Viscous Singular Shock Structure for a Nonhyperbolic Two-Fluid Model

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Abstract. We consider a system of two nonhyperbolic conservation laws modeling incompressible two-phase flow in one space dimension. The purpose of this paper is to justify the use of singular shocks in the solution of Riemann problems. We prove that both strictly and weakly overcompressive singular shocks are limits of viscous structures. Using Riemann solutions we solve Cauchy problems with piecewise constant data for the nonhyperbolic two-fluid model.

1 Introduction

This paper examines a nonhyperbolic system arising in the modeling of incompressible two-fluid flows. The principal result we obtain here is that approximate viscous profiles can be constructed for both strictly and weakly overcompressive singular shocks that are important in solving Riemann problems. In addition, we use this stability result for Riemann solutions to show that a class of Cauchy problems for the nonhyperbolic two-fluid model can be solved.

We consider a system of two equations, $w_t+q(w)_x = 0$, for a state $w = (\beta, v)^\top$, which for the two-fluid model takes the form

$$\begin{aligned} \beta_t + \left(vB_1(\beta) + K\beta\right)_x &= 0 \\ v_t + \left(v^2 B_2(\beta) + Kv\right)_x &= 0, \end{aligned} \tag{1.1}$$

with

$$B_1(\beta) = \frac{(\beta-\rho_1)(\beta-\rho_2)}{\beta}, \qquad B_2(\beta) = \frac{\beta^2-\rho_1\rho_2}{2\beta^2}. \tag{1.2}$$

This system is equivalent to the system of four equations modeling one-dimensional unsteady isothermal incompressible two-phase flow, [2, p 248]. The volume fractions α_1 and $\alpha_2 = 1-\alpha_1$ have been replaced by a density-weighted volume element

$$\beta = \rho_2\alpha_1 + \rho_1\alpha_2,$$

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and the momentum equations replaced by a single equation for the momentum difference,

$$v = \rho_1 u_1 - \rho_2 u_2 - (\rho_1 - \rho_2)K.$$

Here $K(t) = \alpha_1 u_1 + \alpha_2 u_2$ is constant in space as a consequence of conservation of mass. In this paper we take $K = 0$, for convenience. The system (1.1) has been nondimensionalized: the densities ρ_i and β have been scaled by dividing by $\rho_1 - \rho_2$, so we take $\rho_1 - \rho_2 = 1$ throughout and note that ρ_i and β are dimensionless. Hence v has the dimensions of velocity; we may replace v and K by v/v^0 and K/v^0, where v^0 is some characteristic velocity, and simultaneously scale x/t by v^0, making (1.1) dimensionless. Ordinarily, there is a balance term, $G = F_1/\alpha_1 - F_2/\alpha_2$ in the second equation of (1.1); G models buoyancy, drag and other interfacial forces. Research on two-fluid models usually concentrates on correct modeling of the forces F_i. However, the focus of this article is on the mathematical structure of the differential operator on the left side of the system (1.1), and we set G to zero.

It is well known [2, 13] that the system is not hyperbolic, as the characteristics at any mixed-phase state ($\rho_2 < \beta < \rho_1$), are complex. We define $\lambda_j(w)$, $j = 1, 2$, to be the eigenvalues of $dq(w)$, and note that $\lambda_1 = \overline{\lambda}_2$ if $\rho_2 < \beta < \rho_1$, while $\lambda_1 = \lambda_2$ if $\beta = \rho_i$, $i = 1$ or 2. Keyfitz, Sanders and Sever have discussed this equation in earlier papers [4, 6]. Two important facts are that the system, at least by a formal asymptotic expansion, admits solutions of very low regularity, called singular shocks, and that by means of singular shocks one can solve the Riemann problem for (1.1) for all initial discontinuities.

In this paper, we justify the use of singular shocks by showing that they appear as weak limits of smooth solutions to a regularized, viscous system,

$$w_t + q(w)_x = \varepsilon w_{xx}. \tag{1.3}$$

More precisely, we construct a sequence of functions $w^\varepsilon(x, t)$ such that, as $\varepsilon \to 0$,

$$w^\varepsilon(\cdot, t)_t + q(w^\varepsilon(\cdot, t))_x - \varepsilon w^\varepsilon(\cdot, t)_{xx} \rightharpoonup 0, \tag{1.4}$$

weakly in the space of measures on $\mathbf{R}$, pointwise in t. The sequence w^ε converges to a singular shock, in the same sense.

Establishing a relation like (1.4) is a critical step in arguing that singular shock solutions have a physical interpretation for the system modeled by (1.1). As is well known, even for hyperbolic systems of conservation laws, which are linearly well-posed (unlike our nonhyperbolic system), solutions exist in only the weak sense, and establishing well-posedness requires admissibility conditions for shocks. Admissibility conditions can be derived in a number of ways, as described in Smoller's monograph [12], including perturbation of the system by a second-order term representing viscosity, as in (1.3). Another motivation for introducing a viscous approximation like (1.3) is that finite-difference numerical schemes for conservation laws achieve stability by introducing a small amount of dissipation, which can be modeled at the continuous level with a system like (1.3). In principle, one would like to prove admissibility via the use of a physically realistic viscosity, or, equivalently, to prove that the precise form of viscosity does not affect important macroscopic properties of shocks such as their speed and stability. This is important both for establishing physical relevance and for studying the convergence of numerical approximations. The precise form of physical viscosity is not known for the system (1.1). Modeling numerical dissipation is also complicated. 'Artificial' viscosity, as in (1.3), is often chosen in theoretical studies, for convenience. We choose it here, because it is the only choice which leads to relatively simple viscous structures. Finding comparable structures for other forms of viscous perturbation remains an open problem, as we discuss in the concluding section.

In related work, Sanders and Sever construct approximate viscous profiles for nonhyperbolic systems which exhibit a symmetry [9]. The method outlined in [9] applies to the model system (1.1). However, that paper is limited to strictly overcompressive singular shocks, and hence, for the system (1.1), does not include all the shock connections needed to solve Riemann problems. In this paper, we give a complete proof of the construction for (1.1), in both the overcompressive and weakly overcompressive cases.

For a model strictly hyperbolic system originally studied by Keyfitz and Kranzer [5], existence of self-similar shock profiles for singular shocks when self-similar viscosity of the form $\varepsilon t w_{xx}$ is added to the system has recently been proved by Schecter [10]. Schecter uses geometric singular perturbation theory to find smooth solutions for sufficiently small, positive ε. It is possible that a geometric perturbation result can be found for nonhyperbolic systems such as (1.1), perhaps using some new scalings as suggested by this paper. The construction in our paper provides viscous structure for a regularized system with viscosity εw_{xx}, and also gives self-similar viscous structure with viscosity $\varepsilon t w_{xx}$ in some cases, but not in the important case of weakly overcompressive shocks.

Recent work of Kreiss and Yström [7], on the relation of viscous regularizations to their limits, considers a number of model equations, not including (1.1) but with the nonhyperbolic equations of the two-fluid model very much in mind. Their work shows that the existence of certain kinds of energy bounds can prevent unbounded amplitude growth in the parabolic approximations, and suggests that weak convergence to a limit of bounded but high-frequency oscillations may occur instead. Amplitude bounds do not exist for (1.1), and the type of convergence we report here is different from the nonlinear examples in [7]. Numerical experiments by Dinh, Nourgaliev and Theofanous [1] on the model (1.1) confirm the singular behavior predicted by our analysis.

In the next section, we give the definition of a singular shock solution, and also review the singular shock solutions of (1.1), described in detail in [6]. Section 3 states the main theorem, along with the theorem of Sever from [11] which we use to prove it. The construction of $\{w^\varepsilon\}$ is carried out in Section 4, while the proof that this construction provides an approximation as required by the theory in [11] occupies the three sections which follow. As an application, we solve some Cauchy problems for (1.1) in Section 8. The significance of our results is explained in Section 9.

