Transonic regular reflection for the Unsteady Transonic Small Disturbance Equation - details of the subsonic solution

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Abstract We study a Riemann problem for the unsteady transonic small disturbance (UTSD) equation which leads to a regular reflection with subsonic flow behind the reflected shock. The problem is written in self-similar coordinates resulting in a free boundary value problem. A solution is found in a neighborhood of the reflection point using the Schauder fixed point theorem and Schauder estimates for the fixed boundary value problems. The study of the fixed boundary value problem applies to a more general class of operators satisfying certain structural conditions.

1 Introduction

We revisit and give a more detailed presentation of [3] by Čanić, Keyfitz and Kim on a solution to a special Riemann problem for the unsteady transonic small disturbance (UTSD) equation. The Riemann initial data consists of two states in the upper half plane $\{(x,y) : x \in \mathbb{R},\, y \geq 0\}$ separated by an incident shock and results in a regular reflection where the flow behind the reflected shock is subsonic. Written in self-similar coordinates $\xi = x/t$ and $\eta = y/t$, this configuration leads to a system which changes type. We find a solution in the hyperbolic part of the domain using the standard theory of one-dimensional conservation laws and the notion of quasi-one-dimensional Riemann problems developed in [1]. Solution in the elliptic part of the domain is described by a free boundary value problem. The free boundary is given by the position of the reflected shock which is, through the Rankine-Hugoniot relations, coupled to the subsonic state behind the shock. The main idea in solving this free boundary value problem is to fix the position of the reflected shock within some bounded set of admissible curves, solve the fixed boundary value problem and then update the position of the reflected shock using the Rankine-Hugoniot relations.

The novelty of our paper is in the study of the fixed boundary value problem. We consider a more general class of fixed boundary value problems for which the operators in the domain and on the boundary satisfy certain structural conditions. The main tool is the theory developed in Gilbarg & Hormander [7], Gilbarg & Trudinger [8], Lieberman [10]-[13], and Lieberman & Trudinger [14].

1.1 Related work

This approach to solving Riemann problems for two-dimensional systems of hyperbolic conservation laws was first developed by Čanić, Keyfitz and Lieberman [2] in a study of nonlinear stability of transonic shocks for the steady transonic small disturbance equation. The ideas have been extended to the cases of regular reflection for the UTSD equation: with a subsonic state behind the reflected shock in [3] and with a supersonic state immediately behind the reflected shock in [4]. A Riemann problem for the nonlinear wave system (NLWS) which leads to Mach reflection is studied in [5].

The main features of this method in studying two-dimensional Riemann problems for a special class of systems of hyperbolic conservation laws (including the UTSD equation, the NLWS, the isentropic compressible gas dynamics equations, etc.) have been presented in Keyfitz [9]. We also mention the earlier work of Chang & Chen [6] in stating the free boundary value problems modeling regular reflection for the adiabatic gas dynamics equations.

1.2 Summary of the paper

In §2 we formulate a Riemann problem for the UTSD equation leading to a transonic regular reflection. We write the problem in self-similar coordinates (ξ, η) and obtain a system which changes type. We find a solution in the hyperbolic part of the domain and the equation of the reflected shock. The free boundary value problem is stated in Theorem 2.1 and the rest of the paper is devoted to finding its solution.

In §3 we change coordinates to $(\rho = \xi + \eta^2/4, \eta)$. We introduce several cut-off functions to ensure that the free boundary value problem in the elliptic part of the domain is well-posed and suitable for applying the theory of second order elliptic equations developed by Lieberman. This modified free boundary value problem is stated in Theorem 3.1. The main idea in finding its solution is to fix the position of the reflected shock within some set $\mathcal{K}$ of admissible curves, to solve the fixed boundary value problem, and to update the position of the reflected shock. This gives a mapping $J : \mathcal{K} \to \mathcal{K}$.

The fixed boundary value problem is studied in §4. We use the results in [7], [8], [10]-[14], which are valid for a more general class of second order boundary value problems as long as the operators in the domain and on the boundary have some desired properties. This observation motivates our study in §4.4 where, instead of considering only the fixed boundary value problem resulting from the transonic regular reflection for the UTSD equation, we consider a class of fixed boundary value problems satisfying certain structural conditions given in §4.3. This more general fixed boundary value problem is stated in Theorem 4.1. This is a nonlinear problem and first we find a solution to its linearized version in §4.4. Using the fixed point theory we solve the nonlinear problem in §4.4.

In §5 we use the Schauder fixed point theorem to show that the map J, defined on the set $\mathcal{K}$, has a fixed point. This completes the proof of Theorem 3.1.

Finally, the conditions under which the solution to the modified problem in Theorem 3.1 solves the original free boundary value problem of Theorem 2.1 are discussed in §6. Since not all of the cut-offs could be removed entirely, a solution to the original free boundary value problem is found only in a neighborhood of the reflection point.

1.3 Acknowledgments

Research of the first two authors was partially supported by the Department of Energy, Grant DE-FG02-03ER25575. The third author was partially supported by the National Science and Foundation under grants NSF FRG DMS-0244343, NSF DMS-0225948 and NSF DMS-0245513.

We appreciate the hospitality of the Fields Institute during K.J.'s research visits and we also thank the Focused Research Grant on Multi-dimensional Compressible Euler Equations for its encouragement and support.

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2 The statement of the free boundary value problem

In this section we formulate for the UTSD equation a Riemann problem resulting in a transonic regular reflection. We study this phenomenon in self-similar coordinates, which yields a system of mixed type. Using the standard one-dimensional theory of hyperbolic conservation laws and the results on quasi-one-dimensional Riemann problems [1], we find a solution in the hyperbolic part of the region in §2.1. We formulate the free boundary problem in Theorem 2.1 and give the outline of its proof in §2.2.

Consider the UTSD equation

$$\begin{aligned} u_t + u\,u_x + v_y &= 0, \\ -v_x + u_y &= 0, \end{aligned} \tag{2.1}$$

where $U := (u, v) : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$. The Riemann initial data (Figure 1a) is given in the upper half plane $\{(x, y) : y \geq 0\}$ and consists of two states

$$U_0 = (0, 0) \quad \text{and} \quad U_1 = (1, -a), \tag{2.2}$$

separated by a half line $x = a\,y$, $y \geq 0$, with a parameter

$$a > \sqrt{2} \tag{2.3}$$

fixed. We impose symmetry across the $x-$axis, meaning

$$u_y = v = 0 \quad \text{along} \quad y = 0. \tag{2.4}$$

We note that the symmetric Riemann data (2.2), (2.4) posed in the upper half-plane is equivalent to the initial data given in three sectors in the full plane, as depicted in Figure

1b. In some parts of this study, it will be more convenient to consider the Riemann problem in the full plane with states $U_0 = (0,0)$, $U_1 = (1,-a)$ and $\overline{U}_1 = (1,a)$, instead of the original problem (2.1), (2.2), (2.4) in the half-plane.

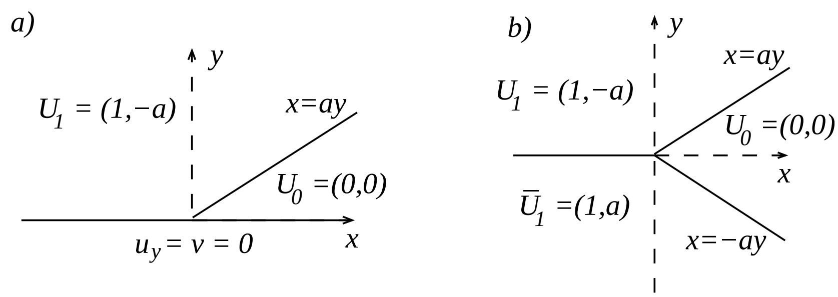

FIGURE 1: The Riemann initial data.

We study the initial-boundary value problem (2.1), (2.2), (2.4) in self-similar coordinates $\xi = x/t$ and $\eta = y/t$. From (2.1) we get

$$\begin{aligned} (u-\xi)\,u_\xi - \eta\,u_\eta + v_\eta &= 0, \\ -v_\xi + u_\eta &= 0. \end{aligned} \tag{2.5}$$

It is clear that when the system (2.5) is linearized about a constant state $U = (u, v)$, the system is hyperbolic outside and elliptic inside the sonic parabola

$$P_U : \quad \xi + \frac{\eta^2}{4} = u. \tag{2.6}$$

Using the Rankine-Hugoniot conditions, the initial discontinuity $x = a\,y$ propagates as a shock given, in (ξ,η)-coordinates, by the equation $\xi = a\,\eta + a^2 + 1/2$. The initial-boundary value problem (2.1), (2.2), (2.4) can be replaced by the system (2.5) with the following boundary conditions

$$\begin{array}{lll} U(\xi,\eta) = U_0 & \text{on} & \{(\xi,\eta) : \xi + \eta^2/4 = C, \xi > a\,\eta + a^2 + 1/2, \eta > 0\}, \\ U(\xi,\eta) = U_1 & \text{on} & \{(\xi,\eta) : \xi + \eta^2/4 = C, \xi < a\,\eta + a^2 + 1/2, \eta > 0\}, \\ u_\eta = v = 0 & \text{on} & \eta = 0, \end{array} \tag{2.7}$$

where C is a large positive constant. In the full plane, the equivalent boundary conditions are

$$\begin{aligned} U(\xi,\eta) = U_0 \text{ on } \{(\xi,\eta) : \xi + \eta^2/4 = C,\ &(\xi > a\,\eta + a^2 + 1/2,\ \eta \geq 0) \text{ or} \\ &(\xi < -a\eta + a^2 + 1/2,\ \eta \leq 0)\}, \\ U(\xi,\eta) = U_1 \text{ on } \{(\xi,\eta) : \xi + \eta^2/4 = C,\ &\xi < a\,\eta + a^2 + 1/2,\ \eta > 0\}, \\ U(\xi,\eta) = \overline{U}_1 \text{ on } \{(\xi,\eta) : \xi + \eta^2/4 = C,\ &\xi > -a\,\eta + a^2 + 1/2, \eta < 0\}. \end{aligned}$$

Let us further denote by P_0 and P_1 the sonic parabolas corresponding to the states U_0 and U_1, respectively. Notice that the sonic parabola for $\overline{U}_1$ coincides with P_1.

2.1 The solution in the hyperbolic region. The position of the reflected shock

In this part of the paper we briefly sketch the solution to the initial-boundary value problem (2.5), (2.7) in the hyperbolic region using the notion of quasi-one-dimensional Riemann problems and we derive the position of the reflected shock. More details on how to solve a quasi-one-dimensional Riemann problem for the UTSD equation can be found in [1].

Let us denote the incident shock separating states U_0 and U_1 by

$$S \,:\, \xi = a\,\eta + a^2 + \frac{1}{2},\, \eta \geq 0.$$

By the choice of the parameter a (see (2.3)), the shock S does not interact with the parabola P_1. Let us denote by

$$\Xi_a := (\xi_a, 0) = \left(a^2 + \frac{1}{2}, 0\right) \tag{2.8}$$

the point where the shock hits the $\xi-$axis. We solve the quasi-one-dimensional Riemann problem at Ξ_a (for details, see [1]). The states on the left and on the right in this Riemann problem are $\overline{U}_1 = (1, a)$ and $U_1 = (1, -a)$, respectively. Since $a > \sqrt{2}$ there are two solutions to this problem, known as *weak* and *strong regular reflection*, each consisting of two shocks, one below and one above the ξ-axis. The intermediate states for the two solutions are given by

$$U_R = (1 + a^2 - a\sqrt{a^2 - 2}, 0) \quad \text{ and } \quad U_F = (1 + a^2 + a\sqrt{a^2 - 2}, 0). \tag{2.9}$$

Here, the subscripts R and F stand for *reflected* and *fast reflected*. Let P_R and P_F denote the sonic parabolas for the states U_R and U_F, respectively. It is clear that the point Ξ_a is inside P_F for any choice of parameter $a > \sqrt{2}$. However, Ξ_a is inside P_R only if $a \in \left(\sqrt{2}, \sqrt{1 + \sqrt{5}/2}\right)$. In this paper we are interested in the case when the point of interaction of the shock S with the $\xi-$axis is inside the sonic parabola for the solution U at this point; namely, we study a *transonic regular reflection*. We denote the value of U at Ξ_a by $U_* = (u_*, v_*)$, and we choose

$$U_* = U(\Xi_a) := \begin{cases} U_R \text{ or } U_F, & \sqrt{2} < a < \sqrt{1 + \sqrt{5}/2} \\ U_F, & a \geq \sqrt{1 + \sqrt{5}/2}. \end{cases} \tag{2.10}$$

Further, we denote the reflected shock by S' and the sonic parabola for the state U_* by P_*. Since the point Ξ_a is within the subsonic region determined by P_*, the shock S' is transonic (Figure 2). By causality, S' cannot cross P_*. The asymptotic analysis in [3] shows that S' approaches the sonic parabola P_1 as $\xi \to -\infty$.

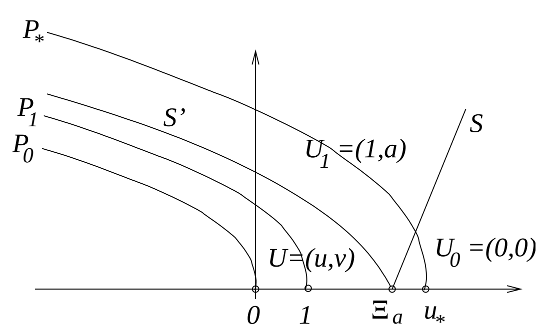

FIGURE 2: The position of the incident shock S and the reflected shock S' in (ξ, η)-coordinates.

REMARK If the reflected shock S' were rectilinear, its equation would be given by $\xi = k_*\eta + a^2 + 1/2$. Here, $k_* = k_R$ in the case of the solution with the intermediate state U_R, and $k_* = k_F$ when the intermediate state is U_F, with

$$k_R = -\frac{1}{a - \sqrt{a^2 - 2}} \quad \text{and} \quad k_F = -\frac{1}{a + \sqrt{a^2 - 2}}. \tag{2.11}$$

Let us assume that the reflected transonic shock S' is given by equation

$$\xi = \xi(\eta),\ \eta \geq 0. \tag{2.12}$$

We denote a solution of the system (2.5) behind the shock S' by $U = (u, v)$. Hence, the curve (2.12) satisfies the Rankine-Hugoniot condition with states U and U_1:

$$\frac{d\xi}{d\eta} = -\frac{[v]}{[u]} \quad \text{and} \quad \frac{d\xi}{d\eta} = \frac{[\frac{1}{2}u^2 - \xi u]}{[v - \eta u]}, \tag{2.13}$$

where $[\cdot]$ denotes the jumps across the shock. We eliminate v in (2.13) and obtain

$$\frac{d\xi}{d\eta} = -\frac{\eta}{2} - \sqrt{\xi + \frac{\eta^2}{4} - \frac{u+1}{2}}. \tag{2.14}$$

The negative sign is appropriate here. Furthermore, by eliminating $d\xi/d\eta$ in (2.13), we obtain the following relation between u and v along the shock S'

$$v = -a + (u - 1)\left(\frac{\eta}{2} + \sqrt{\xi + \frac{\eta^2}{4} - \frac{u+1}{2}}\right). \tag{2.15}$$

2.2 The statement of the main result and the outline of its proof

In this section we give the formulation of the free boundary value problem arising in the transonic regular reflection for the UTSD equation presented above.

