Bounded and Divergent Orbits and Expanding Curves on Homogeneous Spaces

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The main objects

- $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$.
- $g_t$ is $\text{Ad}$-diagonalizable over $\mathbb{R}$:

  $$\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \left\{ X \in \mathfrak{g} : g_tXg_{-t} = e^{\alpha(t)}X \right\}$$

- $u(Y) = \exp(Y)$ for $Y \in \mathfrak{g}$.
- $X$ a topological space and $G \curvearrowright X$. 

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A map \( \varphi : [0, 1] \rightarrow g \) is \( g_t \)-admissible if:

1. \( \varphi \) is \( C^2 \) and \( \dot{\varphi} \neq 0 \).
2. \( g_t \) normalizes \( \dot{\varphi} \): the image of \( \varphi \) is contained in \( g_\alpha \) for some \( \alpha > 0 \).
3. \( \varphi \) commutes with \( \dot{\varphi} \): \( [\varphi, \dot{\varphi}] \equiv 0 \).
Central Questions

- Estimate the Hausdorff dimension of the set of parameters $s \in [0, 1]$:
  1. $g_t u(\varphi(s))x_0$ diverges on average in $X$: for any compact set $K \subseteq X$:

$$\frac{1}{T} \int_0^T \chi_K(g_t u(\varphi(s))x_0) \, dt \to 0$$

  2. $g_t u(\varphi(s))x_0$ remains inside a compact subset of $X$ for all $t > 0$. 

Real Rank One Manifolds

- $T^1M$: unit tangent bundle of a rank 1, locally symmetric manifold of finite volume, $p \in M$.
- $g^t : T^1M \rightarrow T^1M$: the geodesic flow.
- $\varphi : [0, 1] \rightarrow T^1_pM$ a $g^t$-admissible map. (Automatic for $\mathbb{H}^n$).

Theorem (K. ’18)

The Hausdorff dimension of the set of $s \in [0, 1]$ such that

1. $g^t \varphi(s)$ diverges on average is at most $1/2$.
2. $g^t \varphi(s)$ is bounded is equal to 1. (This set is winning).

Remark: (2) was previously obtained by Aravinda and Leuzinger by different methods (ETDS ’95).
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- Remark: (2) was previously obtained by Aravinda and Leuzinger by different methods (ETDS ’95).
• $K = \mathbb{Q}(\alpha)$ a number field of degree $d$, e.g. $K = \mathbb{Q}(\sqrt{2})$.

• $\mathcal{O}_K$ its ring of integers, e.g. $\mathbb{Z}[\sqrt{2}]$.

• $\Sigma$ the set of Galois embeddings of $K$ into $\mathbb{R}$ and $\mathbb{C}$, e.g.

\[
\begin{align*}
a + b\sqrt{2} & \mapsto a + b\sqrt{2}, \quad a + b\sqrt{2} \mapsto a - b\sqrt{2}
\end{align*}
\]
\( K_\Sigma = \mathbb{R}^r \times \mathbb{C}^s, \ r + s = |\Sigma|. \)

\( x = (x_\sigma)_{\sigma \in \Sigma} \in K_\Sigma \) is *badly approximable* by \( K \) if there exists \( c > 0 \), for all \( p, q \in \mathcal{O}_K \):

\[
\max_{\sigma \in \Sigma} \{|\sigma(p) + x_\sigma \sigma(q)|\} \max_{\sigma \in \Sigma} \{|\sigma(q)|\} \geq c
\]
$G = \text{SL}(2, \mathbb{R})^r \times \text{SL}(2, \mathbb{C})^s$.

$\Gamma$ is image of diagonal Galois embedding of $\text{SL}(2, \mathcal{O}_K)$.

$g_t = \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right)_{\sigma \in \Sigma}, \quad u(x) = \left(\begin{pmatrix} 1 & x_{\sigma} \\ 0 & 1 \end{pmatrix}\right)_{\sigma \in \Sigma}$

Einsiedler-Ghosh-Lyttle (Dani’s correspondence in number fields): $x \in K_\Sigma$ is *badly approximable* iff $g_t u(x) \Gamma$ remains bounded in $G/\Gamma$. 
\( K_\Sigma = \mathbb{R}^r \times \mathbb{C}^s \) can be identified with the full unstable manifold of \( g_t \) via \( x \mapsto u(x) \).