2 Definitions

A singular shock solution to a system $w_t + q(w)_x = 0$ is of the form $w = \omega + \gamma$, where ω, the regular part of the solution, is a discontinuous function and γ, the singular part, is a measure supported on the discontinuity set of ω. A self-similar singular shock has the following structure:

$$\omega(x,t) = \begin{cases} w_-, & x < st \\ w_+, & x > st \end{cases}, \tag{2.1}$$

$$\gamma(x,t) = t\, e(w_-, w_+, s)\delta(x - st), \tag{2.2}$$

where

$$e(w_-, w_+, s) = s(w_+ - w_-) - q(w_+) + q(w_-). \tag{2.3}$$

A measure-valued function such as w does not satisfy the conservation law in the usual weak sense. Hence, it can be difficult to assign a significance to singular shocks. Based on a standard approach to analysing conservation laws by studying limits of regularized approximations, we attempt to justify singular shocks by finding them as limits of solutions of approximating systems. This might mean strong limits in an appropriate measure space, as was carried out by Keyfitz and Kranzer for a hyperbolic model system [5]. However, one can work directly with weak limits [11], and that is the approach we adopt here. In [11, Chapter 3, §1], Sever gives a definition of singular shock solutions, which we use in this paper.

Definition 2.1 *A* singular shock solution *of a system $w_t + q(w)_x = 0$ is a measure of the form*

$$w(x,t) = \omega(x,t) + \sum_i M_i(t)\chi_i(t)\delta(x - x_i(t)),$$

where ω is a classical weak solution away from the singularities, χ_i is the characteristic function of an interval $[A_i, B_i)$; $M_i \in L^\infty$, and $x_i \in W^{1,\infty}$. The function w is the weak limit of a sequence w^ε with $w^\varepsilon(\cdot,t) \in L^1_{loc}$ uniformly with respect to ε, pointwise in t; satisfying

$$w^\varepsilon(\cdot,t) \rightharpoonup w(\cdot,t) \quad \textit{and} \quad w^\varepsilon(\cdot,t)_t + q(w^\varepsilon(\cdot,t))_x - \varepsilon(Aw^\varepsilon(\cdot,t)_x)_x \rightharpoonup 0, \tag{2.4}$$

as $\varepsilon \to 0$, weakly in the space of measures on $\mathbf{R}$, pointwise with respect to t, for some positive definite matrix A.

Remark 2.2 A function $f(\cdot,\varepsilon)$ is said to be in L^1_{loc} *uniformly with respect to* ε if, for any set $J \subset \mathbf{R}$ of finite measure, $\int_J |f(\cdot,\varepsilon)| \leq M = M(J)$, where M is independent of ε. In general, a property holds *uniformly with respect to a parameter* if the bound which establishes the property is independent of the parameter.

Remark 2.3 The matrix A need not be constant in this definition. Two important cases are the scalar matrices $A = I$ and $A = It$, corresponding to the standard concept of 'artificial' viscosity. These are the only choices which appear to give simple viscous structures for singular shocks. The significance of this is discussed in the concluding section of the paper.

The measures defined this way generalize the self-similar singular shocks described by (2.1), (2.2) and (2.3); however, for any candidate satisfying (2.1)–(2.3), one must verify (2.4) to establish that the object is indeed a singular shock. In [11], Sever established some important properties of singular shocks. First, the singular mass $M(t)$ evolves according to the Rankine-Hugoniot deficit, defined in equation (2.3) above. Along the i-th singular shock,

$$\frac{dM_i(t)}{dt} = e\big(w(x_i(t)-0,t), w(x_i(t)+0,t), x_i'(t)\big). \tag{2.5}$$

Second, singular shocks are generally overcompressive discontinuities; for equations with real characteristics this means $\lambda_j(w_-) \geq s \geq \lambda_j(w_+)$, where $j = 1, 2$, $s = x_i'$, and $w_\pm$ are the limits of w from the right and left sides of the discontinuity. Furthermore, for systems with real characteristics, the existence of convex entropies implies a direction for the vector e. In a number of hyperbolic model problems this direction picks out a coordinate system such that the first component of e is identically zero, which means that the first Rankine-Hugoniot condition holds.

The last two points do not immediately generalize to our system (1.1) with complex characteristic speeds. However, in [6] it was shown that if we use a generalization of the overcompressibility condition, obtained by examining the real part of the characteristic speeds,

$$\mathrm{Re}\big(\lambda_j(w_-)\big) \geq s \geq \mathrm{Re}\big(\lambda_j(w_+)\big), \quad j = 1, 2, \tag{2.6}$$

and if we satisfy the first Rankine-Hugoniot condition at each singular shock jump, then there exist solutions of the form (2.1), (2.2), (2.3) by means of which one can solve all Riemann problems. The case was also made in [6], based on the derivation of (1.1), that β in the first equation is genuinely conserved, while the rationale for the form of the second equation is weaker. Hence it seems appropriate to satisfy the first Rankine-Hugoniot relation.

In the remainder of this paper, we shall assume condition (2.6), and also take $A = I$ or It in Definition 2.1. An explanation of this choice is given in Section 9.

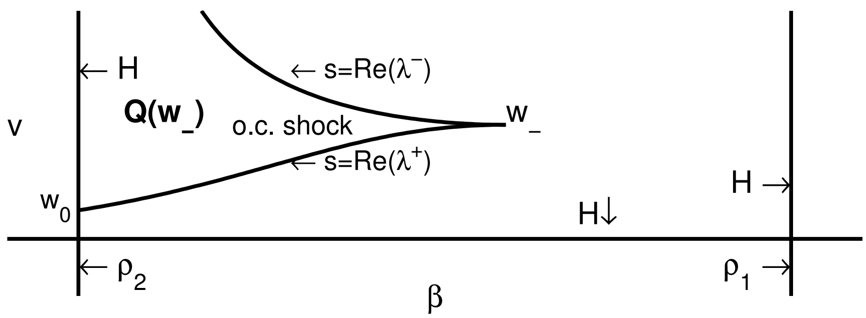

Figure 2.1: The Region Q of Overcompressive Shocks

The Riemann solution of (1.1) is not unique [6]. We assume that the first Rankine-Hugoniot condition is satisfied, so that

$$s = \frac{[vB_1]}{[\beta]} \equiv \frac{v_+ B_1(\beta_+) - v_- B_1(\beta_-)}{\beta_+ - \beta_-} \tag{2.7}$$

is determined by w_- and w_+. For any w_-, we define $Q(w_-)$ to be the set of points w_+ in phase space such that (2.6), which is now a condition on w_+, holds. Then, two solutions exist for each $w_+ \in Q(w_-)$. One solution consists of a single, singular shock, while the other exhibits a complicated wave pattern consisting of singular shocks, rarefactions, and a contact discontinuity [6]. When w_+ is not in $Q(w_-)$, then only the second type of solution can be constructed. The wave in this case is always composite, although if w_- and w_+ are in different vertical half-planes there is only a single rarefaction and no contact discontinuity. Furthermore, in a composite wave, the singular shock connects w_- with a particular state, w_0, the point where ∂Q, the boundary of Q, intersects H, which we define to be the set of points in the physical region with real characteristics. See Figure 2.1. A shock between w_- and a point on ∂Q is only weakly overcompressive.

One detail of the composite Riemann solutions is that two points on the boundary of Q are needed to construct a composite wave: $w_0(w_-)$ and the analogous point, which we called $w_1(w_+)$ in [6], in H, which can be joined to w_+ on the right with a weakly overcompressive singular shock with $\lambda_1(w_1) = \lambda_2(w_1) = s > \mathrm{Re}(\lambda_j(w_+))$, $j = 1, 2$. In this paper, we show that these points in $\partial Q \cap H$ admit viscous profile connections.