First, we restrict the unbounded domain behind the reflected shock S'. More precisely, we introduce a cut-off parameter $\eta^* > 0$, which is fixed throughout the paper. We define $V := (\xi(\eta^*), \eta^*)$ and $W := (\xi(\eta^*), 0)$, the closed vertical line segment $\sigma := [V, W]$, the open horizontal line segment $\Sigma_0 := (W, \Xi_a)$ and the set $\Sigma := \{(\xi(\eta), \eta) : \eta \in (0, \eta^*)\}$, where $\xi(\eta)$, $\eta \geq 0$, is the unknown curve describing the position of the reflected shock S' (recall (2.12)). Further, we denote by Ω the domain whose boundary is $\partial\Omega = \Xi_a \cup \Sigma \cup \sigma \cup \Sigma_0$ (Figure 3a).

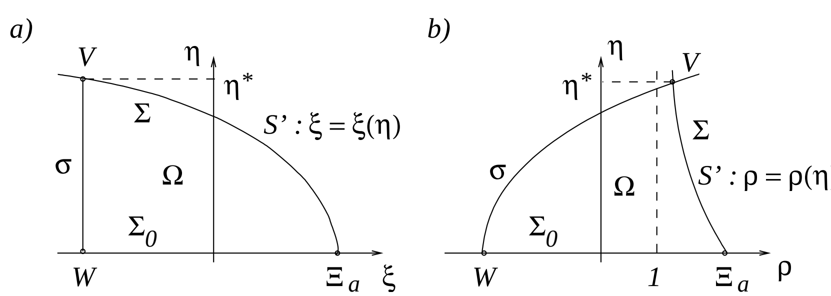

FIGURE 3: The domain Ω and its boundary.

Next, we impose a Dirichlet condition $u = f$ along the vertical boundary σ. We assume that $f : \mathcal{R} \to \mathbb{R}$ is in the Holder space H_γ for a parameter $\gamma \in (0, 1)$ to be determined later (for the definitions of Holder spaces see §4.1), where $\mathcal{R}$ is an open set containing σ, defined in §4.2. Moreover, we impose the following two conditions

$$\begin{aligned} &1+\epsilon_* \leq f(\xi,\eta) \leq u_*, \qquad (\xi,\eta) \in \mathcal{R}, \\ &f(\xi(\eta^*),\eta) > \xi(\eta^*) + \tfrac{\eta^2}{4},\ \eta \in [0,\eta^*], \end{aligned} \tag{2.16}$$

for an arbitrary parameter $\epsilon_* \in (0, u_* - 1)$, which is fixed throughout the paper.

REMARK Note that Σ, Σ_0, σ, the domain Ω, the size of the angles at the corners V and Ξ_a, and the boundary data f along σ depend on the unknown position of the reflected shock $S' : \xi = \xi(\eta)$. However, we will find $\xi(\eta)$ within a certain bounded set $\mathcal{K}$ (whose bounds depend only on $a > \sqrt{2}$, $\eta^* > 0$ and $\epsilon_* \in (0, u_* - 1)$) giving a priori bounds on Σ, and therefore also on σ, Σ_0, Ω and the angles at V and Ξ_a. In particular, the second condition in (2.16) will make sense. For the definition of the set $\mathcal{K}$, see §4.2.

With this notation we prove

Theorem 2.1 *(Free boundary value problem)*
Let $a > \sqrt{2}$, $\eta^* > 0$ and $\epsilon_* \in (0, u_* - 1)$, with u_* specified by (2.10)*, be given. Let f be any function in H_γ such that the inequalities* (2.16) *hold. There exists $\gamma_0 > 0$ depending on the parameters a, η_* and ϵ_*, such that for any $\gamma \in (0, \min\{\gamma_0, 1\})$ and $\alpha_{\mathcal{K}} = \gamma/2$, the problem*

$$\left.\begin{aligned} (u-\xi)u_\xi - \eta u_\eta + v_\eta &= 0 \\ -v_\xi + u_\eta &= 0 \end{aligned}\right\} \quad in \quad \Omega, \tag{2.17}$$

$$\left.\begin{aligned} \frac{d\xi}{d\eta} &= -\frac{\eta}{2} - \sqrt{\xi + \frac{\eta^2}{4} - \frac{u+1}{2}} \\ \frac{d\xi}{d\eta} &= -\frac{v+a}{u-1} \end{aligned}\right\} \quad on \quad \Sigma, \tag{2.18}$$

$$\xi(0) = \xi_a, \tag{2.19}$$

$$v = u_\eta = 0 \quad on \quad \Sigma_0, \tag{2.20}$$

$$u = f \quad on \quad \sigma, \tag{2.21}$$

$$u(\Xi_a) = u_*, \tag{2.22}$$

has a solution $u, v \in H^{(-\gamma)}_{1+\alpha_*}$ in a finite neighborhood of Ξ_a, for all $\alpha_* \in (0, \alpha_{\mathcal{K}}]$. Moreover, the curve $\xi = \xi(\eta)$, $\eta \in (0, \eta^*)$, giving the location of the free boundary Σ, satisfies $\xi \in H_{1+\alpha_{\mathcal{K}}}$.

The Holder spaces are defined in §4.1. The outline of the proof of this theorem and the rest of the paper is as follows.

First, we change coordinates and consider the problem (2.17)-(2.22) in the more convenient (ρ, η)-coordinate system in §3.1. In order to use the elliptic theory by Gilbarg, Lieberman and Trudinger, we reformulate the problem using a second-order free boundary value problem for u and an equation for v in terms of u. We modify the problem so that it is strictly elliptic and well-defined by introducing several cut-off functions in §3.3.

The main idea in solving this modified second-order free boundary value problem for $u(\rho, \eta)$ is: (1) fix the position of the reflected shock within a certain bounded set $\mathcal{K}$ of admissible curves, (2) find a solution of the fixed modified boundary value problem, and (3) update the position of the shock curve. This defines a mapping $J : \mathcal{K} \to \mathcal{K}$ for which we show there is a fixed point in §5.

Given a shock curve within the set $\mathcal{K}$, finding a solution to the fixed modified boundary value problem for $u(\rho, \eta)$ is a challenging task completed in §4. As already mentioned in the introduction, this part of our paper does not depend on the specific form of the fixed boundary value problem arising from the study of the UTSD equation, i.e., the results in §4 apply to a more general class of operators satisfying certain structural conditions.

In §6 we discuss whether and how we could remove the cut-off functions introduced in §3.3 and we complete the proof of Theorem 2.1.

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3 The modified problem

In this section we reformulate the free boundary value problem stated in Theorem 2.1 so that we can solve it using the techniques developed by Gilbarg, Lieberman and Trudinger. We write the problem (2.17)-(2.22) as a second order problem for u and an equation for v. Instead of imposing the Rankine-Hugoniot conditions along the free boundary, we derive an oblique derivative boundary condition for u along Σ and a shock evolution equation. To make sure that the second order problem for u is strictly elliptic, that the operator describing the boundary condition along Σ is strictly oblique and that the shock evolution equation is well-defined, we introduce several auxiliary cut-off functions. This modified problem is stated in Theorem 3.1.

3.1 The (ρ, η)-coordinate system

We define a new variable $\rho = \xi + \frac{\eta^2}{4}$ and in the rest of the paper we work in the (ρ, η)-coordinate system. For simplicity, we use the same notation for the domain Ω and its boundary in the (ρ, η)-coordinates as we do in §2.2 in the (ξ, η)-coordinates (Figure 3b). Under this change of variables, the system (2.5) becomes

$$\begin{aligned} (u-\rho)u_\rho - \tfrac{\eta}{2}u_\eta + v_\eta &= 0, \\ \tfrac{\eta}{2}u_\rho - v_\rho + u_\eta &= 0, \end{aligned} \tag{3.1}$$

the equation (2.14) implies

$$\frac{d\rho}{d\eta} = -\sqrt{\rho - \frac{u+1}{2}}, \tag{3.2}$$

and from (2.15) we have

$$v = -a + (u-1)\left(\frac{\eta}{2} + \sqrt{\rho - \frac{u+1}{2}}\right). \tag{3.3}$$

On the other hand, by eliminating v in (3.1) we obtain the second order equation for u

$$\left((u-\rho)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = 0, \tag{3.4}$$

and, from the first equation in (3.1), we can recover v in terms of u as

$$v(\rho, \eta) = \int_0^\eta \left\{\frac{y}{2}u_y - (u-\rho)u_\rho\right\} dy. \tag{3.5}$$

3.2 The oblique derivative boundary condition along the reflected shock

A condition of the form

$$\beta \cdot \nabla u = 0 \tag{3.6}$$

holds along the reflected shock S'. Here, $\nabla u = (u_\rho, u_\eta)$, $\beta = \beta(u, \rho, \rho') \in \mathbb{R}^2$ and $\rho' = d\rho/d\eta$. To obtain this, we differentiate the equation (3.3) along the shock S' and use equations (3.1) to express the derivatives v_ρ and v_η in terms of u_ρ and u_η (for details of this calculation, see [3]). We obtain

$$\beta = \left(\rho' \left\{\frac{7u+1}{8} - \rho\right\}, \frac{5u+3}{8} - \rho\right). \tag{3.7}$$

3.3 Formulation of the modified free boundary value problem

In this section we reformulate the free boundary value problem (2.17)-(2.22) in (ρ, η) coordinates as a second-order elliptic free boundary value problem for $u(\rho, \eta)$ (using the equation (3.4)).

In §2.1, §3.1 and §3.2 we have shown that if $U = (u, v)$ is a solution to (3.1) in Ω, then the Rankine-Hugoniot condition (2.13) along the reflected shock S' implies the equation (3.2) for the position of S' and the oblique derivative relation (3.6) with β given by (3.7). The operations under which we derived (3.2) and (3.6) from (2.13) can be reversed up to a constant and, hence, if $U = (u, v)$ satisfies (3.2) and (3.6), and if (2.13) holds at one point on the reflected shock S', then the Rankine-Hugoniot condition (2.13) holds at each point along S'. Our idea here is that instead of imposing the condition (2.13) along the reflected shock, we require that the shock curve $\rho(\eta)$, $\eta \geq 0$, satisfies the differential equation (3.2) with an initial condition $\rho(0) = \xi_a$, and that the solution $u(\rho, \eta)$ satisfies the oblique derivative boundary condition (3.6) along the free boundary $\Sigma = \{(\rho(\eta), \eta) : \eta \in (0, \eta^*)\}$.

To study the second-order equation (3.4) in the domain Ω, we introduce three cut-off functions: a function ϕ to ensure that (3.4) is strictly elliptic, ψ to ensure that the shock evolution equation (3.2) is well-defined, and a function χ to ensure that the vector β in (3.6) is nowhere tangential to Σ.

Let us introduce the operator Q by

$$Q(u) := \left((u-\rho)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = (u-\rho)u_{\rho\rho} + u_{\eta\eta} - \frac{u_\rho}{2} + u_\rho^2.$$

To ensure strict ellipticity, we replace Q by the operator

$$\begin{aligned}\tilde{Q}(u) :=& \left(\phi(u-\rho)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} \\ =& \phi(u-\rho)u_{\rho\rho} + u_{\eta\eta} + \left(\frac{1}{2} - \phi'(u-\rho)\right)u_\rho + \phi'(u-\rho)u_\rho^2.\end{aligned} \tag{3.8}$$

Here, ϕ is a function given by

$$\phi(x) = \begin{cases} \delta, & x < \delta \\ x, & x \geq \delta, \end{cases} \tag{3.9}$$

for some positive δ to be specified in §6. Since we will need ϕ' to be continuous in our study, we modify ϕ in a neighborhood of $x = \delta$ to be smooth and such that $\phi'(x) \in [0, 1]$, for all $x \in \mathbb{R}$. Note that the operator $\tilde{Q}$ is strictly elliptic since

$$\lambda := \min\{\phi(u-\rho), 1\} \geq \min\{\delta, 1\} > 0. \tag{3.10}$$

After we derive a priori bounds on a solution u to the problem $\tilde{Q}(u) = 0$ in Ω (see Lemma 4.2), we will show that the operator $\tilde{Q}$ is also uniformly elliptic, i.e., that the ellipticity ratio of $\tilde{Q}$ is bounded from above uniformly in u and $(\rho, \eta) \in \Omega$ (see Proposition 1).

To ensure that the nonlinear shock evolution equation (3.2) is well-defined we replace it by

$$\frac{d\rho}{d\eta} = -\sqrt{\psi\left(\rho - \frac{u+1}{2}\right)}, \tag{3.11}$$

with a function ψ given by

$$\psi(x) = \begin{cases} \delta_*, & x < \delta_* \\ x, & x \geq \delta_*. \end{cases} \tag{3.12}$$

Here, $\delta_* > 0$ is a parameter to be chosen in §5. We will need ψ' to be continuous, and for that we modify ψ to be smooth in a neighborhood of $x = \delta_*$.

