in *Nonlinear Theory of Generalized Functions*, Michael Grosser, Günther Hörmann, Michael Kunzinger, and Michael Oberguggengerger (Editors), Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton, 1999, pp 99-111.

Conservation Laws, Delta Shocks and Singular Shocks

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1 Modeling with Conserved Quantities

The theory of hyperbolic conservation laws was developed to study phenomena in unsteady compressible gas dynamics, especially the propagation of nonlinear waves, and the formation and evolution of shock discontinuities. Comparison with experiments shows that the theory is reasonably successful. Although a rigorous theory of conservation laws encompassing existence, uniqueness, well-posedness, and asymptotic behavior has been achieved only for equations in a single space variable, numerical simulations which have been carried out in higher dimensions also give good correspondence with the three-dimensional world. During the last fifty years, the use of conservation laws has grown to include many kinds of fluid and continuum models.

It is also the case that conservation laws are used as qualitative models. This is indeed part of their pedigree; perhaps the first instance is Burgers' use [2] of the scalar equation which now bears his name to describe the formation and decay of coherent structures in turbulence. The basis for Burgers' model was the competition between the steepening of a wave due to amplitude-dependent propagation speed and its weakening due to dissipation. This is an ingredient of many nonlinear systems, and has been derived using weakly nonlinear geometric optics, [6]. Other well-known models balance the same nonlinear convective term against dispersion (as in the Korteweg-de Vries equation), or against higher-order effects (for example, the Kuramoto-Sivashinsky equation).

Alternatively, it is possible to study examples with a particular mathematical structure, and then to see what phenomena they might describe. This is quite a different approach from the original one. Some of these prototype conservation laws have quite novel solutions. Our focus here is on equations which lead to delta-shocks and to singular shocks; we will give some background on physical situations which may lead to such equations, and we will discuss delta-shocks and singular shocks themselves.

2 Basic Systems and their Use in Physical Modelling

Starting with the same basic equations and constitutive relations, one can derive several systems of conservation laws which differ in their choice of conserved quantities. We give some examples in this section.

The equations for isentropic gas dynamics in Eulerian form,

$$\rho_t + (\rho u)_x = 0 \tag{1}$$

$$(\rho u)_t + (\rho u^2 + A\rho^\gamma)_x = 0\,, \tag{2}$$

for an ideal gas with pressure density relation $p = A\rho^\gamma$, express conservation of mass and momentum. In many applications these are precisely the quantities one expects will be conserved. The parameter γ, which is the ratio of specific heats, is always greater than unity, and the limit $\gamma \to 1$ does not make physical sense in most applications. However, it is noteworthy that the solutions of (1, 2) are qualitatively the same even if γ assumes the unrealistic value of unity. On the other hand, setting $A = 0$ amounts to neglecting the pressure altogether; while this can be done formally, it is not surprising that the equations that result, called the "pressureless gas dynamics equations", have very different solutions from (1, 2).

Now, carrying out the differentiation in (2), subtracting (1) and dividing by ρ results in another equation which may be put in conservation form,

$$u_t + (\tfrac{1}{2}u^2 + q(\rho))_x = 0, \tag{3}$$

using the *specific enthalpy*

$$q(\rho) = \begin{cases} A\frac{\gamma}{\gamma-1}(\rho^{\gamma-1} - 1)\,, & \gamma > 1 \\ A\log\rho\,, & \gamma = 1 \end{cases} . \tag{4}$$

Equation (3) represents "conservation of velocity". Although velocity is not normally a conserved quantity, this equation is used both by Shandarin and Zeldovich