Without loss of generality, we concentrate on the right half of the profile; in particular, we consider only shocks where strict overcompressibility fails at the state on the right. That is,

$$\mathrm{Re}\big(\lambda_j(w_-)\big) \geq s = \mathrm{Re}\big(\lambda_j(w_+)\big), \quad j = 1, 2. \tag{2.8}$$

The conclusions of the paper also apply if strict overcompressibility fails on the left instead, or even if it fails on both right and left, as the convergence arguments at the right and left endpoints proceed independently of each other.

There are three ways that singular shocks which are weakly overcompressive on the right might arise in (1.1). In examining Riemann problems, one finds that states w_- with $\beta_- = \rho_1$ and $v_- > 0$ can be connected to $w_+ = (\rho_2, 0)$ via a weakly overcompressive discontinuity satisfying (2.7) and (2.8) with speed $s = 0$. The most important case we consider, however, is that with w_- in the interior of the physical region and $w_+ \in \partial Q(w_-) \cap H$, so $w_+ = w_0(w_-)$, as in Figure 2.1: these are the singular shocks which appear in composite waves. Finally, it is in principle possible that a state $w_+ \in \partial Q(w_-) \cap \{\rho_2 < \beta < \rho_1\}$ could be joined to w_- via a singular shock. Such connections are not necessary to solve Riemann problems, since composite waves are also available. Singular shocks of this form do not appear to have viscous structure.

A principal difference between strictly and weakly overcompressive shocks, as pointed out in [6], is the rate of convergence of viscous trajectories to their end states, which is exponential in the case of strictly overcompressive shocks, but not in the weakly overcompressive case. In this paper, we show that convergence holds when $w_+ \in H$, and it is strong enough to prove the existence of singular shocks in the sense of Definition 2.1.

3 Viscous Profiles

We review the construction in [11], which generates a sequence of functions satisfying (1.4). To motivate this, recall that the classical traveling wave construction for conservation laws uses the similarity variable $\xi = (x - st)/\varepsilon$ to find solutions of $w_t + q(w)_x = \varepsilon w_{xx}$ in the form

$$w = y(\xi). \tag{3.1}$$

This leads to an ordinary differential equation for y:

$$y'' = (q(y) - sy)'; \tag{3.2}$$

and if we now assume that s is the shock speed corresponding to a pair of states $w_\pm$ which satisfy the Rankine-Hugoniot relation $s(w_+ - w_-) = q(w_+) - q(w_-)$, then we can integrate (3.2) to obtain $y' = (q(y) - sy) + C$, where C is the common value of $sw_\pm - q(w_\pm)$. For given data $w_\pm$ and s, we can assume $C = 0$. Thus the fundamental dynamical system in the study of classical shock profiles is

$$y' = q(y) - sy. \tag{3.3}$$

However, singular shocks connect states which do not satisfy the Rankine-Hugoniot relation, and so the premise underlying (3.1) is flawed. We instead let

$$w^\varepsilon(x, t) = y\left(\frac{x - st}{\varepsilon}, \frac{t}{\varepsilon}\right) \equiv y(\xi, \tau), \tag{3.4}$$

and define

$$z(\xi,\tau) = y_\xi - q(y) + sy; \tag{3.5}$$

the function z measures, in some sense, the extent to which the problem fails to have a classical viscous profile solution. Then the theorem proved in [11] is

Theorem 3.1 (Theorem 3.3.1 of [11]) *Suppose that there exist functions y and z, related by* (3.5)*, such that*

(1) *$z(\cdot,\tau)$ is constant for $|\xi|$ sufficiently large;*

(2) *$\lim_{\xi\to\pm\infty} y(\xi,\tau) = w_\pm$, uniformly in τ;*

(3) *$y_\tau(\cdot,\tau) \in L^1(\mathbf{R})$ uniformly in τ;*

(4) *$\int_{\mathbf{R}} y_\tau(\xi,\tau)\,d\xi = e(w_-,w_+,s) + o(1)$ as $\tau\to\infty$;*

(5) *$y(\cdot,1) \in \mathrm{BV}$ and $z(\cdot,\tau)\in \mathrm{BV}$ uniformly in τ.*

Then the sequence $\{w^\varepsilon\}$ defined by (3.4) *satisfies Definition 2.1 with $A = I$.*

(See Remark 2.2 for the interpretation of uniformity in τ.) Sever [11] also proves that if an additional constraint holds:

$$\int_{\mathbf{R}} |\xi|\big(|y_\xi(\xi,1)| + |y_\tau(\xi,\tau)|\big)\,d\xi < \infty, \tag{3.6}$$

then we can define

$$w^\varepsilon(x,t) = y\left(\frac{x-st}{\varepsilon t}, \frac{1}{\varepsilon}\right), \tag{3.7}$$

which satisfies Definition 2.1 with $A = tI$. (In this sense, there is not much difference between the standard and the self-similar viscosity criteria.)

Using this theorem, the question of finding singular shocks can be reduced to finding a pair of functions y and z which satisfy the hypotheses of Theorem 3.1. The theorem we prove in this paper is

Theorem 3.2 *For initial conditions w_-, $w_+\in Q(w_-)$, and $s = [vB_1]/[\beta]$ satisfying either*

$$\mathrm{Re}(\lambda_j(w_-)) > s > \mathrm{Re}(\lambda_j(w_+)), \quad j = 1,2, \tag{3.8}$$

or $w_+ \in H \cap \partial Q(w_-)$ and

$$\mathrm{Re}(\lambda_j(w_-)) \ge s = \lambda_1(w_+) = \lambda_2(w_+), \quad j = 1,2, \tag{3.9}$$

there exists a sequence $\{w^\varepsilon\}$ of approximate viscous profile connections satisfying (2.4)*. Hence singular shock connections exist.*

4 The Construction of Approximate Viscous Profiles

If one applies a formal asymptotic expansion to a solution concentrated at a singular shock (with β remaining bounded, as is physically reasonable, but with no constraint on v), as was done in [6], then the problem reduces, to dominant order, to equation (3.3) with $s = 0$ (equation (4.2) below). We now show that, as suggested by the formal treatment, there are viscous structures satisfying Definition 2.1.

The construction is based on the ansatz that near the singular shock locus the viscous solution w^ε behaves like a solution to the simple system (3.3), $w' = q(w) - sw$, while far away, as $\xi \to \pm\infty$ it tends to solutions of different systems $w' = q(w) - sw - C_\pm$, where $C_\pm = q(w_\pm) - sw_\pm$. Furthermore, in the central interval, an approximate solution can be found by a simple scale change.

The point of departure of the construction is the observation that the system (1.1) possesses a scaling invariance: If

$$t \mapsto t' = t/\eta, \quad v \mapsto v' = \eta v, \tag{4.1}$$

then solutions of $w_t + q(w)_x = 0$ are mapped to solutions of $(\beta, v')^\top_{t'} + q(\beta, v')^\top_x = 0$. Also to the point is the observation that the flux vector field, $q(w)$, admits a one-parameter family of heteroclinic connections. The dynamical system

$$w' = q(w) \tag{4.2}$$

is just (3.3) with $s = 0$; in Section 4.1, we explain the role of s in comparing (4.2) to (3.3).

The heteroclinic orbits of (4.2) connect points $(\rho_1, 0)$ and $(\rho_2, 0)$; we shall assume $v \geq 0$ and then $W_- = (\rho_1, 0)$, $W_+ = (\rho_2, 0)$. The entire physical region with $v > 0$ and $\rho_2 \leq \beta < \rho_1$ is in the stable set for W_+. There is a similar construction when $v \leq 0$, with $(\rho_1, 0)$ the attractor.

The heteroclinic connections can be found explicitly.