Finally, we define the operator N by

$$N(u) := \beta \cdot \nabla u, \tag{3.13}$$

where $\beta = \beta(u, \rho, \rho')$ is given by (3.7). Let $\nu := (-1, \rho')/\sqrt{1 + (\rho')^2}$ denote the unit inner normal to the boundary Σ. We compute

$$\beta \cdot \nu = \frac{-\rho'(\eta)\,(u-1)}{4\sqrt{1 + (\rho')^2}}.$$

The definition of ψ implies $\rho'(\eta) \leq -\sqrt{\delta_*} < 0$, for all $\eta \in [0, \eta_*]$, and therefore $\beta \cdot \nu = 0$ holds only if $u = 1$. Let us introduce a function $\chi : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\chi(u, \rho, \rho') = \begin{cases} (\rho' \{1 + 7\epsilon_*/8 - \rho\}, 1 + 5\epsilon_*/8 - \rho), & u < 1 + \epsilon_* \\ \beta(u, \rho, \rho'), & u \geq 1 + \epsilon_*, \end{cases} \tag{3.14}$$

where ϵ_* is the same positive parameter as in (2.16). As mentioned in Remark 2.2, we will assume that the reflected shock curve belongs to a certain admissible set of curves $\mathcal{K}$ (see §4.2), imposing a priori bounds on both $\rho(\eta)$ and $\rho'(\eta)$, $\eta \in (0, \eta^*)$, in terms of fixed parameters a, η^* and ϵ_*. This implies

$$\chi \cdot \nu \geq \min\left\{ \frac{1 + \sqrt{\delta_*}}{\sqrt{\xi_a}}, \frac{\sqrt{\delta_*}\,\epsilon_*}{4\sqrt{\xi_a}} \right\} > 0, \quad \text{for all } u \in \mathbb{R} \text{ and } \rho \in \mathcal{K}. \tag{3.15}$$

We define the modified operator

$$\tilde{N}(u) := \chi \cdot \nabla u, \tag{3.16}$$

which is by (3.15) strictly oblique. After we show uniform a priori bounds on a solution u to the problem $\tilde{Q}(u) = 0$ in the domain Ω (see Lemma 4.2), we will show that in fact $\chi = \beta$, i.e., that the cut-off function χ can be removed and that we have $\tilde{N} = N$. Moreover, we will find a uniform lower bound on the obliqueness constant for the operator N, which will imply that the operator N is uniformly oblique (see Proposition 1).

We prove the following theorem for the modified free boundary problem and in §6 we discuss the removal of the remaining cut-off functions ϕ and ψ and we deduce Theorem 2.1.

Theorem 3.1 *(Modified free boundary value problem)*
Let $a > \sqrt{2}$, $\eta^* > 0$ and $\epsilon_* \in (0, u_* - 1)$ be given, and let $\delta > 0$ be arbitrary. Let f be any function in H_γ such that inequalities (2.16) hold. There exists $\gamma_0 > 0$, depending on a, η_*, ϵ_* and δ, such that for any $\gamma \in (0, \min\{\gamma_0, 1\})$ and $\alpha_{\mathcal{K}} = \gamma/2$, the problem

$$\tilde{Q}(u) = \left(\phi(u-\rho)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = 0 \quad \text{in} \quad \Omega, \tag{3.17}$$

$$\frac{d\rho}{d\eta} = -\sqrt{\psi\left(\rho - \frac{u+1}{2}\right)} \quad \text{on} \quad \Sigma, \tag{3.18}$$

$$\rho(0) = \xi_a, \tag{3.19}$$

$$N(u) = \beta \cdot \nabla u = 0 \quad \text{on} \quad \Sigma, \tag{3.20}$$

$$u_\eta = 0 \quad \text{on} \quad \Sigma_0, \tag{3.21}$$

$$u = f \quad \text{on} \quad \sigma, \tag{3.22}$$

$$u(\Xi_a) = u_*, \tag{3.23}$$

has a solution $u \in H^{(-\gamma)}_{1+\alpha_*}$*, for all* $\alpha_* \in (0, \alpha_{\mathcal{K}}]$*. The function* $\rho(\eta)$*,* $\eta \in (0, \eta^*)$*, describing the position of the reflected shock satisfies* $\rho \in H_{1+\alpha_{\mathcal{K}}}$*.*

---

4 The fixed boundary value problem

The goal of this section is to fix the function $\rho = \rho(\eta)$, $\eta \in (0, \eta^*)$, describing the free boundary Σ, within a certain set of admissible functions and to solve the nonlinear fixed boundary problem (3.17), (3.20)-(3.23).

In §4.1 we recall the definitions of Holder norms and spaces that we use in this paper. More details can be found in [7] and in §4 of [8]. We define the set $\mathcal{K}$ of admissible curves in §4.2, and in §4.4 we fix $\rho \in \mathcal{K}$ and consider the fixed boundary value problem. We remark that the results in section §4.4 rely heavily on the study of the second order elliptic mixed boundary value problems (*"mixed"* meaning that we impose different types of boundary conditions - Dirichlet and oblique derivative) in Gilbarg & Trudinger [8], Lieberman [10]-[13] and Lieberman & Trudinger [14]. However, their results do not depend on the particular form of the elliptic operator $\tilde{Q}$ and the oblique derivative boundary operator $\tilde{N}$, defined in equations (3.8) and (3.16), as long as these operators are strictly elliptic and strictly oblique, respectively. With this in mind, we consider a more general class of boundary value problems satisfying certain structural conditions. These conditions are given in §4.3. We solve the linearized version of the problem in §4.4 and then we use a fixed point theorem to solve the nonlinear problem in §4.4.

4.1 Holder norms and Holder spaces

Let $S \subseteq \mathbb{R}^2$ be an open set and let $u : S \to \mathbb{R}$. We define the *supremum norm* for u on the set S to be

$$|u|_{0;S} := \sup_{x \in S} |u(x)|.$$

For $\alpha \in (0, 1)$ we define the *Holder seminorm with exponent* α as

$$[u]_{\alpha;S} := \sup_{x,y \in S,\, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

and the *Holder norm with exponent* α as

$$|u|_{\alpha;S} := |u|_{0;S} + [u]_{\alpha;S}.$$

Let k be a positive integer and $\alpha \in (0, 1)$. We define the $(k + \alpha)$-*Holder norm* to be

$$|u|_{k+\alpha;S} := \sum_{j=0}^{k} |D^j u|_{0;S} + [D^k u]_{\alpha;S},$$

where D^j denotes the $j-$th order derivatives

$$\left\{ \frac{\partial^j}{\partial^{j_1} x\, \partial^{j_2} y} \, : \, j = j_1 + j_2,\, j_1, j_2 \geq 0 \right\}.$$

The space of functions for which the $(k + \alpha)$-Holder norm on the set S is finite is denoted by $H_{k+\alpha;S}$.

REMARK For the boundary condition on σ, we assume $u = f \in H_{\gamma;\mathcal{R}}$, where $\gamma \in (0,1)$. To simplify our notation, we write H_γ. Further, we show in Theorem 2.1 (also in Theorem 3.1) that the function $\xi(\eta)$ (or, equivalently, $\rho(\eta)$) is in the Holder space $H_{1+\alpha_\mathcal{K};(0,\eta^*)}$, where $\alpha_\mathcal{K} \in (0,1)$. For simplicity, we write $H_{1+\alpha_\mathcal{K}}$.

Further, let $T \subseteq \partial S$. For a fixed $\delta > 0$ we define the set

$$S_{\delta;T} := \{x \in S \,:\, \mathrm{dist}(x,T) > \delta\},$$

and for $a > 0$ and b such that $a - b \geq 0$, we define the *weighted interior norm* by

$$|u|^{(-b)}_{a;\overline{S}\setminus T} := \sup_{\delta>0} \delta^{a-b} |u|_{a;S_{\delta;T}}. \tag{4.1}$$

The space of functions on the set S for which the weighted interior norm (4.1) is finite is denoted by $H^{(-b)}_{a;\overline{S}\setminus T}$.

REMARK In our study, the domain of interest is Ω and the distinguished part of the boundary is $\mathbf{V} := \{V, W, \Xi_a\}$, where V, W and Ξ_a are the corners introduced in §2.2. To simplify our notation instead of $H^{(-\gamma)}_{1+\alpha;\overline{\Omega}\setminus\mathbf{V}}$ we write $H^{(-\gamma)}_{1+\alpha}$.

REMARK

- If $0 < a' < a$, it is easy to show that $[u]_{a'} \leq C\,[u]_a$, for a constant C depending on a', a and the diameter of the domain S.
- If $0 < a' < a$, $0 < b' < b$, $a - b \geq 0$ and $a' - b' \geq 0$, we have ([7]): a bounded sequence in $H^{(-b)}_a$ is precompact in $H^{(-b')}_{a'}$, and there exists a constant C, independent of u, such that $|u|^{(-b')}_{a'} \leq C\,|u|^{(-b)}_a$.

4.2 Definition of the set $\mathcal{K}$ of admissible curves

We consider the Banach space $H_{1+\alpha_\mathcal{K}}$, as in Remark 4.1, where $\alpha_\mathcal{K} \in (0,1)$ is a parameter which will be specified in §5. The admissible set $\mathcal{K}$ is defined so that the curve $\rho(\eta)$, $\eta \in [0,\eta^*]$, is in $\mathcal{K}$ if and only if the following four conditions hold

- *smoothness*: $\rho \in H_{1+\alpha_\mathcal{K}}$,
- *initial conditions*:
$$\rho(0) = \xi_a \quad \text{and} \quad \rho'(0) = k_*, \tag{4.2}$$
where ξ_a and k_* are given by (2.8) and (2.11), respectively,
- *monotonicity*:
$$-\sqrt{\xi_a - 1} \leq \rho'(\eta) \leq -\sqrt{\delta_*}, \quad \text{for all } \eta \in (0,\eta^*), \tag{4.3}$$
with δ_* (see also (3.12)) to be specified in §5,
- *boundedness*:
$$\rho_L(\eta) \leq \rho(\eta) \leq \rho_R(\eta), \quad \text{for all } \eta \in [0,\eta^*], \tag{4.4}$$
where the functions ρ_L and ρ_R will be also given in §5.

REMARK The parameter δ_* and the curves ρ_L and ρ_R will be given in terms of $a > \sqrt{2}$, $\eta^* > 0$ and $\epsilon_* \in (0, u_* - 1)$. By the definition (5.3) of δ_*, we have $\sqrt{\delta_*} < 1 < \sqrt{\xi_a - 1}$, for $a > \sqrt{2}$, and so the condition (4.3) makes sense.

REMARK Note that Ω, σ and Σ_0 depend on the choice of the curve $\rho \in \mathcal{K}$ describing the boundary Σ. Hence, the Holder estimates we derive in §4.4, which depend on the size of the domain Ω and its boundary, also depend on ρ. However, the set $\mathcal{K}$ is bounded in terms of the fixed parameters a, η^* and ϵ_*, implying a priori bounds on Σ, Ω, σ and Σ_0. Therefore, our estimates which depend on the size of Ω or the parts of $\partial\Omega$ will be uniform in $\rho \in \mathcal{K}$. Furthermore, the monotonicity property (4.3) implies that the domain Ω satisfies the exterior cone condition defined in [8], page 203, and that the angles of Ω at the corners V and Ξ_a are bounded both from below and from above uniformly in $\rho \in \mathcal{K}$.

REMARK We may also define the set $\mathcal{R}$ in terms of the bounds on ρ_L and ρ_R.

4.3 Structural conditions

As already remarked, once the curve $\rho \in \mathcal{K}$ describing the boundary Σ is fixed, the techniques for solving the nonlinear fixed boundary problem (3.17), (3.20)-(3.23) do not depend on the specific definition of the operator $\tilde{Q}$ in Ω nor on the specific definition of the boundary conditions along $\partial\Omega$. In this section we define a more general class of fixed boundary value problems which we will solve in §4.4.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and connected set as in Figure 3b, so that $\partial\Omega = \Sigma \cup \Xi_a \cup \Sigma_0 \cup \sigma$, where Σ_0 is an open line segment aligned with the ξ-axis, Σ is given by an arbitrary curve $\rho \in \mathcal{K}$, where $\mathcal{K}$ is defined as in §4.2, Ξ_a is a corner and σ is a closed set. We assume that $\tilde{\Sigma} := \Sigma \cup \Sigma_0$ has an inner normal ν at each point and that $\tilde{\Sigma}$ and $\partial\Omega \setminus \tilde{\Sigma} = \sigma \cup \Xi_a$ meet at the set of corners $\mathbf{V}$.

We consider the boundary value problem

$$\begin{array}{lll} \tilde{Q}(u) = 0 & \text{in} & \Omega, \\ \tilde{N}(u) = 0 & \text{on} & \tilde{\Sigma} = \Sigma \cup \Sigma_0, \\ u = \tilde{f} & \text{on} & \partial\Omega \setminus \tilde{\Sigma} = \sigma \cup \Xi_a. \end{array} \tag{4.5}$$

The operators $\tilde{Q}$ and $\tilde{N}$ are given by

$$\tilde{Q}(u) := \sum_{i,j} a_{ij}(u, \rho, \eta) D^{ij} u + \sum_i b_i(u, \rho, \eta) D^i u + \sum_{i,j} c_{ij}(u, \rho, \eta) D^i u D^j u, \tag{4.6}$$

and

$$\tilde{N}(u) = \chi(u, \rho', \rho, \eta) \cdot \nabla u. \tag{4.7}$$

The function $\tilde{f}$ is defined on $\mathcal{R} \cup \Xi_a$, where $\mathcal{R}$ is an open set containing σ. We assume that $\tilde{f}$ is in the Holder space $H_{\gamma;\mathcal{R}}$, for a parameter $\gamma \in (0,1)$ to be determined later, and that

$$m_1 \le \tilde{f} \le m_2 \quad \text{and} \quad \tilde{f} > \rho \text{ on } \sigma \tag{4.8}$$

hold for constants $m_1, m_2 \ge 0$, independent of $\rho \in \mathcal{K}$, $\tilde{f}$ and u. Further, we impose the following structural conditions.

- The coefficients a_{ij}, b_i and c_{ij} are in C^1, and for a fixed curve $\rho \in \mathcal{K}$ we have $\chi_i \in C^2$.
- The operator $\tilde{Q}$ is strictly elliptic, meaning

$$\lambda \ge C_1 > 0, \quad \text{for all } u \text{ and } \rho \in \mathcal{K}, \tag{4.9}$$

where λ denotes the smallest eigenvalue of the operator $\tilde{Q}$. Moreover, we assume a bound on the ellipticity ratio of the form

$$\frac{\Lambda}{\lambda} \leq C_2(|u|_0), \qquad \text{for all } u \text{ and } \rho \in \mathcal{K}, \tag{4.10}$$

where $C_2(|u|_0)$ is a continuous function on $\mathbb{R}^+$. Here, Λ denotes the maximum eigenvalue of $\tilde{Q}$.

- The operator $\tilde{N}$ is strictly oblique, i.e.,

$$\chi \cdot \nu \geq C_3 > 0, \qquad \text{for all } u \text{ and } \rho \in \mathcal{K}. \tag{4.11}$$

Also,

$$|\chi| \leq C_4(|u|_0), \qquad \text{for all } u \text{ and } \rho \in \mathcal{K}, \tag{4.12}$$

holds, where $C_4(|u|_0)$ is a continuous function on $\mathbb{R}^+$.