[17] and by Rosenau [15] in place of (2). (Indeed, Burgers' original model, while not a description of compressible gas dynamics, is an evolution equation for a velocity variable.) Specifically, Shandarin and Zeldovich use (3) in the case $A = 0$, along with (1), to model the emergence of velocity and density inhomogeneities in a model of the universe which ignores gravitation. (See Section II.B.3, 'Sticky dust', in [17]; although viscosity is mentioned, it is set to zero before solving. For a complete solution of this system, see [5].) When $A = 0$ the systems (1, 2) and (1, 3) have only a single characteristic speed, $\lambda = u$, and are formally of parabolic, not hyperbolic, type. (The sole characteristic is linearly degenerate; in the terminology of conservation laws, $r \cdot \nabla\lambda \equiv 0$, where r is the eigenvector of the Jacobian matrix of the system. However, the equations still admit shock formation.) The variable ρ appears only to linear power in either system, and the solution of the Cauchy problem, for bounded measurable data, is not of classical type: a unique solution exists, at least for some initial data, but the expression for ρ contains a Dirac δ-function. A measure-theoretic justification of solutions of this form can be found in Bouchut's paper [1]; see also E, Rykov and Sinai [5].

We note that the appearance of measures as solutions for problems with L^∞ data could be predicted from the explicit solution $\rho(x,t) = \rho_0(x) - tu_0'(x)$, $u(x,t) = u_0(x)$ of the system

$$\rho_t + u_x = 0, \qquad u_t = 0$$

obtained by linearizing (1, 2) or (1, 3) about $\rho = 1$, $u = 0$. The qualitative nature of the solutions of (1, 2) and (1, 3) is the same for either choice of conserved quantity, though solving the system (1, 2) gives rise to more straightforward analysis in [1]. Zheng [20] makes some further observations about parabolically degenerate nonlinear systems.

Shandarin and Zeldovich [17] consider also a model with gravitationally interacting particles, which can be written, in simplified notation [5], as

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ u_t + (\tfrac{1}{2}u^2)_x &= -g_x \\ g_{xx} &= G\rho\,, \end{aligned} \tag{5}$$

(G is a gravitational constant). This system is solved by reducing it to

$$\begin{aligned} h_t + uh_x &= 0 \\ u_t + (u^2/2)_x &= -h \end{aligned}$$

(with $h = g_x$), which is hyperbolic, with a double characteristic, but has the disadvantage of not being in conservation form. By using Volpert's theory of BV spaces, a weak solution is found to this system [5]. The trivial pressure term again has the effect of eliminating acoustic waves.

A system with a form similar to (5) was advanced by Rosenau [15] to exhibit threshold phenomena in the velocity evolution of ion-acoustic waves. Rosenau's system uses the enthalpy function $q(\rho)$ from equation (4):

$$\begin{aligned} n_t + (nu)_x &= 0 \\ u_t + (\tfrac{1}{2}u^2)_x &= -q_x \\ \beta q_{xx} &= \rho - n\,. \end{aligned} \tag{6}$$

In this model, ρ measures the (charge) density of electrons, n the ion density and β is a small parameter, the normalized Debye length. The analogy with (5) breaks down in solving (6), as will be discussed later.

This selection of equations is not meant to be exhaustive. We close the section by drawing attention to the facts that in numerous models an equation represents conservation of velocity, and in some reductions the equations are not all in conservation form.

3 Delta-Shock Waves

As mentioned in the preceding section, some systems with a double eigenvalue, whether or not they have an eigenvector deficiency, exhibit solutions which contain δ-functions or other measure-valued functions, even for bounded initial data. In this section, we discuss a related concept.

The term 'delta-shock waves' was coined by Tan and Zhang in a paper which solved the Riemann problem for a model system of conservation laws in which hyperbolicity fails (because of eigenvalue coincidence and an incomplete set of eigenvectors) on a curve in state space. A prototype is the system

$$u_t + (u^2)_x = 0, \qquad v_t + (uv)_x = 0 \tag{7}$$

first discovered in the Ph.D. dissertation of Dennis Korchinski [12], who solved the Riemann problem using δ-functions and provided some justification for the solutions. A measure-theoretic justification for delta-shocks, along the lines of Bouchut, could be given. In [18], Tan, Zhang and Zheng give a justification based on approximation by viscous equations. The model on which most of their work has focussed is a two-dimensional form of the system:

$$u_t + (u^2)_x + (uv)_y = 0 \quad (8)$$

$$v_t + (uv)_x + (v^2)_y = 0\,, \quad (9)$$

which is formally obtained from the two-dimensional compressible gas dynamics equations by first setting the pressure to zero and ignoring the density equation and then assuming the density to be constant. Thus it is also a sort of zero-pressure gas dynamics, but differs from the Shandarin-Zeldovich model in being hyperbolic for most states. In (7), one of the characteristic speeds is genuinely nonlinear and the other linearly degenerate for all states. Because the δ-functions occur only in the linear family, they can be dealt with by essentially the same means as the degenerate equations of Shandarin and Zeldovich, E, Rykov and Sinai, and Bouchut. The analysis of these models is still incomplete, though, and may be worth pursuing. A feature of systems such as (7) which makes them similar to systems with eigenvector deficiency in an open set is that the eigenvector at points on the coincidence locus in (7) is tangent to the locus. That is, the coincidence locus is a characteristic variable, and this could create a mechanism for resonance. By contrast, some nonstrictly hyperbolic systems with a parabolic degeneracy along a curve which is crossed transversally by the eigenvectors have classical Riemann solutions which do not contain delta-shocks [8, 9]. We return to this later.

4 The Effect of Conserving Different Quantities

We now return to Rosenau's model equations. Following Rosenau, we write $n = \rho + \beta q_{xx}$, invert (4), taking $A = 1$, to obtain $\rho = \rho(q)$ and eliminate n from the first equation of (6) to obtain

$$q_t + uq_x + (\gamma + (\gamma - 1)q)u_x = -\frac{\beta}{\rho'}\Big(q_{xxt} + (uq_{xx})_x\Big) \quad (10)$$

when $\gamma \geq 1$. The term multiplied by β is treated by Rosenau as a perturbation and approximated. The order of differentiation makes this a dispersive rather than dissipative term. However, from the viewpoint of conservation laws we now see that, up to a term that is $\mathcal{O}(\beta)$, n in (6) is the same as ρ in (1, 2) or (1, 3) so the system now derived, consisting of (10) and the second equation of (6), could equally well have been found by adding dispersion to (1, 3). It is interesting that precisely this system, written in the form

$$u_t + uu_x = \sigma_x, \qquad \sigma_t + u\sigma_x = k^2 u_x \tag{11}$$

(so σ, which represents elastic stress, takes the place of $-q$) was extensively studied by Colombeau and Le Roux and others as a prototype nonconservative system (see [4, 14] and the references there). We will return to the question of nonconservative systems.

Since (10) is not in conservation form, Rosenau replaces it, when $\gamma = 1$, by a system of conservation laws, while commenting that the physical interpretation is unclear. The system conserves u (the velocity) and $v = \frac{1}{2}u^2 - q = \frac{1}{2}u^2 - \log\rho$, a convex function of velocity and density. After some algebraic manipulation one obtains

$$\begin{aligned} u_t + (u^2 - v)_x &= 0\,, &(12)\\ v_t + (\tfrac{1}{3}u^3 - u)_x &= 0\,. &(13) \end{aligned}$$

(I have omitted the dispersive term which, after some further approximation, appears as βu_{xxt} in the first equation.)

When $\gamma > 1$, it is possible to obtain an analogous pair of conservation laws; although n and u in Rosenau's model represent density and velocity of ions, they also admit an interpretation as density and velocity in an ordinary γ-law gas. The analogue of (12, 13) when $\gamma > 1$ also conserves u and $v = (1 - \frac{1}{2}\gamma)u^2 - q$:

$$\begin{aligned} u_t + \left(\tfrac{3-\gamma}{2}u^2 - v\right)_x &= 0 &(14)\\ v_t + \left(\tfrac{(3\gamma-5)(\gamma-2)}{6}u^3 - \gamma u + (\gamma-1)uv\right)_x &= 0\,. &(15) \end{aligned}$$

Again, this represents conservation of velocity and a convex function of u and ρ (when $\gamma = 2$ the second conserved quantity is the enthalpy itself).

The second conserved quantity is an "entropy" function in the terminology of conservation laws (a convex function of the variables which is conserved for classical solutions). It differs from the specific energy by a factor ρ, just as velocity differs from momentum.