Proposition 4.1 *For the functions B_1 and B_2 given by equation (1.2), solutions to the system (4.2) in the strip $(\rho_2, \rho_1) \cap \{v > 0\}$ form a one-parameter family with asymptotic behavior*

$$\beta \sim \rho_2 + \frac{4\rho_2^3}{V^2\xi^2} \quad \text{and} \quad v \sim \frac{2\rho_2}{\xi} = -\frac{1}{B_2(\rho_2)\xi}, \quad \text{as} \quad \xi \to \infty, \tag{4.3}$$

$$\beta \sim \rho_1 - \frac{4\rho_1^3}{V^2\xi^2} \quad \text{and} \quad v \sim -\frac{2\rho_1}{\xi} = -\frac{1}{B_2(\rho_1)\xi}, \quad \text{as} \quad \xi \to -\infty, \tag{4.4}$$

where the parameter V is a positive constant.

Proof: We write

$$\frac{d\beta}{dv} = \frac{1}{v}\frac{B_1}{B_2},$$

and solve by separating the variables, recalling that $B_2 = B_1'/2$, to obtain the family of solutions

$$v = V\sqrt{|B_1|} = V\left|\frac{(\beta-\rho_1)(\beta-\rho_2)}{\beta}\right|^{\frac{1}{2}}, \tag{4.5}$$

for any positive constant V. Thus the equation for $\beta(\xi)$ is

$$\frac{d\beta}{d\xi} = -V\left(\frac{(\rho_1-\beta)(\beta-\rho_2)}{\beta}\right)^{3/2}, \tag{4.6}$$

and $\xi(\beta)$ is given by quadrature; since $\beta \to \rho_2$ as $\xi \to \infty$, then from (4.6),

$$\frac{d\beta}{d\xi} \sim -V\left(\frac{\beta-\rho_2}{\rho_2}\right)^{\frac{3}{2}}$$

as $\xi \to \infty$. Integrating, we get the first limit in (4.3), and the second comes from (4.5). By observing that $\beta \to \rho_1$ as $\xi \to -\infty$, we obtain (4.4) in the same way. The solutions are parameterized by V. ■

The system also admits a translation symmetry, $\xi \to \xi - \xi_0$. A convenient normalization for ξ is $\beta(0) = \sqrt{\rho_1\rho_2}$ where B_1 attains its minimum. Then

$$v(0) = V(\sqrt{\rho_1} - \sqrt{\rho_2})$$

is the maximum of v along the trajectory. In particular, we can identify the nontrivial parameter V with $\max v$. A scaling symmetry like (4.1) also holds for the dynamical system: If $\zeta = \xi/\eta$, and $v' = \eta v$, then solutions in ξ are mapped into solutions in ζ. We use this scaling in defining the second variable τ in Theorem 3.1.

As a consequence of the scaling, we can express all solutions of (4.2) in terms of a single function,

$$W_0(\xi) = (\beta_0, v_0) \quad \text{with} \quad v_0(0) = 1. \tag{4.7}$$

For any other solution, we have

$$\big(\beta(\xi), v(\xi)\big) = \big(\beta_0(\eta\xi), \eta v_0(\eta\xi)\big), \tag{4.8}$$

for some η; we take $\eta = \eta(\tau)$ to handle the dependence of solutions on τ.

4.1 The Role of s

In typical constructions of viscous profiles for conservation laws arising in fluid dynamics, one makes a Galilean change of variables $x \mapsto x - st$ to reduce to the case $s = 0$. However, we have already used the change of variables $x \mapsto x - Kt$ in

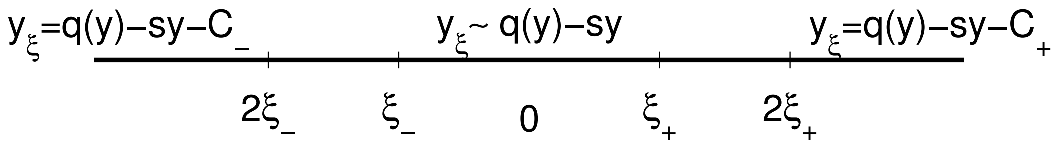

Figure 4.1: Asymptotic Dynamics of the Approximate Profiles

setting $K = 0$ in (1.1). Thus, we must consider both the cases $s = 0$ and $s \neq 0$. We find the difference in the technical detail that (3.3) does not possess the useful heteroclinic connection of (4.2). However, we can use the system (4.2) to replace (3.3) in the neighborhood of $\xi = 0$ by introducing a stretched variable $\overline{\xi}(\xi)$, with $\overline{\xi}(0) = 0$ and $d\overline{\xi}/d\xi > 0$. With respect to a fixed heteroclinic connection, for example $W_0 = (\beta_0, v_0)$, the function

$$y(\xi,\tau) = \left(\beta_0(\eta\overline{\xi}), \eta\frac{d\overline{\xi}}{d\xi}v_0(\eta\overline{\xi})\right)^{\top}, \tag{4.9}$$

satisfies (3.3) approximately. In particular, $y_\xi = q(y) + (0, \eta v_0 d^2\overline{\xi}/d\xi^2)^{\top}$. If the last term were $-sy$, we would have (3.3) and then $z = 0$ in (3.5). However, even if $\eta \to \infty$ as $\tau \to \infty$ (as we find to be the case), only the second component of y becomes unbounded as $\tau \to \infty$, and so z is a function with bounded variation if we choose the second component of y in (4.9) to satisfy the second equation of (3.3). That is, we set

$$\eta v_0 \frac{d^2\overline{\xi}}{d\xi^2} = e_2(-sy) = -s\eta\frac{d\overline{\xi}}{d\xi}v_0 \quad \text{or} \quad \frac{d^2\overline{\xi}}{d\xi^2} = -s\frac{d\overline{\xi}}{d\xi}.$$

(Here $e_2(f)$ refers to the second component of vector f.) Thus, $d\overline{\xi}/d\xi = e^{-s\xi}$; and now, imposing $\overline{\xi}(0) = 0$, we get $\overline{\xi}(\xi) = (1 - e^{-s\xi})/s$. This explains the form of the function we use below in equation (4.12).

4.2 The Five Zones of the Construction

Since the component v of the solution grows unboundedly at the shock as $\varepsilon \to 0$, we use the scaling of $v(0)$ to represent the ε dependence, and, recalling the ansatz at the beginning of Section 4, we match solutions of the three different dynamical systems near $\xi = -\infty$, $\xi = 0$ and $\xi = \infty$. See Figure 4.1. On the central interval $[\xi_-, \xi_+]$, singular behavior occurs and the dynamics are governed by (3.3); the two asymptotic systems hold on $(-\infty, 2\xi_-]$ and $[2\xi_+, \infty)$, and in transition regions, $(-2\xi_-, \xi_-)$ and $(\xi_+, 2\xi_+)$, we go from one system to another. The choice of transition points is essentially arbitrary: analytic and numerical evidence, presented in [6], suggests a rapid transition from the singular shock to the asymptotic regime. We designate by y the candidate for w^ε when scaled as in (3.4) or (3.7), while z, related to y by (3.5), will be shown to be small in the sense of Theorem 3.1.