- For any solution u to the equation $\tilde{Q}(u) = 0$ in Ω we have

$$0 \leq \sum_{i,j} c_{ij}(u, \rho, \eta) D^i u D^j u, \tag{4.13}$$

and there exist $\mu_0, \Phi \in \mathbb{R}$, independent of u, such that

$$|\sum_{i,j} a_{ij}(u, \rho, \eta) D^{ij} u| \leq \lambda \left(\mu_0 \sum_i |D^i u|^2 + \Phi \right). \tag{4.14}$$

REMARK Suppose that there is uniform bound on the supremum norm $|u|_0$, where u is any solution to the equation $\tilde{Q}(u) = 0$ in Ω. Then

- the sup-norms $|a_{ij}|_0$, $|b_i|_0$, $|c_{ij}|_0$ and $|\chi_i|_0$ are uniformly bounded in u and $\rho \in \mathcal{K}$, and a uniform bound on the α-Holder seminorm $[u]_\alpha$ implies that $[a_{ij}]_\alpha$, $[b_i]_\alpha$, $[c_{ij}]_\alpha$ and $[\chi_i]_\alpha$ are uniformly bounded in u and $\rho \in \mathcal{K}$ (here, $\alpha \in (0, 1)$ is arbitrary),
- the operator $\tilde{Q}$ is uniformly elliptic, by (4.10),
- the inequality (4.12) implies a uniform upper bound on $|\chi|$, and using (4.11) we have

$$\frac{\chi \cdot \nu}{|\chi|} \geq \frac{C_3}{C_4(|u|_0)} > 0, \qquad \text{uniformly in } u \text{ and } \rho \in \mathcal{K},$$

so that the operator $\tilde{N}$ is uniformly oblique with an obliqueness constant $C_3/C_4(|u|_0)$, and

- since the matrix $[a_{ij}(u, \rho, \eta)]$ is uniformly positive definite and the coefficients $c_{ij}(u, \rho, \eta)$ are uniformly bounded, there exists $k > 0$, independent of u and $\rho \in \mathcal{K}$, such that

$$\sum_{i,j} c_{ij}(u, \rho, \eta) D^i u D^j u \leq k \sum_{i,j} a_{ij}(u, \rho, \eta) D^i u D^j u. \tag{4.15}$$

Before stating the general boundary value we will solve, Theorem 4.1, we verify that the problem for the UTSD equation satisfies these conditions.

PROPOSITION 1 *For any $\rho \in \mathcal{K}$ fixed, the boundary value problem* (3.17), (3.20)-(3.23) *for the UTSD equation, satisfies the structural conditions* (4.8)-(4.13). *Moreover, for any*

$k > 1/\delta$, where δ is a positive parameter in the definition (3.9) *of the cut-off function ϕ, the inequality* (4.15) *holds.*

PROOF The condition (4.8) holds with $m_1 := 1 + \epsilon_*$ and $m_2 := u_*$.

Recall the inequalities (3.10) and (3.15), and note that the operators $\tilde{Q}$ and $\tilde{N}$, defined by the equations (3.8) and (3.16), satisfy (4.9) and (4.11) with constants C_1 and C_3 depending on the parameters $a > \sqrt{2}$, $\eta^* > 0$ and $\epsilon_* \in (0, u_* - 1)$ which are fixed throughout the paper, and on δ_* and δ which will be specified in §5 and §6, respectively, also in terms of a, η^* and ϵ_*. The Neumann condition $(0,1) \cdot \nabla u = 0$ on Σ_0 is obviously both strictly and uniformly oblique. Therefore, if u is a solution to (3.17), (3.20)-(3.23), Lemma 4.2 implies the uniform bounds $1 + \epsilon^* \leq u \leq u_*$. Hence, the definition (3.14) gives that $\chi = \beta$, for all u, and the operators N and $\tilde{N}$ are identical. Moreover, we have the following uniform bound on the ellipticity ratio for the operator $\tilde{Q}$

$$\begin{aligned}\frac{\Lambda}{\lambda} = \frac{\max\{\phi(u-\rho), 1\}}{\min\{\phi(u-\rho), 1\}} &\leq \frac{\max\{\delta, |u|_0 + |\rho|_0, 1\}}{\min\{\delta, 1\}} \\ &\leq \frac{\max\{\delta, u_* + \xi_a, 1\}}{\min\{\delta, 1\}}, \end{aligned} \tag{4.16}$$

using a priori bounds on both u and $\rho \in \mathcal{K}$ (the left bound ρ_L in (4.4) will be chosen so that $\rho_L \geq 1$). Hence, the operator $\tilde{Q}$ is uniformly elliptic. Note that $|\beta(u)| \leq C\sqrt{1 + (\rho')^2} \leq C\sqrt{\xi_a}$, for a constant C independent of u and $\rho \in \mathcal{K}$, using the definition (3.7) of β and a priori L^∞ bounds on u, ρ and ρ'. Therefore,

$$\frac{\beta \cdot \nu}{|\beta|} \geq \frac{\sqrt{\delta_*}\, \epsilon_*}{4\, C\, \sqrt{\xi_a}} > 0, \tag{4.17}$$

giving a lower bound for the obliqueness constant of the operator N uniformly in u and $\rho \in \mathcal{K}$.

Clearly, the coefficients a_{ij}, b_i and c_{ij} of the operator $\tilde{Q}$ and the coefficients β_i of the operator N have the desired smoothness and their sup- norms (or α-seminorms) are bounded using the uniform bounds on $|u|_0$ (or $[u]_\alpha$), $|\rho|_0$ and $|\rho'|_0$.

From (3.8) we have $c_{11}(u) = \phi'(u - \rho)$ and $c_{12} = c_{21} = c_{22} = 0$. Hence, $\phi'(u-\rho)u_\rho^2 \geq 0$, and, therefore, (4.13) holds.

Further, for a solution u to the equation (3.17) we have

$$\begin{aligned}|\phi(u-\rho)u_{\rho\rho} + u_{\eta\eta}| &\leq |\phi'(u-\rho)||u_\rho|^2 + \left|\frac{1}{2} - \phi'(u-\rho)\right| |u_\rho| \\ &\leq |u_\rho|^2 + \frac{1}{2}|u_\rho| \leq \frac{3}{2}|u_\rho|^2 + \frac{1}{2} \\ &\leq \lambda \left(\frac{3}{2\min\{1,\delta\}}|u_\rho|^2 + \frac{1}{2\min\{1,\delta\}} \right), \end{aligned}$$

implying that (4.14) holds.

Finally, for $k > 1/\delta$, where δ is a positive parameter in the definition of the cut-off function ϕ (see (3.9)) to be determined in §6, we have $k \geq \phi'/\phi$, and therefore

$$\phi'(u-\rho)u_\rho^2 \leq k\phi(u-\rho)u_\rho^2 \leq k\left\{\phi(u-\rho)u_\rho^2 + u_\eta^2\right\}.$$

Hence, (4.15) holds. ∎

4.4 Solution to the fixed boundary value problem

In this section we prove

Theorem 4.1 *(Fixed boundary value problem)*

Suppose that the domain Ω and the operators $\tilde{Q}$ and $\tilde{N}$ satisfy the structural conditions of §4.3. There exists $\gamma_0 > 0$, depending on the size of the opening angles of the domain Ω at the set of corners $\mathbf{V}$ and on the ellipticity ratio of $\tilde{Q}$, such that for every $\gamma \in (0, \min\{\gamma_0, 1\})$, $\alpha_{\mathcal{K}} \in (0, \min\{1, 2\gamma\})$, $\rho \in \mathcal{K}$ and any function $\tilde{f}$ which is in H_γ on an open set containing σ and satisfies inequalities (4.8)*, there exists a solution u to the fixed boundary value problem* (4.5)*. Moreover, $u \in H^{(-\gamma)}_{1+\alpha_*}$, for all $\alpha_* \in (0, \alpha_{\mathcal{K}}]$.*

The proof of Theorem 4.1 is organized as follows. In Lemma 4.2 we assume that a solution to (4.5) exists in $C^1(\Omega)$ and we find its lower and upper bounds using the Maximum Principle and Hopf's Lemma. We solve the linearized problem in §4.4 and we use a fixed point theorem to solve the nonlinear problem (4.5) in §4.4.

Lemma 4.2 *Suppose that $u \in C^1(\Omega)$ solves the fixed boundary value problem*

$$\begin{array}{lll} \tilde{Q}(u) = 0 & in & \Omega, \\ \tilde{N}(u) := \chi \cdot \nabla u = 0 & on & \tilde{\Sigma}, \\ u = \tilde{f} & on & \partial\Omega \setminus \tilde{\Sigma}, \end{array}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded, open and connected set, $\tilde{\Sigma}$ is a finite disjoint union of relatively open sets with an inner normal at every point, and the operators $\tilde{Q}$ and $\tilde{N}$ are strictly elliptic and oblique, respectively. Then

$$\min_{\partial\Omega\setminus\tilde{\Sigma}} \tilde{f} \leq u(\rho, \eta) \leq \max_{\partial\Omega\setminus\tilde{\Sigma}} \tilde{f} \tag{4.18}$$

holds for all $(\rho, \eta) \in \Omega$.

PROOF Since the operator $\tilde{Q}$ is strictly elliptic, by the Maximum Principle, if u has an extremum at the point X, then $X \in \partial\Omega$. To show (4.18), it suffices to show $X \notin \tilde{\Sigma}$.

Suppose $X \in \tilde{\Sigma}$. Then the tangential derivative of u along this part of the boundary must be zero, since X is also an extremum of the function restricted to the boundary. On the other hand, the derivative $\chi \cdot \nabla u$ is zero along $\tilde{\Sigma}$. Since the operator $\tilde{N}$ is oblique, the vector χ is not tangential to $\tilde{\Sigma}$. Therefore, if $X \in \tilde{\Sigma}$, the derivative of u at X is zero at two different directions. This yields $\nabla u(X) = 0$ and, finally, contradicts Hopf's Lemma (Lemma 3.4 in [8]). Hence, $X \in \partial\Omega \setminus \tilde{\Sigma}$. ∎

The linear problem

In this part of our study we solve the linearized version of the fixed boundary value problem (4.5), under conditions (4.8)-(4.14), using Theorem 1 in [11] and we further derive estimates on its solution using Theorem 1 in [12].

Let $\alpha_{\mathcal{K}} \in (0, 1)$ and $\rho \in \mathcal{K}$ be fixed. Let $\tilde{f}$ be in H_γ on an open set containing σ, for an arbitrary $\gamma \in (0, 1)$, and suppose that inequalities (4.8) hold. Let $\gamma_1 \in (0, 1)$ and $\epsilon \in (0, \alpha_{\mathcal{K}}]$ be also arbitrary, and let $z \in H^{(-\gamma_1)}_{1+\epsilon}$ be any function such that $m_1 \leq z \leq m_2$. The role of parameters γ_1 and ϵ is to establish compactness needed in the study of the nonlinear problem in the next section (see Lemma 4.4). We define the linear operators

$$
\begin{aligned}
Lu &:= \sum_{i,j} a_{ij}(z,\rho,\eta)D^{ij}u + \sum_i b_i(z,\rho,\eta)D^i u + \sum_{i,j} c_{ij}(z,\rho,\eta)D^i z D^j u \\
&= \sum_{i,j} a_{ij}(z,\rho,\eta)D^{ij}u + \sum_i \left\{ b_i(z,\rho,\eta) + \sum_j c_{ji}(z,\rho,\eta)D^j z \right\} D^i u \qquad (4.19)\\
&\text{in } \Omega, \qquad (4.20)
\end{aligned}
$$

and

$$Mu := \chi(z,\rho',\rho,\eta)\cdot\nabla u \quad \text{on} \quad \tilde{\Sigma}. \tag{4.21}$$

For convenience, we introduce a new function

$$\tilde{u}(\rho,\eta) := u(\rho,\eta) - \tilde{f}(\Xi_a), \quad (\rho,\eta)\in\Omega, \tag{4.22}$$

and consider the linear fixed boundary value problem

$$
\begin{array}{lll}
L\tilde{u} = 0 & \text{in} & \Omega, \\
M\tilde{u} = 0 & \text{on} & \tilde{\Sigma} = \Sigma\cup\Sigma_0, \\
\tilde{u} = \tilde{f} - \tilde{f}(\Xi_a) & \text{on} & \partial\Omega\setminus\tilde{\Sigma} = \sigma\cup\Xi_a.
\end{array} \tag{4.23}
$$

Theorem 4.3 *(Linear problem)*

Suppose that the domain Ω and the operators $\tilde{Q}$ and $\tilde{N}$ satisfy the conditions of §4.3. Let $\alpha_{\mathcal{K}}\in(0,1)$ and $\rho\in\mathcal{K}$ be fixed, and let $\tilde{f}$ be any function which is in H_γ on an open set containing σ and satisfies (4.8). Suppose that $z\in H_{1+\epsilon}^{(-\gamma_1)}$, for arbitrary parameters $\gamma_1\in(0,1)$ and $\epsilon\in(0,\alpha_{\mathcal{K}}]$, is any function such that

$$m_1 \le z(\rho,\eta) \le m_2, \quad (\rho,\eta)\in\Omega, \tag{4.24}$$

where m_1 and m_2 are as in (4.8), and there exists a constant m so that

$$|b_i(z,\rho,\eta) + \sum_j c_{ji}(z,\rho,\eta)D^j z| \le m\, d_{\mathbf{V}}^{\gamma_1-1}(\rho,\eta). \tag{4.25}$$

Here $\mathbf{V}:=\{V_1,V_2,V_3\}$ denotes the set of corners of Ω, $d_{V_i}(X):=|X-V_i|$ and $d_{\mathbf{V}}(X):=\min_i d_{V_i}(X)$.

Then there exists $\gamma_0>0$, depending on the geometry of Ω and the supremum norm $|z|_0$, so that for any $\gamma\in(0,\min\{\gamma_0,1\})$, there exists a unique solution $\tilde{u}\in H_{1+\alpha_{\mathcal{K}}}^{(-\gamma)}$ of the linear problem (4.23).

Moreover, the following two estimates hold

$$|\tilde{u}|_{1+\alpha_{\mathcal{K}}}^{(-\gamma)} \le C\left\{|\tilde{f}-\tilde{f}(\Xi_a)|_\gamma + \sup_{i,(\rho,\eta)\in\Omega} d_{V_i}^{-\gamma}(\rho,\eta)|\tilde{u}(\rho,\eta)-\tilde{u}(V_i)|\right\} \tag{4.26}$$

and

$$|\tilde{u}|_{1+\alpha_{\mathcal{K}}}^{(-\gamma)} \le C_1\,|\tilde{f}-\tilde{f}(\Xi_a)|_\gamma, \tag{4.27}$$

where C and C_1 depend on $\alpha_{\mathcal{K}}$, $[z]_{\alpha_{\mathcal{K}}}$, $[\chi_i(z,\rho,\rho')]_{\alpha_{\mathcal{K}}}$, $|\rho|_{1+\alpha_{\mathcal{K}}}$, m, $|z|_0$ and the size of the domain Ω.