5 The Use of Generalized Distributions for Nonconservative Systems and Singular Shocks

We can think of (1, 2) with $A = 1$ and $\gamma = 1$, (11) and (12, 13) as three different ways of representing the same equation. Although they appear in the context of different, specific physical models (compressible flow, elasticity, and plasmas, respectively), physics is not offered as a justification for the different forms of the equations. (To be more precise, the elastic model is a simplification of a system which lacks a conservation principle; the plasma model, as we saw, simply uses Burgers' equation for the velocity.)

It is known that such systems have inequivalent weak solutions, even though they have the same classical solutions. A standard example, to fix ideas, is the pair

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0\,, \qquad \left(\tfrac{1}{2}u^2\right)_t + \left(\tfrac{1}{3}u^3\right)_x = 0$$

for which the shock joining the states $u = 1$ and $u = 0$ has speed $s = \frac{1}{2}$ and $s = \frac{2}{3}$ respectively. In applications to conservation laws of the theory of generalized distributions, one states that these equations hold only in the sense of association, [3, 4, 14]. While, in principle, one should use the correct form of the equation, in fact, as we have seen, equations used in qualitative modelling are transformed rather freely.

As we shall now demonstrate, different forms of the system may have qualitatively different solutions. In fact, neither (11) nor (12, 13) can be solved using classical conservation law theory, and neither has been completely solved. However, a theory for Riemann solutions and shock admissibility exists.

In the case of equations in nonconservative form, a theory was developed by Vol'pert [19] and extended by Le Floch and others (see [13]), based on geometric measure theory. At the same time, Colombeau, Le Roux and others (see [3, 4]) have shown that Colombeau's calculus of generalized distributions can be used to give a direct definition of products such as $u\sigma_x$ appearing in (11). Oberguggenberger [14] has reconciled the two approaches. Either approach includes an explicit recognition that the nonconservative form is incomplete as given, and that further conditions, perhaps originating in physics but expressed as mathematical constraints, must be supplied. This is true for the original system also. However, the original system in conservation form will supply a shock speed given left and right states, or will determine that no such solution exists. The nonconservative system requires further information in order to construct shocks.

The third form of the equations, system (12, 13), is exactly the model analysed by Keyfitz and Kranzer [11], as an example of a strictly hyperbolic, genuinely nonlinear system for which the Riemann problem has no classical solution when the left and right states are not close together. We found candidates for nonclassical solutions, although the sense in which they satisfy the equation is still unclear. These solutions are also singular; in fact, v contains a δ-function, like the delta-shocks mentioned earlier, and equation (12) is satisfied in the same sense as the equations of Section 3. However, the function u cannot be constructed as a measure, and requires a more refined space. We exhibited an explicit representation, as a space of weighted measures, in [11]. Another model for the solution space is the class $\mathcal{G}_s$ of generalized distributions [3]. Generalized distributions were used in [11] to supply a template for the singular solutions, which were also shown to converge in a Banach space. Unlike the delta-shocks of Section 3, the singular shocks which are needed to solve (12, 13) are truly nonlinear objects which cannot be defined in the context of classical distribution theory.

Comparing the systems (1, 2) with $\gamma = 1$ and (12, 13), we see that both systems are in conservation form, and are equivalent for smooth solutions, yet one has classical shocks and the other does not. One might conjecture that this is due to the unphysical choice $\gamma = 1$. However, such is not the case. As we have shown, the system (14, 15) is a smooth transformation of (1, 2) when $\gamma > 1$, and we have the following result.

PROPOSITION 1 *In the system (14, 15) for $1 < \gamma < \frac{5}{3}$, the set of states (u_R, v_R) that can be connected to a given state (u_L, v_L) by means of a shock forms a compact set.*

This proposition is easily proved by calculating the Hugoniot locus for shocks. Further investigation of this system shows that its structure is similar to that of the model system (12, 13) for which the Riemann problem was solved using singular shocks in [10]. Hence waves of a similar type will exist. There is an important difference, however: equations (14) and (15) represent a physical system only if $v \leq (1 - \frac{1}{2}\gamma)u^2 + \frac{\gamma}{\gamma-1}$, corresponding to nonnegative densities. If we attempted to construct singular shocks along the lines of [11], they would be limits of approximate solutions, constructed by the method of [10] for example, which contain nonphysical states. (This is not the case when $\gamma = 1$). Thus it would appear that the mathematical solutions which could be constructed in this case do not have a physical interpretation.