If y is to approach the limit states $w_\pm$ at $\xi = \pm\infty$, then, from (3.5), $z(\pm\infty, \tau)$ takes the values $C_\pm = q(w_\pm) - sw_\pm$. We choose z to take these constant values outside $(2\xi_-, 2\xi_+)$, thus satisfying (1) of Theorem 3.1. Then, for large $|\xi|$, the dynamics of y, by (3.5), are the asymptotic dynamics of

$$y_\xi = q(y) - sy - C_\pm. \tag{4.10}$$

The dependence on τ (or ε) is driven by the initial condition $v(0)$ in the central dynamical system in the central interval. It is noteworthy that different systems with singular shocks show quite different scalings; in the system studied in this paper, the scaling is exponential in τ. (This exponential scaling, which differs from the polynomial scaling of the hyperbolic system in [5], seems to be responsible for the failure of Schecter's geometric construction in [10] to carry over immediately to (1.1).) Thus, in the middle interval $[\xi_-, \xi_+]$, beginning with the heteroclinic connection $W_0 = (\beta_0, v_0)$ defined in (4.7), we choose the scaling parameter η in (4.8) to be an increasing function $\eta(\tau)$ with the properties that $\eta(1) = 1$, $\eta(\infty) = \infty$ and

$$\lim_{\tau\to\infty} \frac{\eta'(\tau)}{\eta(\tau)} = L, \tag{4.11}$$

where L will be determined in the proof. (It is through the choice of L that condition (4) of Theorem 3.1— see (2.3) — is satisfied.) Nothing is lost if we simply assume $\eta(\tau) = e^{L(\tau-1)}$. Finally, we introduce $\overline{\xi}$ to obtain the form (4.9), and we have, for $\xi \in [\xi_-, \xi_+]$,

$$y = y_V(\xi, \tau) \equiv \left(\beta_0\left(\eta(\tau)\frac{1-e^{-s\xi}}{s}\right), \eta(\tau)e^{-s\xi}v_0\left(\eta(\tau)\frac{1-e^{-s\xi}}{s}\right)\right)^\top. \tag{4.12}$$

Then (3.5) implies

$$z = z_V(\xi, \tau) \equiv -s\left(\beta_0\left(\eta(\tau)\frac{1-e^{-s\xi}}{s}\right), 0\right)^\top. \tag{4.13}$$

The case $s = 0$ is included in the formulas above, since $\lim_{s\to 0}(1-e^{-s\xi})/s = \xi$, but there are simpler expressions in this case:

$$y_V(\xi, \tau) = \big(\beta_0\big(\eta(\tau)\xi\big), \eta(\tau)v_0\big(\eta(\tau)\xi\big)\big)^\top \quad \text{and} \quad z_V(\xi, \tau) = (0, 0)^\top. \tag{4.14}$$

We define

$$y_\pm = \lim_{\tau\to\infty} y_V(\xi_\pm, \tau). \tag{4.15}$$

From (4.3) and (4.4), explicit formulas for the limits $y_\pm$ give the finite value

$$y_+ = \left(\rho_2, -\frac{e^{-s\xi_+}}{B_2(\rho_2)\overline{\xi}(\xi_+)}\right)^\top. \tag{4.16}$$

From now on, we focus on the right hand intervals only.

According to our ansatz, between ξ_+ and $2\xi_+$, the function y moves from the point $y_V(\xi_+,\tau)$, which tends to y_+ in equation (4.16) as $\tau \to \infty$, to a point p_1 in the stable set $\Omega_0(w_+)$ of w_+. We choose p_1 to be independent of τ and define $y(\xi,\tau)$ for $\xi \in [\xi_+, 2\xi_+]$ as a straight line:

$$y(\xi,\tau) = \left(\frac{2\xi_+ - \xi}{\xi_+}\right) y_V(\xi_+,\tau) + \left(\frac{\xi - \xi_+}{\xi_+}\right) p_1. \tag{4.17}$$

Finally, $y(\xi,\tau)$ in $(2\xi_+,\infty)$ is determined from the asymptotic dynamics, equation (4.10), and continuity at ξ_+, as the solution of

$$y_\xi = q(y) - q(w_+) - s(y - w_+), \text{ for } \xi > 2\xi_+, \quad y(2\xi_+) = p_1. \tag{4.18}$$

Consistent with (4.10), (4.18) is an autonomous system for $\xi > 2\xi_+$, with equilibrium w_+; note that y is independent of τ in this interval.

The auxiliary function z is now determined for $\xi \in (\xi_+,\infty)$ by

$$z(\xi,\tau) = y_\xi - q(y) + sy, \tag{4.19}$$

consistent with (3.5). Note that $z = sw_+ - q(w_+)$ is constant for $\xi > 2\xi_+$. In addition, the behavior of z in $(\xi_+, 2\xi_+)$ is as required for Theorem 3.1, (5).

Proposition 4.2 *In the interval* $[\xi_+, 2\xi_+]$, $z \in \mathrm{BV}$, *uniformly in* τ.

Proof: The function z is given by (4.19), and since y is the straight line (4.17), y_ξ takes the constant value

$$y_\xi = \frac{1}{\xi_+}\Big(p_1 - y_V(\xi_+,\tau)\Big).$$

Since $y_V(\xi_+,\tau)$ tends to the limit y_+ (see equation (4.16)), the value y_ξ is uniformly bounded in τ; hence the total variation of y, and thence $q(y)$ and thence z, is uniformly bounded on this interval. ■

We have now defined the functions y and z for all (ξ,τ). Of the five conditions in Theorem 3.1, the first, (1), on asymptotic behavior of z, holds by construction. We next establish (2), uniform convergence of y to the equilibrium w_+ of (4.10), in Section 5. Then in Section 6 we prove that $y_\tau(\xi,\tau)$ is uniformly integrable for all $\tau \geq 1$, establishing (3). Finally in Section 7 we complete the proof of Theorem 3.2 by showing that conditions (4) and (5) hold in Theorem 3.1.

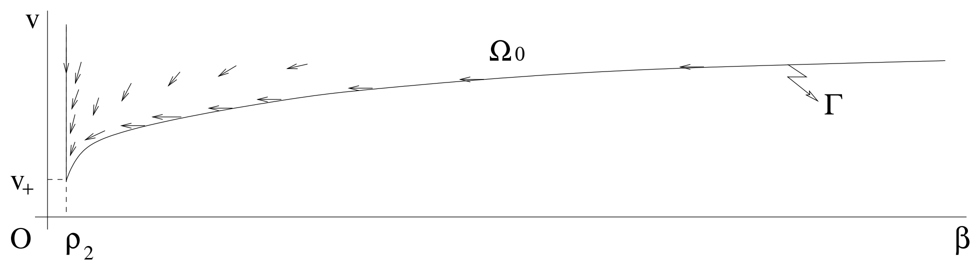

Figure 5.1: The Nullcline $v_\xi = 0$

5 Convergence of the Approximate Solutions at Infinity

For strictly overcompressive connections with $s > \mathrm{Re}(\lambda_j(w_+))$, $j = 1, 2$, the point w_+ is a spiral sink for (4.10), and convergence of y to w_+ takes place at an exponential rate, provided only that p_1 be chosen in the open basin of attraction of w_+.

There are three weakly overcompressive cases:

(1) $w_+ = (\rho_2, 0)^\top$ and $s = 0$;

(2) $w_+ = w_0(w_-) = (\rho_2, v_+)^\top \in \partial Q \cap \{\beta = \rho_2\}$, and $s = 2B_2(\rho_2)v_+ = -v_+/\rho_2$;

(3) $w_+ = (\beta_+, \rho_+)^\top \in \partial Q \cap \{\rho_2 < \beta < \beta_-\}$, and $s = 2B_2(\beta_+)v_+$;

and we prove

Proposition 5.1 *In cases* (1) *and* (2) *above,* $y \to w_+$.

Proof: In case (1), when $v_+ = 0$, then $s = 0$. Since $w_+ = (\rho_2, 0)^\top = W_+$, then equation (4.18) is the same as (4.2) for large ξ. The open set

$$\Omega_0(w_+) = \{(\beta, v) | \rho_2 < \beta < \rho_1, v > 0\},$$

is in the basin of attraction of $w_+ = (\rho_2, 0)$, and y is the solution y_V of (4.2) given by (4.12) for all $\xi > \xi_-$ and all τ. This implies the uniform approach of y to w_+, as can be seen from (4.3). Thus, condition (2) of Theorem 3.1 holds. To establish the integrability of y_τ (item (3) of Theorem 3.1), we choose y to be independent of τ for $\xi > 2\xi_+$. (See the proof of Proposition 6.5.) For this we choose p_1 to be any point, independent of τ, in $\Omega_0(w_+)$.

