gradient of F is orthogonal to any level set, $F(x, y) = c$.

PROBLEM 4 Show that ∇F is orthogonal to the curve $F(x, y) = c$. Hint: Adapt the argument above, for example by considering the function $G(x, y) = F(x, y) - c$, whose zero-set is a level set of F and whose partial derivatives are the same as those of F.

In spaces of higher dimension, there are a number of ways to represent curves and surfaces. A function of n variables, $u(x_1, \ldots, x_n)$, is the equation of a surface in the $n + 1$-dimensional space of points $(x_1, \ldots, x_n, u)$. Typically, an equation of the form $F(x_1, \ldots, x_n) = 0$ gives a surface (or hypersurface) in n-dimensional space. A higher-dimensional version of the Implicit Function Theorem gives conditions for being able to write the surface as $x_n = f(x_1, \ldots, x_{n-1})$.

How do we denote a curve in $\mathbb{R}^3$? One simple way is to write the y and z coordinates as functions of x; that is, $y = Y(x)$, $z = Z(x)$. Then, the points in space which are on the curve are written as $(x, y, z) = (x, Y(x), Z(x))$; notice that this is a special case of a parametric representation, with $x = t$. A general parametric representation of a curve in three-dimensional space is given by $(x(t), y(t), z(t))$, where the parameter, t, lies in an interval of the real line. A curve in n-dimensional space is given by $\mathbf{x} = \mathbf{x}(t)$.

The fact that a curve in three-dimensional space is given by two functions, $Y(x)$ and $Z(x)$, might lead you to guess that in general two equations, $F(x, y, z) = 0$ and $G(x, y, z) = 0$ determine a curve, just as one equation determines a surface. Under appropriate conditions, this is so; one needs a variant of the implicit function theorem to prove it.

A result similar to (1.1) holds in any number of variables. Writing the equation $x_n = f(x_1, \ldots, x_{n-1})$ in $\mathbb{R}^n$, which determines a surface, in implicit form as $F(x_1, \ldots, x_n) = 0$ for some function F of n variables, again we claim that ∇F is normal to the surface. To see this, we calculate the implicit derivatives with respect to each of $x_1, \ldots, x_{n-1}$ of the identity

$$F(x_1, x_2, \ldots, x_{n-1}, f(x_1, x_2, \ldots, x_{n-1})) = 0.$$

We obtain

$$\begin{aligned}
\frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_n}\frac{\partial f}{\partial x_1} &= 0; \\
\frac{\partial F}{\partial x_2} + \frac{\partial F}{\partial x_n}\frac{\partial f}{\partial x_2} &= 0; \\
&\vdots \\
\frac{\partial F}{\partial x_{n-1}} + \frac{\partial F}{\partial x_n}\frac{\partial f}{\partial x_{n-1}} &= 0.
\end{aligned}$$

Now, this says that the dot product of

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots, \frac{\partial F}{\partial x_n}\right)$$

with each of the $n-1$ vectors

$$\left(1, 0, \ldots, 0, \frac{\partial f}{\partial x_1}\right), \left(0, 1, 0, \ldots, 0, \frac{\partial f}{\partial x_2}\right), \ldots, \left(0, \ldots, 0, 1, \frac{\partial f}{\partial x_2}\right)$$

is zero. Each vector in this list is a tangent vector to the surface, since it is tangent to the curve in the surface in one of the coordinate directions. This shows that ∇F is orthogonal to $n-1$ linearly independent vectors in the tangent plane of the surface and so it is normal to the surface. See Figure 1.3.

2.2 Vector Fields

There are many familiar examples of vector fields in physical space. In $\mathbb{R}^3$, we may write a vector field $\mathbf{V}(x, y, z)$ as (P, Q, R), where P, Q and R are functions of x, y and z: $\mathbf{V}$ is a vector-valued function defined on a domain $D \subset \mathbb{R}^3$. One can study the motion determined by the following rule: A point mass at $\mathbf{x}$ moves under the influence of $\mathbf{V}(\mathbf{x})$ along a path (also called an orbit or trajectory) which satisfies the system of first-order ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{x}). \tag{1.2}$$

Example 5 Here is an example involving fluid flow in the plane. Suppose that at any point in the plane, the velocity field $\mathbf{V}(x, y) = (-y, x)$. Then the position $(x(t), y(t))$ of a particle satisfies

$$\begin{aligned}
\frac{dx}{dt} &= -y \\
\frac{dy}{dt} &= x.
\end{aligned}$$

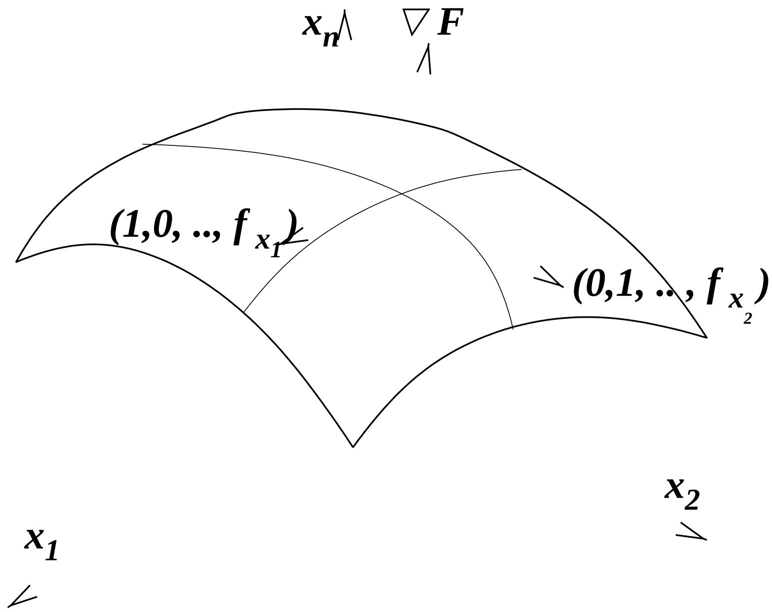

FIGURE 1.3: Tangents and Normal to a Surface

This is a first-order system of ordinary differential equations. By the basic theory of linear ordinary differential equations, the position of the particle is completely determined for all time if we know it at any one time, say $(x(0), y(0)) = (x^0, y^0)$. We outline two ways to solve this system:

1. Write the equations for x and y as a second-order equation in a single variable,
$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(-y) = -\frac{dy}{dt} = -x$$
or $d^2x/dt^2 + x = 0$. This is the familiar 'harmonic oscillator' equation whose general solution is
$$x(t) = A\cos t + B\sin t$$
for parameters A and B. We then recover $y = -dx/dt = -A\sin t + B\cos t$, and then apply the initial conditions
$$x(0) = x^0 = A, \qquad y(0) = y^0 = B$$
to determine A and B from the data.

2. A curve $(x(t), y(t))$ can be written (locally) as $y = f(x)$ or $x = g(y)$ (that is, $F(x, y) = 0$) and so
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x}{-y}.$$

Using this approach, we get a first order ordinary differential equation $dy/dx = -x/y$ or $dx/dy = -y/x$. This equation is not linear, but is a separable equation which can be solved as follows:

$$x\,dx = -y\,dy \qquad \Rightarrow \qquad \frac{1}{2}x^2 = -\frac{1}{2}y^2 + C$$

where C is a constant of integration; from the initial condition, $C = \frac{1}{2}[(x^0)^2 + (y^0)^2]$. We immediately see that the solution lies on a circle; however, we have to do a little more work to find x and y as functions of t. We use the solution to find another first-order equation:

$$\frac{dx}{dt} = -y = -\sqrt{2C - x^2}$$

or

$$\frac{dx}{\sqrt{2C - x^2}} = -dt.$$

Integrating both sides (the left side is seen to yield an inverse trigonometric function upon integration), and solving for x we get the same solution as above.

The next example is set up a little differently.

EXAMPLE 6 The gravitational field due to a point source of mass M at the origin is

$$\mathbf{E}(\mathbf{r}) = -\frac{MG\mathbf{r}}{|\mathbf{r}|^3},$$

where G is the gravitational constant and we let $\mathbf{r}$ denote the point (x, y, z). The domain of $\mathbf{E}$ is $D = \mathbb{R}^3 \backslash \mathbf{0}$. The motion of a particle of unit mass under the influence of gravity is given by

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \mathbf{v} \\ \frac{d\mathbf{v}}{dt} &= \mathbf{E}(\mathbf{r}). \end{aligned}$$

For $d\mathbf{r}/dt = \mathbf{v}(t)$ is the velocity of the particle, while, by Newton's law of gravitation, the acceleration, $d^2\mathbf{r}/dt^2$, is equal to $\mathbf{E}$. This system of six first-order ordinary differential equations is an example of (1.2) in six dimensions, with $\mathbf{x} = (\mathbf{r}, \mathbf{v})$ and $\mathbf{V} = (\mathbf{v}, \mathbf{E})$.

We focus on the position of a solution $\mathbf{x}(t)$ of (1.2). For example, if $\mathbf{x}(t) = (x(t), y(t), z(t))$ is in $\mathbb{R}^3$, then as t varies, this forms a curve in space; as we saw, $(x(t), y(t), z(t))$ is a standard parametric representation of a curve in $\mathbb{R}^3$. In Example 5, we noted that $(x(t), y(t))$ is a parametric representation of a curve in the plane given by $F(x, y) = 0$. Recall that on such a curve we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Now the system

$$\begin{aligned} \frac{dx}{dt} &= P \\ \frac{dy}{dt} &= Q \end{aligned}$$

is the same as

$$\frac{dy}{dx} = \frac{Q}{P}$$

and so we have a relation between the partial derivatives of F and the vector field $\mathbf{V} = (P, Q)$:

$$-\frac{F_x}{F_y} = \frac{Q}{P}$$

or

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} = 0.$$

So we have demonstrated the following proposition.

Proposition 1 *If $F(x, y) = 0$ is an integral curve of the vectorfield $\mathbf{V} = (P, Q)$, then $F(x, y)$ is a solution of the first order partial differential equation*

$$P(x, y)\frac{\partial F}{\partial x} + Q(x, y)\frac{\partial F}{\partial y} = 0$$

Here is an illustration.

Example 7 Applying this in the example above, where $\mathbf{V} = (-y, x)$, the corresponding partial differential equation is

$$-y\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y} = 0,$$

and we saw that $F = x^2 + y^2 - c$ is a solution of this equation for any constant c.

PROBLEM 5 Drill in curves and surfaces.