\[ \varphi = (\varphi_\sigma)_{\sigma \in \Sigma} : [0, 1] \to \mathbb{R}^r \times \mathbb{C}^s \text{ is } C^{1+\varepsilon}. \]

**Maximality Assumption:**

\[ \dot{\varphi}_\sigma \neq 0, \quad \sigma \in \Sigma \]
Theorem (K. ’18)

For all $x_0 \in G/\Gamma$, the Hausdorff dimension of the set of $s \in [0, 1]$ such that

1. $g_t u(\varphi(s)) x_0$ is divergent on average in $G/\Gamma$ is at most $1/2$.
2. $g_t u(\varphi(s)) x_0$ is bounded in $G/\Gamma$ is equal to 1. (The set is winning).

The result for curves remains true for:

1. reducible lattices, or
2. any semisimple algebraic group $G$ and $\Gamma < G$ is an arithmetic lattice of $\mathbb{Q}$-rank equal to 1 under an appropriate maximality condition.
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Remarks:

- $G = \text{SL}(2, \mathbb{R})^r \times \text{SL}(2, \mathbb{C})^s, \Gamma = \Delta(\text{SL}(2, \mathcal{O}_K))$: Dimension of bounded orbits on curves was previously obtained by Einsiedler, Ghosh and Lyttle by different methods (ETDS ’16).

- Y. Cheung (ETDS ’07): the dimension of divergent orbits for $g_t$ in the entire $\text{SL}(2, \mathbb{R})^n/\text{SL}(2, \mathbb{Z})^n$ is $3n - 1/2$ for $n \geq 2$. 

Systems of Linear Forms

- $Y \in \mathcal{M}_{m,n}(\mathbb{R})$ is **badly approximable** if there exists $c > 0$ for all $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n$:
  \[
  \|p + Y \cdot q\|_\infty^m \|q\|_\infty^n \geq c
  \]

- $Y$ is **singular** if for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$; for all $N \geq N_0$, there exists $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n$:
  \[
  \begin{cases}
  \|p + Yq\| \leq \varepsilon/N \\
  0 < \|q\| \leq N^{n/m}
  \end{cases}
  \]
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  $$\begin{cases}
  \|p + Yq\| \leq \varepsilon/N \\
  0 < \|q\| \leq N^{n/m}
  \end{cases}$$
$G = \text{SL}(m+n, \mathbb{R}), \Gamma = \text{SL}(m+n, \mathbb{Z}),$

$$g_t = \begin{pmatrix} e^{nt}I_m & 0 \\ 0 & e^{-mt}I_n \end{pmatrix}, \quad u(Y) = \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix}$$

Dani’s Correspondence: $Y$ is **badly approximable** iff $g_t u(Y) \Gamma$ remains bounded in $G/\Gamma$ and **singular** iff $g_t u(Y) \Gamma$ diverges in $G/\Gamma$. 
Theorem (K. ’18)

Suppose $A \in \text{GL}(n, \mathbb{R})$, $B \in M_{n,n}(\mathbb{R})$ and $\varphi : [0, 1] \rightarrow M_{n,n}(\mathbb{R})$ is given by

$$\varphi(s) = B + sA$$

Then, for any $x_0 \in G/\Gamma$, the Hausdorff dimension of the set of $s \in [0, 1]$ such that

1. $g_t u(\varphi(s))x_0$ diverges on average is at most $1/2$.
2. $g_t u(\varphi(s))x_0$ remains bounded in $G/\Gamma$ is equal to 1. (This set is winning).
Bounded orbits: a very brief history

- Schmidt 1969: the set of badly approximable matrices in $M_{m,n}(\mathbb{R})$ is winning (has full dimension).

- Beresnevich (Invent. Math. ’15): (weighted) badly approximable points on non-degenerate curves in $M_{1,n}(\mathbb{R}) \cong \mathbb{R}^n$ have dimension 1.

- Kleinbock-Weiss (Adv. in Math. ’10, JMD ’13): the set of bounded orbits for a partially hyperbolic algebraic flow on a homogeneous space is winning.
Y. Cheung (Annals ’11): singular vectors in $M_{1,2}(\mathbb{R}) \cong \mathbb{R}^2$ has dimension $4/3$.

Cheung-Chevallier (Duke ’16): singular vectors in $\mathbb{R}^n$ have dimension $n^2/n + 1$.