REMARK

- We are assuming that the function ρ describing the boundary $\Sigma \subset \tilde{\Sigma}$ is such that $\rho \in H_{1+\alpha_{\mathcal{K}}}$, where $\alpha_{\mathcal{K}} \in (0,1)$. To show existence of a solution, we will use Theorem 1 of [11] which assumes more smoothness on ρ. More precisely, this theorem requires that the boundary $\tilde{\Sigma}$, along which we pose the oblique derivative boundary condition, is described by a curve in $H_{2+\alpha}$, with $\alpha \in (0,1)$. This is satisfied for $\Sigma_0 \subset \tilde{\Sigma}$, and our idea is to approximate Σ with a sequence of boundaries Σ_k described by smooth curves $\{\rho_k\} \subset \mathcal{K}$ and to solve the linear problem (4.23) as a limit of problems on regularized domains Ω_k.

- The parameter γ_0 (and therefore γ) in the statement of the theorem depends on the size of angles of the domain Ω at the set of corners $\mathbf{V}$ and on the ellipticity ratio of the linear operator L. By the choice of set $\mathcal{K}$, these angles are bounded uniformly in $\rho \in \mathcal{K}$ (recall Remark 4.2). Also, the ellipticity ratio for L is uniformly bounded, with respect to z, using the assumption (4.10) for $\tilde{Q}$ and condition (4.24) for the choice of z. Hence, γ_0 (and also γ) can be taken independent of both the domain Ω and the function z.

- To find the parameter γ_0 we will use Theorem 1 in [12]. This theorem assumes that the operator M is uniformly oblique. By the assumptions (4.11) and (4.12) on the operator $\tilde{N}$, the lower bound on the obliqueness constant for M depends on $|z|_0$, which is uniformly bounded by the condition (4.24).

PROOF (of Theorem 4.3)

We divide the proof into four steps.

Step 1. Let $\tilde{u} \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$ be a solution to (4.23). Using the standard elliptic theory (for example, Theorem 6.2 in [8]) we have $u \in C^2(\Omega)$, and by Lemma 4.2 we obtain the L^∞ estimate

$$|\tilde{u}|_{0;\Omega} \leq |\tilde{f} - \tilde{f}(\Xi_a)|_0. \tag{4.28}$$

Step 2. In this step we prove that if $\tilde{u} \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$, for an arbitrary $\gamma \in (0,1)$ and an arbitrary $\tilde{f} \in H_\gamma$ on an open set containing σ satisfying (4.8), is a solution to the linear problem (4.23) with boundary Σ described by a smooth curve $\rho \in \mathcal{K}$, then the estimate (4.26) holds. First, we derive weighted local estimates on a particular seminorm of the first derivatives of $\tilde{u}$ inside the domain Ω and on the boundary $\partial\Omega \setminus \mathbf{V}$, where we pose different types of boundary conditions (the oblique derivative boundary condition on $\tilde{\Sigma} = \Sigma \cup \Sigma_0$ and the Dirichlet condition on σ). These local estimates follow from Gilbarg & Trudinger [8], and together with interpolation inequalities establish (4.26). The estimate (4.27) is deduced from (4.26) for the parameter γ sufficiently small using Theorem 1 of [12].

We claim that there exists a constant C, independent of $\tilde{u}$ and the choice of $\rho \in \mathcal{K}$, such that the auxiliary inequality

$$R^{1+\alpha_{\mathcal{K}}}[D\tilde{u}]_{\alpha_{\mathcal{K}};B_R(x_0)\cap\Omega} \leq C\left\{R^\gamma|\tilde{f} - \tilde{f}(\Xi_a)|_\gamma + \sup_{i,(\rho,\eta)\in\Omega} |\tilde{u} - \tilde{u}(V_i)|_{0;B_{2R}(x_0)\cap\Omega}\right\} \tag{4.29}$$

holds in the following three cases

1. $x_0 \in \sigma$ and $B_{2R}(x_0) \cap \tilde{\Sigma} = \emptyset$,
2. $x_0 \in \tilde{\Sigma}$ and $B_{2R}(x_0) \cap \sigma = \emptyset$, and
3. $B_{2R}(x_0) \subseteq \Omega$.

To show (4.29) in case (1) we use the discussion on page 139 in [8] for elliptic problems with Dirichlet boundary conditions. It implies the estimate

$$R^{1+\alpha_{\mathcal{K}}-\gamma}[D\tilde{u}]_{\alpha_{\mathcal{K}};B_R(x_0)\cap\Omega} \leq C\{|\tilde{f}-\tilde{f}(\Xi_a)|_\gamma + |\tilde{u}|_{0;B_R(x_0)\cap\Omega}\}, \tag{4.30}$$

with a constant C depending on $\alpha_{\mathcal{K}}$, the domain Ω and the norms of the coefficients of the operator L defined in (4.19). Using the a priori bounds on the set $\mathcal{K}$ and conditions (4.24) and (4.25), we have that the constant C in (4.30) does not depend on the solution $\tilde{u}$ nor on the choice of $\rho \in \mathcal{K}$ describing the boundary Σ. The estimate (4.30) together with the L^∞ bound (4.28) gives

$$R^{1+\alpha_{\mathcal{K}}-\gamma}[D\tilde{u}]_{\alpha_{\mathcal{K}};B_R(x_0)\cap\Omega} \leq C\,|\tilde{f}-\tilde{f}(\Xi_a)|_\gamma,$$

and, clearly, (4.29) follows.

In case (2) we use Theorem 6.26 in [8] for the oblique derivative boundary value problems. For convenience we consider this theorem for the functions $\tilde{u}-\tilde{u}(V_i)$, $i \in \{1,2,3\}$, also satisfying the linear differential equation in Ω and the linear oblique derivative boundary condition on $\tilde{\Sigma}$. For each i we obtain the estimate

$$\begin{aligned} R^{1+\alpha_{\mathcal{K}}}[D\tilde{u}]_{\alpha_{\mathcal{K}};B_R(x_0)\cap\Omega} &= R^{1+\alpha_{\mathcal{K}}}[D(\tilde{u}-\tilde{u}(V_i))]_{\alpha_{\mathcal{K}};B_R(x_0)\cap\Omega} \\ &\leq C\,|\tilde{u}-\tilde{u}(V_i)|_{0;B_{2R}(x_0)\cap\Omega}, \end{aligned}$$

with a constant C depending on $\alpha_{\mathcal{K}}$ and the bound on the obliqueness constant of the linear oblique derivative operator M. Therefore, (4.29) holds with C depending on the parameter $\alpha_{\mathcal{K}}$ and the supremum norm $|z|_0$.

In case (3) we use Theorem 8.32 in [8]. Again we use this theorem for functions $\tilde{u}-\tilde{u}(V_i)$, $i \in \{1,2,3\}$, also solutions to the differential equation $L\tilde{u}=0$ in Ω. For each i, we obtain

$$\begin{aligned} [D\tilde{u}]_{\alpha_{\mathcal{K}};B_R(x_0)\cap\Omega} &= [D(\tilde{u}-\tilde{u}(V_i))]_{\alpha_{\mathcal{K}};B_R(x_0)\cap\Omega} \\ &\leq C\,|\tilde{u}-\tilde{u}(V_i)|_{0;B_{2R}(x_0)}, \end{aligned}$$

and (4.29) follows for $R \in (0,1]$. Here, the constant C depends on supremum norms of the coefficients of L, $\alpha_{\mathcal{K}}$-seminorms of the coefficients of L and M and the size of the domain Ω.

Note that cases (1)-(3) cover all points $x_0 \in \overline{\Omega}\backslash\mathbf{V}$. Let $R := \tan\left(\frac{\theta}{4}\right)\frac{d_{\mathbf{V}}(x_0)}{\mathrm{diam}\Omega}$, where θ stands for the corner angle. We multiply the estimate (4.29) by $R^{-\gamma}$ and use the interpolation inequalities (6.8)-(6.9) in [8] to obtain (4.26).

Next, we derive the estimate (4.27) from (4.26). Note that for each $i \in \{1,2,3\}$ we have

$$\sup_{(\rho,\eta)} d_{V_i}(\rho,\eta)^{-\gamma}|\tilde{u}(\rho,\eta)-\tilde{u}(V_i)| \leq |\tilde{u}|_{\gamma;\Omega\backslash\mathbf{V}}. \tag{4.31}$$

On the other hand, Theorem 1 of [12] gives that there exist positive constants γ_0 and a_0, depending on the size of angles of the domain Ω at the set of corners $\mathbf{V}$ and on the ellipticity ratio for the operator L, such that for all $\gamma \in (0,\gamma_0)$ and all $a \in (1, 1+a_0)$ we have

$$|\tilde{u}|_a^{(-\gamma)} \leq C\{|\tilde{f}-\tilde{f}(\Xi_a)|_\gamma + |\tilde{u}|_0\}.$$

We fix $\gamma \in (0, \min\{\gamma_0, 1\})$. Since

$$|\tilde{u}|_{\gamma;\Omega\backslash\mathbf{V}} = |\tilde{u}|_\gamma^{(-\gamma)} \leq C(a,\gamma,\mathrm{diam}(\Omega))|\tilde{u}|_a^{(-\gamma)},$$

holds for any $a > \gamma$, we obtain

$$|\tilde{u}|_{\gamma;\Omega\backslash\mathbf{V}} \leq C\{|\tilde{f}-\tilde{f}(\Xi_a)|_\gamma + |\tilde{u}|_0\} \leq C|\tilde{f}-\tilde{f}(\Xi_a)|_\gamma,$$

using the L^∞ bound (4.28). Together with (4.26) and (4.31), this establishes the estimate (4.27).

Step 3. We approximate the boundary Σ, specified by $\rho \in \mathcal{K}$, by a sequence of boundaries $\{\Sigma_k\}$ given by smooth curves $\rho_k \in \mathcal{K}$. This leads to a sequence $\{\Omega_k\}$ of the domains approximating the domain Ω, and the sequences $\{\sigma_k\}$ and $\{\Sigma_{0,k}\}$ approximating boundaries σ and Σ_0, respectively. Let $\tilde{f}$ be in H_γ on an open set $\mathcal{R}$ containing σ, for $\gamma \in (0,1)$ arbitrary, such that the inequalities (4.8) hold. Since $\tilde{f}$ satisfies the second inequality in (4.8) for ρ and σ, by continuity we have that $\tilde{f}$ satisfies the second condition in (4.8) for ρ_k and σ_k, where $k \geq k_0$. We use Theorem 1 of [11] for the linear problem (4.23) in Ω_k and get a unique solution $\tilde{u}_k \in C^2(\Omega_k \cup \Sigma_k) \cap C(\overline{\Omega}_k)$. By step 2 we have that $\tilde{u}_k \in H^{(-\gamma)}_{1+\alpha_\mathcal{K}}$ and that the sequence $\{\tilde{u}_k\}$ satisfies the estimates (4.26) and (4.27) uniformly in k. (Recall Remark 4.4 and that since we have uniform bounds on the geometry of the domains Ω_k, we can take both parameters γ_0 and γ independent of k.)

Step 4. In this step we show that the sequence $\{\tilde{u}_k\}$ has a convergent subsequence and that its limit is a unique solution to the linear boundary value problem (4.23) in the domain Ω.

As $k \to \infty$, we have $\Sigma_k \to \Sigma$, $\Omega_k \to \Omega$ and $\sigma_k \to \sigma$. Since the estimate (4.27) holds uniformly in k, the sequence $\{\tilde{u}_k\}$ is uniformly bounded in $H^{(-\gamma)}_{1+\alpha_\mathcal{K}}$ and by the Arzela-Ascoli Theorem, it contains a subsequence $\{\tilde{u}_{j_k}\}$ which converges uniformly to a function $\tilde{u} \in H^{(-\gamma)}_{1+\alpha_\mathcal{K}}$. It is clear that the estimates (4.26) and (4.27) also hold for $\tilde{u}$ with the same constants C and C_1.

Next, we show that $\tilde{u}$ is a solution to the linear boundary value problem (4.23). Since $|\tilde{u}_k|^{(-\gamma)}_{1+\alpha_\mathcal{K}}$ is uniformly bounded, $\tilde{u}_k$ and $D\tilde{u}_k$ are equicontinuous on compact subsets of Ω, implying that $\tilde{u}$ satisfies the differential equation $L\tilde{u} = 0$ weakly in the domain Ω. Further, let $x_0 \in \Sigma$ and let $x_k \in \Sigma_k$ be such that $x_k \to x_0$. By the uniform convergence of equicontinuous sequences in $H^{(-\gamma)}_{1+\alpha_\mathcal{K}}$ we obtain that $\chi(z, \rho_{j_k}, \rho'_{j_k}) \cdot \nabla \tilde{u}_{j_k}(x_{j_k}) \to \chi(z, \rho, \rho') \cdot \nabla \tilde{u}(x_0)$. Hence, the oblique derivative boundary condition $M\tilde{u} = 0$ holds on Σ. Similarly, the oblique derivative boundary condition on Σ_0 holds. The Dirichlet condition at the corner point Ξ_a is clearly satisfied, and to show the Dirichlet condition on σ we also use continuity of $\tilde{f}$ in an open set containing σ and σ_k, for $k \geq k_0$ and k_0 is sufficiently large. Therefore, the function $\tilde{u}$ solves the linear problem (4.23) in the domain Ω.

Since $\tilde{u} \in C^2(\Omega)$, we use Lemma 4.2 and linearity of the operators L and M to conclude that $\tilde{u}$ is the unique solution of the linear problem (4.23). ∎

We note that uniqueness of the solution $\tilde{u}$ implies that the whole sequence $\{\tilde{u}_k\}$ in the previous proof converges.

The nonlinear problem

Let $\alpha_\mathcal{K} \in (0,1)$ and $\rho \in \mathcal{K}$ be given. Let $\gamma_1 \in (0,1)$ and $\epsilon \in (0, \alpha_\mathcal{K}]$ be arbitrary. By Theorem 4.3 and Remark 4.4, there exists a parameter $\gamma_0 > 0$ such that for any fixed $\gamma \in (0, \min\{\gamma_0, 1\})$, any function $z \in H^{(-\gamma_1)}_{1+\epsilon}$ satisfying conditions (4.24) and (4.25) and any function $\tilde{f} \in H_\gamma$ on an open set containing σ and satisfying (4.8), there exists a unique solution $\tilde{u} \in H^{(-\gamma)}_{1+\alpha_\mathcal{K}}$ to the linear problem (4.23). Let us define a mapping T so that

$$Tz := \tilde{u} + \tilde{f}(\Xi_a). \tag{4.32}$$

In this section we show that we can choose the parameter $\alpha_\mathcal{K} \in (0,1)$, depending on γ, so that the mapping T has a fixed point. This will complete the proof of Theorem 4.1.