1. Sketch the graph of the function $y = x^3 - 3x$.
2. Sketch the graph of the function $y = x^3 - 3x + 2$.
3. What is the domain of $y^2 = x^3 - 3x$? Sketch the curve. What functions are defined by this equation?
4. Sketch the graph of $y = \tan x$.
5. Sketch the graph of $y = \tanh x$.
6. Sketch the level curves of $y = x^2 + y^2$.
7. Sketch the level curves of $y = x^2 - y^2$.

PROBLEM 6 Drill in ODE.

1. Find the general solution of $y'' + 4y' + 5y = 0$.
2. Find the general solution of $y'' + 5y' + 4y = 0$.
3. Solve the initial value problem $y'' + 9y = 0$, $y(0) = 1$, $y'(0) = 3$.
4. Find the general solution of $y'' + \omega^2 y = t$.
5. Solve $\frac{dy}{dx} = \frac{y}{x}$.
6. Solve $\frac{dy}{dx} = \frac{x}{y}$.
7. Solve the system
$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -y \\ \dot{z} &= 1. \end{aligned}$$
8. Solve the system
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \\ \dot{z} &= x. \end{aligned}$$

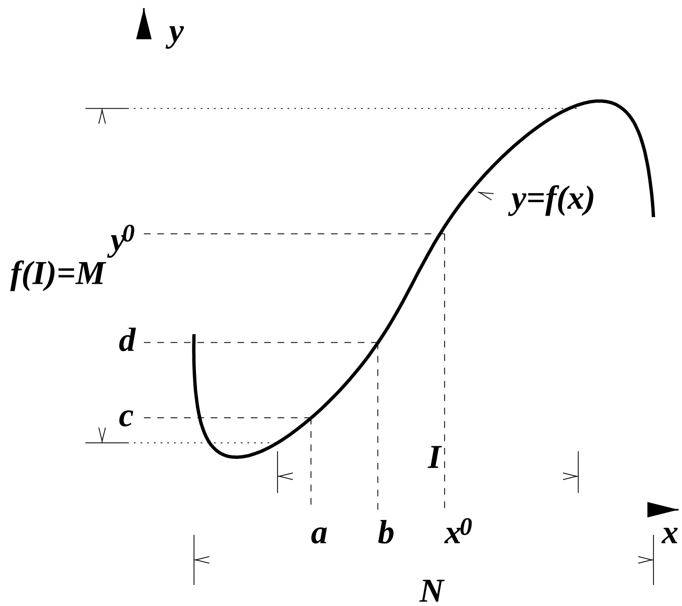

Figure 1.4: The Inverse Function Theorem in One Variable

2.3 The Implicit Function Theorem

We discuss several related theorems which are very useful in solving PDE. They go by the names of 'implicit' and 'inverse' function theorems. They tell us when a function $y = f(x)$ can also be written as $x = f(y)$ (for example $y = 2x$ is the same as $x = y/2$), in one or many dimensions, and also when an equation $f(x, y) = 0$ can be solved for $y(x)$; again, we are interested in cases where x and y may be vector-valued. For our purposes here, it is more important to understand that these theorems do not always hold, and that the hypotheses need to be checked, than to follow the details of how they are proved, and so we will prove only the simplest case, the inverse function theorem in one variable, which uses only results from calculus. Proofs of all the theorems can be found in texts on advanced calculus.

Theorem 1 (Inverse Function Theorem) *Given a $\mathcal{C}^1$ function $y = f(x)$ for x in a neighborhood N of x^0, with the property $f'(x^0) \neq 0$, then, denoting $f(x^0)$ by y^0, there is a neighborhood M of y^0 and a function $g \in \mathcal{C}^1(M)$ such that $x = g(y)$ for $y \in M$ if and only if $y = f(x)$.*

Proof We can assume $f'(x^0) > 0$. Since f' is continuous, there is an open interval $I \subset N$ in which $f'(x) > 0$. (See Figure 1.4.) Considering f as a mapping, then f maps I to an image, which we shall call $f(I)$. Since f' is positive, then f is monotonic, and so the mapping is $1-1$. Also, the image, $f(I)$ is also an interval. We see this by using use the intermediate value theorem, from calculus, as follows: take any closed interval $[a, b]$ inside I;

let $f(a) = c$ and $f(b) = d$; then $c < d$ and by the intermediate value theorem for any $y \in [c, d]$, there is an $x \in [a, b]$ for which $f(x) = y$, so y is in $f(I)$, which is thus connected and forms an interval. Since f is one-to-one, the value x associated with each y is unique, and so defines a function, $g(y) = x$, the inverse mapping; g maps $f(I) \equiv M$ to I. To complete the proof, we need to show that g is monotonic, continuous, and differentiable.

Monotonicity is proved along the lines above: for any c and d in M with $c < d$, find a and b such that $g(c) = a$ and $g(d) = b$; then if $b < a$ we will have $d = f(b) < f(a) = c$, a contradiction.

We prove that g is continuous by using the mean-value theorem for f. Let c be the point at which we want to prove g is continuous, and let d be another point; we want to show $\lim_{d\to c} g(d) = g(c)$. Now, let $g(c) = a$ and $g(d) = b$; we apply the mean-value theorem to f:

$$f(b) - f(a) = (b - a)f'(k), \tag{1.3}$$

where k is a value between a and b. Since we are choosing all points to lie in the interval I, we know that $f'(k) \neq 0$, so we can divide by it. Now rewrite (1.3) expressing $f(b)$ as d, and so on: we obtain

$$\frac{1}{f'(k)}(d - c) = g(d) - g(c).$$

Finally, we take the limit $d \to c$. The quantity $\frac{1}{f'(k)}$ will vary with d, but it is bounded (in fact, as $d \to c$, it approaches the limit $\frac{1}{f'(c)}$), and so $g(d) \to g(c)$.

We can now find g' directly by calculating the difference quotient. For any $y \in [c, d]$, we want to find

$$\lim_{k\to 0} \frac{g(y + k) - g(y)}{k}.$$

Let $x = g(y)$, and define $h = g(y + k) - g(y)$, so $g(y + k) = x + h$ and $y + k = f(x + h)$ or $k = f(x + h) - f(x)$. Then the quantity above can be written as

$$\frac{g(y + k) - g(y)}{k} = \frac{h}{f(x + h) - f(x)}.$$

Now, by the continuity of g, when $k \to 0$ then also $h \to 0$. But

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

exists, since this is by definition $f'(x)$, and it is nonzero, so the limit of its reciprocal also exists and is $\frac{1}{f'(x)}$. Thus $g'(y)$ exists and is equal to $\frac{1}{f'(x)}$. ■

We have given this proof in some detail so that the reader can see where each hypothesis is used. The implicit function theorem in two variables reads as follows:

Theorem 2 (Implicit Function Theorem) *If $F = F(x, y) \in \mathcal{C}^1(\Omega)$, and $F(x^0, y^0) = 0$ while $\partial F(x^0, y^0)/\partial y \neq 0$, then there is a function $f = f(x) \in \mathcal{C}^1$ with $f(x^0) = y^0$, and a neighborhood N of x^0 in $\mathbb{R}^1$ such that $F(x, y(x)) = 0$ for all $x \in N$.*

Notice that Theorem 1, which we have just proved, is a special case of Theorem 2 with the function F defined by $F(x, y) = y - f(x)$. However, the proof of Theorem 2 in two or more variables is quite intricate. One way to prove Theorem 2 is to use a two-dimensional version of Theorem 1. The context now is inverting a mapping $T : (x, y) \mapsto (u, v)$ of two variables. Such mappings are more complicated than one-dimensional mappings (functions).

Example 8 Cartesian to polar coordinates. For $(x, y) \in \mathbb{R}^2$, polar coordinates are defined implicitly by

$$x = r\cos\theta, \quad y = r\sin\theta,$$

with $r \geq 0$ and a convention such as $0 \leq \theta < 2\pi$. This defines a mapping $T(r, \theta) \mapsto (x, y)$ from a half-strip in $\mathbb{R}^2$ to $\mathbb{R}^2$. See Figure 1.5. An inverse is defined by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \begin{cases} \cos^{-1}(x/r), & y > 0 \\ 2\pi - \cos^{-1}(x/r), & y < 0 \end{cases},$$

where a careful definition of θ is necessary to ensure that θ represents the angle between the vector (x, y) and the positive x-axis for all points (x, y), including those in the lower half-plane. Even with this care, the inverse mapping has two unavoidable difficulties: first, the origin $(x, y) = (0, 0)$ is singular, in that there is no way of defining θ there. In fact, the entire interval $I = \{r = 0, 0 \leq \theta < 2\pi\}$ is mapped by T to $(0, 0)$, so the mapping T is not one-to-one there. In addition, the inverse mapping is not continuous across the positive x-axis: points just above the axis are mapped to points (r, θ) with θ near 0, while their neighbors just below the axis are mapped to points with θ near 2π. One could redefine the original mapping T to make the domain of θ the interval $(-\pi, \pi]$, but then the discontinuity just moves to the negative x axis. Alternatively, one could allow θ to take all real values; then T is no longer one-to-one, but a continuous inverse can be defined in a neighborhood of any non-zero point (x, y).

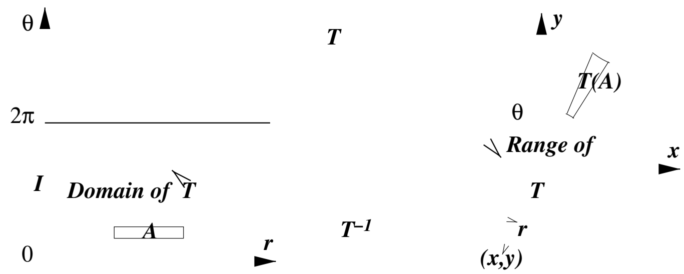

FIGURE 1.5: Polar and Cartesian Coordinates

To uncover the correct hypotheses for inverting a mapping, consider the case of a linear mapping, in which case T is given by a 2×2 matrix A; we may suppose that (x, y) and (u, v) are column vectors, and T acts by matrix multiplication by A:

$$T(x, y) = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Now, in this case, the mapping is invertible if and only if A is nonsingular, that is, $\det(A) \neq 0$. We can use Taylor's Theorem for vector functions to find the linear map that best approximates a nonlinear mapping at a point (x^0, y^0); it is given by the Jacobian matrix

$$A = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

where all partial derivatives are evaluated at (x^0, y^0). The determinant of A, the Jacobian determinant, is often written $J(x^0, y^0) = |\partial(u, v)/\partial(x, y)|$, where the entries are evaluated at (x^0, y^0).

EXAMPLE 9 In Example 8, noting that $T(r, \theta) = (x, y)$, we have

$$A = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and $\det A = r$. Thus the singularity at $r = 0$, or $(x, y) = (0, 0)$, is predicted by the fact that the Jacobian determinant is zero there.