Case (2) requires a more careful analysis of equation (4.18), an autonomous system with equilibrium (ρ_2, v_+), for large ξ. We write

$$\begin{aligned}
\beta_\xi &= vB_1(\beta) + \frac{v_+}{\rho_2}(\beta - \rho_2) = \frac{\beta - \rho_1}{\beta}(\beta - \rho_2)(v - v_+) + \frac{\rho_1 v_+}{\rho_2 \beta}(\beta - \rho_2)^2, \\
v_\xi &= v^2 B_2(\beta) - v_+^2 B_2(\rho_2) + \frac{v_+}{\rho_2}(v - v_+) = v^2 B_2(\beta) + \frac{v_+}{\rho_2}\left(v - \frac{v_+}{2}\right) \\
&= B_2(\beta)(v - v_+)^2 + \frac{\rho_1 v_+}{\rho_2 \beta^2}\left(v - \frac{v_+}{2}\right)(\beta + \rho_2)(\beta - \rho_2).
\end{aligned} \tag{5.1}$$

We note that w_+ has an open basin of attraction. A nullcline $\{v_\xi = 0\}$ of equation (5.1) is given by

$$\Gamma = \left\{ B_2(\beta) = -\frac{v_+}{\rho_2 v^2}\left(v - \frac{v_+}{2}\right)\right\}, \tag{5.2}$$

which can be solved for $\beta(v)$, $v \geq v_+$. On Γ, we have

$$\frac{d\beta}{dv} = \frac{\beta^3(v - v_+)}{v^3 \rho_1 \rho_2},$$

which is zero at (ρ_2, v_+); so in the phase plane (β, v), Γ is tangent to the line $\beta = \rho_2$. The open set

$$\Omega_0(w_+) = \{(\beta, v) | \rho_2 < \beta < \beta_1(v), v > v_+\},$$

between Γ and the line $\beta = \rho_2$, is in the stable set of w_+. See Figure 5.1. By choosing p_1 independent of τ in the interior of Ω_0 we ensure that we have $y(2\xi_+, \tau)$ in Ω_0. Then solutions of (5.1) for $\xi > 2\xi_+$ have the property that $y(\xi, \tau) \to w_+$ as $\xi \to \infty$. ∎

For the third type of potential connection, points in the interior, $\rho_2 < \beta_+ < \beta_-$, on $\partial Q(w_-)$, the construction of viscous profiles does not seem possible. To study the type of the equilibrium $w_+ \in \partial Q(w_-)$ of the system, we can fix w_- and consider a family of states w_+ crossing ∂Q. Then when w_+ is on $\partial Q(w_-)$, the Hopf bifurcation theorem [3, Theorem 3.4.2] implies that one of three things must happen: either w_+ is a center, or supercritical or subcritical Hopf bifurcation takes place at w_+. Calculation of the normal form of the equilibrium up to third order is inconclusive. Numerical simulations indicate that w_+ is a center.

6 Integrals of the Approximate Solutions

An important technical feature of the construction is that the L^1 integral of $y_\tau(., \tau)$ is bounded uniformly in τ. Absolute integrability of y_τ may not be an intrinsic requirement of viscous profiles. However, we need this condition in order to apply Theorem 3.1. Potential sources of difficulty occur near the origin, where $y \to \infty$ with τ, and near $\xi = \infty$, where in the weakly overcompressive case y does not tend exponentially fast to w_+. For the system studied in this paper, we have eliminated the difficulty near infinity by choosing approximate solutions which are independent of τ outside a finite interval.

In this section we prove integrability near $\xi = 0$, in Theorem 6.1. Part (iii) of Theorem 6.1 will be used to obtain condition (4) of Theorem 3.1.

In addition, in Proposition 6.5, we establish the uniform integrability of y_τ on the remaining interval, $[\xi_+, 2\xi_+]$.

We first prove the result when $s = 0$; then as Corollary 6.4 we show it follows for any s.

Theorem 6.1 *Let y be the function y_V given by (4.14). For given $\xi_- < 0 < \xi_+$, and with* $\lim_{\tau\to\infty} \eta'/\eta = L$,

(i) $\int_{\xi_-}^{\xi_+} |y_\tau(\xi,\tau)|\, d\xi$ *is bounded uniformly in* τ*;*

(ii) *For any* $\xi \in (\xi_-, \xi_+)$ *with* $\xi \neq 0$, $\lim_{\tau\to\infty} y_\tau(\xi,\tau) = 0$*;*

(iii) $\lim_{\tau\to\infty} \int_{\xi_-}^{\xi_+} \beta_\tau(\xi,\tau)\, d\xi = 0$, *and* $\lim_{\tau\to\infty} \int_{\xi_-}^{\xi_+} v_\tau(\xi,\tau)\, d\xi = 2L(\rho_1+\rho_2)$.

We first prove some lemmas about the functions (β, v) defined by (4.14).

Lemma 6.2 *For any fixed $\tau > 0$, $v_\tau(\xi,\tau)$ changes sign at most once when $\xi < 0$ and once when $\xi > 0$.*

Proof: We may assume $v_\tau(0,\tau) > 0$, since $v(0,\tau) = \eta(\tau)v_0(0) \to \infty$ as $\tau \to \infty$. Differentiating $v(\xi,\tau)$ with respect to τ, we have

$$v_\tau(\xi,\tau) = \eta'(\tau)v_0(\eta\xi)\big(1 + \eta\xi B_2(\beta_0(\eta\xi))v_0(\eta\xi)\big). \tag{6.1}$$

The factor $1+\eta\xi B_2(\beta_0)v_0$ can be rewritten $T(\zeta) = 1 + \zeta B_2(\beta_0(\zeta))v_0(\zeta)$, whose behavior depends only on the fixed solution (β_0, v_0). Let $\zeta = \delta$ be a point where $T(\zeta) = 0$. Then at $\zeta = \delta$ (using $\delta B_2 v_0 = -1$ and the differential equation (4.2)) we have

$$T'(\zeta) = B_2 v_0 + \zeta B_2' \beta_0' v_0 + \zeta B_2 v_0' = \delta v_0^2 B_1 B_2';$$

and $B_1 < 0$ and $B_2' > 0$ imply that $\operatorname{sgn} T' = -\operatorname{sgn} \delta$. In other words, as a function of ζ, T can go only from positive to negative when $\zeta > 0$ and the other way when $\zeta < 0$. Thus, either $T > 0$ everywhere or there are unique values $\delta_- < 0 < \delta_+$ at which T changes sign. ■

The function T can be computed explicitly using a standard solver, and changes sign twice. We define $\xi_{\delta_\pm}(\tau) = \delta_\pm/\eta(\tau)$ as the points where v_τ changes sign. Now we prove

Lemma 6.3 *The integral $\int_{\xi_{\delta_-}(\tau)}^{\xi_{\delta_+}(\tau)} |v_\tau(\xi,\tau)|\, d\xi$ is bounded uniformly in τ.*

Proof: From Lemma 6.2, we may drop the absolute value sign, and from (6.1), for a given $\tau > 0$, we write

$$\int_{\xi_{\delta_-}(\tau)}^{\xi_{\delta_+}(\tau)} v_\tau(\xi,\tau)\, d\xi = \frac{\eta'(\tau)}{\eta(\tau)} \int_{\xi_{\delta_-}(\tau)}^{\xi_{\delta_+}(\tau)} v_0(\eta\xi) + \eta\xi v_0'(\eta\xi)\, d(\eta\xi).$$

Carrying out the integration gives

$$\int_{\xi_{\delta_-}}^{\xi_{\delta_+}} v_\tau\, d\xi = \frac{\eta'}{\eta}\eta\xi v_0(\eta\xi)\Big|_{\xi_{\delta_-}}^{\xi_{\delta_+}} = \frac{\eta'}{\eta}\left(\xi_{\delta_+}\eta v_0(\eta\xi_{\delta_+}) - \xi_{\delta_-}\eta v_0(\eta\xi_{\delta_-})\right).$$

Now, from $T(\delta_\pm) = 0$ the last expression becomes

$$\frac{\eta'}{\eta}\left(-\frac{1}{B_2(\beta_0(\delta_+))} + \frac{1}{B_2(\beta_0(\delta_-))}\right),$$

which is uniformly bounded in τ. ■

We can now prove Theorem 6.1.