The fixed point result we use is the following

Theorem. (Theorem 11.3 in [8]) Let T be a compact mapping of a Banach space $\mathcal{B}$ into itself and suppose that there exists a constant M such that

$$\|u\|_{\mathcal{B}} \leq M, \text{ for all } u \in \mathcal{B} \text{ and } \tau \in [0,1] \text{ satisfying } u = \tau T u. \tag{4.33}$$

Then T has a fixed point.

The verification of the conditions of this fixed point theorem consists of two parts. In Lemma 4.4 we select an appropriate Banach space $\mathcal{B}$ so that $T(\mathcal{B}) \subseteq \mathcal{B}$ and that the mapping T is compact. Using Lemma 4.5 we choose the parameter $\alpha_{\mathcal{K}} \in (0,1)$, in terms of γ, so that there exists M, independent of $\tilde{u}$, for which the inequality (4.33) holds.

Lemma 4.4 *Let $\gamma \in (0, \min\{\gamma_0, 1\})$, where $\gamma_0 > 0$ is as in Theorem 4.3, and let $\alpha_{\mathcal{K}} \in (0,1)$ be arbitrary. If*

$$\epsilon = \frac{\alpha_{\mathcal{K}}}{2} \qquad \text{and} \qquad \gamma_1 = \frac{\gamma}{2}, \tag{4.34}$$

then for $\mathcal{B} := H^{(-\gamma_1)}_{1+\epsilon}$, the mapping T given by (4.32) is precompact and $T(\mathcal{B}) \subseteq \mathcal{B}$.

PROOF Theorem 4.3 implies $T(H^{(-\gamma_1)}_{1+\epsilon}) \subseteq H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$. To ensure $T(\mathcal{B}) \subseteq \mathcal{B}$, we choose γ_1 and ϵ, so that $0 < \epsilon \leq \alpha_{\mathcal{K}}$ and $0 < \gamma_1 \leq \gamma$. In order for the map T to be compact we need these inequalities to be strict (see Remark 4.1), and in particular the choices (4.34) suffice. ∎

Lemma 4.5 *Let $\gamma \in (0, \min\{\gamma_0, 1\})$, where $\gamma_0 > 0$ is as in Theorem 4.3, and let $\alpha_{\mathcal{K}} \in (0,1)$ be arbitrary. Let ϵ and γ_1 be as in (4.34). There exists $M > 0$ such that if $\tilde{u} \in H^{(-\gamma_1)}_{1+\epsilon}$ and*

$$\tilde{u} + \tilde{f}(\Xi_a) = \tau T(\tilde{u} + \tilde{f}(\Xi_a)), \tag{4.35}$$

for some $\tau \in [0,1]$, then

$$|\tilde{u} + \tilde{f}(\Xi_a)|^{(-\gamma)}_{1+\alpha_*} \leq M, \tag{4.36}$$

where $\alpha_* := \min\{\alpha_{\mathcal{K}}, \gamma\}$. The constant M depends on the geometry of Ω, the bounds on the ellipticity ratio and the minimal eigenvalue of the operator $\tilde{Q}$, the bound on the obliqueness constant for $\tilde{N}$, the sup-norm $|\tilde{u}|_0$ and the Holder norm $|\tilde{f}|_{\gamma;\mathcal{R}}$.

PROOF We divide the proof into four steps.

Step 1. In this step we obtain an L^∞ bound on $\tilde{u} \in H^{(-\gamma_1)}_{1+\epsilon}$ satisfying (4.35).

Using the definition (4.32) of the map T, the assumption (4.35) implies that $\tilde{u}$ solves the following nonlinear fixed boundary problem

$$\begin{aligned} \sum_{i,j} a_{ij}(\tilde{u} + \tilde{f}(\Xi_a)) D^{ij}\tilde{u} + \sum_i b_i(\tilde{u} + \tilde{f}(\Xi_a)) D^i \tilde{u} & \\ + \sum_{i,j} c_{ij}(\tilde{u} + \tilde{f}(\Xi_a)) D^i(\tilde{u} + \tilde{f}(\Xi_a)) D^j \tilde{u} = 0 & \quad \text{in} \quad \Omega, \\ \chi(\tilde{u} + \tilde{f}(\Xi_a)) \cdot \nabla \tilde{u} = 0 \quad \text{on} \quad \tilde{\Sigma}, & \\ \tilde{u} = \tau(\tilde{f} - \tilde{f}(\Xi_a)) \quad \text{on} \quad \partial\Omega \setminus \tilde{\Sigma}. & \end{aligned} \tag{4.37}$$

Since we have $\tilde{u} \in C^2(\Omega)$, by Lemma 4.2 we obtain the L^∞ bound

$$|\tilde{u}|_0 \leq \tau |\tilde{f} - \tilde{f}(\Xi_a)|_0 \leq |\tilde{f} - \tilde{f}(\Xi_a)|_0. \tag{4.38}$$

Step 2. In this part of the proof we find an estimate for the term

$$\sup_{i,(\rho,\eta)\in\Omega} d^{-\gamma}_{V_i}(\rho,\eta) |\tilde{u}(\rho,\eta) - \tilde{u}(V_i)|$$

which is independent of $\tilde{u}$. The idea is to construct two linear problems independent of $\tilde{u}$ (with the same linear elliptic and linear oblique derivative boundary operators) and to show that we can bound $\tilde{u}$ both from below and from above using the solutions of these linear problems. Once these bounds are established, we use Lemma 4.1 in [12] giving corner barriers for the linear elliptic and linear oblique derivative boundary operators.

Let us define the linear operators $\overline{L}$ and $\overline{M}$ by

$$\overline{L}v := \sum_{i,j} a_{ij}(\tilde{u} + \tilde{f}(\Xi_a))D^{ij}v + \sum_i b_i(\tilde{u} + \tilde{f}(\Xi_a))D^i v \tag{4.39}$$

and

$$\overline{M}v := \beta(\tilde{u} + \tilde{f}(\Xi_a)) \cdot \nabla v. \tag{4.40}$$

First, we consider the linear problem

$$\begin{array}{lll} \overline{L}v = 0 & \text{in} & \Omega, \\ \overline{M}v = 0 & \text{on} & \tilde{\Sigma}, \\ v = \tau(\tilde{f} - \tilde{f}(\Xi_a)) & \text{on} & \partial\Omega \setminus \tilde{\Sigma}. \end{array} \tag{4.41}$$

Note that

$$\overline{L}\tilde{u} = \tilde{Q}(\tilde{u} + \tilde{f}(\Xi_a)) - \sum_{i,j} c_{ij}(\tilde{u} + \tilde{f}(\Xi_a))D^i\tilde{u}D^j\tilde{u} \le 0,$$

since $\tilde{u}$ is a solution to the nonlinear problem (4.37) and the left inequality in (4.15) holds. Therefore, $\tilde{u}$ is a supersolution for (4.41), meaning, $v \le \tilde{u}$. Further, consider the linear problem

$$\begin{array}{lll} \overline{L}w = 0 & \text{in} & \Omega, \\ \overline{M}w = 0 & \text{on} & \tilde{\Sigma}, \\ w = \frac{1}{k}\left(e^{k\tau(\tilde{f} - \tilde{f}(\Xi_a))} - 1\right) & \text{on} & \partial\Omega \setminus \tilde{\Sigma}, \end{array} \tag{4.42}$$

where $k \ge 0$ is such that (4.15) holds. Note that for $w_{sub} := \frac{1}{k}\left(e^{k\tilde{u}} - 1\right)$ we have

$$\begin{aligned} \overline{L}w_{sub} &= e^{k\tilde{u}}\left(\tilde{Q}(\tilde{u} + \tilde{f}(\Xi_a)) + \sum_{i,j}\left(ka_{ij}(\tilde{u} + \tilde{f}(\Xi_a)) - c_{ij}(\tilde{u} + \tilde{f}(\Xi_a))\right)D^i\tilde{u}D^j\tilde{u}\right) \\ &\ge 0, \end{aligned}$$

since $\tilde{Q}(\tilde{u} + \tilde{f}(\Xi_a)) = 0$ and the right hand inequality in (4.15) holds. This implies that w_{sub} is a subsolution for the problem (4.42), meaning, $w_{sub} \le w$. On the other hand, from the definition of w_{sub}, clearly $w_{sub} > \tilde{u}$. Therefore, if $\tilde{u}$ solves the problem (4.37), then the inequalities

$$v \le \tilde{u} \le w \tag{4.43}$$

hold, where v and w denote arbitrary solutions of the linear problems (4.41) and (4.42), respectively.

Next we use Lemma 4.1 of [12] which gives a corner barrier function for the linear operators $\overline{L}$ and $\overline{M}$, defined in (4.39) and (4.40), respectively. By this lemma, there exist positive constants h_0 and γ_0 (depending on the size of the opening angles of the domain Ω at the set of corners $\mathbf{V}$, on the ellipticity ratio of the linear operator $\overline{L}$ and on the bounds on Σ and $\tilde{u}$) such that for every fixed parameter $\gamma \in (0, \gamma_0)$ there exist a constant $c_1 \in (0, 1]$ and a function $g \in C^2(\cup_i\overline{\Omega_i(h_0)} \setminus \mathbf{V}) \cap C(\cup_i\overline{\Omega_i(h_0)})$ (depending on the same parameters as γ_0 and also on γ) with property that for each $i \in \{1, 2, 3\}$ we have

$$\begin{array}{lll} \overline{L}g \le 0 & \text{in} & \Omega_i(h_0), \\ c_1 d_{V_i}^{\gamma} \le g \le d_{V_i}^{\gamma} & \text{in} & \Omega_i(h_0), \\ \overline{M}g \le 0 & \text{on} & \tilde{\Sigma}_i(h_0). \end{array} \tag{4.44}$$

Here, $\Omega_i(h_0)$ and $\tilde{\Sigma}_i(h_0)$ denote the subsets of Ω and $\tilde{\Sigma}$, respectively, on which $d_{V_i} < h_0$. We remark that the parameter γ_0 provided by Lemma 4.1 of [12] is the same γ_0 as in Theorem 4.3, step 2, and, as before, we take $\gamma \in (0, \min\{\gamma_0, 1\})$. Further, (4.44) also implies

$$\begin{aligned} \overline{L}(-g) \geq 0 \quad &\text{in} \quad \Omega_i(h_0), \\ \overline{M}(-g) \geq 0 \quad &\text{on} \quad \tilde{\Sigma}_i(h_0). \end{aligned}$$

We multiply g by a positive constant C^* so that

$$C^* g + \tilde{u}(V_i) \geq \frac{1}{k}\left(e^{\tau(\tilde{f}-\tilde{f}(\Xi_a))} - 1\right) \qquad \text{on} \qquad \Omega_i(h_0) \setminus \tilde{\Sigma}_i(h_0),$$

for each $i \in \{1, 2, 3\}$. Note that the constant C^* does not depend on $\tilde{u}$. This gives

$$C^* g + \tilde{u}(V_i) \geq w \quad \text{in} \quad \Omega_i(h_0),$$

where w is a solution to the linear problem (4.42). With the right hand inequality in (4.43) and (4.44), we get

$$\tilde{u} - \tilde{u}(V_i) \leq w - \tilde{u}(V_i) \leq C^* g \leq C^* d_{V_i}^{\gamma} \quad \text{in} \quad \Omega_i(h_0),$$

and, hence, for each i we have

$$d_{V_i}^{-\gamma}(\tilde{u} - \tilde{u}(V_i)) \leq C^* \quad \text{in} \quad \Omega_i(h_0). \tag{4.45}$$

Similarly, we multiply $-g$ by a positive constant C_* so that

$$-C_* g + \tilde{u}(V_i) \leq \tau(\tilde{f} - \tilde{f}(\Xi_a)) \quad \text{on} \quad \Omega_i(h_0) \setminus \tilde{\Sigma}_i(h_0),$$

for each $i \in \{1, 2, 3\}$. Again, the constant C_* is independent of $\tilde{u}$. This yields

$$-C_* g + \tilde{u}(V_i) \leq v \quad \text{in} \quad \Omega_i(h_0),$$

for a solution v of (4.41). Recalling the left-hand inequality in (4.43) and (4.44) we obtain

$$-C_* d_{V_i}^{\gamma} \leq -C_* g \leq v - \tilde{u}(V_i) \leq \tilde{u} - \tilde{u}(V_i) \quad \text{in} \quad \Omega_i(h_0),$$

and, therefore, for each i we have

$$d_{V_i}^{-\gamma}(\tilde{u} - \tilde{u}(V_i)) \geq -C_* \quad \text{in} \quad \Omega_i(h_0). \tag{4.46}$$

Note that on $\Omega \setminus (\cup_i \Omega_i(h_0))$, we have that for each i

$$d_{V_i}^{-\gamma}|\tilde{u} - \tilde{u}(V_i)| \leq h_0^{-\gamma}\, |\tilde{u} - \tilde{u}(V_i)|_0 \leq 2h_0^{-\gamma}\, |\tilde{f} - \tilde{f}(\Xi_a)|_0,$$

using the L^∞ bound (4.38) for $\tilde{u}$. Together with inequalities (4.45) and (4.46), this gives

$$\sup_{i,(\rho,\eta)\in\Omega} d_{V_i}^{-\gamma}(\rho, \eta)|\tilde{u}(\rho, \eta) - \tilde{u}(V_i)| \leq C, \tag{4.47}$$

for a constant C independent of $\tilde{u}$, as desired.

Step 3. In this step we show that there exist positive constants δ^* and C, depending on the geometry of Ω, the bounds on the minimal eigenvalue and the ellipticity ratio of the operator $\tilde{Q}$, on the obliqueness constant for the operator $\tilde{N}$, and the bounds on $\tilde{u}$ and $\tilde{f}$ such that

$$|\tilde{u}|_{\delta^*} \leq C. \tag{4.48}$$

(By assumptions (4.8)-(4.12) and the inequality (4.38), all of these bounds are uniform in $\tilde{u}$ and $\rho \in \mathcal{K}$.)