The idea behind the use of the Jacobian determinant is exactly the same in any number of dimensions, and forms a hypothesis of the inverse function theorem:

THEOREM 3 (HIGHER DIMENSIONAL VERSION OF IFT) *Given a $\mathcal{C}^1$ mapping $T : (x_1, \ldots, x_n) \mapsto (u_1, \ldots, u_n)$ for $\mathbf{x}$ in a neighborhood N of $\mathbf{x}^0$, with the property that the Jacobian determinant $J(\mathbf{x}^0) \neq 0$, then, denoting $T(\mathbf{x}^0)$ by $\mathbf{u}^0$, there is a neighborhood M of $\mathbf{u}^0$ and a mapping $S \in \mathcal{C}^1(M)$ such that $\mathbf{x} = S(\mathbf{u})$ for $\mathbf{u} \in M$ if and only if $T(\mathbf{x}) = \mathbf{u}$.*

We omit the proof of this Theorem, but we show how the proof of Theorem 2 follows from it:

Proof of Theorem 2 Define a mapping T by

$$T(x, y) = (u, v) = (x, F(x, y));$$

T is defined on the domain Ω of the theorem. At (x^0, y^0), the Jacobian is

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix}$$

and its determinant is $\partial F/\partial y(x^0, y^0) \neq 0$ by hypothesis. Thus there is an inverse mapping $S : (u, v) \mapsto (x, y)$, which we will write as $x = X(u, v)$, $y = Y(u, v)$ (that is, X and Y are the functions given by Theorem 3). Also S is defined on a neighborhood of $(u^0, v^0) = T(x^0, y^0) = (x^0, 0)$. Now, T composed with S gives the identity:

$$(u, v) = T(S(u, v)),$$

which means

$$F(x, y) = v = F(u, Y(u, v))$$

and this identity holds for all v near 0. In particular, it holds at $v = 0$, which means

$$0 = F(u, Y(u, 0)) = F(x, Y(x, 0)).$$

This equation shows that the function $f(x)$ we seek is $f(x) = Y(x, 0)$, where Y is the function given by Theorem 3. ■

2.4 Linearity

Throughout these notes, we will talk about linearity in different contexts: linear functions, linear spaces, linear mappings, linear operators, linear dependence and independence, and so on. Examples of linear spaces, also called vector spaces, are $\mathbb{R}^n$, real Euclidean space, for any n. The properties which define a vector space are given in any linear algebra text; we recall here the two fundamental properties that vectors in the same space can be added, and that a vector can always be multiplied by a scalar. In linear algebra, one speaks of linear transformations on finite dimensional vector spaces. These are mappings which can be described by matrices: if $\mathbf{x}$ and $\mathbf{u}$ are vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and A is an $m \times n$ matrix, then a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by $\mathbf{u} = A\mathbf{x}$. This mapping has the important properties of superposition:

1. If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$ then $A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y})$;
2. If $\mathbf{x} \in \mathbb{R}^n$ and α is any number, then $A(\alpha\mathbf{x}) = \alpha A(\mathbf{x})$.

One consequence of linearity is that $A(\mathbf{0}) = \mathbf{0}$ — that is, a linear transformation always maps the origin in the domain space to the origin in the image space. Another consequence is that if a linear transformation is defined for a set of vectors, it is then defined on the entire vector space spanned by those vectors. Yet another is that the set of solutions $\mathbf{x}$ of an equation $A\mathbf{x} = \mathbf{0}$ forms a linear space, the null space of A.

There is also a relation between the solution set of a system of linear equations $A\mathbf{x} = \mathbf{b}$ and the associated homogeneous system $A(\mathbf{x}) = \mathbf{0}$: if N (for null space) denotes the vector space of solutions to $A(\mathbf{x}) = \mathbf{0}$ and $\mathbf{x}^0$ is one solution of $A\mathbf{x} = \mathbf{b}$, then $X = N + \mathbf{x}^0$ is the space of solutions to $A\mathbf{x} = \mathbf{b}$. The plus sign in the definition of X means that X consists of all vectors of the form $\mathbf{x} + \mathbf{x}^0$, with $\mathbf{x} \in N$. The space X is not a vector space (superposition does not apply and the space does not contain the origin); however, it has many other properties in common with a vector space; it is called an affine space.

In ODE, one learns that the general solution of a homogeneous linear equation is the set of linear combinations of some basic solutions. Similarly, the general solution of an inhomogeneous linear equation is found by adding a particular solution to the general solution of the homogeneous equation. In fact, one can think of the basic solutions as vectors which form a basis for the vector space of solutions to the homogeneous equation. Then the solutions to the inhomogeneous equation form an affine space in the same way as in linear algebra. As we saw in Example 1, a linear differential equation can be written as a differential operator acting on a function; for a second-order equation, the operator looks like

$$L(y) = (aD^2 + bD + c)y.$$

This operator is linear in the same way that linear transformations on $\mathbb{R}^n$ are linear: the two superposition principles hold. Notice that this is true whether the coefficients a, b and c are constant or are functions of the independent variable, t, but it is not true if they are functions of y.

Problem 7 Find the general solution of $y'' + \omega^2 y = 0$. What is a basis for the vector space of solutions? What is the dimension of the space? Find the general solution of $y'' + \omega^2 y = t$ and describe it as an affine space.

Problem 8 Show that the operator $L = t^2D^2 + 2tD + \sin t$ is linear. Show that the operator $N(y) = D^2y + yDy$ is not linear by showing that superposition does not work.

In these notes, we will study many PDE which are linear, and we will discover how the linear structure of the solution space helps to solve the equation. We will also study many nonlinear problems, and we will find ways to compensate for the lack of linear structure. Here is one more example from ODE showing how linearity can emerge from a different aspect of the problem; we will see much more like this in PDE.

Example 10 A two-point boundary-value problem for an ODE consists of an equation, say $y''+\omega^2 y = 0$, which holds on an interval, say $0 \leq t \leq \pi$, and boundary conditions at the end points. If we take as boundary conditions $y(0) = 0$ and $y(\pi) = 0$, then one can verify that

1. $y \equiv 0$ is always a solution, and

2. $y \equiv 0$ is the only solution unless ω is an integer, say $\omega = n$, and then $y(t) = c \sin nt$ is a solution for any constant c.

Now, we can write the equation in terms of a homogeneous linear operator: $L(y) \equiv (D^2 + \omega^2)y = 0$, and we can also write the boundary conditions as homogeneous equations by defining linear operators $B_0(y) = y(0)$ and $B_\pi(y) = y(\pi)$. Then the boundary conditions become $B_0(y) = 0$, and $B_\pi(y) = 0$. Now, because the operator and the boundary conditions are all homogeneous, their solution sets are vector spaces. Depending on the value of ω, the solution space has dimension 0 or dimension 1.

Problem 9 Let $L(y) = (aD^2 + bD + c)y$ be a linear ordinary differential operator, and let B_0 and B_π be the boundary operators above. Without solving the two-point boiundary-value problem, show that the set of solutions to $L)y) = 0$, $B_0(y) = 0$ and $B_\pi(y) = 0$ forms a linear space — that is, that superposition applies.

Chapter 2

First-Order Equations

In this chapter we study first-order equations; specifically, we examine equations which involve a single unknown function u of two or more variables (x, y) or $(x_1, x_2, \ldots, x_n)$ and its first derivatives. If we let $\mathbf{x}$ denote the vector variable and ∇u denote the vector gradient of u, $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$, then the notation for a first-order equation is

$$F(u, \nabla u, \mathbf{x}) = 0. \tag{2.1}$$

We often express the PDE as an equation in $2n + 1$ variables by writing $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ to represent the gradient ∇u.

Example 11 The function $F(u, \mathbf{p}, \mathbf{x}) = p_1^2 + p_2^2 - 1$ gives the equation

$$\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 = 1;$$

the function $F(u, \mathbf{p}, \mathbf{x}) = x_1 p_1 + \ldots x_u p_n + u$ corresponds to

$$x_1 \frac{\partial u}{\partial x_1} + \ldots + x_n \frac{\partial u}{\partial x_n} + u = 0.$$

We shall see that there is a solution method which applies to all such equations, and which determines precisely which equations have solutions, and when the solutions break down. This can sometimes lead to an explicit formula for solutions, using techniques of ordinary differential equations. Even when the equations cannot be solved explicitly, the theory of ordinary differential equations, coupled with the implicit function theorem, tell us when solutions exist (and so could be found by numerical or other approximation) and what obstacles might be present to existence of solutions.

In having this feature, the ability of a single, overarching technique to give a general solution of all equations of the form (2.1), whether linear or nonlinear and no matter how many independent variables they contain or what sort of boundary or initial data is prescribed, first-order equations differ from the typical situation of higher-order equations or systems of equations. (A student familiar with ordinary differential equations will recognize that, at least for linear constant-coefficient equations, the passage from a single first-order equation to a system of first-order equations is effected by introducing linear algebra. No such unifying theory is available for partial differential equations.) In this sense first-order equations are atypical, and it might be considered misleading to begin the book with them. We start here, however, for four reasons:

1. To give students some experience in manipulating functions of several variables and their derivatives,

2. Because for at least some types of higher-order equations and systems, knowledge of a related first-order equation is useful in constructing the solution,

3. Nonlinear first-order equations can be solved by modifications of the technique which applies to linear equations, and so first-order equations provide a case in which we can, by explicit examples, contrast linear and nonlinear equations, and

4. There are interesting examples and applications, which we consider at the end of this chapter.

The equation (2.1) is linear if the unknown u and its derivatives appear only in linear combination with coefficients depending on $\mathbf{x}$. Thus the form of a linear first-order equation is

$$P(x,y)\frac{\partial u}{\partial x} + Q(x,y)\frac{\partial u}{\partial y} + C(x,y)u = G(x,y)$$

in two independent variables, or

$$a_1(\mathbf{x})\frac{\partial u}{\partial x_1} + a_2(\mathbf{x})\frac{\partial u}{\partial x_2} + \ldots a_n(\mathbf{x})\frac{\partial u}{\partial x_n} + b(\mathbf{x})u = f(\mathbf{x})$$

with n independent variables. A linear equation is called constant-coefficient if the coefficients (P, Q, C or a_i and b) multiplying the unknown function and its derivatives are constants (that is, independent of x), and homogeneous if the unknown or one of its derivatives appears in every term (that is, the terms G and f are absent).