Proof: We begin with (ii), pointwise bounds for β_τ and v_τ. For any $\xi > 0$, $\beta_0(\eta\xi) \to \rho_2$ and $\eta\xi v_0(\eta\xi) \to -1/B_2(\rho_2)$ as $\tau \to \infty$, by Proposition 4.1; so we have

$$\beta_\tau(\xi,\tau) = \frac{\eta'(\tau)}{\eta(\tau)}\eta\xi v_0(\eta\xi)B_1(\beta_0(\eta\xi)) \to 0$$

as $\tau \to \infty$. We note also that, since $\eta\xi v_0(\eta\xi) = \zeta v_0(\zeta)$ is bounded, β_τ is in fact uniformly bounded. For v_τ, we use (6.1), multiplying and dividing by η and inserting limits from Proposition 4.1:

$$v_\tau(\xi,\tau) \to -L\frac{1}{B_2(\rho_2)\xi}\left(1 - \xi B_2(\rho_2)\frac{1}{B_2(\rho_2)\xi}\right) = 0,$$

as $\tau \to \infty$. The proof for $\xi < 0$ is the same.

Now we establish (iii). The formula $\beta_\tau(\xi,\tau) = \eta'(\tau)\xi\beta_0'(\eta(\tau)\xi) = \eta'\xi v_0 B_1$ gives

$$\operatorname{sgn}\,\beta_\tau(\xi,\tau) = -\operatorname{sgn}\,\xi. \tag{6.2}$$

For any $\xi_0 > 0$ with $\xi_0 < \min\{-\xi_-, \xi_+\}$, using (6.2) we have

$$\begin{aligned}\int_{-\xi_0}^{\xi_0} \beta_\tau\,d\xi = \frac{\eta'(\tau)}{\eta(\tau)}\int_{-\xi_0}^{\xi_0}\eta\xi\beta_0'(\eta\xi)\,d\xi &\geq \frac{\eta'}{\eta}\xi_0\left[\int_{-\eta\xi_0}^{0}\beta_0'(\zeta)\,d\zeta + \int_0^{\eta\xi_0}\beta_0'(\zeta)\,d\zeta\right]\\ &= \frac{\eta'}{\eta}\xi_0\left[\beta_0(\xi_0\eta) - \beta_0(-\xi_0\eta)\right] \geq \frac{\eta'}{\eta}\xi_0(\rho_2 - \rho_1).\end{aligned}$$

We may assume that $\eta'/\eta \leq 2L$ for all τ. For any $\epsilon > 0$, since the integral is negative, take ξ_0 small enough that

$$\left|\int_{-\xi_0}^{\xi_0}\beta_\tau\,d\xi\right| < \frac{\epsilon}{2}.$$

By (ii), given $\epsilon > 0$ and $\xi_0 > 0$ we can find $T > 0$ such that, if $\tau > T$,

$$\left|\int_{\xi_-}^{-\xi_0}\beta_\tau\,d\xi\right| < \frac{\epsilon}{4} \quad\text{and}\quad \left|\int_{\xi_0}^{\xi_+}\beta_\tau\,d\xi\right| < \frac{\epsilon}{4}.$$

Therefore if $\tau > T$,

$$\left|\int_{\xi_-}^{\xi_+}\beta_\tau(\xi,\tau)\,d\xi\right| < \epsilon.$$

Now we check the second component of y_τ. As in the proof of Lemma 6.3 we integrate v_τ:

$$\int_{\xi_-}^{\xi_+} v_\tau(\xi,\tau)\,d\xi = \frac{\eta'(\tau)}{\eta(\tau)}\Big(\eta\xi_+ v_0(\eta\xi_+) - \eta\xi_- v_0(\eta\xi_-)\Big).$$

By (4.3) and (4.4),

$$\eta\xi_+ v_0(\eta\xi_+) \to 2\rho_2 \quad \text{and} \quad \eta\xi_- v_0(\eta\xi_-) \to -2\rho_1, \tag{6.3}$$

as $\tau\to\infty$, while $\eta'/\eta \to L$. The second relation in (iii) follows.

Finally, we prove (i). For any τ we have

$$\begin{aligned}\int_{\xi_-}^{\xi_+} |\beta_\tau(\xi,\tau)|\,d\xi &= \int_{\xi_-}^{0} \beta_\tau(\xi,\tau)\,d\xi - \int_{0}^{\xi_+} \beta_\tau(\xi,\tau)\,d\xi \\ &= \frac{\eta'(\tau)}{\eta(\tau)}\left[\int_{\eta\xi_-}^{0} \xi\beta_0'(\eta(\tau)\xi)\,d(\eta\xi) - \int_{0}^{\eta\xi_+} \xi\beta_0'(\eta(\tau)\xi)\,d(\eta\xi)\right] \\ &\le 2L(\rho_1-\rho_2)\max\{-\xi_-,\xi_+\}.\end{aligned}$$

This is the desired result for β_τ. We separate the v_τ integral into its positive and negative parts:

$$\int_{\xi_-}^{\xi_+} v_\tau(\xi,\tau)\,d\xi = \int_{\xi_-}^{\xi_{\delta_-}} v_\tau(\xi,\tau)\,d\xi + \int_{\xi_{\delta_-}}^{\xi_{\delta_+}} v_\tau(\xi,\tau)\,d\xi + \int_{\xi_{\delta_+}}^{\xi_+} v_\tau(\xi,\tau)\,d\xi.$$

As we have shown in (iii) that the left side tends to $2L(\rho_1+\rho_2)$ while from Lemma 6.3 the middle integral is uniformly bounded, we have uniform bounds on the two outer integrals for large τ, and

$$\int_{\xi_-}^{\xi_+} |v_\tau(\xi,\tau)|\,d\xi = -\left(\int_{\xi_-}^{\xi_{\delta_-}} v_\tau(\xi,\tau)\,d\xi + \int_{\xi_{\delta_+}}^{\xi_+} v_\tau(\xi,\tau)\,d\xi\right) + \int_{\xi_{\delta_-}}^{\xi_{\delta_+}} v_\tau(\xi,\tau)\,d\xi$$

is also uniformly bounded.