Kadyrov-Kleinbock-Lindenstrauss-Margulis (J. d’Analyse ’17): singular matrices in $M_{m,n}(\mathbb{R})$ have dimension at most $mn - \frac{mn}{m+n}$. 
Ingredients of the proof

1. Contraction Hypothesis $\implies$ Dimension Estimates.

2. Establish the Contraction Hypothesis.
Recall our set up

- $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$.
- $g_t$ is Ad-diagonalizable over $\mathbb{R}$:
  
  $$g = \bigoplus g_\alpha, \quad g_\alpha = \left\{ X \in \mathfrak{g} : g_t X g^{-t} = e^{\alpha(t)} X \right\}$$

- $u(Y) = \exp(Y)$ for $Y \in \mathfrak{g}$.
- $G \ltimes \mathcal{X}$, a topological space (not necessarily a homogeneous space for $G$).
Recall our set up

Definition

A map $\varphi : [0, 1] \rightarrow g$ is $g_t$-admissible if:

1. $\varphi$ is $C^2$ and $\dot{\varphi} \neq 0$.
2. $g_t$ normalizes $\dot{\varphi}$: the image of $\varphi$ is contained in $g_\alpha$ for some $\alpha > 0$.
3. $\varphi$ commutes with $\dot{\varphi}$: $[\varphi, \dot{\varphi}] \equiv 0$. 
$f : X \to [0, \infty]$ is a **height function**:

1. $f$ is proper and finite on compact subsets of $X \setminus \{f = \infty\}$.

2. $f$ is **log-smooth**: for every bounded set $\mathcal{O} \subset G$, there exists $C \geq 1$, for all $g \in \mathcal{O}$ and all $x \in X \setminus \{f = \infty\}$,

   \[ C^{-1}f(x) \leq f(gx) \leq Cf(x) \]

3. $\{f = \infty\}$ is $G$-invariant.
For $M > 0$, $\chi_M$ indicator function of $\{f \leq M\}$.

For $x \in X$, we say

1. $g_t x$ diverges on average if for all $M > 0$:

$$\frac{1}{T} \int_0^T \chi_M(g_t x) \, dt \to 0$$

2. $g_t x$ is bounded if

$$\sup_{t>0} f(g_t x) < \infty$$
The Contraction Hypothesis

Definition

\( \varphi \) satisfies the **first order \( \beta \)-contraction hypothesis** on \( X \) if there exists a height function \( f \) and \( 0 < \beta < 1 \) such that for all \( t > 0 \):

\[
\int_0^1 f(g_t u(r \dot{\varphi}(s))x) \, dr \leq ce^{-\beta \alpha(t)}f(x) + b
\]

for some constants \( c, b > 0 \).

In words, \( g_t \) orbits starting from points on \( \varphi \) are biased towards sublevel sets of \( f \): when \( f(x) \gg 1 \)

\[
\int_0^1 f(g_t u(r \dot{\varphi}(s))x) \, dr \ll e^{-\beta \alpha(t)}f(x)
\]
The Contraction Hypothesis

Definition

ϕ satisfies the **first order β-contraction hypothesis** on X if there exists a height function f and 0 < β < 1 such that for all t > 0:

\[ \int_0^1 f(g_t u(r\varphi(s))x) \, dr \leq c e^{-\beta \alpha(t)} f(x) + b \]

for some constants c, b > 0.

In words, \( g_t \) orbits starting from points on \( \varphi \) are biased towards sublevel sets of \( f \): when \( f(x) \gg 1 \)

\[ \int_0^1 f(g_t u(r\varphi(s))x) \, dr \ll e^{-\beta \alpha(t)} f(x) \]
Theorem (K. ’18)

Suppose $\varphi$ is a $g_t$-admissible curve satisfying the 1st order $\beta$-contraction hypothesis. Then, for all $x_0 \in X \setminus \{f = \infty\}$, the Hausdorff dimension of the set of $s \in [0, 1]$ such that

1. $g_t u(\varphi(s))x_0$ is divergent on average is at most $1 - \beta$.
2. $g_t u(\varphi(s))x_0$ remains bounded in $X$ is equal to 1.
An example: $\text{SL}(2, \mathbb{R})/\text{SL}(2, \mathbb{Z})$

$f : \text{SL}(2, \mathbb{R})/\text{SL}(2, \mathbb{Z}) \to \mathbb{R}_+$ is given by the $y$-coordinate in the upper half plane model.