1. Linear First Order Equations

We begin with a special kind of linear equation: a homogeneous equation in which the term $b(\mathbf{x})u$ is missing. That is, the equation looks like

$$a_1(\mathbf{x})\frac{\partial u}{\partial x_1} + a_2(\mathbf{x})\frac{\partial u}{\partial x_2} + \ldots a_n(\mathbf{x})\frac{\partial u}{\partial x_n} = 0. \tag{2.2}$$

We will solve this equation by generalizing Proposition 1 from the Introduction. Recall that this Proposition states that if $u(x, y)$ satisfies

$$P(x, y)u_x + Q(x, y)u_y = 0 \tag{2.3}$$

then the curve $u(x, y) = 0$ is an integral curve of

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y).$$

In fact, because of the form of equation (2.3), if u is a solution then so is $u - c$ for any constant c, so the curve $u(x, y) = c$ is also an integral curve of the same vectorfield. For simplicity, suppose that the vectorfield $\mathbf{V} = (P, Q)$ is defined for all (x, y) in the plane and is never zero. Then through every point in the plane there is exactly one integral curve. Suppose the value of $u(x, y)$ is known at one point on such a curve, say at (x^0, y^0). Then, since u is constant along the curve, the value of u is that same value everywhere on the curve. Thus, for this equation, in order to determine a solution u at every point (x, y) in the plane, it is necessary and sufficient to specify u at exactly one point on each integral curve of $\mathbf{V}$. In order to do this, we need to know something about the vectorfield.

Example 12 Suppose $P = P^0$ and $Q = Q^0$ are nonzero constants. Then the integral curves are straight lines, which can be written parametrically as

$$x = x^0 + P^0 t, \quad y = y^0 + Q^0 t.$$

Furthermore, we may choose for (x^0, y^0) a one-parameter family of points such that one point is on each integral curve. This can be done in many ways; since we assumed $P^0 \neq 0$ and $Q^0 \neq 0$, we could take $y^0 = 0$, $-\infty < x^0 < \infty$; or $x^0 = 0$, $-\infty < y^0 < \infty$; or we could take all the points on a line through the origin perpendicular to $\mathbf{V}$: that is, the set of all (x^0, y^0) such that $P^0x^0 + Q^0y^0 = 0$. In fact, we could take any curve of points $(x^0(s), y^0(s))$ which is transverse to the vector field; the condition is that

$$\left(\frac{dx^0}{ds}, \frac{dy^0}{ds}\right)$$

must not be parallel to (P^0, Q^0). This can also be written

$$P^0(y^0)'(s) - Q^0(x^0)'(s) \neq 0. \tag{2.4}$$

Along this curve, we can assign values of u in any way we wish, say $u = u^0(s)$ at the point $(x^0(s), y^0(s))$. Then the solution of the problem is

$$u(x^0(s) + P^0t, y^0(s) + Q^0t) = u^0(s).$$

This is not completely satisfactory; what we would like to know is $u(x, y)$; how do we find the value of s which corresponds to a given point (x, y)? We need to solve the system

$$x = x^0(s) + P^0t, \qquad y = y^0(s) + Q^0t \tag{2.5}$$

for s and t. Note that the Jacobian of this system,

$$\frac{\partial(x, y)}{\partial(s, t)} = \det \begin{pmatrix} (x^0)'(s) & P^0 \\ (y^0)'(s) & Q^0 \end{pmatrix} \neq 0$$

for all s and t by the transversality condition. By the inverse function theorem, this is enough to imply that we can always find s and t for a given x and y in a sufficiently small set. However, in the case that P^0 and Q^0 are constant, as we are assuming in this example, we can find the set. We do this by solving explicitly for t (since equations (2.5) are linear in t):

$$t = \frac{x - x^0(s)}{P^0} = \frac{y - y^0(s)}{Q^0}$$

and then writing a single equation involving s:

$$P^0y^0(s) - Q^0x^0(s) = P^0y - Q^0x. \tag{2.6}$$

The left side of this equation is a function of s, $g(s)$, say; by the relation (2.4), it is a monotonic function, since $g'(s) = P^0(y^0)'(s) - Q^0(x^0)'(s) \neq 0$, and hence it has an inverse, so s is uniquely determined for each (x, y) such that $P^0y - Q^0x$ is in the range of g; and hence $u^0(s)$ is also uniquely determined for each (x, y) in this set. Thus we have proved that for the case P and Q constant, the PDE has a unique solution; in fact we can write it as $u(x, y) = u^0(g^{-1}(P^0y - Q^0x))$ where g^{-1} is the inverse of g. The details of g^{-1} depend on how we selected the curve $(x^0(s), y^0(s))$. For example, it is reasonable to select the curve so that it crosses every integral curve of the vector field. We can also say something quite general: every solution of this constant-coefficient problem is of the form $u = f(P^0y - Q^0x)$ for some function f of a single variable; and every function of this form is a solution of $P^0 u_x + Q^0 u_y = 0$. Thus we are justified in calling $u = f(P^0 y - Q^0 x)$ the general solution. We also note that the general solution of this equation in two independent variables is given by an arbitrary function of a single variable.

Problem 10 Find the solution of

$$5u_x + 3u_y = 0$$

which takes the value $u = s^2$ on the curve

$$x^0(s) = s, \quad y^0(s) = \frac{1}{s}, \qquad 0 < s < \infty.$$

For what values of x and y is the solution defined?

Example 12 can be generalized to a constant-coefficient equation in any number of variables, of the form

$$P_1 \frac{\partial u}{\partial x_1} + \ldots + P_n \frac{\partial u}{\partial x_1} = 0, \tag{2.7}$$

where $\mathbf{P} = (P_1, \ldots, P_n)$ is constant.

Problem 11 Consider the equation (2.7) above, and assume that $P_n \neq 0$. Show that the integral curves of the vectorfield $\mathbf{V} = \mathbf{P}$ are the straight lines

$$x_i = P_i t + x_i^0, \quad i = 1, \ldots, n \tag{2.8}$$

and that the plane in $\mathbb{R}^n$ given by $x_n^0 = 0$ is an $(n-1)$-parameter surface, parameterized by $x_1^0, \ldots, x_{n-1}^0$, transversal to the integral curves. (That is, show that the Jacobian determinant

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(x_1^0, \ldots, x_{n-1}^0, t)}$$

is nonzero.) Noting that with $x_n^0 = 0$ in (2.8) one can easily solve for $t = x_n / P_n$ and thence invert the mapping $(x_1^0, \ldots, x_{n-1}^0, t) \mapsto (x_1, \ldots, x_n)$ explicitly, show that any solution of (2.7) can be written in the form

$$u(\mathbf{x}) = u^0 \left(x_1 - \frac{P_1}{P_n} x_n, \ldots, x_{n-1} - \frac{P_{n-1}}{P_n} x_n \right)$$

for an arbitrary function u^0 of $n-1$ variables.

PROBLEM 12 Carry out the construction in Problem 11 for the equation

$$u_x + 2u_y - u_z = 0,$$

assuming arbitrary data given on the x,y-plane:

$$u(x, y, 0) = u^0(x, y).$$

When the vectorfield is not constant, its integral curves may not fill the plane smoothly, even when it is linear. Typically, the condition $\mathbf{V} \neq 0$ does not hold in the entire plane.

EXAMPLE 13 Returning to $n = 2$, if P and Q are linear or affine functions of x and y, the equations $\dot{x} = P$, $\dot{y} = Q$ can be written as a linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

(or $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$). The solutions of this equation have different qualitative behavior depending on the matrix A; if A is invertible, then there is an equilibrium at $\mathbf{x} = -A^{-1}\mathbf{b}$, which may be a saddle, sink, source, or center, and the integral curves behave in one of a number of ways which are classified in the study of ODE. If A is not invertible there may be a number of equilibria, or none.

EXAMPLE 14 We looked at the vectorfield $\mathbf{V} = (-y, x)$ in Examples 5 and 7 of Chapter 1. The integral curves are circles centered at the origin; the origin itself is an equilibrium of the system $\dot{\mathbf{x}} = \mathbf{V}$. There are no integral curves through the origin in this case. Furthermore, every other integral curve is a closed curve. Writing the general solution as

$$x(t) = A\cos t + B\sin t, \quad y(t) = A\sin t - B\cos t,$$

we see that choosing a curve transversal to the vector field corresponds to choosing a one-parameter family $A(s), B(s)$. Some notation is necessary to convey the information that the solution depends both on t and on the choice of curve. For example, suppose that at $t = 0$ we get the positive x-axis, by letting $A = s$, $B = 0$, where s ranges through the open interval $(0, \infty)$. We then write $\mathbf{x}^0(s) = (A(s), -B(s)) = (s, 0)$ to indicate the data curve. Then the solution to the vectorfield becomes

$$x(t, s) = x(t; \mathbf{x}^0(s)) = s\cos t, \quad y(t, s) = y(t; \mathbf{x}^0(s)) = s\sin t. \tag{2.9}$$

PROBLEM 13 In Example 14, what does equation (2.9) become if the curve is chosen to be the positive y-axis and the parameterization is $y = e^s$ instead?

When we try to solve the PDE, in this variable-coefficient case, we find two difficulties. The first is that finding the integral curves of $\mathbf{V}$ is more difficult, and so is establishing transversality. The second is that manipulating the implicitly given solution to eliminate the parameters is also harder. The next theorem, which uses the inverse function theorem again, shows that in principle this can always be done, but possibly not in the entire x,y-plane.

THEOREM 4 *Suppose that P and Q are $\mathcal{C}^1$ functions of x and y in a connected open set D in the x,y-plane, and that they do not simultaneously vanish there. Then the equation*

$$P(x,y)u_x + Q(x,y)u_y = 0$$

has a $\mathcal{C}^1$ solution $u(x,y)$ in D. Furthermore, we can find a solution that takes a given value $u^0(s)$ on a given curve $(x^0(s), y^0(s))$ which is a $\mathcal{C}^1$ function of s and is transverse (not tangent) to $\mathbf{V}$ at every point. The solution with this data is unique on a subset of D.