This completes the proof of the theorem. ∎

Corollary 6.4 *Let y be the solution y_V given by (4.12) with $s\neq 0$. Then the relations in Theorem 6.1 hold.*

Proof: When $s\neq 0$, we have

$$\beta(\xi,\tau) = \beta_0\left(\eta(\tau)\frac{1-e^{-s\xi}}{s}\right), \quad v(\xi,\tau) = \eta(\tau)e^{-s\xi}v_0\left(\eta(\tau)\frac{1-e^{-s\xi}}{s}\right).$$

As in Section 4.1, let $\overline{\xi} = (1-e^{-s\xi})/s$. Then

$$\beta(\xi,\tau) = \beta_0\big(\eta\overline{\xi}\big) = \bar\beta(\overline{\xi},\tau), \quad v(\xi,\tau) = (1-s\overline{\xi})\eta v_0\big(\eta\overline{\xi}\big) = (1-s\overline{\xi})\bar v(\overline{\xi},\tau),$$

where $(\bar\beta, \bar v)$ are given by the formulas in (4.14). Hence the estimates in Theorem 6.1 apply to $(\bar\beta, \bar v)$. Since any points $\xi_\pm$ are just mapped to points $\overline{\xi}(\xi_\pm)$, while the factor $1 - s\overline{\xi}$ is bounded on $[\overline{\xi}(\xi_-), \overline{\xi}(\xi_+)]$, these bounds hold also for (β, v). Finally, the nonzero estimate in (iii) is also valid since $\int \overline{\xi} v_\tau \to 0$ as $\tau \to \infty$. ∎

Proposition 6.5 *When $w_+ = W_+ = (\rho_2, 0)$, or $w_+ = (\rho_2, v_+)$, and if p_1 is chosen in the stable set $\Omega_0(w_+)$, independent of τ, then $\int_{\xi_+}^\infty |y_\tau|\, d\xi \to 0$ as $\tau \to \infty$.*

Proof: The only contribution to this integral is from the interval $\xi_+ < \xi < 2\xi_+$, since $y_\tau = 0$ for $\xi > 2\xi_+$. Since y and y_τ are linear in ξ, from (4.17) we have

$$\int_{\xi_+}^{2\xi_+} |y_\tau|\, d\xi = \frac{\xi_+}{2} \left| \frac{\partial}{\partial \tau} y_V(\xi_+, \tau) \right|,$$

and this expression tends to zero as $\tau \to \infty$, by Theorem 6.1, (ii). ∎

We have now established condition (3) of Theorem 3.1.

7 Completion of the Proof of Theorem 3.2

It remains to show that conditions (4) and (5) of Theorem 3.1 hold. Using Proposition 6.5 and Theorem 6.1, (iii), we have

$$\lim_{\tau\to\infty} \int_{\mathbf{R}} y_\tau(\xi, \tau)\, d\xi = \lim_{\tau\to\infty} \int_{\xi_-}^{\xi_+} y_\tau(\xi, \tau)\, d\xi = \begin{pmatrix} 0 \\ 2L(\rho_1 + \rho_2) \end{pmatrix};$$

and letting L be the nonzero component of $e(w_-, w_+, s)/2(\rho_1 + \rho_2)$ (see equation (2.3)) yields (4).

Finally, from the construction of y and z, $y(\xi, \tau)$ is Lipschitz continuous for $\xi \in \mathbf{R}$, $\tau \geq 1$ and approaches constants at $\pm\infty$, so $y(\cdot, 1)$ is of bounded variation. To show that $z(\xi, \tau)$ is of bounded variation uniformly in τ, we compute (noting $\beta_0' < 0$) the first component z_1 from equation (4.13):

$$\int_{\xi_-}^{\xi_+} |z_{1,\xi}|\, d\xi = -s \int_{\xi_-}^{\xi_+} \beta_0'(\eta\overline{\xi}) e^{-s\xi}\, d\xi \leq -s \int_{-\infty}^{\infty} \beta_0'(\overline{\xi})\, \frac{d\overline{\xi}}{\eta},$$

which tends to zero as $\tau \to \infty$. Proposition 4.2 gives uniformly bounded variation for $z \in [\xi_+, 2\xi_+]$, and z is constant for $\xi > 2\xi_+$. Thus, (5) holds.

Thus we can apply Theorem 3.1 to obtain Theorem 3.2.

In the case of strictly overcompressive shocks, the condition (3.6) holds whenever the conditions of Theorem 3.1 are met, and we can conclude that both ordinary (viscosity εw_{xx}) and self-similar (viscosity $\varepsilon t w_{xx}$) viscous structures exist. However, in the weakly overcompressive cases of Theorem 3.2, a rough calculation of the solutions near the attractor w_+ indicates that convergence of the integral of $|\xi y_\xi|$ is not rapid enough to give (3.6).

8 The Cauchy Problem with Piecewise Constant Data

Here we take $K = 0$, $G = 0$, and use the dependent variable Z given by

$$Z = \begin{pmatrix} \beta \\ (\beta - \rho_2)(\rho_1 - \beta)v \end{pmatrix}, \tag{8.1}$$

noting that Z uniquely determines the volume fraction and the velocity of any phase present [6]. The region H (in which all phases present have zero velocity) corresponds to $Z_2 = 0$.

The entropy Riemann solutions obtained in [6] have the following properties.

(i) All of the intermediate states created are in H.

(ii) An admissible discontinuity separating a region where the state is $Z_H \in H$ from a region where the state is $Z \notin H$ moves towards the region where the state is Z. (Points in space change only from state Z to Z_H.)

(iii) For a given $Z \notin H$, there is a positive δ such that all admissible discontinuities connecting Z to any point $Z_H \in H$ have speed greater than δ.

(iv) An admissible discontinuity connecting two states in H has speed zero.

Now consider the Cauchy problem with piecewise constant initial data $Z(\cdot, 0)$, assuming only a finite number of different values and with a locally finite set of points of discontinuity. Solutions are easily constructed by wave front tracking. Indeed, in general a plethora of corresponding solutions exists, as at any point (x, t) where Z is continuous and $Z(x, t) \notin H$, one may choose the nontrivial Riemann solution of the trivial Riemann problem.

However, all such solutions have a very simple form.

From (i), it follows that for any $t > 0$

$$\{Z(\cdot, t)\} \subseteq \{Z(\cdot, 0)\} \cup H. \tag{8.2}$$

Denote by $S(t)$ the interior points of the set $\{x \mid Z(x, t) \in H\}$. Then from (ii) and (iv), for any $x \in S(t_0)$ it follows that

$$Z(x, t) = Z(x, t_0), \quad \forall t > t_0. \tag{8.3}$$

From (8.2), the assumption that $Z(., 0)$ assumes only a finite number of different values, and (iii), there is a positive δ such that no admissible discontinuities of speed less than δ connect states Z_H in H to states not in H.

Assume now that the closure of $S(t)$ is not empty and is not all of $\mathbf{R}$. Then the boundary of $S(t)$ necessarily contains a point at which an admissible discontinuity is moving out of $S(t)$ at a speed of at least δ. So if the complement of $S(0)$ has finite measure, then the closure of $S(t)$ will be all of $\mathbf{R}$ in finite time, independent of which solution is chosen. After this finite time, from (8.3) the solution is independent of t; from (1.1), recalling that $K = 0$, we deduce that v vanishes identically. Thus, this model predicts that if there is a finite amount of mixed phase fluid initially, it will separate into single phase states in finite time. In other words, absent any interfacial momentum transfer terms (the balance terms we have omitted), the mathematical solution correctly predicts that the Bernoulli effect will dominate [8].

9 Conclusions

Singular shocks represent a novel type of solution to conservation law systems, and may provide solutions in situations, such as large data for some hyperbolic systems as well as the present context, in which classical shock solutions do not suffice to solve all initial value problems. Because they are not yet well understood, it is important to apply to them the same tests that have been used, classically, to establish the admissibility and stability of weak solutions of conservation law systems.

In this paper we have established viscous structure for singular shocks which can be used to solve Riemann problems for a nonhyperbolic model system. This exercise is not based on a physically realistic form of fluid-dynamic viscosity. Nonetheless, it is an important step in justifying the use of singular shocks to solve the model problem. In particular, it shows that there is a sense in which solutions to the ill-posed system (1.1) can be recovered as limits of solutions of well-posed systems of the form (1.3). Since (1.1) is equivalent to a standard two-fluid model in widespread use, a better understanding of the mathematical structure of its solutions will have significant application.

Many questions remain. It would be valuable to understand how singular shocks can be approximated when different forms of viscous regularization are used. This question has not been answered even in the simpler case that the underlying equation is hyperbolic. In a second direction, critically important to the application to two-phase flow, we would like to understand what happens to singular shock solutions of (1.1) when balance terms are added to the second equation of (1.1). The technique of this paper suggests adding both viscous perturbations and balance terms, and seeking scale-invariant or asymptotic solutions. We are currently studying this problem.

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