First we quote several results from Gilbarg & Trudinger [8] and Lieberman & Trudinger [14] which give local Holder estimates on the parts of the boundary $\partial\Omega \setminus \mathbf{V}$ where we impose different types of boundary conditions. More precisely, we derive local Holder estimates on $\tilde{\Sigma}$, where we assume the oblique derivative condition, and on $\partial\Omega \setminus (\mathbf{V} \cup \tilde{\Sigma})$, where we have Dirichlet condition. For the Holder estimate at the set of corners $\mathbf{V}$, we use the inequality (4.47) established in step 2.

For the estimate on $\tilde{\Sigma}$ we use Theorem 2.3 in [14]. This theorem is proved when the considered part of the boundary has smoothness C^2. However, the authors of [14] remark that it suffices that the boundary is $H_{1+\alpha}$, for $\alpha \in (0,1)$, which is the case for $\tilde{\Sigma} = \Sigma \cup \Sigma_0$. The assumptions of Theorem 2.3 in [14] are that the operator $\tilde{Q}$ satisfies the structure condition (4.14), that the operators $\tilde{Q}$ and $\tilde{N}$ are uniformly elliptic and uniformly oblique, respectively, and that the supremum norm of $\tilde{u}$ is uniformly bounded. This theorem implies that there exist α_0 and C such that

$$[\tilde{u}]_{\alpha_0} \leq C, \tag{4.49}$$

in a neighborhood of $\tilde{\Sigma}$. Here, α_0 depends on the bounds for the ellipticity ratio of $\tilde{Q}$ and the obliqueness constant for $\tilde{N}$, and $\mu_0|\tilde{u}|_0$, where μ_0 is the constant from the structure condition (4.14). The constant C depends also on Ω.

The estimate on the Dirichlet part of the boundary $\partial\Omega \setminus (\tilde{\Sigma} \cup \mathbf{V}) = \sigma \setminus \mathbf{V}$ follows from the assumption that $\tilde{u} = \tau(\tilde{f} - \tilde{f}(\Xi_a))$ on σ and that $\tilde{f} \in H_\gamma$ on an open set $\mathcal{R}$ containing σ. Hence, $[\tilde{u}]_\gamma \leq [\tilde{f}]_{\gamma;\mathcal{R}}$ on $\sigma \setminus \mathbf{V}$.

For the local estimate at the set of corners $\mathbf{V}$ we use the inequality (4.47) which implies $[\tilde{u}]_\gamma \leq C$.

Next, we take $\overline{\alpha} := \min\{\alpha_0, \gamma\}$ and note that we have shown that

$$\operatorname{osc}_{\partial\Omega \cap B_R(x_0)}(\tilde{u}) \leq K R^{\overline{\alpha}}, \qquad \text{for every } x_0 \in \partial\Omega \text{ and } R > 0, \tag{4.50}$$

where osc stands for the oscillation and $K = [\tilde{u}]_{\overline{\alpha}}$. Again, the parameter $\overline{\alpha}$ in (4.50) does not depend on $\tilde{u}$. More precisely, $\overline{\alpha}$ depends on the size of Ω (which can be estimated in terms of a priori bounds on $\rho \in \mathcal{K}$), the bounds on the ellipticity ratio for the operator $\tilde{Q}$ and the obliqueness constant of the operator $\tilde{N}$, and on the bounds on $|\tilde{u}|_0$ (and these bounds can be estimated uniformly in $\tilde{u}$ and $\rho \in \mathcal{K}$).

Finally, with the inequality (4.50) being satisfied we have that the assumptions of Theorem 8.29 in [8] hold. This theorem implies that there exist positive δ^* and C such that the desired estimate (4.48) holds. Here, the parameter δ^* depends on the ellipticity ratio for $\tilde{Q}$, the minimal eigenvalue of $\tilde{Q}$ and the parameter $\overline{\alpha}$ in the inequality (4.50), while the constant C also depends on $|\tilde{u}|_0$. Notice that (4.48) also implies $\tilde{u} \in H_{\delta^*}$.

Step 4. Having $\tilde{u} \in H_{\delta^*}$ and the uniform estimate (4.48), we use Theorem 4.3 with z replaced by $\tilde{u}$ and $\alpha_{\mathcal{K}}$ replaced by $\min\{\delta^*, \alpha_{\mathcal{K}}, \gamma\}$. (Recall again that since we are not concerned with the existence of a solution to the problem (4.37), we can treat (4.37) as a linear problem with $z := \tilde{u} + \tilde{f}(\Xi_a)$.) The estimate (4.26) of Theorem 4.3 gives

$$|\tilde{u}|^{(-\gamma)}_{1+\min\{\delta^*, \alpha_{\mathcal{K}}, \gamma\}} \leq C. \tag{4.51}$$

Note that the seminorms $[\tilde{u}]_{\min\{\delta^*, \alpha_{\mathcal{K}}, \gamma\}}$ and $[\chi_i(\tilde{u}, \rho, \rho')]_{\min\{\delta^*, \alpha_{\mathcal{K}}, \gamma\}}$ are bounded independently of $\tilde{u}$ by (4.48), as well as the term

$$\sup_{i,(\rho,\eta)\in\Omega} d_{V_i}^{-\gamma}(\rho,\eta)|\tilde{u}(\rho,\eta) - \tilde{u}(V_i)|$$

by (4.47). Therefore, we have that the constant C in the estimate (4.51) does not depend on $\tilde{u}$. We use the inequality

$$|\tilde{u}|_\gamma = |\tilde{u}|^{(-\gamma)}_\gamma \leq C(\delta^*, \alpha_{\mathcal{K}}, \gamma, \mathrm{diam}(\Omega))|\tilde{u}|^{(-\gamma)}_{1+\min\{\delta^*,\alpha_{\mathcal{K}},\gamma\}},$$

and the estimate (4.51) to get $\tilde{u} \in H_\gamma$.

To eliminate δ^* in (4.51), we repeat step 4 with δ^* replaced by γ and obtain the estimate analogous to (4.51); that is, we get

$$|\tilde{u}|^{(-\gamma)}_{1+\min\{\alpha_{\mathcal{K}},\gamma\}} \leq C.$$

Therefore, (4.36) follows. ∎

Finally, for $\gamma_0 > 0$ from Theorem 4.3, we take

$$\gamma \in (0, \min\{\gamma_0, 1\}) \quad \text{and} \quad \alpha_{\mathcal{K}} \in (0, \min\{1, 2\gamma\}),$$

and recall the choices (from Lemma 4.4):

$$\epsilon = \frac{\alpha_{\mathcal{K}}}{2} \quad \text{and} \quad \gamma_1 = \frac{\gamma}{2}.$$

With the notation $u := \tilde{u} + \tilde{f}(\Xi_a)$ and using Remark 4.1 and the estimate (4.36), we obtain

$$|u|^{(-\gamma_1)}_{1+\epsilon} = |u|^{(-\gamma/2)}_{1+\alpha_{\mathcal{K}}/2} \leq C|u|^{(-\gamma)}_{1+\min\{\alpha_{\mathcal{K}},\gamma\}} \leq C\,M, \tag{4.52}$$

for a constant C depending on $\alpha_{\mathcal{K}}$, γ and the diameter of Ω and the constant M is as in (4.36). Hence, the hypotheses of the quoted fixed point theorem at the beginning of §4.4 (Theorem 11.3 in [8]) are satisfied. Therefore, the map T defined in (4.32) has a fixed point $u \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$. This fixed point u solves the fixed boundary value problem (4.5) and since $H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}} \subseteq H^{(-\gamma)}_{1+\alpha_*}$, for any $\alpha_* \in (0, \alpha_{\mathcal{K}}]$, the proof of Theorem 4.1 is completed.

REMARK Note that by (4.52) we also have a uniform estimate of the γ-Holder norm of u, a solution to the fixed boundary value problem (4.5), on $\Omega \cup \tilde{\Sigma}$. Namely, since

$$|u|_\gamma = |u|^{(-\gamma)}_\gamma \leq C(\gamma, \mathrm{diam}(\Omega))|u|^{(-\gamma)}_{1+\min\{\alpha_k,\gamma\}},$$

we have

$$u \in H_{\gamma;\Omega\cup\Sigma} \quad \text{and} \quad |u|_\gamma \leq C, \tag{4.53}$$

for a constant C depending on γ, the size of the domain Ω, bounds on the ellipticity ratio and on the minimal eigenvalue of the operator $\tilde{Q}$, the bounds on the obliqueness constant of the operator $\tilde{N}$ and on the supremum norm $|u|_0$, and the Holder seminorm $[\tilde{f}]_{\gamma;\mathcal{R}}$.

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5 Solution to the modified free boundary value problem

In this section we prove Theorem 3.1. The main idea is to fix the function $\rho(\eta)$, $\eta \in [0, \eta^*]$, specifying the boundary $\Sigma = \{\rho(\eta), \eta) : \eta \in (0, \eta^*)\}$, find a solution $u(\rho, \eta)$ of the fixed boundary value problem (3.17), (3.20)-(3.23) using Theorem 4.1, and then, update the boundary Σ to $\{\tilde\rho(\eta),\eta) : \eta \in (0,\eta^*)\}$, using the shock evolution equation (3.11). More precisely, we find $\tilde\rho$ as a solution of the following initial value problem

$$\frac{d\tilde\rho}{d\eta} = -\sqrt{\psi\left(\tilde\rho(\eta) - \frac{u(\rho(\eta),\eta)+1}{2}\right)}, \tag{5.1}$$

$$\tilde\rho(0) = \xi_a. \tag{5.2}$$

We define a map J so that $J(\rho) = \tilde\rho$. To prove Theorem 3.1, we show that the map J has a fixed point. We use the following

Theorem. (Corollary 11.2 in [8]) Let $\mathcal{K}$ be a closed and convex subset of a Banach space $\mathcal{B}$ and let $J : \mathcal{K} \to \mathcal{K}$ be a continuous mapping so that $J(\mathcal{K})$ is precompact. Then J has a fixed point.

We choose the space $\mathcal{B} = H_{1+\alpha_{\mathcal{K}}}$, and we take the set $\mathcal{K} \subset \mathcal{B}$ as in §4.2. In this section we further specify the parameters γ and $\alpha_{\mathcal{K}}$, and δ_*, ρ_L and ρ_L in the definition (4.3)-(4.4) of the set $\mathcal{K}$ so that the hypothesis of this fixed point theorem are satisfied.

Let $a > \sqrt{2}$, $\eta^* > 0$, $\epsilon_* \in (0, u_* - 1)$ and $\delta > 0$ be arbitrary. We make the following choices for δ_*, ρ_L and ρ_R:

- δ_* depends on the particular value u_* we consider so that

$$\begin{aligned} &\text{- if } U_* = U_R, \text{ then } 0 < \delta_* < \min\left\{\tfrac{a^2-1+a\sqrt{a^2-2}}{2}, \left(\tfrac{\epsilon_*}{2\eta^*}\right)^2\right\}, \\ &\text{- if } U_* = U_F, \text{ then } 0 < \delta_* < \min\left\{\tfrac{a^2-1-a\sqrt{a^2-2}}{2}, \left(\tfrac{\epsilon_*}{2\eta^*}\right)^2\right\}, \end{aligned} \tag{5.3}$$

- the definition of ρ_L depends on η^*, and

$$\begin{aligned} &\text{- if } \eta^* \in \left(0, \sqrt{\xi_a - 1}\right], \text{ then } \rho_L(\eta) := \xi_a - \eta\sqrt{\xi_a - 1}, \eta \in [0,\eta^*], \\ &\text{- if } \eta^* > \sqrt{\xi_a - 1}, \text{ then } \rho_L(\eta) := \begin{cases} \xi_a - \eta\sqrt{\xi_a-1}, & \eta \in [0, \sqrt{\xi_a - 1}], \\ 1, & \eta \in (\sqrt{\xi_a-1}, \eta^*], \end{cases} \end{aligned} \tag{5.4}$$

- ρ_R is defined by

$$\rho_R(\eta) := \xi_a - \eta\sqrt{\delta_*}, \quad \eta \in [0,\eta^*]. \tag{5.5}$$

Clearly, the set $\mathcal{K}$ defined in §4.2 with the above specifications of the parameter δ_* and the curves $\rho_L(\eta)$ and $\rho_R(\eta)$, $\eta \in [0,\eta^*]$, is a well-defined, closed and convex subset of the Banach space $\mathcal{B}$.

Lemma 5.1 *Let $a > \sqrt{2}$, $\eta^* > 0$ and $\epsilon_* \in (0, u_* - 1)$ be given. If δ_* is chosen as in* (5.3), *then the curve $\tilde\rho$ given by* (5.1) *and* (5.2) *is decreasing and satisfies the following lower bound*

$$\tilde\rho(\eta) > 1, \text{ for all } \eta \in [0,\eta^*]. \tag{5.6}$$

PROOF By the definition (3.12) of the function ψ and the equation (5.1) for $\tilde\rho'$, we have

$$\tilde\rho'(\eta) \le -\sqrt{\delta_*} < 0, \qquad \text{for all } \eta \in (0,\eta^*), \tag{5.7}$$

implying the desired monotonicity of the curve $\tilde\rho$.

Next we show (5.6). If $\eta_0 \in [0,\eta^*]$ is such that

$$\tilde\rho(\eta_0) - \frac{u(\rho(\eta_0),\eta_0)+1}{2} \ge \delta_*, \tag{5.8}$$

then certainly $\tilde{\rho}(\eta_0) - \frac{u(\rho(\eta_0),\eta_0)+1}{2} > 0$, implying

$$\tilde{\rho}(\eta_0) - 1 > \frac{u(\rho(\eta_0),\eta_0) - 1}{2} \geq \frac{\epsilon_*}{2}, \tag{5.9}$$

by Lemma 4.2. It is easy to check that (5.8) holds for $\eta_0 = 0$, using the expression (2.9) for $u_* = u(\xi_a, 0)$ and the bounds (5.3) on δ_*. Further, note that the change of $\tilde{\rho}$ over the values of η_0 for which (5.8) does not hold is $\sqrt{\delta_*}$ per unit interval in η. This implies that the total change of $\tilde{\rho}$ over the interval $[0,\eta^*]$ is bounded above by $\sqrt{\delta_*}\eta^*$. However, our choice (5.3) of δ_* implies

$$\sqrt{\delta_*}\eta^* < \frac{\epsilon_*}{2\eta^*}\eta^* = \frac{\epsilon_*}{2}. \tag{5.10}$$

From (5.9) and (5.10) we get that $\tilde{\rho}(\eta) - 1 > 0$, for all $\eta \in [0,\eta^*]$. ▯

REMARK Note that for every $\eta \in [0,\eta^*]$ we have the following lower bound

$$\rho(\eta) - 1 > \frac{\epsilon_*}{2} - \sqrt{\delta_*}\eta^* > 0,$$

which can be estimated in terms of only a, η^* and ϵ_* using (5.3).