PROOF By hypothesis, the vectorfield $\mathbf{V} = (P,Q) \neq \mathbf{0}$ in D, and hence the integral curves of $\dot{x} = P$, $\dot{y} = Q$ form a set of $\mathcal{C}^1$ curves which fill the region D: through each point in D there is a unique curve; no two curves intersect, and each curve can be continued until it leaves D (or tends to infinity, if D is unbounded in some directions). As in the example, denote by $x(t;\mathbf{x}^0)$, $y(t;\mathbf{x}^0)$ the integral curve through $\mathbf{x}^0 = (x^0, y^0)$. Now let $\mathbf{x}^0(s) = (x^0(s), y^0(s))$, be the data curve. The transversality condition, corresponding to (2.4), is $P(y^0)'(s) - Q(x^0)'(s) \neq 0$ where P and Q, functions of x and y, are now evaluated at points on the curve. This can be written

$$\det \begin{pmatrix} P(x^0(s), y^0(s)) & Q(x^0(s), y^0(s)) \\ (x^0)'(s) & (y^0)'(s) \end{pmatrix} \neq 0.$$

We let u take the value $u^0(s)$ at the point $(x^0(s), y^0(s))$; then $u = u^0$ at every point on the integral curve $x(t;\mathbf{x}^0)$, $y(t;\mathbf{x}^0)$ through $(x^0(s), y^0(s))$. We can write this as

$$u(x(t;\mathbf{x}^0(s)), y(t;\mathbf{x}^0(s))) = u^0(s).$$

Once again, as in the example, we have a formula which gives a solution, but it is expressed in terms of parameters s and t instead of directly as a function of x and y. We have a mapping $(t,s) \mapsto (x,y)$ given by the formula

$$(x,y) = (x(t;\mathbf{x}^0(s)), y(t;\mathbf{x}^0(s))).$$

By the inverse function theorem, the condition for inverting this mapping is that the Jacobian $\partial(x,y)/\partial(t,s)$ be nonzero. Now, $\partial\mathbf{x}/\partial t = (P,Q)$; however, the derivatives of $\mathbf{x}$ with respect to s are complicated functions. As these are derivatives of the solution of a differential equation with respect to the initial data, their existence is affirmed by the continuous dependence theorem for ODE. However, the only place where it is straightforward to calculate them is at $t=0$: since

$$\mathbf{x}(0,s) = \mathbf{x}(0;\mathbf{x}^0(s)) = \mathbf{x}^0(s),$$

we can write the Jacobian at $(0,s)$ as

$$\frac{\partial(x,y)}{\partial(t,s)}(0,s) = \det\begin{pmatrix} P(x^0(s),y^0(s)) & Q(x^0(s),y^0(s)) \\ (x^0)'(s) & (y^0)'(s) \end{pmatrix}$$

and by construction this is nonzero. Since we assumed that P, Q and $\mathbf{x}^0$ were continuously differentiable functions, this expression is continuous, and so it is nonzero in an open set containing the curve $\mathbf{x}^0$. Now the inverse function theorem says that we can find (t,s) for each (x,y) in some (possibly smaller) set. This is good enough: by construction, we now have a unique solution in a neighborhood of the data curve $\mathbf{x}^0$. But we can now choose any other $\mathcal{C}^1$ curve which is transverse to $\mathbf{V}$ in this neighborhood. Since the solution u is known there, we can take this as data, and apply the construction again to get a unique solution in a neighborhood of this curve. In fact, beginning with the original data curve $\mathbf{x}^0(s)$, we can construct the solution this way all along the integral curve through $\mathbf{x}^0(s)$, for each s, until the curve leaves the domain D (or all the way out to infinity, if it remains in D). The solution so found is unique in the entire region comprising the collection of integral curves through $\mathbf{x}^0(s)$ until they leave D. If there are parts of D that are not covered by the curves through $\mathbf{x}^0(s)$, or if the curves exit from D and re-enter it, then this construction will not find the solution everywhere in D, but only in a subset of D. However, the solution can always be continued (albeit not uniquely) to the rest of D. For if there is a part of D which is not included in the region, R, say, swept out by the integral curves coming from points in $\mathbf{x}^0(s)$, then the boundary between that part and R consists of an integral curve. Then we can take any curve transverse to $\mathbf{V}$ that crosses the boundary, and extend the solution defined along it in R in any smooth way as a function $u(s)$ to the part of D where it has not been defined yet. Now perform the construction again using this curve as a base curve. Finally, we note that if we were not given a data curve in the first place, we could begin with any smooth curve transverse to $\mathbf{V}$, and so we have proved the first part of the theorem as well. ■

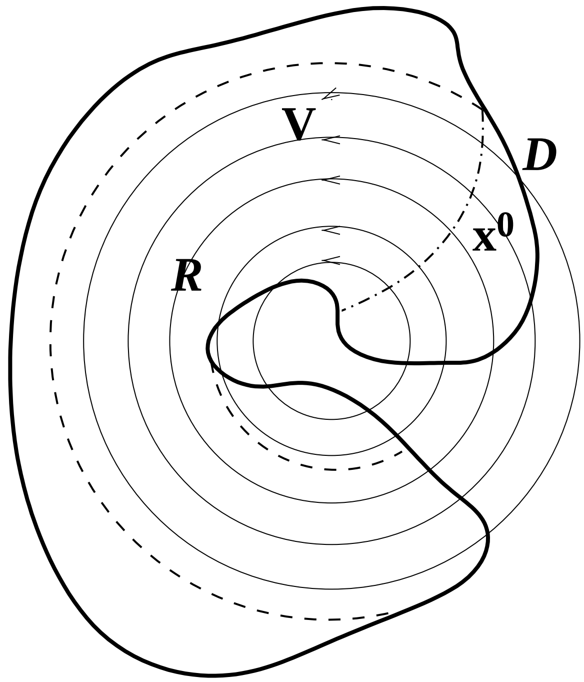

FIGURE 2.1: An Illustration of the Theorem

In Figure 2.1, we sketch a typical example: in this case the integral curves are circles, D is a simply connected domain not containing the origin, and $\mathbf{x}^0(s)$ a curve transverse to the integral curves. It can be seen that there are several parts of D in which the solution is not uniquely defined by data given on $\mathbf{x}^0(s)$. We may add a comment on the hypotheses of this theorem. It is clear that difficulties will arise if $\mathbf{V}$ has zeros, for the integral curves through such points will not be well defined. Furthermore, smoothness of $\mathbf{x}^0(s)$ is required in order for us to calculate whether the curve is transverse to the vectorfield. The smoothness of P and Q is not needed for this, but is needed in order to apply the fundamental existence and continuous dependence theorems of ordinary differential equations. For this, we could get away with a slightly weaker condition: in fact we need only that P and Q be Lipschitz continuous. The condition we gave is simpler. (We shall often state results with simple, rather than optimal, hypotheses.)

Now, this procedure for solving equations of the special form (2.2) works in any number of dimensions. For practice in the use of subscripts, we will describe the general method; then we will do some examples in three dimensions.

THEOREM 5 *Suppose that* $a_1, \ldots, a_n$ *are* $\mathcal{C}^1$ *functions of* $\mathbf{x}$ *in a connected open set* D *in* $\mathbb{R}^n$, *and that* $\mathbf{V} = (a_1, \ldots, a_n) \neq 0$ *there. Then the equation* (2.2) *has a* $\mathcal{C}^1$ *solution* $u(x, y)$ *in* D. *Furthermore, we can find a solution* *that takes a given value $u^0(\mathbf{s})$ on a given $\mathcal{C}^1$ surface $\mathbf{x}^0(\mathbf{s})$ parameterized by $\mathbf{s} \in \mathbb{R}^{n-1}$ where $u^0(\mathbf{s})$ is a $\mathcal{C}^1$ function of $\mathbf{s}$ and the surface is transverse to $\mathbf{V}$ at every point. The solution with this data is unique on a subset of D.*

Note the dimensions of all the sets involved: a surface in an n-dimensional space can be written parametrically in terms of $n-1$ parameters. One fact that was apparent in the simple Example 12 but may have been obscured in the statements of Theorems 4 and 5 is that the general solution of the equation is a function of $n-1$ variables. That is exactly the role of the function $u^0(\mathbf{s})$. Notice that this is an exact generalization of the case of a first-order ODE: the number of independent variables is one, in this case, and the general solution contains an arbitrary function of zero variables — that is to say, a constant. The reader may want to guess at this point that the solution of a second-order PDE in n variables will involve two arbitrary functions of $n-1$ variables. However, as we shall see, this is true only for a very limited set of equations: when we begin to look at second-order equations, we shall find that they are much less like ODE than are first-order equations.

Theorem 5 is proved in exactly the same way as Theorem 4: we first construct a function of t and $\mathbf{s}$ by finding the integral curve through $\mathbf{x}^0(\mathbf{s})$, which we call $\mathbf{x}(t;\mathbf{x}^0(\mathbf{s}))$; then, since u is constant along that curve, we have

$$u(t,\mathbf{s}) = u^0(\mathbf{x}(t;\mathbf{x}^0(\mathbf{s}))).$$

Next we find out when we can invert the mapping $(t,\mathbf{s}) \mapsto \mathbf{x}(t;\mathbf{x}^0(\mathbf{s})$. Again, this depends on the Jacobian of the mapping, and this can be calculated at any point on the surface $\mathbf{x}^0(\mathbf{s})$, and is

$$\frac{\partial \mathbf{x}}{\partial(t,\mathbf{s})}(0,\mathbf{s}) = \det \begin{pmatrix} \mathbf{V}(\mathbf{x}^0(s) \\ \frac{\partial \mathbf{x}^0}{\partial \mathbf{s}} \end{pmatrix}.$$

Now, the point is that the transversality condition implies that this determinant is nonzero. The argument is that a row of $\partial\mathbf{x}^0/\partial\mathbf{s}$, say the i-th row, is $\partial\mathbf{x}^0/\partial s_i$ is a tangent vector to the surface. The condition that the vectorfield be transverse to the surface is exactly a statement that $\mathbf{V}$ is not tangent to the surface and hence not a linear combination of the tangent vectors $\partial\mathbf{x}^0/\partial s_i$. (The condition that $\mathbf{x}^0(\mathbf{s})$ is a surface also implies that there are $n-1$ linearly independent tangent vectors at each point.)