$f(x + iy) = y^\delta$
Example 2: $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z}) \leftrightarrow \{\text{unimodular lattices in } \mathbb{R}^n\}$

$$f(x) = \max_{1 \leq i \leq n} \max \left\{ \frac{1}{\|\Lambda\|} : \Lambda \text{ is a subgroup of } x \text{ of rank } i \right\}$$
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Markov Chains and Stochastic Stability

SECOND EDITION

\[ \| P^n(x, \cdot) - \pi \|_f \to 0 \]

\[ \pi(f) < \infty \]

\[ \Delta V(x) \leq -f(x) + b\mathbb{1}_C(x) \]

Sean Meyn and Richard L. Tweedie

Cambridge
Eskin-Margulis-Mozes: averaging over $SO(p) \times SO(q) < SL(p + q, \mathbb{R})$.

Eskin-Margulis, Benoist-Quint: random walks on homogeneous spaces.

Eskin-Masur: recurrence of Teichmüller flow orbits in strata of quadratic differentials.

Eskin-Mirzakhani-Mohammadi: recurrence away from proper affine submanifolds.
Contraction in higher rank: the enemy

- $G = \text{SL}(3, \mathbb{R})$ and $\Gamma = \text{SL}(3, \mathbb{Z})$:

$$
\begin{align*}
    g_t &= \begin{pmatrix}
        e^{2t} & 0 & 0 \\
        0 & e^{-t} & 0 \\
        0 & 0 & e^{-t}
    \end{pmatrix}, \\
    u_s &= \begin{pmatrix}
        1 & s & 0 \\
        0 & 1 & 0 \\
        0 & 0 & 1
    \end{pmatrix}
\end{align*}
$$

- Mahler’s compactness criterion: a subset $K$ of unimodular lattices inside $G/\Gamma$ is bounded iff for all lattices $\Lambda \in K$, $\Lambda \cap B_{\varepsilon}(0) = \{0\}$ for some $\varepsilon > 0$.

$$
\begin{align*}
    g_t u_s \begin{pmatrix}
        0 \\
        0 \\
        1
    \end{pmatrix} &= \begin{pmatrix}
        0 \\
        0 \\
        e^{-t}
    \end{pmatrix} \xrightarrow{t \to \infty} 0
\end{align*}
$$
A uniform first order contraction hypothesis is not possible!

A higher order form of the contraction hypothesis can be established:

\[ \int_0^1 f(g(t) \Phi(r)x) \, dr \leq af(x) + b \]

for some \( 0 < a < 1 \) and \( b > 0 \) and \( \Phi \) a certain Taylor polynomial for the curve \( \varphi \).
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A higher order form of the contraction hypothesis can be established:

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\int_{0}^{1} f(g_t \Phi(r)x) \, dr \leq af(x) + b
\]

for some \(0 < a < 1\) and \(b > 0\) and \(\Phi\) a certain Taylor polynomial for the curve \(\varphi\).
Higher order contraction

- \( G = SL(m + n, \mathbb{R}) \), \( \Gamma = SL(m + n, \mathbb{Z}) \) and \( X = G/\Gamma \).
- \( Y \in M_{m,n}, (r, s) = (r_1, \ldots, r_m, s_1, \ldots, s_n) \in \mathbb{R}_+^n \) with \( \sum r_i = 1 = \sum s_j \):

\[
u(Y) = \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix}, \quad g_t^{r,s} = \text{diag}(e^{r_1 t}, \ldots, e^{r_m t}, e^{-s_1 t}, \ldots, e^{-s_n t})
\]

**Theorem (K. 18)**

Suppose \( \varphi : [0, 1] \to M_{m,n} \) is a strongly non-planar curve and \((r, s)\) is any weight with \( \sum r_i = 1 = \sum s_j \). Then, for all \( x \in X \),

\[
\sup_{t>0} \int_0^1 f \left( g_t^{r,s} u(\varphi(s))x \right) \, ds < \infty
\]

Moreover, the supremum can be taken to be uniform as \( x \) varies in compact subsets of \( X \).
This implies very well approximable points have measure 0. (Kleinbock-Margulis 1998, Kleinbock-Margulis-Wang 2010).

The approach uses the $(C, \alpha)$-good theory of polynomials only.
Thanks!