6 Conclusions

We have examined a number of model conservation law systems which do not have classical solutions. Since, in one space dimension, strictly hyperbolic systems always have solutions for sufficiently small data, the difficulties arise for one of two reasons: failure of strict hyperbolicity or problems associated with large data. In a number of examples where strict hyperbolicity fails, specifically cases with a parabolic degeneracy on an open set in state space or on a submanifold, the solutions to initial value problems with bounded data contain measure-valued functions. Construction of solutions in such cases requires careful identification of a suitable class of test functions, and application of the Radon-Nikodym theorem. In all the cases of this type which have appeared so far, these solutions are essentially linear and do not require multiplication of distributions.

However, in working with conservation law models, one finds two types of problems which go beyond this, and cannot be handled without some type of nonlinear theory. Both have arisen in problems showing a loss of strict hyperbolicity, but this is probably a coincidence. The first type, discussed only briefly in this paper, consists of quasilinear systems which are not in conservation form. Discontinuities will arise from smooth data in such equations, and continuing the solution beyond the time when shocks form requires new theory. The calculus of generalized distributions has proved to be a valuable technique in this problem.

A second class of problem involves some examples of conservation law systems which do not possess classical weak solutions in the large. The example for which the Riemann problem has been extensively studied shows that any candidate for a weak solution will lie in quite a complicated function space. The theory of generalized distributions provides a useful setting for weak solutions, and, in fact, gave one representation for the solution in [11]. We have no convincing explanation for why some forms of a conservation law system have singular shocks while others do not. Zheng and others have suggested that some notion of "lack of hyperbolicty in the large" may be associated with such systems. However, all forms of the equation have the same characteristic structure, and some do not show the same difficulties with large data. Some time ago, Herbert Kranzer and I found a condition, which we refer to as “opposite variation”, which guarantees existence of Riemann solutions in the large for a system of two conservation laws [7]. The condition states that the two characteristic speeds should vary in opposite directions along the eigenvector corresponding to one of them. Different forms of a system may have different nonlinear properties; however, as it happens, the system (1, 2) with $\gamma = 1$ does not have the opposite variation property, and yet it still has global Riemann solutions. Nonetheless, some refinement of this concept may operate here. In a later paper which studied some examples of nonstrictly hyperbolic systems with a parabolic degeneracy on a submanifold [9], Kranzer and I noted that the examples we could solve all exhibited opposite variation of the characteristic variables across the parabolic line, and that a prototype example with the complementary property, “same variation”, did not have even a local solution. Schaeffer, Schecter and Shearer [16] later solved the Riemann problem for this system, also using singular shocks. As we noted in Section 3, when a characteristic variable coincides with the parabolic line one may expect delta-shocks to figure in the solution. Thus one is tempted to conjecture that singular shocks are a manifestation of a sort of “nonlinear resonance”. At the moment, though, we have no clear physical interpretation of the condition on characteristics, and no explanation of why it does not always lead to singular solutions. These are interesting open questions.

In both problems involving nonservative quasilinear equations and problems where there are no bounded shock solutions, the nonlinear theory of generalized distributions has played a role.

A physical interpretation of singular shocks has not been found, nor is it even clear in what sense the weak solutions satisfy the equation. However, as we have shown in this paper, systems with singular shocks may be closely related (by a nonlinear change of variables) to common, classical systems. Thus, we conclude that conservation laws systems are quite sensitive to the choice of conserved variable, and choices not motivated by physical principles may lead to singular and possibly nonphysical solutions.

7 Acknowledgements

This research was supported in part by the Department of Energy under Grant DE-FG03-94ER25222. It is a pleasure to thank the Erwin Schrödinger Institute in Vienna for their hospitality and the opportunity to visit and pursue this research during the program on *Nonlinear Theory of Generalized Functions* organized by Michael Oberguggenberger.

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