Example 15 We do an example in $\mathbb{R}^3$. For the equation

$$xu_x + yu_y + u_z = 0,$$

the vectorfield $\mathbf{V} = (x, y, 1)$ is never zero, so we may take $D = \mathbb{R}^3$. Integrating the vectorfield equations

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= y \\ \dot{z} &= 1, \end{aligned}$$

introducing a parameter, t, say, and constants of integration, we get

$$\begin{aligned} x &= c_1 e^t \\ y &= c_2 e^t \\ z &= c_3 + t, \end{aligned}$$

or $(x, y, z) = (c_1 e^t, c_2 e^t, c_3 + t)$. Now we want to give data u^0 on a surface which is transverse to $(x, y, 1)$. Conveniently, the x, y-plane has this property, and it is conveniently parameterized by $x = c_1$, $y = c_2$, $z = 0$. That is, working in dimension 3, we have a two-parameter family representing the surface. Now we may suppose u has a given value, say $u = u^0(c_1, c_2)$ at every point in the x, y-plane, and so we have the solution expressed parametrically as $u(x, y, z) = u^0(c_1, c_2)$ where $(x, y, z) = (c_1 e^t, c_2 e^t, t)$ since this is the solution of $\dot{\mathbf{x}} = \mathbf{V}$ with the initial data $(c_1, c_2, 0)$. The Jacobian of the mapping is

$$\frac{\partial(x, y, z)}{\partial(t, c_1, c_2)} = \det \begin{pmatrix} c_1 e^t & e^t & 0 \\ c_2 e^t & 0 & e^t \\ 1 & 0 & 0 \end{pmatrix} = e^{2t}.$$

In this example, we can calculate the Jacobian at every point, not merely at points on the surface $\mathbf{x}^0$. In fact, we can invert the mapping explicitly:

$$t = z, \quad c_1 = xe^{-z}, \quad c_2 = ye^{-z},$$

and so we can see directly that the inverse is defined for every point in $\mathbb{R}^3$. Now we can write the solution explicitly as well:

$$u(x, y, z) = u^0(xe^{-z}, ye^{-z}),$$

and this formula substantiates the claim made earlier that the general solution is given by a function of two variables.

We introduce some terminology.

Definition 1 *In a first-order linear equation of the form* $\mathbf{a}(\mathbf{x}) \cdot \nabla u = 0$, *the integral curves of the vectorfield equation* $\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x})$ *are called* bicharacteristic curves *or* characteristics.

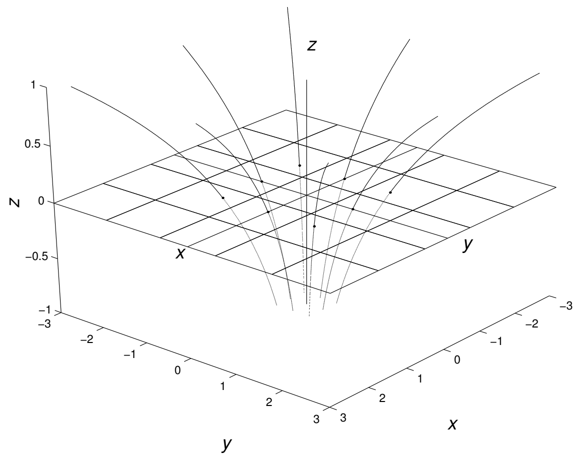

FIGURE 2.2: The Bicharacteristic Curves in Example 15

The procedure we have developed for solving first-order equations of this form is called the *method of characteristics*. In Example 15, the bicharacteristic curves are the curves $(x, y, z) = (c_1e^t, c_2e^t, c_3 + t)$, parameterized by t, or the curves given by $x = Ae^z$, $y = Be^z$. These are sketched in Figure 2.2. Notice that in this example each bicharacteristic curve lies in a plane $x/y = A/B$.

EXAMPLE 16 This example will appear in the next section on quasilinear equations. The equation $zu_x + u_y = 0$ is an example of a linear equation in three independent variables, (x, y, z) in which one variable, z appears only as a parameter in the equation. The method of characteristics can be applied to this equation. The vectorfield is $\mathbf{V} = (z, 1, 0)$ and the equations for the bicharacteristic curves are

$$\begin{aligned} \dot{x} &= z \\ \dot{y} &= 1 \\ \dot{z} &= 0; \end{aligned}$$

we solve by integrating, beginning with the last equation, whose solution is $z = z^0$. Choose $y^0 = 0$ to make the data curve the x-z-plane. This is a logical choice since the x-z-plane is transverse to the vectorfield $\mathbf{V}$. Then $y = t$ and $x = z^0t + x^0$, so the bicharacteristic curves are $(x(t), y(t), z(t)) = (z^0t + x^0, t, z^0)$. Furthermore the mapping $(t, x^0, z^0) \mapsto (x, y, z)$ has an inverse defined for all points in $\mathbb{R}^3$: $t = y$, $x^0 = x - yz$, and $z^0 = z$. Thus the general solution is

$$u(x, y, z) = u^0(x^0, z^0) = u^0(x - yz, z)$$

where u^0 is an arbitrary function of two variables.

PROBLEM 14 Show that the bicharacteristic curves of of $zu_x + u_y + u_z = 0$ are the curves $(x, y, z) = (x^0 + z^0 t + t^2/2, t, t + z^0)$ and that the general solution is a function of the two variables $x - yz + y^2/2$ and $y - z$.

We now greatly expand the range of equations which can be solved using the method of characteristics by noting that the equation reduces to an ODE along each bicharacteristic curve. The method we are about to describe applies to any equation of the form

$$\mathbf{a}(\mathbf{x}) \cdot \nabla u = g(\mathbf{x}, u). \tag{2.10}$$

If g depends on $\mathbf{x}$ only, or is of the form $-b(\mathbf{x})u + f(\mathbf{x})$, then the equation is linear (homogeneous or nonhomogeneous), of the form given at the beginning of this chapter. The method applies more generally, though we shall see a difference between linear and nonlinear equations in the form and properties of their solutions. Here is the idea. Take a solution $u(\mathbf{x})$ to (2.10), and differentiate u along a single bicharacteristic curve, $\mathbf{x}(t)$, with respect to t. To emphasize that we are restricting to a single curve, along which all variables depend on t only, we shall write $u(t)$ for $u(\mathbf{x}(t))$, and evaluate the (ordinary) derivative of u along the curve, using the chain rule and the equation $\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x})$ for the characteristics:

$$\begin{aligned} \frac{du}{dt} = \dot{u}(t) = \frac{d}{dt} u(x_1(t), x_2(t), \ldots, x_n(t)) = u_{x_1}\dot{x_1} + u_{x_2}\dot{x_2} + \ldots + u_{x_n}\dot{x_n} \\ = u_{x_1}a_1(\mathbf{x}(t)) + u_{x_2}a_2(\mathbf{x}(t)) + \ldots + u_{x_n}a_n(\mathbf{x}(t)) = \mathbf{a} \cdot \nabla u = g(\mathbf{x}(t), u(t)); \end{aligned}$$

in short, since $\mathbf{x}$ is a known function of t along the curve, we have an ODE of the form $\dot{u} = G(t, u)$ along the curve. This equation is linear in u if g is linear, and if g is independent of u it reduces to an integration in t. Whatever the form of g, the theory of ordinary differential equations gives existence of a unique solution which takes a given value $u^0(\mathbf{s})$ on a data surface transverse to the bicharacteristic curves as in Theorem 5. This gives an extension of Theorem 5.

THEOREM 6 *Suppose that $a_1, \ldots, a_n$ are $\mathcal{C}^1$ functions of $\mathbf{x}$ in a connected open set D in $\mathbb{R}^n$, and that $\mathbf{V} = (a_1, \ldots, a_n) \neq 0$ there. Suppose also that g is a $\mathcal{C}^1$ function of $\mathbf{x}$ and u for $\mathbf{x} \in D$ and for u in an interval $I \in \mathbb{R}$. Then the equation (2.10) has a $\mathcal{C}^1$ solution $u(x, y) \in I$ in a neighborhood of any point $\mathbf{x}$ in D. We can find a solution that takes a given value $u^0(\mathbf{s}) \in I$ on a given $\mathcal{C}^1$ surface $\Gamma = \{\mathbf{x}^0(\mathbf{s})\}$ parameterized by $\mathbf{s} \in \mathbb{R}^{n-1}$ where $u^0(\mathbf{s})$ is a $\mathcal{C}^1$ function of $\mathbf{s}$ and the surface is transverse to $\mathbf{V}$ at every point. The solution with this data exists and is unique on a neighborhood of Γ in D.*

Notice that we assert only local existence; this is because we have allowed the function g to be nonlinear, and existence theory for nonlinear ODE is only local. If g is a linear or affine function of u, then again we get a conclusion similar to Theorem 5.

COROLLARY 1 *If in* (2.10) g *has the form* $-b(\mathbf{x})u + f(\mathbf{x})$, *then there exists a solution which is* $\mathcal{C}^1$ *in* D *and takes the value* $u^0(\mathbf{s})$ *on a given* $\mathcal{C}^1$ *surface* Γ *transverse to* $\mathbf{V}$ *at every point.*

We illustrate with several examples before giving the proof.

EXAMPLE 17 An equation of the form $cu_x + u_y = g$ is a transport equation with a source term g; we suppose now that y represents the time variable and x a space variable. When u is the density of the substance being transported down a pipe, then g may depend on space and time if the substance is being injected (or removed) at places along the pipe at some given, possibly variable, rates; g will depend on u if the rate of injection or removal at a point in space and time depends also on the amount u of the substance present at that point. Suppose for example that the equation is $2u_x + u_y = x$, in which a nonhomogeneous source is applied to the original transport equation. The vectorfield is constant; the bicharacteristic curves, parameterized by t, are $(x(t), y(t)) = (x^0 + 2t, t)$. Here we have chosen the data curve to be $y = 0$, and y coincides with the parameter t. We write $u(t)$ for $u(x(t), y(t)) = u(x^0 + 2t, t)$, noting that u also depends on x^0, a fact not reflected in the notation. Along a characteristic, the equation for $u(t)$ is

$$\dot{u} = x = x^0 + 2t$$

and we find u by integration. If we compute the definite integral, then

$$u(t) - u(0) = x^0 t + t^2. \tag{2.11}$$

Now $u(0)$ means $u(x^0, 0)$, and we suppose as usual that we are given a function $u^0(x^0)$ on the data curve $y = 0$ (initial data in this case). Writing the solution as $u(x(t), y(t)) = u(t)$ we have

$$u(x(t), y(t)) = u(x^0 + 2t, t) = u^0(x^0) + x^0 t + t^2,$$

from (2.11) and we complete the problem by inverting the mapping $(t, x^0) \mapsto (x, y) = (x^0 + 2t, t)$. Thus, $t = y$ and $x^0 = x - 2y$, so

$$u(x, y) = u^0(x - 2y) + (x - 2y)y + y^2 = u^0(x - 2y) + (x + y)y.$$

It can be checked that this is a solution of the equation. The solution is a superposition of $u^0(x - 2y)$, the solution of the homogeneous equation with data $u(x,0) = u^0(x)$, and a particular solution, $(x+y)y$, to the nonhomogeneous equation, which is zero at $y = 0$. This solution is clearly defined for all x and y.