Lemma 5.2 *Let $a > \sqrt{2}$, $\eta^* > 0$, $\epsilon_* \in (0, u_* - 1)$ and $\delta > 0$ be given and suppose that δ_*, ρ_L and ρ_R are chosen as in* (5.3)-(5.5).

There exists a parameter $\gamma_0 > 0$, depending on a, η^*, ϵ_* and δ, such that for any $\gamma \in (0, \min\{\gamma_0, 1\})$ and $\alpha_{\mathcal{K}} = \frac{\gamma}{2}$ we have

(a) $J(\mathcal{K}) \subseteq \mathcal{K}$, and

(b) the set $J(\mathcal{K})$ is precompact in $H_{1+\alpha_{\mathcal{K}}}$.

PROOF Let the boundary Σ be given by a curve $\rho \in \mathcal{K}$ and let $u(\rho,\eta) \in H^{(-\gamma)}_{1+\alpha_{\mathcal{K}}}$ be a solution of the fixed boundary value problem found by Theorem 4.1. Recall that $\gamma \in (0, \min\{\gamma_0, 1\})$ is arbitrarily chosen, where γ_0 is a parameter depending on the size of opening angles of the domain Ω at the corners, and on the bound on the ellipticity ratio for $\tilde{Q}$. By Remark 4.4, γ_0 depends only on the fixed parameters a, η^*, ϵ_* and δ. Recall also that $\alpha_{\mathcal{K}} \in (0, \min\{1, 2\gamma\})$ is arbitrary. To prove this lemma we will take γ_0 smaller, still depending only on the a priori fixed parameters a, η^*, ϵ_* and δ, and we will specify $\alpha_{\mathcal{K}} = \gamma/2$.

Let $\tilde{\rho}(\eta)$, $\eta \in [0,\eta^*]$, be a solution of the initial-value problem (5.1), (5.2). To show part (a) we need to show that $\tilde{\rho} \in \mathcal{K}$.

Clearly, $\tilde{\rho}(0) = \xi_a$ and

$$\tilde{\rho}'(0) = -\sqrt{\psi\left(a^2 + \frac{1}{2} - \frac{u_* + 1}{2}\right)} = k_*,$$

by taking u_* and k_* as in (2.9) and (2.11), with respect to the particular value of the parameter a as in (2.10). This shows (4.2).

That the curve $\tilde{\rho}$ satisfies the right inequality of (4.3) has been shown in (5.7). To show the left side of (4.3) note that

$$\begin{aligned}\max_{\eta\in[0,\eta^*]} \left\{\tilde{\rho}(\eta) - \frac{u(\rho(\eta),\eta)+1}{2}\right\} &\leq \max_{\eta\in[0,\eta^*]} \tilde{\rho}(\eta) - \min_{\eta\in[0,\eta^*]} \frac{u(\rho(\eta),\eta)+1}{2} \\ &\leq \xi_a - \frac{2+\epsilon_*}{2} < \xi_a - 1,\end{aligned}$$

by Lemma 4.2 and the choice of f (see (2.16)). Therefore, $\tilde{\rho}'(\eta) \geq -\sqrt{\xi_a - 1}$, for all $\eta \in (0, \eta^*)$.

Next we check that $\tilde{\rho}$ satisfies (4.4). Note that the left side of (4.3) implies $\tilde{\rho}(\eta) \geq \xi_a - \eta\sqrt{\xi_a - 1}$, $\eta \in [0, \eta^*]$. Since (5.6) also holds, we get $\tilde{\rho}(\eta) \geq \rho_L(\eta)$, $\eta \in [0, \eta^*]$, where ρ_L is given by (5.4). On the other hand, the inequality (5.7) implies $\tilde{\rho}(\eta) \leq \xi_a - \eta\sqrt{\delta_*} = \rho_R(\eta)$, for all $\eta \in [0, \eta^*]$.

To complete the proof of (a), it is left to show

$$\tilde{\rho} \in H_{1+\alpha_{\mathcal{K}}}. \tag{5.11}$$

We recall the estimate (4.49) which is independent of u, and we replace the parameter γ_0 by $\min\{\gamma_0, \alpha_0\}$. Again we note that both γ_0 and α_0 depend only on the a priori fixed parameters a, η^*, ϵ_* and δ. Let

$$0 < \gamma < \min\{\gamma_0, 1\}, \tag{5.12}$$

Using the differential equation (5.1), it follows that $|\tilde{\rho}'|_\gamma \leq C$ and, hence,

$$|\tilde{\rho}|_{1+\gamma} \leq C\eta^*. \tag{5.13}$$

Therefore, $\tilde{\rho} \in H_{1+\gamma}$ and to ensure (5.11), we need to take $\alpha_{\mathcal{K}} \in (0, \gamma]$.

Moreover, since (5.13) holds uniformly in $\tilde{\rho}$, we have that $J(\mathcal{K})$ is contained in a bounded set in $H_{1+\gamma}$. To show (b) it suffices to choose $\alpha_{\mathcal{K}} \in (0, \gamma)$. We take $\alpha_{\mathcal{K}} := \gamma/2$. ∎

We note that the map $J : \mathcal{K} \to \mathcal{K}$ given by (5.1)-(5.2) is also continuous. Therefore, by taking the parameter γ as in (5.12) and choosing $\alpha_{\mathcal{K}} = \gamma/2$, we have that the hypotheses of the fixed point theorem from the beginning of this section (Corollary 11.2 in [8]) are satisfied. Hence, the map J has a fixed point $\rho \in \mathcal{K}$. We use this curve $\rho(\eta)$, $\eta \in [0, \eta^*]$, to specify the boundary Σ in Theorem 4.1 and we get a solution $u \in H^{(-\gamma)}_{1+\alpha_*}$, for all $\alpha_* \in (0, \alpha_{\mathcal{K}}]$. This completes the proof of Theorem 3.1.

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6 The proof of Theorem 2.1

In order to derive Theorem 2.1 from Theorem 3.1, we need to see whether we can remove the cut-off functions we have introduced in Theorem 3.1. More precisely, we need to investigate under which conditions it is possible to replace the functions ϕ and ψ by the identity function and to replace χ by β.

Let $u \in H^{(-\gamma)}_{1+\alpha_*}$, for $\alpha_* \in (0, \alpha_{\mathcal{K}}]$, be a solution to the modified free boundary value problem (4.5) in Theorem 3.1, and suppose that $v \in H^{(-\gamma)}_{1+\alpha_*}$ is recovered using the equation (3.5).

First, we recall from Proposition 1 that a priori bounds on u imply $\chi = \beta$, meaning that the operators N and $\tilde{N}$ are the same.

To remove the cut-off ϕ we prove the following

Lemma 6.1 *Suppose that $u \in C^1(\Omega)$ is a solution to the fixed boundary value problem* (4.5) *and define a function*

$$w(\rho, \eta) := u(\rho, \eta) - \rho, \quad (\rho, \eta) \in \Omega. \tag{6.1}$$

Then

(a) w attains its minimum on $\sigma \cup \Sigma \cup \Xi_a$, and

(b) w cannot attain a non-positive minimum on Σ.

PROOF First we show (a). Using the equation $\tilde{Q}(u) = 0$ in (4.5) and the definition (6.1), we obtain the following second order uniformly elliptic equation for w

$$\phi(w)w_{\rho\rho} + w_{\eta\eta} + \left(\phi'(w) + \frac{1}{2}\right) w_\rho + \frac{1}{2} + \phi'(w)(w_\rho)^2 = 0.$$

Using the Minimum Principle (Theorem 3.5 in [8]), we have that w must attain its minimum on $\partial\Omega$. Suppose there is a minimum $X_0 \in \Sigma_0$. By (4.5) we have $w_\eta(X_0) = 0$, which contradicts Hopf's Lemma (Lemma 3.4 in [8]). Hence, the minimum must occur on $\partial\Omega\backslash\Sigma_0 = \sigma \cup \Sigma \cup \Xi_a$.

Next we show (b). Assume there is $X_0 \in \Sigma$ so that $w(X_0) = \min_\Omega w$. Since X_0 is the minimum of w, the tangential derivative of w along Σ at X_0 must be zero, i.e.,

$$0 = (w_\rho(X_0), w_\eta(X_0)) \cdot (\rho'(X_0), 1) = \rho'(X_0)(u_\rho(X_0) - 1) + u_\eta(X_0),$$

yielding

$$u_\eta(X_0) = -\rho'(X_0)(u_\rho(X_0) - 1). \tag{6.2}$$

On the other hand, Hopf's Lemma (Lemma 3.4 in [8]) implies that the derivative of w in the direction of an outward normal to Σ at X_0 must be negative, meaning

$$0 > (w_\rho(X_0), w_\eta(X_0)) \cdot (1, -\rho'(X_0)) = u_\rho(X_0) - 1 - \rho'(X_0)u_\eta(X_0). \tag{6.3}$$

We substitute (6.2) in (6.3) to find

$$u_\rho(X_0) < 1. \tag{6.4}$$

Next we use the oblique derivative boundary condition in (4.5) with β given by (3.7) and substitute (6.2) to obtain

$$u_\rho(X_0)\rho'(X_0)\frac{u(X_0) - 1}{4} + \rho'(X_0)\left(\frac{5u(X_0) + 3}{8} - \rho(X_0)\right) = 0.$$

Since $u > 1$ on Ω (see Proposition 1), $\rho' < 0$ on Σ (see the definition of set $\mathcal{K}$ at the beginning of §4.2) and (6.4) holds, we have

$$\rho'(X_0)\frac{u(X_0) - 1}{4} + \rho'(X_0)\left(\frac{5u(X_0) + 3}{8} - \rho(X_0)\right) < 0,$$

and using that $u(X_0) = \rho(X_0) + w(X_0)$, this yields

$$\rho'(X_0)\frac{-\rho(X_0) + 7w(X_0) + 1}{8} < 0.$$

Since $\rho'(X_0) < 0$, we obtain

$$w(X_0) > \frac{\rho(X_0) - 1}{7}. \tag{6.5}$$

We recall that $\rho > 1$ on Σ (see (5.6)), and, hence, (6.5) implies $w(X_0) > 0$, as desired. ∎

Note that the cut-off function ϕ differs from the identity function only if $u(\rho, \eta) - \rho < \delta$. In the previous lemma we showed that the function $u(\rho, \eta) - \rho$, $(\rho, \eta) \in \Omega$ attains its minimum on $\sigma \cup \Sigma \cup \Xi_a$. It is easy to calculate that at Ξ_a we have

$$u(\Xi_a) - \xi_a = u_* - \left(a^2 + \frac{1}{2}\right) =: m_1(a) > 0,$$

for both choices of u_* in (2.10). Further, on σ we have

$$u(\rho,\eta)-\rho = f(\eta)-\left(\xi(\eta^*)+\frac{\eta^2}{4}\right) =: m_2(a,\eta^*,\epsilon_*) > 0,$$

by the definition (2.16) of f and the bounds on the closed set $\mathcal{K}$ (see Remark 2.2). Next, in the previous lemma we showed that if $u(\rho,\eta)-\rho$ attains its minimum at $X_0 \in \Sigma$, then this minimum must be positive. More precisely, using (6.5) and Remark 5 we have

$$u(X_0)-\rho(X_0) > \frac{\rho(X_0)-1}{7} =: m_3(a,\eta^*,\epsilon_*) > 0.$$

We choose δ in the definition of ϕ so that

$$0 < \delta < \min\{m_1, m_2, m_3\},$$

with m_1, m_2 and m_3 as above and depending only on a, η^* and ϵ_*. Therefore, the function ϕ is the identity and the operators Q and $\tilde{Q}$ are the same.

Finally, we note that the cut-off function ψ is identity as long as

$$\rho \geq \frac{u_*+1}{2} + \delta_*. \tag{6.6}$$

Using the values (2.9) for u_* and (5.3) for δ_* we obtain $\frac{u_*+1}{2} + \delta_* < \xi_a = \rho(\Xi_a)$. Since it is possible that for some choices of a and δ_* we have $\frac{u_*+1}{2} + \delta_* > 1$, the cut-off ψ can be removed only in a neighborhood of the reflection point Ξ_a.

This completes the proof of Theorem 2.1.

References

[1] S. Čanić, B. L. Keyfitz, *Riemann problems for the two-dimensional unsteady transonic small disturbance equation*, SIAM Journal on Applied Mathematics, **58** (1998), 636–665.

[2] S. Čanić, B. L. Keyfitz, G. M. Lieberman, *A proof of existence of perturbed steady transonic shocks via a free boundary problem*, Communications on Pure and Applied Mathematics, **LIII** (2000), 484–511.

[3] S. Čanić, B. L. Keyfitz, E. H. Kim, *Free boundary problems for the unsteady transonic small disturbance equation: transonic regular reflection*, Methods and Applications of Analysis, **7** (2000), 313–336.

[4] S. Čanić, B. L. Keyfitz, E. H. Kim, *A free boundary problem for a quasi-linear degenerate elliptic equation: regular reflection of weak shocks*, Communications on Pure and Applied Mathematics, **LV** (2002), 71–92.

[5] S. Čanić, B. L. Keyfitz, E. H. Kim, *Free boundary problems for nonlinear wave systems: Mach stems for interacting shocks*, submitted.

[6] T. Chang, G. Q. Chen, *Diffraction of planar shocks along compressive corner*, Acta Mathematica Scientia, **3** (1986), 241–257.

[7] D. Gilbarg, L. Hormander, *Intermediate Schauder estimates*, Arch. Rational Mech. Anal., **74** (1980), no.4 297–318.

[8] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer - Verlag, New York, 2nd edition, 1983.

[9] B. L. Keyfitz, *Self-Similar Solutions of Two-Dimensional Conservation Laws*, Journal of Hyperbolic Differential Equations, **1** (2004), 445–492.

[10] G. M. Lieberman, *The Perron process applied to oblique derivative problems*, Advances in Mathematics, **55** (1985), 161–172.

[11] G. M. Lieberman, *Mixed boundary value problems for elliptic and parabolic differential equations of second order*, Journal of Mathematical Analysis and Applications, **113** (1986), 422–440.

[12] G. M. Lieberman, *Optimal Holder regularity for mixed boundary value problems*, Journal of Mathematical Analysis and Applications, **143** (1989), 572–586.

[13] G. M. Lieberman, *Local estimates for subsolutions and supersolutions of oblique derivative problems for general second order elliptic equations*, Transactions of the American Mathematical Society, **304** (1987), no.1 343–353.

[14] G. M. Lieberman, N. S. Trudinger, *Nonlinear oblique boundary value problems for nonlinear elliptic equations*, Transactions of the American Mathematical Society, **295** (1986), no.2 509–546.