Now look at $2u_x + u_y = u^2$, which is not linear. Using the method of characteristics, the ODE for u is now

$$\dot{u} = u^2,$$

a 'separable' first-order equation, which can be written as

$$\frac{1}{u^2}\frac{du}{dt} = 1,$$

and integrated:

$$\int_0^t \frac{1}{u^2}\frac{du}{dt}\,dt = \int_{u(0)}^{u(t)} \frac{1}{u^2}\,du = t.$$

So

$$-\frac{1}{u(t)} + \frac{1}{u(0)} = t; \tag{2.12}$$

letting $u(0) = u(x(0), y(0)) = u(x^0, 0) = u^0(x^0)$ be the given data at $y = 0$ and solving (2.12) for $u(t)$, we have

$$u(t) = u(x,y) = \frac{u^0(x^0)}{1 - tu^0(x^0)} = \frac{u^0(x-2y)}{1 - yu^0(x-2y)}, \tag{2.13}$$

where we have again solved for x^0 and t in terms of x and y. Again, we have a solution, as can be checked, and the solution is unique because every step is reversible (no additional assumptions were made). However, unlike the choice of $g = x$ in this example, this solution may not be defined for all x and y. If $yu^0(x-2y) = 1$ for any values of x and y, then the denominator of (2.13) becomes zero and since the numerator cannot be zero at the same point, the function $u(x,y)$ is undefined there. To be precise, if there is any value x^0 at which $u^0(x^0)$ is nonzero, then we have $yu^0(x-2y) = 1$ for $y = 1/u^0(x^0)$ and $x = 2y + x^0$. In fact, typically there is a curve of such points. By way of illustration, take $u^0(x^0) = e^{-x^0}$, so the solution is

$$u(x,y) = \frac{e^{2y-x}}{1 - ye^{2y-x}},$$

which becomes undefined on the curve $x = 2y + \log y$ for $y > 0$. This is illustrated in Figure 2.3. Note that while the function $u(x,y)$ may be defined on the other side of the data curve, Γ, from the singular curve, and will still satisfy the PDE, it is no longer a $\mathcal{C}^1$ solution to the problem with

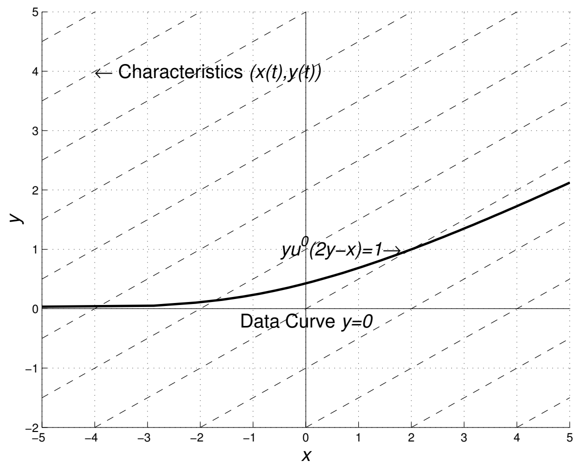

FIGURE 2.3: The Domain of Existence for a Nonlinear Equation

data given on Γ. Also note the difference in structure of the solution in the linear as distinct from the nonlinear problem: in the nonlinear problem we have just solved, one can see from the form of (2.13) that superposition does not hold.

PROBLEM 15 Show that the principle of superposition fails for the problem $2u_x + u_y = u^2$; that is, show that if u and v are solutions with data $u(x,0) = u^0(x)$ and $v(x,0) = v^0(x)$ respectively, then $u+v$ is not generally a solution.

PROBLEM 16 Show that if u_1 and u_2 are solutions of $2u_x + u_y = g$ with $g = g_1(x,y)$ and $g = g_2(x,y)$ respectively, then $u_1 + u_2$ is a solution of $2u_x + u_y = g_1 + g_2$. Use this to solve $2u_x + u_y = x + y^2$.

PROBLEM 17 For what values of x and y is the solution to $2u_x + u_y = u^2$ with $u(x,0) = 1$ defined? What is the solution to $2u_x + u_y = u^2$ with $u(x,0) = x$, and for what (x,y) is it defined? (Consider both positive and negative values of y.)

EXAMPLE 18 Here is a linear equation, but one with variable coefficients:

$$-yu_x + u_y = u.$$

We studied the vectorfield $\mathbf{V} = (-y, x)$ in Example 14, and found that the characteristics through $(s,0)$ for $s > 0$ are $(x,y) = (s\cos t, s\sin t)$. Along a characteristic, the ODE for u is

$$\dot{u} = u,$$

and its solution is $u(t) = u(0)e^t$, from which we deduce that

$$u(x, y) = u^0(s)e^t,$$

where (s, t) are polar coordinates for (x, y); thus $s = \sqrt{x^2 + y^2}$ and $t = \cos^{-1}(x/\sqrt{x^2 + y^2})$ when $y \geq 0$ and $t = 2\pi - \cos^{-1}(x/\sqrt{x^2 + y^2})$ when $y < 0$. With this determination of t the solution is discontinuous as (x, y) tends to the positive x-axis from below; a different choice for t is possible, but there is necessarily a discontinuity somewhere along each circular characteristic. Note that in this example, superposition does hold.

PROBLEM 18 Show that the characteristics of

$$xu_x + yu_y = g(x, y, u) \tag{2.14}$$

are radial lines through the origin, and use the method of characteristics to solve (2.14) with $g = -u$ and data $u = u^0(x^0)$ given on the curve $(x, y) = (x^0, 1)$. Verify that the function you have found solves the equation. Where is the solution defined?

We now prove Theorem 6 and its Corollary.

PROOF From the proof of Theorem 5, we can construct the bicharacteristic curve $\mathbf{x}(t; \mathbf{x}^0(\mathbf{s}))$ through each point $\mathbf{x}^0$ of the data surface Γ. This curve is defined for t in an open interval, and thus the curves fill out a neighborhood of Γ. On each curve, parameterize u as $u(t) = u(\mathbf{x}(t; \mathbf{x}^0))$; u satisfies

$$\dot{u} = g\big(\mathbf{x}(t; \mathbf{x}^0), u(t)\big) = g(t, u).$$

Since g is a $\mathcal{C}^1$ function of u, this equation, with the initial data $u(0) = u^0(\mathbf{x}^0)$, has a unique solution in an interval $a < t < b$, where the endpoints depend on the value of $u^0(\mathbf{x}^0)$ and the size of the interval I where g is a smooth function of u. The size of the interval (a, b) depends also on the interval of t for which $\mathbf{x}(t)$ remains in D, since otherwise $\mathbf{x}$ and hence g may not be defined. However, for each $\mathbf{s}$ and hence for each value of $\mathbf{x}^0$, a unique solution $u(t)$ exists on an open interval in t on which the map $(t, \mathbf{s}) \mapsto \mathbf{x}$ can be inverted to give $t = t(\mathbf{x})$ and thus the unique solution $u(\mathbf{x})$ has been found.

Finally, to prove the Corollary we note that if g is a linear function of u then the solution of the ODE $\dot{u} = g$ exists for all t and so u can be found on the entire length of each characteristic exactly as in Theorem 5. ∎

EXAMPLE 19 Here is an example based on Example 15, in $\mathbb{R}^3$. Consider

$$xu_x + yu_y + u_z = \frac{xy^2}{u},$$

with data $u(x, y, 0) = u^0(x, y)$ on the x-y-plane. We found the characteristics in Example 15 to be $(x(t), y(t), z(t)) = (c_1e^t, c_2e^t, t)$ and so the equation for u is

$$\frac{du}{dt} = \frac{c_1e^t(c_2e^t)^2}{u},$$

or

$$u\frac{du}{dt} = c_1c_2^2e^{3t},$$

which gives the solution

$$\frac{1}{2}u^2(t) - \frac{1}{2}u^2(0) = \frac{1}{3}c_1c_2^2(e^{3t} - 1).$$

Now solving for $t = z$, $c_1 = xe^{-z}$ and $c_2 = ye^{-z}$, and recalling that $u(0) = u(c_1, c_2, 0) = u^0(c_1, c_2)$, we get

$$\frac{1}{2}\big(u(x, y, z)\big)^2 = \frac{1}{2}\big(u^0(xe^{-z}, ye^{-z})\big)^2 + \frac{xy^2}{3}\big(1 - e^{-3z}\big),$$

which we can easily solve for u by multiplying the equation by 2 and taking the square root, noting that we must take the positive or negative square root according as u^0 was positive or negative, in order to get a solution which is equal to the initial data at $z = 0$. Thus

$$u(x, y, z) = \pm\sqrt{\big(u^0(xe^{-z}, ye^{-z})\big)^2 + \frac{2}{3}xy^2(1 - e^{-3z})},$$

where the correct choice of sign is determined as above. The nonlinearity of the equation has apparent consequences: This solution exists only for values of x, y and z for which the quantity under the square root remains positive. In addition, the principle of superposition clearly does not hold.

PROBLEM 19 Find the solution of $u_x + 2u_y - u_z + u = x^2$ with data $u(x, y, 0) = u^0(x, y)$. For what values of (x, y, z) is the solution defined? Where is it unique?

PROBLEM 20 Answer the same questions as in Problem 19 for $u_x + 2u_y - u_z + u^2 = 0$.

PROBLEM 21 Find the solution to $2u_x + u_y = x$ which takes the value $u = 1$ on the line $x + y = 1$.

Problem 22 Solve $2u_x + u_y = x$ with the side condition $u(x, x) = x$.

Problem 23 Show that the solution to $2u_x + u_y = x$ with $u(2y, y) = y^2$ is not unique, and that there is no solution to the equation which satisfies $u(2y, y) = y$.

2. Quasilinear First-Order Equations

In the previous section, we showed how to solve any linear first-order equation of the form

$$a_1(\mathbf{x})\frac{\partial u}{\partial x_1} + a_2(\mathbf{x})\frac{\partial u}{\partial x_2} + \ldots a_n(\mathbf{x})\frac{\partial u}{\partial x_n} + b(\mathbf{x})u = f(\mathbf{x}).$$

In fact, we did a little more: we solved equations of the form

$$a_1(\mathbf{x})\frac{\partial u}{\partial x_1} + a_2(\mathbf{x})\frac{\partial u}{\partial x_2} + \ldots a_n(\mathbf{x})\frac{\partial u}{\partial x_n} = g(\mathbf{x}, u), \tag{2.15}$$

using the method of characteristics. An equation like (2.15) is called *semilinear*; this terminology reflects the property, which we have observed, that although the equation is not linear and superposition does not hold, the fact that the coefficients of the highest-order part of the equation do not depend on u means that characteristics are defined and the method of characteristics can easily be adapted to solve the equation.

In this section, we move further in classifying nonlinear first-order PDE according to the kind of nonlinearity they exhibit by defining *quasilinear* first-order equations.

Definition 2 *A first-order equation is called* quasilinear *if it is of the form*

$$a_1(\mathbf{x}, u)\frac{\partial u}{\partial x_1} + a_2(\mathbf{x}, u)\frac{\partial u}{\partial x_2} + \ldots a_n(\mathbf{x}, u)\frac{\partial u}{\partial x_n} = g(\mathbf{x}, u). \tag{2.16}$$

In a quasilinear equation, the partial derivatives of u, the components of ∇u, appear in linear combination, but the coefficients may depend on u as well as on the independent variable $\mathbf{x}$.

Example 20 The most famous example is the Hopf, or inviscid Burgers', equation $u_t + uu_x = 0$. We will study this, along with similar equations of the form $u_t + a(u)u_x = 0$, which are used, for different functions $a(u)$, to model traffic flow. On the other hand, the equation

$$u_x^2 + u_y^2 = 1,$$

which governs the propagation of light rays, is *fully nonlinear*. We will study fully nonlinear equations in the next section.

The following proposition shows that solutions to quasilinear equations can be obtained from solutions to linear equations.

PROPOSITION 2 *Given equation* (2.16)*, define the linear equation*

$$a_1(\mathbf{x}, z)\frac{\partial w}{\partial x_1} + a_2(\mathbf{x}, z)\frac{\partial w}{\partial x_2} + \ldots a_n(\mathbf{x}, z)\frac{\partial w}{\partial x_n} + g(\mathbf{x}, z)\frac{\partial w}{\partial z} = 0. \qquad (2.17)$$

If $w(\mathbf{x}, z)$ is any solution to (2.17) *with $w_z \neq 0$, then the function $z = z(\mathbf{x})$ obtained by solving $w(\mathbf{x}, z) = 0$ for z is a solution to* (2.16).

EXAMPLE 21 The linear equation corresponding to $uu_x + u_y = 1$ is $zw_x + w_y + w_z = 0$. The method of characteristics for $zw_x + w_y + w_z = 0$ gives (see Problem 14)

$$w(x, y, z) = w^0(x - yz + y^2/2, z - y).$$

Now, according to Proposition 2, a solution to $uu_x + u_y = 1$ is given implicitly by the equation

$$w^0\big(x - yu(x, y) + y^2/2, u(x, y) - y\big) = 0, \qquad (2.18)$$

which can be solved for u as long as $-yw^0_1 + w^0_2 \neq 0$, where the subscripts 1 and 2 indicate partial derivatives of w^0 with respect to its first and second arguments. Suppose that we want a solution of $uu_x + u_y = 1$ which takes the value $u(x, 0) = u^0(x)$ (thinking of y as a time variable). Then, setting $y = 0$ in (2.18), we want w^0 to satisfy

$$w^0(x, u^0(x)) = 0,$$

so an appropriate choice for w^0 is $w^0(a, b) = u^0(a) - b$. In that case the solution (2.18) becomes

$$u^0(x - yu(x, y) + y^2/2) - (u(x, y) - y) = 0,$$

and this defines $u(x, y)$ implicitly as long as $-y(u^0)' - 1 \neq 0$. This condition always holds at $y = 0$ and for y sufficiently close to zero, but fails when $y = -1/(u^0)'$.

PROBLEM 24 The linear equation corresponding to $uu_x + u_y = 0$ is $zw_x + w_y = 0$; this is an equation for $w(x, y, z)$ in which z appears as a parameter. It was solved in Example 16. Using that solution, $w(x, y, z) = w^0(x - yz, z)$, find a formula which gives implicitly the solution $u(x, y)$ to $uu_x + u_y = 0$, with $u(x, 0) = u^0(x)$. Show that the solution breaks down for the same values of y as in Example 21.

The proof of Proposition 2 is a straightforward calculation.

PROOF Given a solution $w(\mathbf{x}, z)$ of (2.17), then the condition $w_z \neq 0$ means we can solve $w(\mathbf{x}, z) = 0$ for $z = u(\mathbf{x})$, say. We need to show that u satisfies (2.16), and for that, we compute the partial derivatives of u by differentiating the equation

$$w(\mathbf{x}, u(\mathbf{x})) = 0$$

with respect to each x_i in turn. Denoting $\partial f/\partial x_i$ by f_i, we obtain

$$w_i + w_z u_i = 0$$

and hence

$$\frac{\partial u}{\partial x_i} = -\frac{w_i}{w_z}, \tag{2.19}$$

for $i = 1, \ldots, n$. Now, always assuming $w_z \neq 0$, simply divide every term in (2.17) by w_z, note from (2.19) that the ratios w_i/w_z are the partial derivatives of u (with a sign change), and note that $z = u$ in (2.17), to conclude that the equation becomes (2.16), and so u, found from solving $w(\mathbf{x}, z) = 0$ for $z = u(\mathbf{x})$ is a solution of (2.16). Furthermore, as in Example 21, we may seek a solution with data $u = u^0(\mathbf{s})$ on the surface $\mathbf{x}^0(\mathbf{s})$ by choosing $w(\mathbf{x}, z) = w^0\big(x^0(\mathbf{s}), u^0(\mathbf{s})\big)$. ■

PROBLEM 25 Show that two solutions of

$$xu_x + yu_y + z^2 u_z = 0$$

are $u_1 = x/y$ and $u_2 = 1/z + \log x$, and use them to find the general solution of

$$xw_x + yw_y = w^2.$$

Find also the solution w satisfying $w(1, y) = y^2$.

PROBLEM 26 Use Proposition 2 and the solution of Example 15 to find the solution of

$$xw_x + yw_y = 1$$

which is zero on the line $x + y = 1$. Where is the solution defined? Justify your answer by reference to the characteristics.

PROBLEM 27 What is the linear equation of the form (2.17) which corresponds to the quasilinear equation $u^2 u_x + u_y = 0$? Find its characteristics, and find, in implicit form, the solution with data $u(x, 0) = u^0(x)$.

A significant difficulty with quasilinear equations, as indicated already in Example 21, is that solving the equation which gives $u(\mathbf{x})$ implicitly may not be possible. Using Proposition 2, we see that we can set up the method of characteristics for a quaslinear equation (2.16) without actually writing down the corresponding linear equation (2.17). For, given (2.16), the equations for the bicharacteristic curves in (2.17) are

$$\begin{aligned} \dot{x}_1 &= a_1(\mathbf{x}, u) \\ \dot{x}_2 &= a_2(\mathbf{x}, u) \\ &\vdots \\ \dot{x}_n &= a_n(\mathbf{x}, u) \\ \dot{u} &= g(\mathbf{x}, u). \end{aligned} \tag{2.20}$$

Example 22 For the Hopf equation, $uu_x + u_y = 0$, we have

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= 1 \\ \dot{u} &= 0. \end{aligned}$$

Comparing the system (2.20) to the vectorfield equations $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ for a linear equation, we see that a significant difference is that the integral curves of (2.20) depend on the dependent variable u as well as on the independent variables $\mathbf{x}$. We extend the definition of bicharcteristic curves to the quasilinear case.

Definition 3 *The integral curves of the system* (2.20) *are called the* bicharacteristic curves *of the quasilinear system* (2.16). *A surface* $\mathbf{x} \in \Gamma \subset \mathbb{R}^n$ *is called* noncharacteristic *for data* $u^0(\mathbf{x})$ *if the vectorfield* $\mathbf{V}(\mathbf{x}, u^0) = (a_1(\mathbf{x}, u^0), a_2(\mathbf{x}, u^0), \dots, a_n(\mathbf{x}, u^0)$ *is transversal to* Γ.

Problem 28 Find the bicharacteristic curves of the equation $e^u u_x + u_y = 0$. Show that both the x-axis and the y-axis are noncharacteristic.

Unlike linear or semilinear equations, data surfaces for quasilinear equations may or may not be characteristic depending on the data assigned there.

Example 23 We looked at the equation $uu_x + u_y = 1$, for which $\mathbf{V} = (u, 1)$, in Example 21. The x-axis is always noncharacteristic, since if $\Gamma = \{(x, 0)\}$ then the vector $(1, 0)$ is tangent to Γ and this is never parallel to $\mathbf{V}$. However, if we give data on the y-axis, then $\Gamma = \{(0, y)\}$ has tangent $(0, 1)$ and this is characteristic at any point $(0, y)$ where the data $u(0, y) = u^0(y)$ is zero. Indeed, if we try to solve this equation with the data $u(0, y) = y$, say, then we find, integrating

$$\dot{x} = u, \quad \dot{y} = 1, \quad \dot{u} = 1,$$

with respect to t, with the initial conditions

$$x(0) = 0, \quad y(0) = y^0, \quad u(0) = y^0,$$

that

$$x = y^0 t + t^2/2, \quad y = y^0 + t, \quad u = y^0 + t.$$

Now it is clear by inspection that $u(x, y) = y$ for all points (x, y) which lie on any characteristic through Γ. However, trying to invert the mapping $(t, y^0) \mapsto (x, y) = (y^0 t + t^2/2, y^0 + t)$ results in $t = y \pm \sqrt{y^2 - 2x}$ and $y^0 = \mp\sqrt{y^2 - 2x}$, and this is defined only if $x \leq y^2/2$. (The + sign in the expression for t corresponds to $y < 0$ and the negative sign is for $y > 0$; the signs are reversed in the expression for y^0.) In fact, all one can say about the solution of this problem is that $u(x, y) = y$ outside the parabola $x = y^2/2$. It is of course the case that $u(x, y) = y$ is a solution for all x and y but so for example is the function

$$u(x, y) = \begin{cases} y, & x \leq y^2/2 \\ \alpha(x + y + y^2/2)/(1 + y) + (1 - \alpha)y, & x > y^2/2 \end{cases}$$

for every α. This function has the correct values on the y-axis and is continuous everywhere. Thus the failure of the data curve to be noncharacteristic at even a single point can have enormous consequences.

Problem 29 Suppose that the initial data in the preceding example are such that $u(0, y) \neq 0$; take $u(0, y) = 1$ for concreteness. Show that a unique solution is now defined for all $x > 0$; show, however, that the solution found by the method of characteristics does not exist for $x < -1/2$.

Problem 30 Use the method of characteristics to find the an implicit formula for the solution to $uu_x + u_y = 0$ which takes the value $u(0, y) = e^y$. Show that this solution is defined in a neighborhood of $x = 0$. Is it defined for all (x, y)?

Problem 31 What are the characteristics of the equation $uu_x + u^2 u_y = 1$? Show that neither the x not the y axis is noncharacteristic for all data. Are there any lines $y = ax + b$ which are noncharacteristic for all choices of data u?